Symmetric nuclear matter calculations: A variational approach

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We have studied nuclear matter with a new M3Y-type effective nucleon-nucleon interaction based on the lowest order constrained variational approach (LOCV) as the principal investigating tool. The chosen interaction, called B3Y-Fetal, has been used in its DDM3Y1, BDM3Y0, BDM3Y1, BDM3Y2, and BDM3Y3 density-dependent versions to reproduce the saturation properties of cold nuclear matter at the saturation density, $\rho = 0.17\ \mathrm{fm}^{-3}$. Such properties as the binding energy per nucleon, also called the equation of state (EOS), and the incompressibility of symmetric nuclear matter have been computed, with results proving to be acceptably in agreement with previous work done with the M3Y-Paris and M3Y-Reid interactions by other researchers. Insightful information on symmetric nuclear matter obtained from the folding analysis in this work has shown it to be possibly governed by a soft EOS.

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I. INTRODUCTION

Effective interactions are a computational necessity for calculations involving finite nuclei and nuclear matter. They occupy an important place in nuclear physics, their fundamental role being the explanation of the properties of atomic nuclei in terms of the basic interactions between nucleons [1]. Their derivation has resulted from the inability of free nucleon-nucleon (NN) potentials to describe nuclei consisting of a large number of nucleons, where many-body effects cannot be neglected. Therefore, effective NN interactions, developed to include the effects of complicated many-body correlations, have been found more profitable for nuclear matter and finite nuclei studies [2,3].

A very useful feature of the effective interactions is that analytical expressions for many interesting quantities in both symmetric and asymmetric nuclear matter are contained in them; so, when used in mean-field studies, they are usually and generally adjusted to various properties of nuclear matter and finite nuclei [1]. They have been used successfully for describing the properties of nuclei near the valley of stability as well as the properties of exotic nuclei with large neutron or proton excess [4].

Of all the effective interactions, the finite-range M3Y interaction and its various density-dependent upgrades are known to have become the most versatile, working well in many nuclear models to produce very reliable results. The derivation of this interaction was pioneered by Bertsch and his co-workers at the Michigan State University by obtaining the G-matrix elements of the Reid potential [5] in an oscillator basis and fitting these matrix elements to a sum of three Yukawa terms, popularly referred to as "Michigan three Yukawa" (M3Y) interaction [6,7]. In their G-matrix-based work, this effective interaction, composed of the central, tensor, and spin-orbit components, has been shown to give unambiguous predictions of inelastic scattering [6]. Since its derivation, the M3Y effective interaction has become the most versatile and most popularly used potential in nuclear matter calculations and in elastic and inelastic reactions. Although different versions of the M3Y effective interaction have come into use at one time or another, the most popular versions presently in use are M3Y-Reid [6] based on the G matrix of the Reid potential [5] and M3Y-Paris [8] based on the G matrix of the Paris potential.

Following the work of Bertsch *et al.* [6], a similarly motivated potential was derived based on the lowest order constrained variational (LOCV) principle by Fiase *et al.* [9], who investigated the mass dependence of the M3Y-type effective interactions about fifteen years ago. They restricted their study strictly to the mass dependence of the M3Y-type effective interactions and the effects of tensor correlations, and their results were similar to those of [6] in most reaction channels. Their work covered the nuclear systems $A = 16$, 24, 40, and 90. In this study, an M3Y-type effective interaction for a nuclear system with mass number $A = 16$ is chosen for the study of nuclear matter properties. The M3Y-type effective interaction of choice is called B3Y-Fetal [1] in this work, with the letter B in the label B3Y representing Botswana, where the effective interaction was developed, while "Fetal" in the label B3Y-Fetal represents Fiase *et al.* [9], who derived it.

It is common knowledge that the saturation of density and energy is a basic property of nuclei. Therefore, in developing effective interactions, the basic requirement is to reproduce the saturation properties of nuclear matter. Symmetric nuclear matter, considered an important testing ground and source of invention of new tools with which to treat the quantitative relationship between the two-body forces and nuclear properties [10], is characterized by a saturation density $\rho_0 = 0.166 \pm 0.018\ \mathrm{fm}^{-3}$ and an energy per particle of 16 ± 1 MeV given by the bulk term of the well-known Bethe-Weizsacker mass formula [11]. The reproduction of these values together with the compression modulus has long been a fundamental and meaningful test for all effective interactions and techniques for many-body problems to pass in order to be used for a successful prediction of nuclear properties of finite nuclei. But the original density-independent M3Y interaction is known to have failed to give the saturation of nuclear matter within Hartree-Fock (HF) calculations of nuclear matter [12–14]. It was, therefore, thought wise to modify it by introducing a realistic density dependence into it to enable it describe the known nuclear matter properties. Consequently, in the last three decades, different density-dependent versions of the M3Y-type effective interaction have been successfully used in HF calculations of symmetric and asymmetric nuclear matter [4,12,15–18] in the mean-field studies of nuclear ground states as well as in numerous folding model studies of nucleon-nucleus and nucleus-nucleus scattering [17,19,20]. The various density dependences are the DDM3Y, BDM3Y, and CDM3Y, out of which the CDM3Y is the latest version [20].

Our intended purpose in this work is to pass the B3Y-Fetal effective interaction through the test for viability of all effective interactions, the reproduction of the well-known saturation properties of symmetric nuclear matter, using the DDM3Y and BDM3Y density-dependent versions. In this regard, this work is, among other things, meant to establish the position of the new effective interaction in relation to other theoretical models. We used the zero-range pseudopotential approximation in our earlier paper [21] and found the results to be in good agreement with theoretical and experimental standards. For this reason, we are determined to use the full-exchange potential in this study. We are hopeful that the emanating results will give a definite form and character of this new effective interaction. The success of this work will pave the way for the application of the new effective interaction in asymmetric nuclear matter and folding analyses. As it is, the B3Y-Fetal is the principal probing effective interaction, but it is used alongside M3Y-Reid and M3Y-Paris, which are meant to serve as standards to compare it with. With this in mind, we organize the paper in the following manner. Section II gives a succinct overview of the procedure for the derivation of B3Y-Fetal in addition to a discussion of the M3Y density-dependent versions used in this study. The computation of the nuclear matter equation of state using the full-exchange potential is undertaken in Sec. III. Section IV is devoted to presentation of results and discussion, and in Sec. V we make concluding remarks.

II. THE B3Y-FETAL EFFECTIVE INTERACTION

The matrix elements of the two-body effective interaction leading to the B3Y-Fetal interaction were calculated in a harmonic oscillator basis using the lowest-order constrained variational (LOCV) method. The details of the calculation were reported in [9,22], where the matrix elements have been shown to be of the form

$$E_2' = \langle\Phi| \sum_{i>j} f(ij) V_{ij} f(ij) |\Phi\rangle, \tag{1}$$

where $\langle\Phi|$ represents the two-body (harmonic oscillator) wave function and $f(ij)$ are the correlation operators which are meant to take care of the effect of the strong repulsion of the nucleon-nucleon interaction, making the matrix elements finite at short internucleon distances, and V_{ij} is the Reid soft-core potential [5]. The LOCV method, while very popular some decades ago, is now well known to be very approximate; however, the success of the method parallels the G-matrix approach or any other sophisticated approach to the microscopic study of effective interactions. We had shown [22] in an earlier LOCV calculation that our matrix elements are quite similar to G-matrix calculations if the tensor force is allowed to operate in its proper angular momentum channels. However, there is no method which suffers no limitations and approximation. In our LOCV method, we had made a cluster expansion of the expectation value of the Hamiltonian into two-body, E_2, three-body, E_3, and higher-order energy terms.

To lowest order, we minimized the two-body energy term E_2 with respect to the functional variations of the two-body correlation functions such that only two-body cluster energy terms were important. We did this by choosing a convergence parameter, K which was required to be $K \ll 1$. This was achieved in our calculations. However, without calculating the three-body cluster terms explicitly, our LOCV results were open to the doubt that perhaps higher order cluster effects might destroy this encouraging similarity with G-matrix calcualtions. Irvine [23], who first introduced this method, had investigated the convergence of the LOCV calculations by calculating the three-body cluster term E_3, and had found this to be negligibly small.

Now, the effective nucleon-nucleon interaction suitable for calculations involving nuclear matter and finite nuclei has been defined in [9] to have a central (V_C), a spin-orbit (V_{LS}), and a tensor (V_T) component expressed as

$$\begin{aligned} V_C &= \sum_k V_k Y\left(\frac{r_{ij}}{R_k}\right), \\ V_{LS} &= \sum_k V_k Y\left(\frac{r_{ij}}{R_k}\right) \boldsymbol{L}\cdot\boldsymbol{S}, \\ V_T &= \sum_k V_k r_{ij}^2 Y\left(\frac{r_{ij}}{R_k}\right) \boldsymbol{S}_{ij}, \end{aligned} \tag{2}$$

where $Y(\frac{r_{ij}}{R_k})$ is a Yukawa potential function of the form [6,9]

$$Y\left(\frac{r_{ij}}{R_k}\right) = \frac{\exp\left(-\frac{r_{ij}}{R_k}\right)}{\left(\frac{r_{ij}}{R_k}\right)}. \tag{3}$$

V_k in Eq. (2) are the strengths of the interaction to be determined by fitting the two-body matrix elements of Eq. (1) to those of the sum of Yukawa functions with different ranges. R_k are the ranges, which are chosen to be 0.25 and 0.40; 0.40 and 0.70; and 0.25, 0.40, and 1.414 fm for the spin-orbit, tensor, and central components, respectively. r_{ij} is the separation between the ith and jth nucleons. The tensor operator $\boldsymbol{S}_{ij}$ is [24]

$$\boldsymbol{S}_{ij} = 3(\boldsymbol{\sigma}_i . r_{ij})(\boldsymbol{\sigma}_j . r_{ij}) - \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j \tag{4}$$

with $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$ representing the Pauli spin matrices. The spin-orbit operator $\boldsymbol{L}\cdot\boldsymbol{S}$ has an expectation value proportional to [10,25]

$$2\langle \boldsymbol{L}\cdot\boldsymbol{S}\rangle = j(j+1) - l(l+1) - s(s+1), \tag{5}$$

where $L = \sqrt{l(l+1)}$, $S = \sqrt{s(s+1)}$, and $J = \sqrt{j(j+1)}$.

The strengths of the central component of the effective interaction, V_k were determined and separated into various angular momenta channels; namely, the singlet even (SE), singlet odd (SO), triplet even (TE), and triplet odd (TO) channels. The M3Y-type effective interaction arising from these strengths of the central component is B3Y-Fetal, and its radial shape is expressed in terms of three Yukawa functions as [6,26]:

$$v_{00}^{D(\mathrm{EX})}(r) = \sum_{k=1}^{3} Y_{00}^{D(\mathrm{EX})}(k)\frac{\exp(-\mu_k r)}{\mu_k r}, \tag{6}$$

where $\mu_k = 1/R_k$ and the functions $Y_{00}^{D(\mathrm{EX})}$ are represented in terms of SE, TE, SO, and TO channels as [6,26]

$$\begin{aligned} Y_{00}^{D} &= \tfrac{1}{16}[3t^{(\mathrm{SE})} + 3t^{(\mathrm{TE})} + 1t^{(\mathrm{SO})} + 9t^{(\mathrm{TO})}],\\ Y_{00}^{\mathrm{EX}} &= \tfrac{1}{16}[3t^{(\mathrm{SE})} + 3t^{(\mathrm{TE})} - 1t^{(\mathrm{SO})} - 9t^{(\mathrm{TO})}], \end{aligned} \tag{7}$$

where t_k are the strengths of the interaction in SE, TE, SO, and TO channels. The interaction strengths used for constructing B3Y-Fetal were taken from Table V of the work of Fiase *et al.* [9].

In general, the direct (v_D) and exchange (v_{EX}) components of the central part of the M3Y NN effective interaction, in terms of spin $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ and isospin $\boldsymbol{\tau}, \boldsymbol{\tau}'$ of the nucleons, are expressed as [19]

$$\begin{aligned} v^{D(\mathrm{EX})}(r) = {}& v_{00}^{D(\mathrm{EX})}(r) + v_{10}^{D(\mathrm{EX})}(r)(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}') + v_{01}^{D(\mathrm{EX})}(r)(\boldsymbol{\tau}\cdot\boldsymbol{\tau}')\\ & + v_{11}^{D(\mathrm{EX})}(r)(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')(\boldsymbol{\tau}\cdot\boldsymbol{\tau}'), \end{aligned} \tag{8}$$

where r is the internucleon distance, ρ is the nuclear density around the interacting nucleon pair, and $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ are the spins and $\boldsymbol{\tau}, \boldsymbol{\tau}'$ are the isospins of the two nucleons participating in the interaction. In this work, the dominant contribution to the study of cold symmetric nuclear matter, which is spin-saturated, is $v_{00}^{D(\mathrm{EX})}(r)$.

Thus, the radial strengths (in MeV) of the B3Y-Fetal effective interaction are given in terms of three Yukawa potentials as [9]

$$\begin{aligned} v_{00}^{D}(r) &= \frac{10472.13e^{-4r}}{4r} - \frac{2203.11e^{-2.5r}}{2.5r},\\ v_{00}^{\mathrm{EX}}(r) &= \frac{499.63e^{-4r}}{4r} - \frac{1347.77e^{-2.5r}}{2.5r} - \frac{7.8474e^{-0.7072r}}{0.7072r}. \end{aligned} \tag{9}$$

Since it is intended in this work to compare the results of our calculation with previous work done with the famous M3Y-Reid and M3Y-Paris interactions, their explicit radial forms are shown in Eqs. (10) and (11) respectively.

M3Y-Reid [6,27]:

$$\begin{aligned} v_{00}^{D}(r) &= \frac{7999.00e^{-4r}}{4r} - \frac{2134.25e^{-2.5r}}{2.5r},\\ v_{00}^{\mathrm{EX}}(r) &= \frac{4631.375e^{-4r}}{4r} - \frac{1787.125e^{-2.5r}}{2.5r} - \frac{7.8474e^{-0.7072r}}{0.7072r}. \end{aligned} \tag{10}$$

M3Y-Paris [8,27]:

$$\begin{aligned} v_{00}^{D}(r) &= \frac{11061.625e^{-4r}}{4r} - \frac{2537.5e^{-2.5r}}{2.5r},\\ v_{00}^{\mathrm{EX}}(r) &= \frac{-1524.25e^{-4r}}{4r} - \frac{518.75e^{-2.5r}}{2.5r} - \frac{7.8474e^{-0.7072r}}{0.7072r}. \end{aligned} \tag{11}$$

For a correct description of the saturation properties of nuclear matter within the non-relativistic HF scheme, it has been shown [12,16] that the introduction of a density dependence into the original M3Y interaction is a necessary and sufficient solution. This is because even the most sophisticated G-matrix calculations with the inclusion of two- and three-nucleon correlations are unable to describe simultaneously the equilibrium density and binding energy of normal nuclear matter. The inclusion of higher-order correlations as well as relativistic effects is shown to improve this situation. One approach to solving this difficulty is to derive an effective interaction as is done in our LOCV approach. The higher-order correlation effects are then parametrized in terms of density dependence and included in the effective interaction in order to obtain a good description of normal nuclear matter. Therefore, the density-dependent M3Y-type effective interaction becomes

$$v_{00}^{D(\mathrm{EX})}(\rho, r) = F_0(\rho)v_{00}^{D(\mathrm{EX})}(r), \tag{12}$$

where $F_0(\rho)$ is the density-dependent factor. In the present work, the explicit forms of the density dependences used are [12,17,27]

$$\begin{aligned} F_0(\rho) &= C(1 + \alpha e^{-\beta\rho}), \qquad \text{DDM3Yn},\\ F_0(\rho) &= C(1 - \alpha\rho^{\beta}), \qquad \text{BDM3Yn}. \end{aligned} \tag{13}$$

For the DDM3Yn effective interaction, $n = 1$, whereas $n = 0, 1, 2, 3$ for the BDM3Yn effective interaction; and the parameters C, α, and β of the density dependences are adjusted to reproduce the saturation properties of nuclear matter at density $\rho_0 = 0.17\ \mathrm{fm}^{-3}$ with a binding energy $E/A = 16$ MeV within the HF calculations. This class of density-dependent M3Y effective interactions has continued to be applied, with great success, to numerous nuclear reactions including folding analysis of nucleon-nucleus, nucleus-nucleus, and charge-exchange reactions [12,27,28].

The DDM3Yn effective interaction was used first by Kobos and his co-researchers [12] to reproduce the density and energy dependence of the microscopic nucleon optical potential obtained by Jeukenne, Lejeune, and Mahaux (JLM) [29] with parameters that were dependent on the nucleon incident energy, which was positive [12]. Having realized that the same parameters were not suitable for calculations involving nucleons embedded in the nuclear matter with negative energies ($k \leqslant k_F$), Khoa and Oertzen, [12] adjusted the parameters to get a reasonable limit for negative energies, leading to the attainment of the saturation condition at the equilibrium density $\rho_0 = 0.07\ \mathrm{fm}^{-3}$ and an equilibrium binding energy $E/A \simeq 15.9$ MeV; but the associated incompressibility K of 129.2 MeV indicated a very soft equation of state (EOS) compared with the standard soft and hard EOS having K

values of 200 and 380 MeV respectively. Therefore, they readjusted the parameters to reproduce the empirical values of the saturation binding energy ($E/A \simeq 16$ MeV) and density ($\rho_0 = 0.17\ \mathrm{fm}^{-3}$) respectively. The density-dependent M3Y interaction arising from the readjustment was dubbed DDM3Y1, and it gave the incompressibilities $K = 170$ and 176 MeV with the M3Y-Reid and M3Y-Paris effective interactions respectively. The exponential nature of the density dependence, however, made a further readjustment of the parameters for a higher value of K impossible. The only way to have a harder EOS was to try to use some other form of density dependence. Consequently, they tried using the BDM3Yn interaction.

The BDM3Yn interaction was first introduced and used by Myers in single-folding calculations [12]. The density dependence is known to change the sign of the interaction at high densities, making it crucially important in fulfilling the saturation condition as well as giving different K values for the nuclear EOS. With a β value of $2/3$ used by Myers originally, the authors in [12] fitted the other two parameters to the saturation condition as for DDM3Y1, dubbing the new density dependent interaction BDM3Y0 and obtaining a harder EOS with an incompressibility of 190 MeV based on the M3Y-Reid effective interaction. In their quest for better and harder equations of state, they went further to use integer values of β ranging from 1 to 3, obtaining the BDM3Y1, BDM3Y2, and BDM3Y3 interactions with higher incompressibility values. These interactions were subsequently used for folding calculations of heavy-ion (HI) optical potentials with insightful results.

The choice of the DDM3Yn and BDM3Yn effective interactions in this work is inspired by the approach of Khoa and co-workers [12,16,17,19]. To be able to compare the performance of B3Y-Fetal with that of the M3Y-Reid and M3Y-Paris effective interactions, nuclear matter calculations with the DDM3Y1, BDM3Y0, BDM3Y1, BDM3Y2, and BDM3Y3 versions are carried out in this paper following their approach.

III. THE NUCLEAR MATTER EQUATION OF STATE (EOS)

Understanding the nuclear matter EOS through microscopic calculations that utilize a model of the nuclear force duly incorporating low-energy two-nucleon scattering data and properties of light nuclei [11,30] has always been a challenging theoretical problem. Therefore, we attempt to understand and solve the nuclear matter EOS herein using the density-dependent B3Y-Fetal effective interaction whose derivation has resulted from microscopic calculations. We have first chosen the symmetric case as our starting point because most nuclear matter calculations usually start with the EOS for this normal nuclear matter and, afterwards, extrapolate the theory developed mainly for it to nuclear matter with extremely high isospin and high densities.

Symmetric nuclear matter is considered in this study as a Fermion system enclosed in an infinite volume Ω, having Z protons $=$ N neutrons [7,26], an infinite mass number $A = N + Z$, a finite density $\rho = A/\Omega$, and a total ground-state energy E at the absolute zero temperature expressed as the sum of a kinetic energy part and a potential energy part [18,31]. The mathematical representation of this energy in Hartree-Fock (HF) approximation is [32]

$$\begin{aligned} E &= \sum_i \langle i| - \frac{\hbar^2}{2m}\nabla^2|i\rangle + \frac{1}{2}\sum_{i\neq j}\sum_j \langle ij|v|ij\rangle \\ &= E_{\mathrm{kin}} + E_{\mathrm{pot}}, \end{aligned} \tag{14}$$

where v is a generalized two-body effective NN interaction. The factor $1/2$ in the total potential energy is meant to avoid double counting of the two-body mutual interactions. Now, a good description of the nuclear matter properties begins with the evaluation of the nuclear matter EOS, also called energy per particle [33], whose mathematical expression follows from Eqs. (12) and (14) as [12]

$$\frac{E}{A}(\rho) = \frac{3\hbar^2 k_F^2}{10m} + F_0(\rho)\frac{\rho}{2}\left(J_0^D + \int [\hat{j}_1(k_F r)]^2 v_{00}^{\mathrm{EX}}(r) d^3r\right), \tag{15}$$

where m is the bare nucleon mass, $F_0(\rho)$ is the density dependence of the isoscalar component of the density-dependent interactions, J_0^D is the volume integral of the direct part of the effective interaction, and $\hat{j}_1(x) = 3j_1(x)/x$, with $j_n(x)$ the nth-order spherical Bessel function.

The equilibrium density of nuclear matter is determined from the saturation condition:

$$\begin{aligned} &\frac{\delta}{\delta\rho}\left(\frac{E}{A}\right)\bigg|_{\rho=\rho_0} = 0 \\ &\quad = \frac{\hbar^2 k_F^2}{5m}\rho + \frac{J_0^D}{2}A_0(\rho) + \frac{1}{2}\int v_{00}^{\mathrm{EX}}(r)[A_0(\rho)[\hat{j}_1(k_F r)]^2 \\ &\qquad - 2F_0(\rho)\hat{j}_1(k_F r)j_2(k_F r)]d^3r\Big|_{\rho=\rho_0} = 0, \end{aligned} \tag{16}$$

where $A_0(\rho) = \rho\frac{dF_0(\rho)}{d\rho} + F_0(\rho)$.

The pressure of symmetric nuclear matter, expressed as [26]

$$P_0(\rho) = \rho^2\frac{\delta}{\delta\rho}\left(\frac{E}{A}(\rho)\right), \tag{17}$$

must be zero at saturation.

With the indicated operation properly carried out,

$$\begin{aligned} P_0(\rho) = \frac{\hbar^2 k_F^2}{5m} &+ A_0(\rho)\rho^2\int v_{00}^{\mathrm{EX}}(r)[\hat{j}_1(k_F r)] \\ &\times [A_0(\rho)[\hat{j}_1(k_F r)] - 2F_0(\rho)j_2(k_F r)]d^3r, \end{aligned} \tag{18}$$

where $A_0(\rho)$ is as defined in Eq. (16).

The incompressibility, or compression modulus of symmetric nuclear matter is a measure of the curvature of the nuclear EOS at saturation. It is of special interest in this study because it characterizes the stiffness of nuclear EOS in a definite manner; and plays a major role in the study of properties of nuclei, supernovae collapse, neutron stars and heavy-ion collisions. It is defined as [12]:

$$K_0(\rho) = 9\rho^2 \frac{\delta^2}{\delta\rho^2}\left(\frac{E}{A}(\rho)\right)\Big|_{\rho=\rho_0}$$

$$= \frac{3\hbar^2 k_F^2}{5m} + B_0(\rho)J_0^D + \int v_{00}^{\rm EX}(r)[B_0(\rho)[\hat{j}_1(k_F r)]^2 - C_0(\rho)\hat{j}_1(k_F r)j_2(k_F r) + 9\rho F_0(\rho)([j_2(k_F r)]^2 + j_1(k_F r)j_3(k_F r))]d^3r|_{\rho=\rho_0}, \tag{19}$$

where $B_0(\rho) = 4.5\rho^3 \frac{d^2F_0(\rho)}{d\rho^2} + 9\rho^2 \frac{dF_0(\rho)}{d\rho}$ and $C_0(\rho) = 18\rho^2 \frac{dF_0(\rho)}{d\rho} + 15\rho F_0(\rho)$.

The DDM3Y1, BDM3Y0, BDM3Y1, BDM3Y2, and BDM3Y3 interactions used for nuclear matter calculations in this study have corresponding K values, based on the M3Y-Paris, M3Y-Reid, and B3Y-Fetal effective interactions, which distinguish different nuclear equations of state in nuclear reactions. This has been suggested by Khoa and collaborators [12,16,17] as a useful way of obtaining vital information on the nuclear matter EOS. To this end, a step is taken further in this work to try out the new effective interaction in the refractive scattering of the $^{16}O + ^{16}O$ nuclear system at an incident energy of 145 MeV so as to observe the connection of EOS with the associated K values. The success of this will, additionally, help us ascertain the viability of the new interaction for optical model analyses of nuclear reactions involving different nuclei. The double folding procedure developed by Khoa *et al.* [16,17] is used for this purpose. To do this successfully, the density-dependent effective interaction in Eq. (12) has an energy dependence introduced into it in the form

$$v_{00}^{D({\rm EX})}(E,\rho,r) = g(E)F_0(\rho)v_{00}^{D({\rm EX})}(r). \tag{20}$$

The inclusion of the energy dependence is to enable the reproduction of the empirical energy dependence of the nucleon optical potential [27]. For simplicity, the energy-dependent interaction is assumed here to be the original M3Y-type effective interaction multiplied by an energy-dependent factor $g(E)$, where E is the incident nucleon energy, and $g(E) \simeq 1-0.002E$ for M3Y-Reid interaction [12] as well as B3Y-Fetal [21]. Seeing that the dominant focus of this paper is nuclear matter calculations, the details of the folding procedure, contained in [16,17,19,20,34], are not included herein; but we report the results of the computation in this work.

TABLE I. Parameters of density dependence and nuclear incompressibility at equilibrium for M3Y-Paris. The results obtained in Refs. [16,27] are in brackets.

Density dependent version	C	α	β	K (MeV)
DDM3Y1-Paris	0.2963	3.7231	3.7384	176
	(0.2963)	(3.7231)	3.7384	(176)
BDM3Y0-Paris	1.4366	1.2624	2/3	221
	(1.4366)	(1.2627)	2/3	(218)
BDM3Y1-Paris	1.2511	1.7445	1	270
	(1.2521)	(1.7452)	1	(270)
BDM3Y2-Paris	1.0656	6.0295	2	418
	(1.0664)	(6.0296)	2	418
BDM3Y3-Paris	1.0037	25.112	3	566
	(1.0045)	(25.115)	3	(566)

TABLE II. Parameters of density dependence and nuclear incompressibility at equilibrium for M3Y-Reid. The results obtained in Refs. [16,27] are in parentheses.

Density-dependent version	C	α	β	K (MeV)
DDM3Y1-Reid	0.2816	3.6693	2.9605	171
	(0.2816)	(3.6693)	2.9605	(171)
BDM3Y0-Reid	1.3817	1.1132	2/3	191
	(1.3817)	(1.1132)	2/3	(190)
BDM3Y1-Reid	1.2244	1.5118	1	232
	(1.2253)	(1.5124)	1	(232)
BDM3Y2-Reid	1.0670	5.1067	2	353
	(1.0678)	(5.1069)	2	(353)
BDM3Y3-Reid	1.0146	21.070	3	475
	(1.0153)	(21.073)	3	(475)

IV. RESULTS AND DISCUSSION

In order to firmly establish the performance of the B3Y-Fetal effective interaction, the saturation properties of symmetric nuclear matter (NM) were computed with the M3Y-Paris and M3Y-Reid effective interactions first. When the results obtained with these interactions in their DDM3Y1, BDM3Y0, BDM3Y1, BDM3Y2, and BDM3Y3 density-dependent versions were compared with the results of [16,27] and found to be exact, the B3Y-Fetal effective interaction was substituted for them in the same computational procedure. The results obtained from the numerical calculation, shown in Tables I–III and Figs. 1–3, have clearly revealed the B3Y-Fetal interaction to be in good agreement with the M3Y-Reid and M3Y-Paris interactions.

Figure 1 presents the curves of the EOS of cold symmetric nuclear matter obtained with the DDB3Y1-, BDB3Y0-, BDB3Y1-, BDB3Y2-, and BDB3Y3-Fetal interactions. Each of these interactions is shown to have reproduced correctly the saturation of nuclear matter at density $\rho = 0.17$ fm^{-3} and binding energy per nucleon $\frac{E}{A} = 16$ MeV. Thus, all the curves have the same shape and value at the saturation point, demonstrating good agreement with the work of [12]. As nuclear density increases, the differences in the performances of the interactions become increasingly clear, with DDB3Y1-Fetal being the weakest and BDB3Y3-Fetal being the strongest. The

TABLE III. Parameters of density dependence and nuclear incompressibility at equilibrium for B3Y-Fetal.

Density-dependent version	C	α	β	K (MeV)
DDB3Y1-Fetal	0.2986	3.1757	2.9605	176
BDB3Y0-Fetal	1.3045	1.0810	2/3	196
BDB3Y1-Fetal	1.1603	1.4626	1	235
BDB3Y2-Fetal	1.0160	4.9169	2	351
BDB3Y3-Fetal	0.9680	20.250	3	467

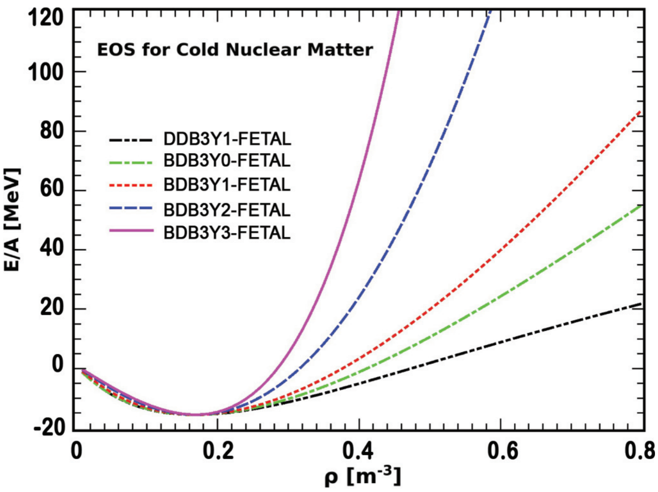

FIG. 1. Equations of state of cold NM calculated with DDB3Y1-, BDB3Y0-, BDB3Y1-, BDB3Y2-, and BDB3Y3-Fetal interactions.

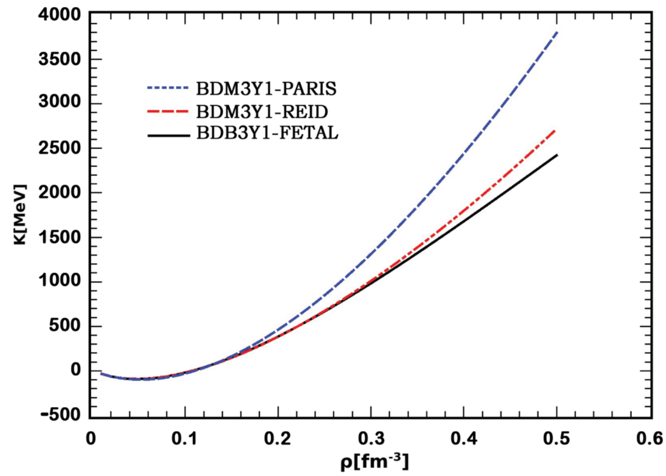

FIG. 3. Nuclear incompressibilities of cold NM calculated with BDB3Y1- Fetal, BDM3Y1-Reid, and BDM3Y1-Paris interactions.

curves also show that the EOS described by the BDB3Y-Fetal effective interaction—whose values are shown in Table III—becomes harder with increasing β; this is, again, in agreement with [12,16,27].

With regard to nuclear matter incompressibility, the results obtained in this work with the M3Y-Paris and M3Y-Reid effective interactions, shown in Tables I and II, respectively, have been observed to be exact when compared with those obtained by Khoa and co-researchers [16,17,27], giving the certainty and confidence that the results obtained with B3Y-Fetal in Table III through the same computational procedure are acceptably accurate. With the very soft DDM3Y1 density-dependent version, the performance gaps displayed by the three effective interactions in Fig. 2 are essentially marginal up to high nuclear matter densities. However, with the BDM3Y1 density-dependent version, it is obvious from the results that the M3Y-Paris interaction predicts a stiffer EOS than the B3Y-Fetal and M3Y-Reid interactions, which are seen in Fig. 3 to exhibit overlapping performances in which B3Y-Fetal is stronger at low nuclear densities and weaker than M3Y-Reid at high nuclear densities.

Altogether, the DDM3Y1 and BDM3Y1 versions of the three density-dependent effective interactions used in the calculation have given for symmetric nuclear matter (SNM) at equilibrium incompressibility $K_0 \simeq 171$–270 MeV, from which B3Y-Fetal predicts $K_0 \simeq 176$–235 MeV; M3Y-Reid gives $K_0 \simeq 171$–232 MeV, and M3Y-Paris predicts $K_0 \simeq 176$–270 MeV. When this is compared with the experimental estimate of $K_0 = 240 \pm 20$ MeV from giant monopole resonances [15,35] and the theoretical estimates of $K_0 = 220 \pm 50$ [12] MeV and $K_0 \simeq 250$–270 MeV, based on nonrelativistic and relativistic mean-field models respectively [15], for SNM, it is evidently clear that the prediction of the B3Y-Fetal interaction is in good agreement.

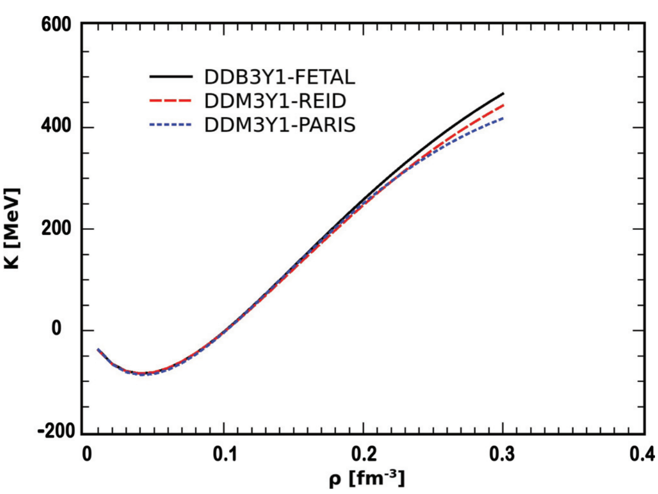

FIG. 2. Nuclear incompressibilities of cold NM calculated with DDB3Y1-Fetal, DDM3Y1-Reid, and DDM3Y1-Paris interactions.

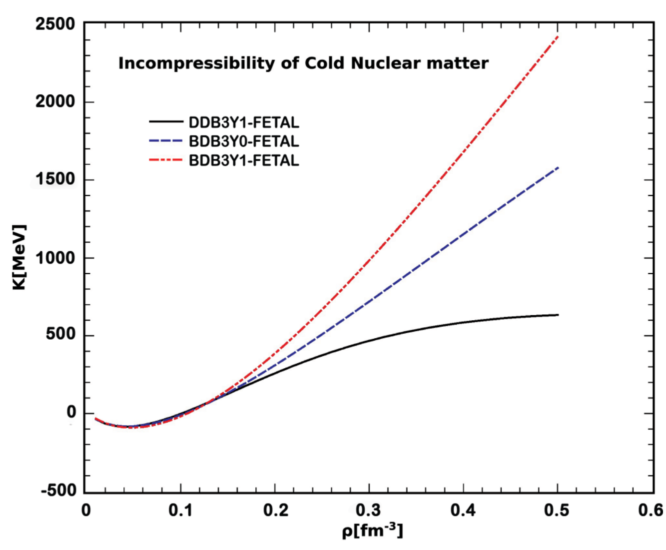

FIG. 4. Nuclear incompressibilities of cold NM calculated with DDB3Y1-Fetal, BDB3Y0-Fetal, and BDB3Y1-Fetal interactions.

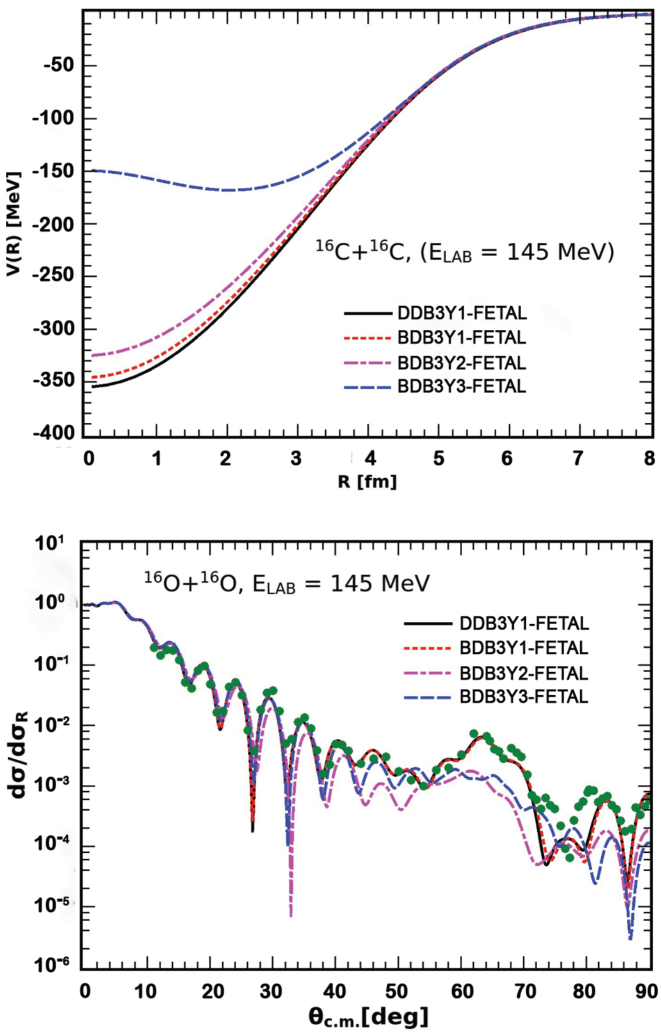

FIG. 5. Fits to the elastic data of $^{16}O + ^{16}O$ at $E_{\rm Lab} = 145$ MeV obtained with B3Y-Fetal-based (upper part) and M3Y-Reid-based (lower Part) optical potentials.

In Fig. 4, a comparative plot of the incompressibilities given by DDB3Y1-Fetal, BDB3Y0-Fetal, and BDB3Y1-Fetal shows the BDB3Y1-Fetal interaction to predict a stiffer EOS at all nuclear densities than DDB3Y1-Fetal, with the BDB3Y0-Fetal version in between them; this is in excellent agreement with the findings of [12,17,19] for their corresponding versions based on M3Y-Reid and M3Y-Paris effective interactions.

To have additional and insightful information on the EOS of symmetric nuclear matter (SNM), the results of the folding analysis of the nuclear reaction involving the $^{16}O + ^{16}O$ nuclear system at 145 MeV, using the density-dependent B3Y-Fetal interaction, are shown in Fig. 5, with the radial shapes of the real folded potentials derived from B3Y-Fetal in the upper region while the associated fits are in the lower region. The radial shapes show DDB3Y1-Fetal to be the most attractive, especially at smaller internuclear distances, whereas BDB3Y3-Fetal is very repulsive. It is also evident from Fig. 5 that the fits produced by DDB3Y1- and BDB3Y1-Fetal have the same shape and are the best in the sense that they demonstrate a better agreement with the elastic data of $^{16}O + ^{16}O$ than those produced by the BDB3Y2- and BDB3Y3-Fetal folded potentials. This observation is also true for the folded potentials derived from the M3Y-Reid (DDM3Y1-, BDM3Y1-, BDM3Y2-, and BDM3Y3-Reid potentials) effective interaction in [12]. When these results are brought to bear on the results of nuclear matter calculations shown in Tables I–III, a strong factual link for establishing the nature of the EOS of SNM using incompressibility values clearly evolves. Now, the incompressibilities of 351 and 467 MeV produced by BDB3Y2-Fetal and BDB3Y3-Fetal are known to represent stiff EOS of SNM [12,15]. These interactions have also been shown in Fig. 5 to give a very bad description of the elastic data of the $^{16}O + ^{16}O$ system whereas DDB3Y1- and BDB3Y1-Fetal, with the incompressibilities of 173 and 235 MeV respectively, are shown to be better and in good agreement. Since the well-known standard is that the theoretical equations of state which predict higher K_0 values of about 300 MeV are often called "stiff" whereas those which predict smaller K_0 values of about 200 MeV are said to be "soft" [12,15], the agreement demonstrated by DDB3Y1- and BDB3Y1-Fetal herein shows that cold nuclear matter possibly has an underlying soft EOS.

V. CONCLUSION

In this work, a new M3Y-type effective interaction derived from variational calculations, called B3Y-Fetal, has been applied in its various density-dependent forms, alongside the M3Y-Paris and M3Y-Reid effective interactions, to the study of symmetric nuclear matter, with results that have shown it to demonstrate good agreement with these well-known M3Y effective interactions. The prediction of $K_0 \simeq 176$–235 MeV by B3Y-Fetal has been found to tally well with the experimental standard where $K_0 = 240 \pm 20$ MeV [15,35] and theoretical estimates of $K_0 = 220 \pm 50$ MeV based on nonrelativistic calculations [16,27]. In addition, evidence available from a combination of folding analysis and nuclear matter calculations here shows the possibility that SNM has an underlying soft EOS in agreement with [16].

Finally, seeing that nuclear matter has remained a trusted testing ground for the viability of an effective interaction as well as the many-body technique or theory involved, and the results of our computation have proved B3Y-Fetal to be in excellent agreement with the M3Y-Reid and M3Y-Paris effective interactions, it is necessary to affirm that this work serves to validate B3Y-Fetal as an effective interaction that can be relied on for a correct explanation of nuclear matter and its associated phenomena. This forms a strong basis for future application of this effective interaction to asymmetric nuclear matter and nuclear reactions.

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