Extrapolation of scattering data to the negative-energy region. III. Application to the p-^{16}O system

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The problem of analytic continuation of the scattering data to the negative-energy region to obtain information on asymptotic normalization coefficients (ANCs) of bound states is discussed. It is shown that a recently suggested Δ method [O. L. Ramírez Suárez and J.-M. Sparenberg, Phys. Rev. C **96**, 034601 (2017)] is not strictly correct in the mathematical sense since it is not an analytic continuation of a partial-wave scattering amplitude to the region of negative energies. However, it can be used for practical purposes for sufficiently large charges and masses of colliding particles. Both the Δ method and the standard method of continuing of the effective range function are applied to the p-^{16}O system, which is of interest for nuclear astrophysics. The ANCs for the ground $5/2^+$ and excited $1/2^+$ states of ^{17}F are determined.

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I. INTRODUCTION

Using scattering data may give valuable information on the features of bound states, particularly on asymptotic normalization coefficients (ANCs), which, in contrast to binding energies, cannot be directly measured. The ANCs are fundamental nuclear characteristics that are important, for example, for evaluating cross sections of peripheral astrophysical nuclear reactions [1–4]. One of the direct ways to extract ANCs from experimental data is the analytic continuation in the energy plane of the partial-wave elastic scattering amplitudes, obtained by the phase-shift analysis, to the pole corresponding to a bound state. Such a procedure, in contrast to the method of constructing optical potentials fitted to scattering data, allows one to circumvent an ambiguity problem associated with the existence of phase-equivalent potentials [5,6].

The conventional procedure for such extrapolation is the analytic approximation of the experimental values of the effective-range function (ERF) $K_l(E)$ with the subsequent continuation to the pole position (l is the orbital angular momentum). The ERF method has been successfully employed to determine the ANCs for bound (as well as resonant) nuclear states in a number of works (see, e.g., [7–9] and references therein).

The ERF is expressed in terms of scattering phase shifts. In the case of charged particles, the ERF for the short-range interaction should be modified. Such modification generates additional terms in the ERF. These terms depend only on the Coulomb interaction and may far exceed, in the absolute value, the informative part of the ERF containing the phase shifts. This fact may hamper the practical procedure of the analytic continuation and affect its accuracy. It was suggested in Ref. [10] to use for the analytic continuation the quantity $\Delta_l(E)$ (which is defined below in Sec. II) rather than the ERF $K_l(E)$. The $\Delta_l(E)$ function does not contain the pure Coulomb terms. We call the continuation method, which uses the Δ function, the Δ method. In [11] this method is called the reduced ERF method.

Note that the validity of employing $\Delta_l(E)$ was not obvious, which resulted in some discussion. The authors of Refs. [12,13] claimed that they proved the mathematical correctness of the Δ method. However, this assertion contradicts the results of Refs. [11,14].

In the present work, we consider the question of the validity and applicability of the Δ method. It is shown that the Δ method in the strict mathematical sense is not an analytic continuation of a partial-wave scattering amplitude to the region of negative energies; however, it can be used for practical purposes for sufficiently large charges and masses of colliding particles. Then both ERF and Δ methods are employed to analyze the p-^{16}O system and determine the ANCs for ground and excited states of ^{17}F in the p-^{16}O channel. Note that the knowledge of these ANCs is important for evaluating the astrophysical S factor of the $^{16}\mathrm{O}(p,\gamma)^{17}\mathrm{F}$ reaction, which is one of the processes of the CNO cycle of nucleosynthesis in stars [15]. The analysis is based on using the experimental phase shifts with corresponding experimental errors. It is demonstrated here that the extrapolation of the elastic scattering data to the bound state poles provides a practical method to determine the ANCs. The ANCs, which are determined by the extrapolation of the elastic scattering data to the bound state poles, can be called experimental ANCs because they are obtained from experimental data.

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The paper is organized as follows. Section II contains the general formalism of the elastic scattering for the superposition of a short-range and the Coulomb interactions which is necessary for the subsequent discussion. The validity and applicability of the Δ method is discussed in Sec. III. Experimental p-^{16}O phase shifts are used to determine the ANCs for ^{17}F in Sec. IV. Throughout the paper we use the system of units in which $\hbar = c = 1$.

II. BASIC FORMALISM

In this section, we recapitulate basic equations which are necessary for the subsequent discussion. The Coulomb-nuclear amplitude of the elastic scattering of particles 1 and 2 is given by

$$f_{NC}(\mathbf{k}) = \sum_{l=0}^{\infty}(2l+1)\exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}P_l(\cos\theta). \tag{1}$$

Here $\mathbf{k}$ is the relative momentum of particles 1 and 2, θ is the center of mass (c.m.) scattering angle, $\sigma_l = \arg\Gamma(l+1+i\eta)$ and δ_l are the pure Coulomb and Coulomb-nuclear phase shifts, respectively, and $\Gamma(z)$ is the gamma function. The Coulomb parameter for the 1+2 scattering state is given as

$$\eta = Z_1 Z_2 e^2 \mu/k, \tag{2}$$

where the relative momentum k is related to the relative energy of these particles E by $k = \sqrt{2\mu E}$, $\mu = m_1 m_2/(m_1+m_2)$, and m_i and $Z_i e$ are the mass and the electric charge of particle i, $i = 1, 2$.

The behavior of the Coulomb-nuclear partial-wave amplitude $f_l = [\exp(2i\delta_l)-1]/2ik$ is irregular near $E = 0$. Therefore, one can introduce renormalized Coulomb-nuclear partial-wave amplitude $\tilde{f}_l$ [16–18] according to

$$\tilde{f}_l = \exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}\left[\frac{l!}{\Gamma(l+1+i\eta)}\right]^2 e^{\pi\eta}. \tag{3}$$

Equation (3) can be rewritten as

$$\tilde{f}_l = \frac{\exp(2i\delta_l)-1}{2ik}C_l^{-2}(\eta), \tag{4}$$

where $C_l(\eta)$ is the Coulomb penetrability factor (or Gamow factor) determined by

$$C_l(\eta) = \left[\frac{2\pi\eta}{\exp(2\pi\eta)-1}v_l(\eta)\right]^{1/2}, \tag{5}$$

$$v_l(\eta) = \prod_{n=1}^{l}(1+\eta^2/n^2)\ (l>0), \quad v_0(\eta) = 1. \tag{6}$$

It was shown in Ref. [16] that the analytic properties of $\tilde{f}_l$ on the physical sheet of E are analogous to the ones of the partial-wave scattering amplitude for the short-range potential, and it can be analytically continued into the negative energy region.

The amplitude $\tilde{f}_l$ can be expressed in terms of the Coulomb-modified ERF $K_l(E)$ [16,18] by

$$\tilde{f}_l = \frac{k^{2l}}{K_l(E) - 2\eta k^{2l+1}h(\eta)v_l(\eta)} \tag{7}$$

$$= \frac{k^{2l}}{k^{2l+1}C_l^2(\eta)(\cot\delta_l - i)} \tag{8}$$

$$= \frac{k^{2l}}{v_l^2 k^{2l}\Delta_l(E) - ik^{2l+1}C_l^2(\eta)}, \tag{9}$$

where

$$K_l(E) = k^{2l+1}\left[C_l^2(\eta)(\cot\delta_l - i) + 2\eta h(k)v_l(\eta)\right], \tag{10}$$

$$h(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta), \tag{11}$$

$$\Delta_l(E) = kC_0^2(\eta)\cot\delta_l, \tag{12}$$

and $\psi(x)$ is the digamma function. $\Delta_l(E)$ is the Δ function introduced in [10]. It was shown in [16] that function $K_l(E)$ defined by Eq. (10) is analytic near $E = 0$ and can be expanded into a Taylor series in E. In the absence of the Coulomb interaction ($\eta = 0$), $K_l(E) = k^{2l+1}\cot\delta_l(k)$.

If the $1+2$ system has a bound state $3 = (1\,2)$ with the binding energy $\varepsilon > 0$ in the partial wave l, then the amplitude $\tilde{f}_l$ has a pole at $E = -\varepsilon$. The residue of $\tilde{f}_l$ at this point is expressed in terms of the ANC $C^{(l)}_{3\to 1+2}$ [17] as

$$\operatorname{res}\tilde{f}_l(E)|_{E=-\varepsilon} = \lim_{E\to-\varepsilon}[(E+\varepsilon)\tilde{f}_l(E)] \tag{13}$$

$$= -\frac{1}{2\mu}\left[\frac{l!}{\Gamma(l+1+\eta_b)}\right]^2\left[C^{(l)}_{3\to 1+2}\right]^2, \tag{14}$$

where $\eta_b = Z_1 Z_2 e^2\mu/\kappa$ is the Coulomb parameter for the bound state 3 and κ is the bound-state wave number.

III. ON THE VALIDITY AND APPLICABILITY OF THE Δ METHOD

In this section, we discuss general properties of the Δ method suggested in Ref. [10]. Within this method, one uses for the analytic continuation the quantity $\Delta_l(E)$ given by Eq. (12) rather than the ERF $K_l(E)$ of Eq. (10). The reasons for introducing the Δ method are outlined above in the introduction. However, the validity of employing $\Delta_l(E)$ was not obvious since $\Delta_l(E)$, in contrast to $K_l(E)$, possesses an essential singularity at $E = 0$.

For brevity, the subsequent formulas in this section are written for the s-wave case and index $l = 0$ is omitted. Nevertheless, all reasonings are valid for arbitrary l.

Consider the partial-wave amplitude $\tilde{f}$. We write

$$\tilde{f}(E) = \frac{1}{D(E)}, \tag{15}$$

where

$$D(E) = kC^2(\eta)\cot\delta - ikC^2(\eta) \equiv \Delta(E) - ig(E). \tag{16}$$

If the Coulomb interaction is switched off, then

$$C^2(\eta) = 1, \quad D(E) = k\cot\delta - ik. \tag{17}$$

Denote $E = E_+$ if $E > 0$ and $E = E_-$ if $E < 0$.

Note that $ig(E_+)$ is purely imaginary. At $E = 0$ the latter has the essential and square-root singularities. On the other hand, $ig(E_-)$ is complex. Also, Im $\Delta(E_+) = 0$ and, at $E = 0$, $\Delta(E)$ possesses the essential singularity. For E_- the imaginary parts of $\Delta(E_-)$ and $ig(E_-)$ cancel each other and the essential singularity in Eq. (15) is canceled as well. As a result, Im $D(E_-) = 0$.

It should be emphasized that $D(E_+)$ and $D(E_-)$ are different parts of the same analytic function. The analytic continuation of $D(E)$ from E_+ to E_- implies, as in the case of neutral particles [see Eq. (17)], that the whole function $D(E_+) = \Delta(E_+) - ig(E_+)$ should be continued rather than only $\Delta(E_+)$. Note that in the Δ method $\Delta(E_+)$ is approximated by polynomials or rational functions in E and then continued to E_- where the approximated $\Delta(E_-)$ is equated to the whole denominator $D(E_-)$ and the position of the pole of $\tilde{f}(E)$ corresponding to a bound state is determined by the condition $\Delta(E_-) = 0$.

Obviously, such a procedure cannot be regarded as mathematically correct. In particular, it does not reproduce the square-root singularity (the normal threshold) of $\tilde{f}(E)$ at $E = 0$. The analytic continuation of $\Delta(E_-)$ thus obtained back to E_+ results in a wrong equation, Im $\tilde{f}(E_+) = 0$.

We note, however, that in the case of a purely short-range interaction Im $D(E_+)$ decreases as $\sqrt{E}$ at $E \to 0$, and in the presence of a repulsive Coulomb potential it decreases exponentially:

$$\text{Im } D(E_+)|_{E\to 0} \sim e^{-\gamma/\sqrt{E}}, \quad \gamma = \pi\sqrt{2\mu}Z_1Z_2e^2. \tag{18}$$

And not only Im $D(E_+)$ but all its derivatives tend to zero at $E \to +0$, which is distinct from the case of neutral particles scattering. Hence, in the presence of the Coulomb interaction there is a range of values of E in the vicinity of $E = 0$ in which one can neglect Im $D(E_+)$ and consider that $D(E_+) \approx \Delta(E_+)$. Within this range $D(E_+)$ can be approximated by a polynomial or a rational function of E and then continued to E_-. The size of this range can be qualitatively determined by the condition

$$|E| \ll \gamma^2. \tag{19}$$

The problem of the validity and applicability of the Δ method was discussed in Refs. [11,14,19]. It was stated in Refs. [11,14] that the Δ method can be employed to obtain information on bound states if their energy and the energy of scattering states used to approximate the Δ function satisfy the condition

$$|E| \leqslant (Z_1Z_2e^2)^2\mu/2. \tag{20}$$

As is noted in [11], the right-hand side of (20) is just the nuclear Rydberg energy: 1 Ry $= (Z_1Z_2e^2)^2\mu/2$. For systems $d + \alpha$ and $\alpha + {}^{12}$C considered in [14], 1 Ry = 0.13 MeV and 10.7 MeV, respectively. These values clearly illustrate the conclusion made in [14] that the Δ method is quite appropriate for $\alpha + {}^{12}$C but fails for $d + \alpha$ due to a very narrow range of allowed energy values.

Inference. In the strict mathematical sense the Δ method is not an analytic continuation of the denominator of the amplitude $\tilde{f}$ from the region $E > 0$ to the region $E < 0$, but it can still be used for practical purposes for sufficiently large charges and masses of colliding particles. The assertion about a strict mathematical proof of the correctness of the Δ method [12] is incorrect. This inference agrees with the results obtained in [11,14].

The Δ method was used in [13] to obtain ANCs for resonant nuclear states. In this regard, we would like to note that no special methods are needed for this purpose. Both the ERF and Δ methods were introduced to overcome the problem of the Coulomb singularity at $E = 0$. However, the Coulomb-nuclear scattering amplitude does not possess the Coulomb singularity in the vicinity of resonances. Hence, one can simply continue analytically $\cot\delta_l$ from the real positive half-axis of E to the resonance pole.

TABLE I. The experimental proton ANCs C_0 for the excited state and C_2 for the ground state of ^{17}F.

C_0 (fm$^{-1/2}$)	C_2 (fm$^{-1/2}$)	Reference
75.5 ± 15	1.1 ± 0.33	[21]
81 ± 26	1.1 ± 0.10	[15]
73.0	1.0	[22]
77.21	0.91	[23]
79.3 ± 3.9		[24]

IV. The p-^{16}O SYSTEM

Consider the p-^{16}O system. For this system $m_1 = m_p = 938.272$ MeV, $m_2 = m_{^{16}\text{O}} = 14\,895.079$ MeV, $Z_1Z_2 = 8$. ^{17}F nucleus has two bound states: the ground state $5/2^+$ ($l = 2$) and the excited state ^{17}F*(0.4953 MeV; $1/2^+$), $l = 0$. The binding energies ε of ^{17}F(ground) and ^{17}F*(0.4953 MeV) in the p-^{16}O channel are 0.6005 MeV and 0.1052 MeV, respectively [20].

In this section we present the proton ANCs of ^{17}F for the first excited state and for the ground state obtained by extrapolation of the ERF and Δ functions to the bound state poles of ^{17}F. They should be compared with the experimental proton ANCs C_0 for the virtual decay $^{17}\text{F} \to {}^{16}\text{O}(2s_{1/2^+}) + p$ and C_2 for the virtual decay $^{17}\text{F} \to {}^{16}\text{O}(1d_{5/2^+}) + p$ shown in Table I. These ANCs are obtained from analyses of the astrophysical $S_{1\,16}$ factors [21], the peripheral proton transfer reactions populating the ground and excited states of ^{17}F [15,22], and the radiative capture $^{16}\text{O}(p,\gamma)^{17}\text{F}$ reaction [23]. The table also shows C_0 determined from fitting the effective field theory (EFT) S factor to the experimental one [24]. Similar results for C_0 and C_2 were also obtained in Ref. [25]. Below we explore the extrapolation of the elastic scattering data to the bound states of ^{17}F to obtain the proton's ANCs of its excited and ground states. We demonstrate that the method of the extrapolation of the elastic scattering data to the negative energy region considered here can serve as another very useful practical method to extract the ANCs from the experimental data.

The proton ANCs of ^{17}F were also calculated using various theoretical approaches; see, for example, [26,27]. In particular, the results of microscopic calculations [26] are as follows: $C_0 = 91.14$ fm$^{-1/2}$, $C_2 = 0.97$ fm$^{-1/2}$ for the V2

TABLE II. ANC C_0 for the excited state of ^{17}F.

N	ERF method		Δ method	
	C_0 (fm$^{-1/2}$)	χ^2	C_0 (fm$^{-1/2}$)	χ^2
1	121.65596	0.070	54.06743	0.7911
2	101.86426	0.061	89.13841	0.0789
3	101.86559	0.065	89.13140	0.0846
4	101.86559	0.070	89.13140	0.0911
5	101.86559	0.076	89.13140	0.0986

TABLE III. ANC C_2 for the ground state of ^{17}F.

N	ERF method		Δ method	
	C_2 (fm$^{-1/2}$)	χ^2	C_2 (fm$^{-1/2}$)	χ^2
1	0.71537	0.16	0.52260	0.36
2	0.87884	0.18	2.35850	0.19
3	0.87881	0.20	2.33879	0.22
4	0.87881	0.23	2.33876	0.26
5	0.87881	0.28	2.33876	0.31

potential and $C_0 = 86.42\,\text{fm}^{-1/2}$, $C_2 = 1.10\,\text{fm}^{-1/2}$ for the MN potential.

According to [11,14], the larger the charges and masses of colliding particles are, the less the error associated with the use of the Δ method. The numerical parameter, which characterizes the accuracy of the Δ method, is the value of the Rydberg energy of the given system (see Sec. III above). For the p-^{16}O system 1 Ry = 1.503 58 MeV. This value is between the values 0.13 MeV and 10.7 MeV corresponding to the Rydberg energies for the $d + \alpha$ and $\alpha + {}^{12}$C systems, respectively. Recall that the Δ method turned out to be quite successful for $\alpha + {}^{12}$C but failed for $d + \alpha$ [14].

The ANC is obtained by analytic approximation of the ERF and Δ function by polynomials in E and the subsequent analytic continuation of these polynomials to the negative energy region. The coefficients of the polynomials are determined by the χ^2 method using the experimental phase shifts for p-^{16}O elastic scattering. To ensure the correct experimental position of a bound-state pole, the values of ERF and Δ function at $E = -\varepsilon$ are added as fitting parameters to their values at positive energies: $K_l(E)|_{E=-\varepsilon} = 2\eta k^{2l+1} h(\eta) v_l(\eta)|_{E=-\varepsilon}$, $\Delta_l(E)|_{E=-\varepsilon} = 0$.

To employ the χ^2 criterion, the errors equal to $\pm 1^\circ$ are applied to phase shifts $\delta_l(E)$. If $\delta_l + 1^\circ$ exceeds 180°, the value $179.999\,999\,99^\circ$ is used instead of $\delta_l + 1^\circ$. We use Eqs. (13) and (14) to find the ANCs.

A. ANC for the excited state of ^{17}F

We begin with the analysis of the $1/2^+$ state of the p-^{16}O system ($l = 0$). For this state, we use the results of the latest phase shift analysis obtained in Ref. [28], in which 16 values of δ_0 in the range of $E = 0.3628$–1.8738 MeV are presented. First, let us consider the approximation of the ERF $K_0(E)$. Our calculations are presented in the second and third columns of Table II. In this table, as well as in the following Table III, N denotes the power of the approximating polynomial. One sees that the obtained ANC C_0 is convergent with increasing N. Convergence is achieved already with $N = 3$. Hence we can consider the variant $N = 3$ as sufficient.

Figure 1 shows the polynomial approximation of the function $K_0(E)$. Note that the value of $K_0(E)$ at the origin is not zero but is very small and cannot be distinguished from zero in the scale of this figure.

Consider now the analytic continuation of the Δ function. Function $\Delta_0(E)$ is approximated by polynomials in the same way as for $K_0(E)$. The polynomial approximation of $\Delta_0(E)$ is shown in Fig. 2 and the results of the calculations are given in the fourth and fifth columns of Table II. It is seen that, similarly to the case of $K_0(E)$, the ANC C_0 converges rapidly with increasing N. Convergence is reached also with $N = 3$ and the result is $C_0 = 89.131\,40\,\text{fm}^{-1/2}$. This value does not deviate much from $C_0 = 101.865\,59\,\text{fm}^{-1/2}$ obtained using polynomial approximation of $K_0(E)$. The difference between these values can be related to the approximate nature of the Δ method. Note that the upper bound of the used energy interval ($E = 1.8738$ MeV) slightly exceeds the value 1 Ry = 1.503 58 MeV for the p-^{16}O system. As was mentioned in Sec. III, 1 Ry can be considered as an upper bound for employing the Δ method. Note that the extrapolated ANCs are in a reasonable agreement with the experimental ANCs from Table I.

B. ANC for the ground state of ^{17}F

Owing to the absence of more recent phase shift analyses of $p + {}^{16}$O scattering in the $5/2^+$ state, we use here the rather

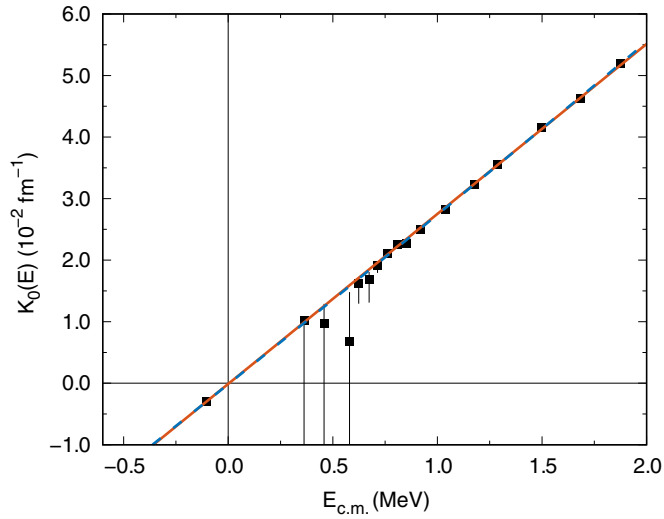

FIG. 1. Polynomial approximation of $K_0(E)$ for $p + {}^{16}$O scattering in the $J^\pi = 1/2^+$ state. Black squares with error bars are the results obtained from the experimental scattering phase shifts [28]. The solid red line is the polynomial approximation with $N = 1$, the dashed blue line is the polynomial approximation with $N = 2$. The lines corresponding to higher N are practically indistinguishable from the $N = 2$ line due to fast convergence. Therefore, they are not shown.

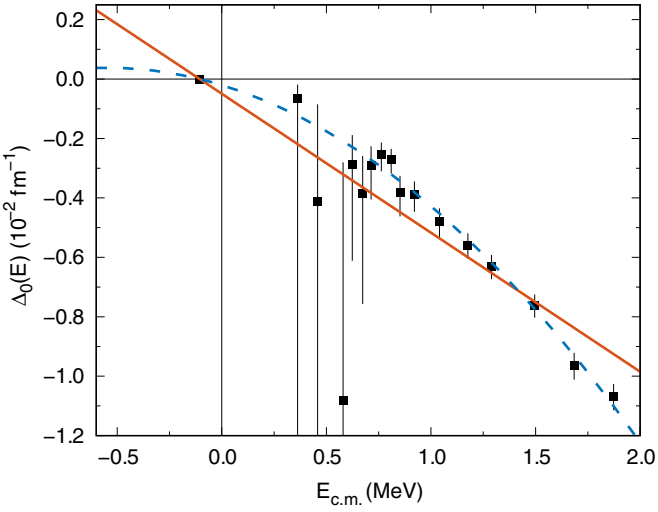

FIG. 2. Polynomial approximation of $\Delta_0(E)$ for $p + {}^{16}$O scattering in the $J^\pi = 1/2^+$ state. The notations are the same as in Fig. 1.

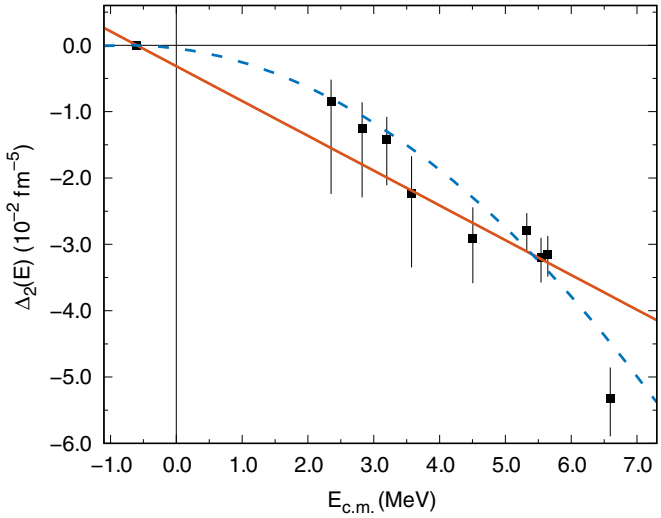

FIG. 4. Polynomial approximation of $\Delta_2(E)$ for $p + {}^{16}$O scattering in the state $J^\pi = 1/2^+$. The notations are the same as in Fig. 3.

old results of the phase shift analysis [29] in which nine values of $\delta_2(5/2)$ in the interval of $E = 2.35$–6.60 MeV were presented. The procedure is analogous to the one used for the excited state described above. The corresponding ANC is denoted by C_2. The results of the polynomial approximation of the ERF are shown in the second and third columns of Table III and in Fig. 3.

It is seen that, similar to the case of the excited state of ^{17}F, the ANC C_2 quickly converges with increasing N. The convergent result for ANC of $C_2 = 0.88$ fm$^{-1/2}$ is achieved with $N = 3$. Note that the ANC obtained using $K_0(E)$ polynomial approximation is close to the ANCs from Table I.

The results of the polynomial approximation of $\Delta_2(E)$ are shown in third and fourth columns of Table III and in Fig. 4.

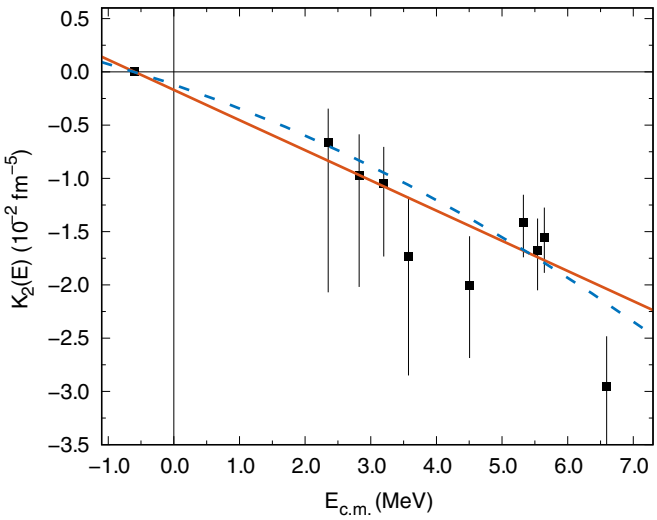

FIG. 3. Polynomial approximation of $K_2(E)$ for $p + {}^{16}$O scattering in the $J^\pi = 5/2^+$ state. Black squares with error bars are the results obtained from the experimental scattering phase shifts [29]. Other notations are the same as in Fig. 1.

Although the results appear to converge, they converge to an obviously wrong value. Most likely, this is due to the energy interval used for the approximation ($E = 2.35$–6.60 MeV) far exceeding the applicability limit of the Δ method for the p-^{16}O system of 1 Ry = 1.503 58 MeV as discussed above.

V. CONCLUSIONS

It is shown that the Δ method suggested in [10] is not strictly correct in the mathematical sense since it is not an analytic continuation of a partial-wave scattering amplitude to the region of negative energies. However, it can be used for practical purposes for sufficiently large charges and masses of colliding particles. It was demonstrated in the previous paper [14] that this method was effective for the α-^{12}C system ($Z_1Z_2 = 12$) but failed for the d-α system ($Z_1Z_2 = 2$). In the present work, both the ERF and Δ methods of analytic continuation of scattering data are applied to the p-^{16}O system ($Z_1Z_2 = 8$) which can be considered as intermediate between d-α and α-^{12}C systems. Both methods are used to determine the ANCs for the ground $5/2^+$ and excited $1/2^+$ states of ^{17}F nucleus in the p-^{16}O channel. Possible errors are added to experimental phase shifts.

The values of the ANC C_0 for the excited $1/2^+$ state of ^{17}F obtained in the present paper on the basis of the phase-shift analysis of Ref. [28] are 101.9 fm$^{-1/2}$ and 89.1 fm$^{-1/2}$ for the ERF and Δ methods, respectively. They are not much different from each other. Note that both ANCs are in reasonable agreement with the experimental ANCs; see Table I. The ANC C_2 for the ground state $5/2^+$ extracted using the phase-shift analysis of [29] is 0.88 fm$^{-1/2}$ for the ERF method and 2.34 fm$^{-1/2}$ for the Δ method. The value 0.88 fm$^{-1/2}$ is close to the experimental ANCs; see Table I. The polynomial approximation of $\Delta_2(E)$ for p-^{16}O scattering in the state $J^\pi = 5/2^+$ leads to the ANC $C_2 = 2.34$ fm$^{-1/2}$, which is significantly higher than the range of this ANC available in the literature and should be considered as erroneous. Such a large discrepancy between the results of the ERF and Δ methods

most likely is due to the fact that the energy interval used for the polynomial approximation of $\Delta_2(E)$ function far exceeds the limit of the applicability of the Δ method.

Summarizing, in this paper we demonstrated that the polynomial extrapolation of the ERF and Δ functions with the preset experimental binding energy gives converging and very reliable results for the proton ANCs of the ground and first excited states of ^{17}F. We presented a practical tool for experimentalists to determine the ANCs from the measured elastic scattering phase shifts. In nuclear astrophysics one needs to know the neutron ANCs. However, it is difficult to accurately measure the neutron elastic scattering phase shifts. Using the methods described here one can determine the proton ANCs from the proton elastic scattering phase shifts and then using the mirror symmetry determine the neutron ANCs of the mirror nuclei [30,31]. The same method can be used to determine the α-particle ANC on an unstable nucleus if the mirror α-particle ANC on a stable nucleus can be determined using elastic scattering data. Another very promising application of the extrapolation method addressed here is the effective field theory. In the EFT the elastic scattering data are analyzed at positive energies and parametrized in terms of the EFT parameters [24,32]. These parameters can be related to the EFR ones and can be used to extrapolate the elastic scattering phase shifts to bound state poles to determine the ANCs [24].

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[1] A. M. Mukhamedzhanov and N. K. Timofeyuk, Yad. Fiz. **51**, 679 (1990) [Sov. J. Nucl. Phys. **51**, 431 (1990)].
[2] H. M. Xu, C. A. Gagliardi, R. E. Tribble, A. M. Mukhamedzhanov, and N. K. Timofeyuk, Phys. Rev. Lett. **73**, 2027 (1994).
[3] A. M. Mukhamedzhanov and R. E. Tribble, Phys. Rev. C **59**, 3418 (1999).
[4] R. E. Tribble, C. A. Bertulani, M. La Cognata, A. M. Mukhamedzhanov, and C. Spitaleri, Rep. Prog. Phys. **77**, 106901 (2014).
[5] L. D. Blokhintsev and V. O. Eremenko, Phys. At. Nucl. **71**, 1219 (2008).
[6] L. Blokhintsev, Y. Orlov, and D. Savin, *Analytic and Diagram Methods in Nuclear Reaction Theory* (Nova Science, New York, 2017).
[7] L. D. Blokhintsev, V. I. Kukulin, A. A. Sakharuk, D. A. Savin, and E. V. Kuznetsova, Phys. Rev. C **48**, 2390 (1993).
[8] J.-M. Sparenberg, P. Capel, and D. Baye, Phys. Rev. C **81**, 011601 (2010).
[9] B. F. Irgaziev and Yu. V. Orlov, Phys. Rev. C **91**, 024002 (2015).
[10] O. L. Ramírez Suárez and J.-M. Sparenberg, Phys. Rev. C **96**, 034601 (2017).
[11] D. Gaspard and J.-M. Sparenberg, Phys. Rev. C **97**, 044003 (2018)
[12] Yu. V. Orlov, B. F. Irgaziev, and Jameel-Un Nabi, Phys. Rev. C **96**, 025809 (2017).
[13] B. F. Irgaziev and Yu. V. Orlov, Phys. Rev. C **98**, 015803 (2018).
[14] L. D. Blokhintsev, A. S. Kadyrov, A. M. Mukhamedzhanov, and D. A. Savin, Phys. Rev. C **97**, 024602 (2018).
[15] C. A. Gagliardi, R. E. Tribble, A. Azhari, H. L. Clark, Y.-W. Lui, A. M. Mukhamedzhanov *et al.*, Phys. Rev. C **59**, 1149 (1999).
[16] J. Hamilton, I. Øverbö, and B. Tromborg, Nucl. Phys. B **60**, 443 (1973).
[17] L. D. Blokhintsev, A. M. Mukhamedzhanov, and A. N. Safronov, Fiz. Elem. Chastits At. Yadra **15**, 1296 (1984) [Sov. J. Part. Nucl. **15**, 580 (1984)].
[18] S. König, Effective quantum theories with short- and long-range forces, Dissertation, Universität Bonn, August 2013 (unpublished).
[19] L. D. Blokhintsev, A. S. Kadyrov, A. M. Mukhamedzhanov, and D. A. Savin, Phys. Rev. C **95**, 044618 (2017).
[20] D. R. Tilley, H. R. Weller, and C. M. Cheves, Nucl. Phys. A **564**, 1 (1993).
[21] S. V. Artemov, S. B. Igamov, Q. I. Tursunmakhatov, and R. Yamukhamedov, Bull. RAN. Ser. Phys. [Izv. RAN. Ser. Fiz.] **73**, 176 (2009).
[22] S. V. Artemov *et al.*, Yad. Fiz. **59**, 454 (1996) [Phys. At. Nucl. **59**, 428 (1996)].
[23] J. T. Huang, C. A. Bertulani, and V. Guimarães, At. Data Nucl. Data Tables **96**, 824 (2010).
[24] E. Ryberg, C. Forssén, H.-W. Hammer, and L. Platter, Ann. Phys. **367**, 13 (2016).
[25] R. Yarmukhamedov and D. Baye, Phys. Rev. C **84**, 024603 (2011).
[26] D. Baye, P. Descouvemont, and M. Hesse, Phys. Rev. C **58**, 545 (1998).
[27] J. Okolowicz, N. Michel, W. Nazarewicz, and M. Ploszajczak, Phys. Rev. C **85**, 064320 (2012).
[28] S. Dubovichenko, N. Burtebayev, A. Dzhazairov-Kakhramanov, D. Zazulin, Z. Kerimkulov *et al.*, Chin. Phys. C **41**, 014001 (2017).
[29] R. A. Blue and W. Haeberli, Phys. Rev. **137**, B284 (1965).
[30] N. K. Timofeyuk, R. C. Johnson, and A. M. Mukhamedzhanov, Phys. Rev. Lett. **91**, 232501 (2003).
[31] A. M. Mukhamedzhanov, Phys. Rev. C **86**, 044615 (2012).
[32] S. K. Bogner, T. T. S. Kuo, and A. Schwenk, Phys. Rep. **386**, 1 (2003).