Manifestation of the important role of nuclear forces in the emission of photons in pion scattering off nuclei

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The bremsstrahlung of photons emitted during the scattering of π^+ mesons off nuclei is studied for the first time. The role of interactions between π^+ mesons and nuclei in the formation of the bremsstrahlung emission is analyzed in detail. We discover an essential contribution of photons emitted from nuclear part of the Johnson–Satchler potential to the full spectrum, in contrast with the optical Woods–Saxon potential. We observe an unusual essential influence of the nuclear part of both potentials on the spectrum at high photon energies. This phenomenon opens a new experimental way to study and check non-Coulomb and nuclear interactions between pions and nuclei via measurements of the emitted photons. We provide predictions of the bremsstrahlung spectra for pion scattering off ^{44}Ca.

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I. INTRODUCTION

The nature of the pion-nucleus interaction in the energy region from 0 to 1 GeV has been the subject of considerable theoretical and experimental investigations [1]. The motivation for such research is to use pions in nuclear reactions as a probe to obtain a deeper understanding of nuclear structure [2–4]. Particular attention has been paid to understanding the discrete and (low-lying) collective states of nuclei in elastic, inelastic, and quasielastic scattering, resonant scattering (with and without polarized nuclei), single- and double-charge-exchange reactions, knock-out reactions, and absorption reactions. Moreover, a lot of interest was also focused on understanding the pion-nucleon $\Delta_{3,3}$ ($J = \frac{3}{2}$, $T = \frac{3}{2}$) resonance and its role in the pion-nucleus interaction. The experimental measurements have been performed at the CERN Synchro-cyclotron [5], in the $M13$ pion channel using the quadrupole-quadrupole-dipole pion spectrometer at TRIUMF (Vancouver) [6] (for example, see a detailed description of the channel and the spectrometer in Refs. [7,8]), at the 7 GeV proton synchrotron NIMROD at Rutherford Laboratory [9], in the pion channel and spectrometer (EPICS) at the Clinton P. Anderson Meson Physics Facility (LAMPF, Los Alamos) [9–12], and at the pion spectrometer facility at the Swiss Institute for Nuclear Research (SIN, Switzerland) [13,14]. An experimental program (for a wide range of experiments) has been proposed at SIN to study the pion-nucleus interactions in a systematic way (see Ref. [15] for details). To date, we have experimental data of cross sections for ^{6,7}Li, ^{9}Be, ^{12}C, ^{16}O, ^{28}Si, ^{32,34}S, ^{40,42,44,48}Ca, ^{54}Fe, ^{90}Zr, ^{208}Pb, etc. at the incident pion energy T_π from 30 MeV up to 860 MeV [5,6,9–14,16].

The optical model [6,12,13,17–19], α-cluster model [20], DWIA collective model [10,11,13,16],and microscopical form-factor model [11] have been used to analyze (and predict in some cases) total- and differential-cross-section measurements. The parameters in pion-nucleus interactions in these models were determined by fitting procedures. The ground-state neutron density root-mean-square radii (for ^{40,42,44,48}Ca and ^{54}Fe) was estimated [16]. Neutron and proton multipole matrix elements for excited states (of ^{42,44,48}Ca; see Ref. [11]) have also been experimentally tested. Deformation lengths in inelastic scattering were extracted from experimental data [lengths for (2^+; 4.44 MeV) and (3^-; 9.64 MeV) states of ^{12}C [20], etc.]. These characteristics confirm the perspectives of the study of pion-nucleus interactions (reactions) when electromagnetic measurements do not exist.

Theoretically, in describing the pion-nucleus interaction, there are two prevailing approaches. The first was introduced by Satchler in Ref. [21] and developed within the frameworks of an optical model with the nonrelativistic Schrödinger equation and the phenomenological local potential of Woods–Saxon shape (see Ref. [18]). The second, which was introduced by Johnson and Satchler in Ref. [22] on the basis of the Klein–Gordon equation, uses a nonlocal Kisslinger-type potential at low and resonance energies. Via Krell–Ericson transformation, the latter formalism is modified to the optical model formalism with local potentials and transformed wave functions (see Refs. [17,19]). Note that the key point in an effective description of pion-nucleus interactions is the proper nuclear interaction, which is deeply investigated.

Bremsstrahlung emission of photons accompanying nuclear reactions has been attracting a lot of interest for a long time (see reviews [23,24] and books [25,26]). This is because

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photons provide independent information about the nuclear processes and can be used as an independent probe of nuclear interactions. However, the nature of emission of hard photons in the nucleus-nucleus collisions is an open question, because of the complicated many-nucleon interactions. Because of this, before attempts to resolve the fully many-nucleon problem of nuclear scattering with emission of photons, people go to simpler bremsstrahlung calculations with the proton chosen as one scattering nuclear object. However, even for the proton-nucleus scattering, a clear understanding of the influence of nuclear interactions on the spectrum of photons has not been obtained [27].

In this paper, we find that our calculated bremsstrahlung spectra are sufficiently sensitive to the shape of this type of the potential. Thus, we propose that the emitted bremsstrahlung photons can be a suitable tool to study the non-Coulomb interactions (i.e., nuclear forces, Coulomb corrections, etc.) in the scattering of the positively charged pions off nuclei. However, the bremsstrahlung emission by real pions scattered off nuclei has never been studied, either theoretically or experimentally. Note that emission of the coherent photons by virtual pions, which are produced in nuclear matter during the proton-nucleus scattering, was studied [28] in the Δ-resonance-energy region. However, in the formation of photons during this reaction, the Coulomb forces outside the nucleus-target (which give the largest contribution to the coherent and incoherent bremsstrahlung emission in the pion-nucleus scattering) are not taken into consideration.[1] In this paper we report our analysis of the question above. We find that nuclear interactions play an important role in the formation of bremsstrahlung spectra, which can be tested experimentally (at the high-energy region of photons) via further measurements of bremsstrahlung photons.

II. MODEL

In construction of our bremsstrahlung formalism (for proton-nucleus scattering and proton decay [29–31], see α decay [32–39], spontaneous fission [40], ternary fission [41]) we use the following logic: We begin the formalism with the Dirac equation for one nucleon. Then, we generalize this equation to describe many nucleons (we take radius vector of the center of mass of the full system as a summation of all radius vectors of nucleons). As the next step, we apply the approximation (known in QED for one fermion), which allows us to generalize the Pauli equation for many nucleons, as a first approximation (we obtain a logic of how to determine the next approximations; see pp. 33–35 in Ref. [42], which is reduced mass, the relation of such a formalism with relativistic classical mechanics with action; see pp. 35 and 36 in Ref. [42], etc.). In this step, we introduce vector potentials of the electromagnetic field (in the standard way of QED via gauge) which describe the emission of photons by each nucleon. This formalism should be transformed to the known Pauli equation after imposing the needed approximations. We formulate initially the Hamiltonian of the full many-nucleon nuclear system in the laboratory frame. We use space variables for each nucleon. On its basis, the full operator of emission of the evolving nuclear system is defined in the laboratory frame. Simply put, the relative acceleration between nucleons forms the emission of bremsstrahlung photons. Thus, we need relative distances between nucleons for analysis, and we rewrite the full Hamiltonian and full wave function via these relative distances. As result, we obtain a term in the operator of emission corresponding to motion of the full nuclear system in the laboratory frame (for example, see the first term in Eq. (8) in Ref. [31] for proton-nucleus scattering). In construction of the wave function of the full nuclear system, we separate the factor describing the motion of the full nuclear system [for example, see Eq. (10) in Ref. [31] for the proton-nucleus scattering]. And we separate the factor of relative motion, because we estimate that the relative motion between two nuclear fragments gives the largest contribution to the full bremsstrahlung spectrum (we knead in this factor in calculations).

In this paper we generalize this formalism for pion-nucleus scattering. For the first estimations of the bremsstrahlung photons emitted during the scattering of pions off nuclei, we put the main forces to determine a leading contribution of emission to the full spectrum. Such a term is based on relative motion of the nucleus and pion. We shall not study the internal structure of these two objects in this paper, because they should give smaller effects (but will be estimated as the next step in this research).[2]

The emission of the bremsstrahlung photons can be introduced to the formalism of $\pi^{\pm}$-nucleus scattering via the Coulomb gauge for each electromagnetic charge in the system as $\mathbf{p}_i \to \mathbf{p}_i - \frac{z_i e}{c}\mathbf{A}(\mathbf{r}_i, t)$, where $\mathbf{p}_i = -i\hbar\nabla_i$ is the momentum of the pion or nucleon with number i, $\mathbf{A}(\mathbf{r}_i, t)$ is the vector potential of the electromagnetic field formed by motion of pion or nucleon with number i, z_i is the electromagnetic charge of pion or nucleon with number i. The modified Hamiltonian is written as $\hat{H} = \hat{H}_0 + \hat{H}_\gamma$, where $\hat{H}_\gamma$ is a new operator describing the emission of photons. A leading part of the emission operator of the system composed of $\pi^{\pm}$ and a nucleus in the laboratory frame (neglecting terms at $\mathbf{A}_i^2$, $A_{i,0}$, and spinor terms; see Appendix A for a detailed derivation of the operator of emission of evolving system

[1]To obtain convergence in calculations of the matrix elements of the bremsstrahlung emission during scattering of the real pion off nucleus, we must include the space region of integration with an external boundary up to the atomic shells of the nucleus. From here, one can see the essential role of the Coulomb forces in estimating bremsstrahlung spectral. Reference [28] investigated the essentially different process of emission of photons without such a Coulomb contribution.

[2]We explained that the observed hump-shaped plateau in the experimental bremsstrahlung spectra in the proton-nucleus scattering in the intermediate- and high-energy regions, by not-minor incoherent emission of photons (formed by interactions between spinor properties of individual nucleons and their momenta) [31]. Such behavior is described as an addition to the leading contribution of the emitted coherent photons. Thus, in this paper, we neglect such a noncoherent effect (and, thus, the spinor properties of nucleons).

of nucleons and pion, and needed definitions) has the form

$$\hat{H}_\gamma = -e\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\exp\left\{-i\mathbf{k}\cdot\left[\mathbf{R}-\frac{m_\pi}{m_A+m_\pi}\mathbf{r}\right]\right\}\left\{\frac{1}{m_A+m_\pi}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}z_\pi+\sum_{j=1}^{A}z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{P}\right.$$
$$\left.+\left[e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{z_\pi}{m_\pi}-\frac{1}{m_A}\sum_{j=1}^{A}z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{p}+\sum_{j=1}^{A-1}\frac{z_{Aj}}{m_{Aj}}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\tilde{\mathbf{p}}_{Aj}-\frac{1}{m_A}\left[\sum_{j=1}^{A}z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\right\}, \quad (1)$$

where $\mathbf{P} = -i\hbar d/d\mathbf{R}$, $\mathbf{p} = -i\hbar d/d\mathbf{r}$, $\tilde{\mathbf{p}}_{Aj} = -i\hbar d/d\boldsymbol{\rho}_{Aj}$, $\mathbf{R}$ are coordinates of center of mass of complete nuclear system of the nucleus and pion, $\mathbf{r}$ is relative distance between center-of-mass of the nucleus and pion, $\boldsymbol{\rho}_{Aj}$ is relative distance between center-of-mass of the nucleus and nucleon with number j of this nucleus, $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\rho}_{Aj}$ are defined in Eq. (A4), star denotes complex conjugation, m_i and z_i are the mass and electromagnetic charge of nucleon with number i ($i = 1, \ldots, A$) or mass and electromagnetic charge of $\pi^\pm$-mesons ($i = A+1$), m_π and m_A are masses of $\pi^\pm$ and nucleus, $\mathbf{e}^{(\alpha)}$ are unit vectors of polarization of the photon emitted, $\mathbf{k}$ is the wave vector of the photon, $w_{\rm ph} = kc = |\mathbf{k}|c$. The diagrammatic representation of the emission of the bremsstrahlung photons in the pion-nucleus scattering is depicted in Fig. 1.

The Hamiltonian of the evolving system of the nucleus and $\pi^\pm$ mesons is $\hat{H}_0 = -\frac{\hbar^2}{2m}\triangle + V_N(\mathbf{r}) + V_C(\mathbf{r})$, where V_C is the Coulomb potential of interactions between the pion and nucleus, V_N is the potential of non-Coulomb interactions between the pion and nucleus, m is the reduced mass of the pion and nucleus [17]. The component V_N can be written in the local Kisslinger-type form [17,19] along the formalism of Johnson and Satchler [22] as (we add upper indices "*JS*" and "*WS*" to denote the type of formalism)

$$\begin{aligned} V_N^{(JS)}(\mathbf{r}) &= U_N(r) + \Delta U_C(r), \\ U_N(r) &= \frac{(\hbar c)^2}{2w}\left\{\frac{q(r)}{1-\alpha(r)} - \frac{k^2\alpha(r)}{1-\alpha(r)}\right. \\ &\quad \left. - \frac{\nabla^2\alpha(r)}{2[1-\alpha(r)]} - \left[\frac{\nabla\alpha(r)}{2[1-\alpha(r)]}\right]^2\right\}, \\ \Delta U_C(r) &= \frac{\alpha(r)V_C(r) - V_C^2(r)/2w}{1-\alpha(r)}, \end{aligned} \quad (2)$$

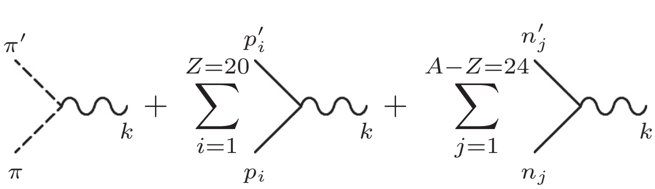

FIG. 1. Diagrammatic representation of the emission of the bremsstrahlung photons during the scattering of the π^+ meson off the ^{44}Ca nucleus. Dashed and solid lines refer to pions and nucleons, respectively. Wavy lines indicate the outgoing bremsstrahlung photons.

or in the local Woods–Saxon form along the optical model formalism [18] as[3]

$$\begin{aligned} V_N^{(WS)}(\mathbf{r}) &= -Uf(r) - iWg(r), \\ f(r) &= \{1 + \exp[(r - R_u)/a_u]\}^{-1}, \\ g(r) &= \{1 + \exp[(r - R_w)/a_w]\}^{-1}. \end{aligned} \quad (3)$$

Here, w is the total energy of the pion in the center-of-mass frame, $q(r)$ and $\alpha(r)$ are results of the s-wave part and p-wave part of pion-nucleon interactions (see Refs. [17,22] for details). U and W are strengths of the potential (3), R_u and R_w are radius-parameters of the potential (3), a_u and a_w are diffuseness (see Table 2 in Ref. [18]).

The leading matrix element of emission is (see Appendix B for its derivation and definition of the wave function of a pion-nucleons system)

$$\begin{aligned} \langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1 = &-\frac{e}{m}\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\langle\Phi_{\pi\text{-nucl,f}}(\mathbf{r}) \\ &\times|Z_{\rm eff}(\mathbf{k},\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}|\Phi_{\pi\text{-nucl,i}}(\mathbf{r})\rangle, \end{aligned} \quad (4)$$

where we introduce the *effective charge of the $\pi^\pm$-nucleus system* as

$$Z_{\rm eff}(\mathbf{k},\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\left\{\frac{m_A z_\pi}{m_A+m_\pi} - e^{i\mathbf{k}\cdot\mathbf{r}}\frac{m_\pi Z_{\rm A}(\mathbf{k})}{m_A+m_\pi}\right\}, \quad (5)$$

[3]In this paper we restrict ourselves by only the real part of the potential in calculations of the bremsstrahlung spectra. Motivations for such calculations are the following: The imaginary part of the potential is related to the internal (inelastic) mechanisms inside the nucleus, which are essentially more difficult to analyze. Inclusion of the inelastic mechanisms will give additional new questions in this tasks (for example, the role of such mechanisms in the scattering in dependence on energy of emitted photons; such mechanisms can be related with producing of incoherent emission, etc.). Thus, we want initially to construct proper basis for description of the bremsstrahlung in the elastic scattering. In the next step, it is sensible to estimate inelastic mechanisms by using the already-constructed basis for bremsstrahlung in the elastic scattering. We find that the not-insignificant role of the nuclear part of the real potential in determining the spectra at high energies of photons is an interesting new effect. The importance of studying this effect will be after adding the imaginary part of the potential to calculations.

the *charged form factor of the nucleus* as

$$Z_{\rm A}(\mathbf{k}) = \langle \psi_{\rm nucl,f}(\boldsymbol{\rho}_{A1}, \ldots, \boldsymbol{\rho}_{AA})| \sum_{j=1}^{A} z_{Aj} e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}} \times |\psi_{\rm nucl,i}(\boldsymbol{\rho}_{A1}, \ldots, \boldsymbol{\rho}_{AA})\rangle. \quad (6)$$

Here, Ψ_s is the wave function of the full pion-nuclear system, $s = i$ or f (indexes i and f denote the state before emission of photon and the state after emission of photon), $\Phi_{\pi\text{-nucl,s}}(\mathbf{r})$ is function describing relative motion (with tunneling) of the $\pi^{\pm}$ related to the nucleus (without description of internal relative motions of nucleons inside the nucleus), $\psi_{\rm nucl,s}(\beta)$ is the many-nucleon function describing internal states of nucleons in the nucleus (it determines space states on the basis of relative distances $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_A$ of nucleons of the nucleus concerned with its center-of-masses, and spin-isospin states also), and β_A is a set of numbers $1, \ldots, A$ of nucleons of the nucleus.

The simplest determination of the matrix element is obtained, if (i) to apply dipole approximation for the effective charge of the full system, and (ii) to neglect relative displacements of nucleons inside the nucleus in calculations of the form factor (see Appendix C for derivation of the matrix elements). At such an approximation, we define the cross section of the emitted photons in the laboratory frame in frameworks of the formalism given in Ref. [30] and we do not repeat it in this paper [see Eq. (49) in that paper]:

$$\frac{d^2\sigma(\theta_f)}{dw_{\rm ph} d\cos\theta_f} = \frac{e^2}{2\pi c^5} \frac{w_{\rm ph} E_i}{m^2 k_i} \left\{ p_{\rm el} \frac{dp_{\rm el}^{*}(\theta_f)}{d\cos\theta_f} + {\rm c.c.} \right\}, \quad (7)$$

where "c.c." means complex conjugation (see Appendix C for the derivation of the matrix elements). We calculate radial wave functions numerically concerned with the chosen potential of interaction between the $\pi^{\pm}$ and the spherically symmetric core [see Eqs. (2) for the nuclear potential and the corrections along the Johnson and Satchler formalism; see Eqs. (3) for the nuclear potential along the Woods–Saxon formalism].

III. DISCUSSIONS

We apply the approach presented above to estimate the spectra of the bremsstrahlung photons emitted during the scattering of the π^+ meson off nuclei. We begin with calculations of the bremsstrahlung photons where interactions between π^+ mesons and nuclei are defined in the optical model formalism with the Woods–Saxon nuclear potential. We estimate the influence of the nuclear part of the potential on the spectrum. Such calculations of the bremsstrahlung cross sections in scattering of π^+ mesons of the ^{44}Ca nuclei at the bombarding-meson energy $E_{\pi^+} = 116$ MeV are presented in Fig. 2(a). One can see clear difference between the spectra at high-energy region of the emitted photons caused by variations of strength U of the nuclear potential (3). This is a manifestation of the important role of nuclear interactions in the formation of photon emission. This phenomenon is more visible in the high-energy region.

We analyze how much the bremsstrahlung emission changes if the interaction between pions and nuclei are defined in the Johnson–Satchler formalism. Such calculations for the ^{44}Ca nucleus at the bombarding energy $E_{\pi^{\pm}} = 116$ MeV are presented in Fig. 2(b) in comparison with previous results given in Fig. 2(a). One can see a principally different behavior between the spectra.

At present, there is no experimental information about bremsstrahlung photons emitted during the scattering of pions off nuclei (we have even not found any idea about such investigations in the literature). Thus, there is no experimental basis for conclusions about the realistic nuclear parameters of the pion-nucleus potential by analyzing the bremsstrahlung emission. At the same time, such photons should be emitted in this reaction. We show the sensitivity of the bremsstrahlung spectra to the parameter U. We conclude that an analysis of the bremsstrahlung emission has good perspective to obtain estimates of this parameter from experimental study.

Without analysis of bremsstrahlung photons, some investigations of the interacting potential in frameworks of optical model were given in Ref. [18]. We use results of the work of those people. Here, there are the following parameters for reaction $\pi^+ + ^{44}$Ca at the energy of the pion beam of $E_{\pi^+} = 116$ MeV used in Figs. 1(a) and 1(b) (see Table 2, p. 760 in Ref. [18]):

$$U = 24.15 \text{ MeV}, \quad R_u = 1.50 \text{ fm}, \quad a_u = 0.20 \text{ fm}. \quad (8)$$

For calculations in Fig. 2(a) we fix these parameters R_u, a_u and vary U for the different spectra. The case of $U = 24.15$ MeV is presented by the upper spectrum (see solid blue line in this figure). The same spectrum for calculations by the optical model is presented in Fig. 2(b) [see red dashed line in this figure, parameters are in Eq. (8)].

As we noted, for bremsstrahlung in the proton-nucleus scattering, the understanding of influence of nuclear interactions on the spectrum of photons has not been obtained [27]. The leading contribution to the spectrum in this reaction can be estimated via optical model calculations (where we obtain coherent photons). Similarity in the bremsstrahlung calculations based on the optical model between the proton-nucleus scattering and the pion-nucleus scattering can be found. Thus, it could be useful to clarify if there is similar sensitivity of the bremsstrahlung spectrum (at high energies of photons) on nuclear potential for the proton-nucleus scattering. We observe a stable dependence of the bremsstrahlung spectra on nuclear strength V_R of the proton-nucleus potential in the high-energy region (see Fig. 3). Note that such a dependence of the spectra on the nuclear parameters of proton-nucleus potential has never been found previously.

In the proton-nucleus scattering we define the potential along formalism in Ref. [31] [see Eqs. (46) and (47) in that paper; here we follow the detailed study of Ref. [45], which is highly cited and has been tested by many people]. In calculating the bremsstrahlung spectra in Figs. 3(a) and 3(b), we vary the nuclear strength V_R. The best agreement between calculations and experimental data corresponds to values of V_R, defined along the formalism of Ref. [31] (for both reactions). In this research we focus on clarifying not-small differences between spectra at varying V_R. The clear understanding of the

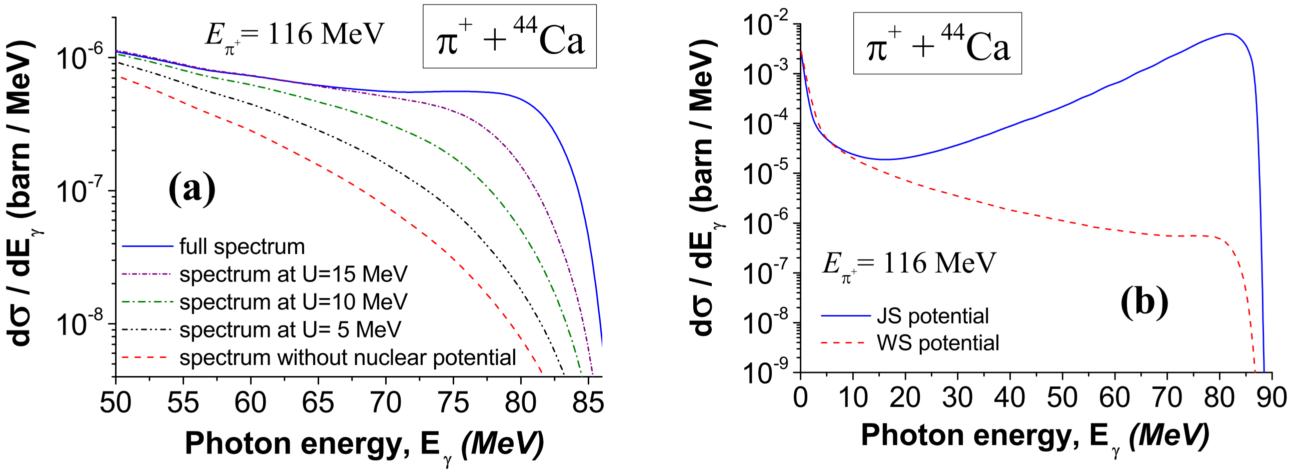

FIG. 2. The bremsstrahlung cross sections of photons emitted during the scattering of the π^+ mesons off the ^{44}Ca nucleus at the bombarding energy of 116 MeV [parameters of calculations: the potential $V_N^{(WS)}$ and $V_N^{(JS)}$ are defined in Eqs. (3) and (2), parameters of these potentials are taken from Ref. [18] and Table 1 in Ref. [22], the angle θ_γ between directions of the photon emission and the π-meson motion is 90°]. (a) The bremsstrahlung spectra, where interactions between $\pi^\pm$ mesons and nuclei are described by the Woods–Saxon optical model. We show changes of the spectrum as a function of strength U of the nuclear potential (3). One can see the stability of the calculations. (b) The bremsstrahlung spectra, where interactions between $\pi^\pm$ mesons and nuclei are defined in the Johnson–Satchler formalism (see blue solid line) and the Woods–Saxon optical model formalism (see red dashed line). One can see principally different behavior between the spectra. This difference is because two theories used in the basis of calculations of these bremsstrahlung spectra are essentially different (the Coulomb interactions and centrifugal potential terms are the same): the Johnson–Satchler formalism is based on the relativistic Klein–Gordon equation, while the optical model formalism is based on the nonrelativistic Schrödinger equation. Note that both spectra obtained by the Johnson–Satchler formalism and the optical model formalism with the Woods–Saxon potential coincide at low photon energies, which confirms our formalism and calculations (calculations are not normalized on any data point).

dependence of the spectra on the nuclear parameter can be obtained, if the incoherent bremsstrahlung is omitted. Thus, we do not use such terms in the current calculations. We find that it is enough to establish not-small dependence of the spectra on the parameter V_R. But, it is not enough to conclude about realistic values for this parameter; it would be better to include terms of the incoherent emission in analysis (that can change the spectra at low energies of photons). There is also a spin-orbital term in the proton-nucleus potential [see $v_{\rm so}(r)$ in Eqs. (46) in Ref. [31]; in contrast to the pion-nucleus

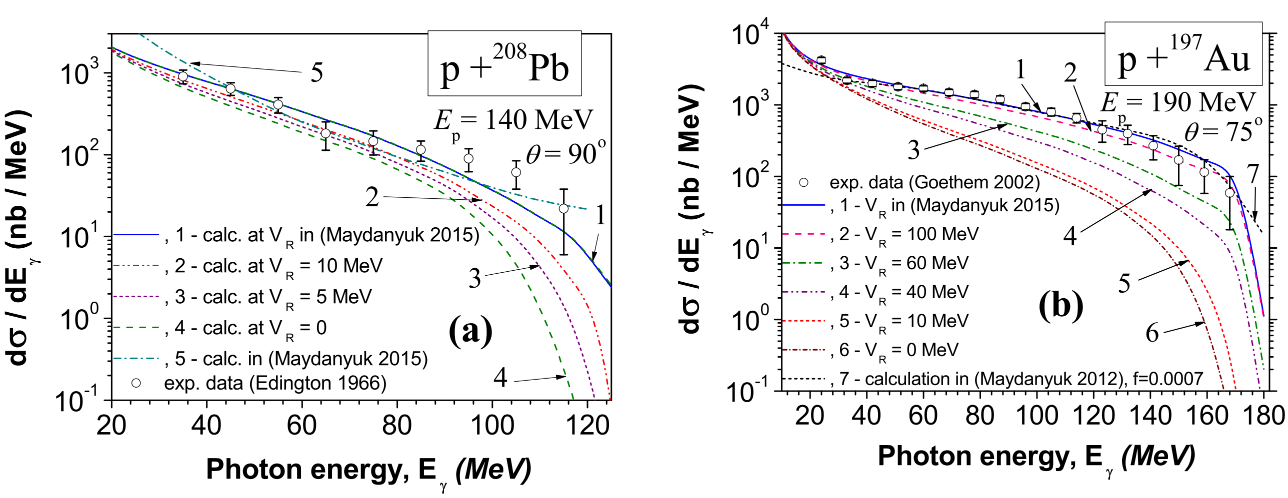

FIG. 3. The bremsstrahlung cross sections of the emitted photons during the scattering of (a) $p + {}^{208}$Pb at the proton bombarding energy $E_{\rm p} = 140$ MeV and (b) $p + {}^{197}$Au at the proton bombarding energy $E_{\rm p} = 190$ MeV in comparison with experimental data (Edington, 1966 [43]) and (Goethem, 2002 [44]) (circular points) The calculations follow the formalism (Maydanyuk 2012 [30]) with electric component $p_{\rm el}$ defined in Eqs. (36) in that paper, without magnetic terms $p_{{\rm mag},1}$ and $p_{{\rm mag},2}$ in Eqs. (36) in Ref. [30], and without incoherent bremsstrahlung studied in Ref. [31]. The parameters of the proton-nucleus potential are taken from Eqs. (46) and (47) in (Maydanyuk, 2015 [31]) with the approximation $r_c = r_R = 0.95$ fm. One can see the change of the spectra in the high-energy region as a function of nuclear strength V_R of the proton-nucleus potential.

scattering]. But, we estimate small bremsstrahlung emission formed by such a term.

IV. PERSPECTIVES TO STUDY INCOHERENT BREMSSTRAHLUNG

In Eq. (1) the spins of the nucleons of the nucleus are not included. We studied the role of spins of individual nucleons of full evolving nuclear system in the formation of bremsstrahlung emission during scattering of protons off nuclei [30,31] and during α decay [39]. For the proton-nucleus scattering, the full bremsstrahlung emission can be separated on incoherent bremsstrahlung (i.e., bremsstrahlung formed due to the interaction of the incoming proton with the nucleus as a whole), and coherent bremsstrahlung (i.e., bremsstrahlung formed due to the interaction of the incoming proton with the individual nucleons in the nucleus; we omit possible interference terms).

In previous studies [30,31,39] we took into account both aspects. The full matrix element of emission has a term corresponding to the incoherent emission and terms corresponding to different aspects of the coherent emission. This allows us to estimate contributions of each component to the full bremsstrahlung spectrum. That was realized for the proton-nucleus scattering and α decay (partially), but not for pion-nucleus scattering. However, in the current research we would like to restrict ourselves to the study of the coherent emission for the following reasons:

(1) Study of the incoherent bremsstrahlung is a more complicated task from the mathematical and numerical points of view. We do not find any information about the emission of bremsstrahlung photons during pion-nucleus scattering in the literature. Thus, our calculations with results of other people cannot be compared (we study bremsstrahlung in pion-nucleus scattering for the first time). We estimate that the incoherent emission (its contribution to the full spectrum) in the pion-nucleus scattering is essentially sensitive basing on the coherent bremsstrahlung calculations. Because of this, we want initially to obtain a proper basis for the description (and estimation) of coherent bremsstrahlung in the pion nucleus scattering. Thus, we put our main focus on constructing a model of the coherent bremsstrahlung in the pion-nucleus scattering.

(2) Study of spins of nucleons and the role of individual nucleons of the nucleus are included in our interest in this topic. We plan to do it in this research area, as a natural next step. We plan to begin with formalism [31] [see Secs. E, F, Eqs. (33–37, 41, 43–45) in that paper, which includes spin terms] with the needed generalization.

(3) If we return back to our previous study of role of individual nucleons of the nucleus and spin of scattered proton, then one can find the following: The experimental bremsstrahlung spectra [44,46,47] in the proton-nucleus scattering have the hump-shaped plateau inside the middle-energy region. And they decrease to the kinematic high-energy limit of the photons (these are data for the scattering of $p + {}^{208}$Pb at the proton-beam energies of 140 and 145 MeV, the scattering of $p + {}^{12}$C, $p + {}^{58}$Ni, $p + {}^{107}$Ag, and $p + {}^{197}$Au at the proton-beam energy of 190 MeV). However, the experimental bremsstrahlung spectra [34,35,48,49] in the α decay of the ^{210,214}Po, ^{226}Ra nuclei have shape of the logarithmic type without such a humped-shaped form. By using the unified formalism [30,31,39], we explained this difference between the bremsstrahlung spectra in proton-nucleus scattering and α decay. In proton-nucleus scattering, nonzero spin of the scattered proton gives a new type of interaction, which is based on its relations with momenta of nucleons of the nucleus [see p_4 in Eqs. (43) and (44), in Ref. [31]]. This produces the incoherent type of bremsstrahlung emission, which is absent in the α decay. Inclusion of the matrix element of incoherent emission p_4 for proton-nucleus scattering to calculations changes the bremsstrahlung spectra (a humped-shaped plateau appears in the spectra) and allows us to describe experimental data well.

According to results of Refs. [39], at higher energies of photons the internal structure of the α-particle in the α decay has a more important role in interactions between the α particles and nuclei. This would produce an incoherent contribution to the full bremsstrahlung spectrum.

So, if to consider the logic above and take zero spin for the pion into account, then the small incoherent bremsstrahlung emission in pion-nucleus scattering can be established. However, this type of emission is increased at higher energies of photons and pions. But this task is more complicated, and it is sensible to focus on it in further research (after obtaining an understanding of coherent emission in pion-nucleus scattering as a proper basis).

V. COMPARISON OF OUR MODEL WITH APPROACH FROM REF. [28]

As we noted above (see Sec. I), in Ref. [28] emission of the coherent photons by virtual pions was studied, which are produced in nuclear matter during proton-nucleus scattering. However, we find that this type of emission is essentially different from photons studied in our work. In formation of photons during the reaction studied in Ref. [28], the Coulomb forces outside the nucleus target are not taken into consideration. But the term of the bremsstrahlung photons emitted by such Coulomb forces outside the nucleus gives the largest contribution to the full coherent and incoherent bremsstrahlung emission in the scattering of real pions (in beams) off nuclei. If to include such a contribution in calculations of full bremsstrahlung spectrum, then the role of the nuclear forces will be essentially suppressed (i.e., it will be essentially more difficult to estimate the role of nuclear interactions via analysis of the bremsstrahlung spectra). But, in our paper we find that this is also possible.

Comparing our formalism with that of Ref. [28], we find the following:

(i) We base our calculations of the matrix elements on the wave functions of the pion-nucleus system, along to the main positions of quantum mechanics. These wave functions are complex and continuous in the full space region of definition. They are defined for the boundary conditions chosen for the studied reaction. Because of this, we take into account possible interference effects in calculations of bremsstrahlung cross sections (in contrast to the approach of Ref. [28]). This shall become important if we study contributions of different types of emission of photons to the full bremsstrahlung spectrum and estimate nuclear interactions from analysis of experimental bremsstrahlung data.

(ii) We obtain direct correspondence between interactions for different nuclei and wave functions. In some cases, these wave functions can be essentially different for (1) different nuclei or different isotopes in the scattering, (3) the same nuclei in the scattering (it depends on the chosen internal mechanisms inside the compound nuclear system), i.e., description of nuclear matter can be essentially different for bound states of nuclei without the scattering, and for unbound states of nuclei in the scattering, etc.

(iii) For example, in Refs. [50,51] we show that a more accurate stationary consideration of formation of a compound nuclear system during the scattering (even without inclusion of inelastic mechanisms) can change cross sections essentially (up to four times, i.e., more than 100%, for reactions $\alpha + {}^{40,44,48}$Ca), even for the same full wave functions with the same boundary conditions. The reason is in a more accurate description of internal mechanisms inside the full nuclear system, which can be characterized via different penetrabilities (for the same wave functions). Such quantum effects also exist in the scattering of protons and pions off nuclei. The approach of Ref. [28] does not take such a not-small effect into account (our formalism allows us to include a description of such effects completely).

(iv) Our formalism is based on the Hartree approximation in determination of the full wave function (of pion-nucleus and proton-nucleus systems). Antisymmetrization of one-nucleon wave functions in construction of the wave function of the nucleus is used in a more complete way (which reflects the Pauli exclusion principle for nucleons). This aspect can be taken into account in our description of emission of photons.

Thus, our model uses a more complete basis in the quantum description of the scattering, in comparison with the formalism [28]. This way allows us to describe the scattering process accurately, based on experimental information which we know and which is extracted on the basis of the theory of nuclear reactions. If we want to extract information about interactions between the nuclei and pions (or protons) from the bremsstrahlung measurements, this aspect becomes more important. But, in the current research, we do not analyze the resonances studied in Ref. [28]. This question is subject of perspective inclusion to further research.

VI. CONCLUSIONS

In this paper we have performed the first investigations of emission of the bremsstrahlung photons during the scattering of pions off nuclei. A motivation of this research is our supposition that such photons can be used as a new independent probe of the non-Coulomb part of the pion-nucleus interactions. We construct a new model of the bremsstrahlung photons emitted in this reaction. To describe the interactions between the $\pi^{\pm}$ mesons and nuclei, we use two nuclear potentials: (i) the Kisslinger-type potential along Johnson–Satchler formalism obtained by the Krell–Ericson transformation from Klein–Gordon equation for pion scattering [22], and (ii) the Woods–Saxon potential used by Akhter *et al.* in optical model calculations [18]. We find that the emission of photons formed due to nuclear part of the Johnson–Satchler potential gives an essential contribution to the full spectrum. Thus, according to formalism [22], nuclear interactions play an important (large) role in the formation of the bremsstrahlung spectra. Moreover, they can be studied experimentally via measurements of the bremsstrahlung photons.

The importance of such a result is reinforced to remind that, in exception with Ref. [39], it has never been possible to extract any information about nuclear parameters of optical models in nuclear reactions from the analysis of existing experimental data of the accompanying bremsstrahlung. This is because the Coulomb interactions play a larger role in the forming the bremsstrahlung than the nuclear interactions. Our results in pion-nucleus scattering show that bremsstrahlung spectra are essentially sensitive to non-Coulomb and nuclear parameters in the high-energy region of photons. For the first time, we observe a similar dependence of the spectra on the strength of nuclear optical potential for proton-nucleus scattering (see Fig. 3). Thus, possible measurements of the bremsstrahlung photons could be a good tools for obtaining new information about the non-Coulomb and nuclear interactions between π mesons and nuclei.

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APPENDIX A: OPERATOR OF EMISSION FOR EVOLVING MANY-NUCLEON SYSTEM AND $\pi^{\pm}$ MESONS

Emission of the bremsstrahlung photons can be introduced to the formalism of the $\pi^{\pm}$-nucleus scattering via application of the Coulomb gauge for each electromagnetic charge in the system as

$$\mathbf{p}_i \to \mathbf{p}_i - \frac{z_i e}{c}\mathbf{A}(\mathbf{r}_i, t), \tag{A1}$$

where $\mathbf{p}_i = -i\hbar\nabla_i$ is the momentum of the pion or nucleon with number i, $\mathbf{A}(\mathbf{r}_i, t)$ is the vector potential of the electromagnetic field formed by motion of pion or nucleon with number i, z_i is the electromagnetic charge of pion or nucleon with number i. The modified Hamiltonian is written as $\hat{H} = \hat{H}_0 + \hat{H}_\gamma$, where $\hat{H}_\gamma$ is a new correction (i.e., operator of emission) describing emission of the bremsstrahlung photons.

We begin with the generalization of the Pauli equation on the system composed of A nucleons and $\pi^{\pm}$, describing scattering of $\pi^{\pm}$ off a nucleus with A nucleons with Hamiltonian constructed as [see Eqs. (1) and (2) in Ref. [31]; see also Ref. [30] for the proton-nucleus scattering]

$$\hat{H} = \sum_{i=1}^{A+1} \left\{ \frac{1}{2m_i}\Big(\mathbf{p}_i - \frac{z_i e}{c}\mathbf{A}_i\Big)^2 - \frac{z_i e\hbar}{2m_i c}\boldsymbol{\sigma}\cdot \mathbf{rot}\mathbf{A}_i + z_i e A_{i,0} \right\} + V(\mathbf{r}_1, \dots, \mathbf{r}_{A+1}) = \hat{H}_0 + \hat{H}_\gamma, \tag{A2}$$

where

$$\hat{H}_0 = \sum_{i=1}^{A+1} \frac{1}{2m_i}\mathbf{p}_i^2 + V(\mathbf{r}_1, \dots, \mathbf{r}_{A+1}),$$
$$\hat{H}_\gamma = \sum_{i=1}^{A+1} \left\{ -\frac{z_i e}{m_i c}\mathbf{p}_i\mathbf{A}_i + \frac{z_i^2 e^2}{2m_i c^2}\mathbf{A}_i^2 - \frac{z_i e\hbar}{2m_i c}\boldsymbol{\sigma}\cdot\mathbf{rot}\mathbf{A}_i + z_i e A_{i,0}\right\}. \tag{A3}$$

Here, m_i and z_i are mass and electromagnetic charge of nucleon with number i $(i = 1, \dots, A)$ or mass and electromagnetic charge of $\pi^{\pm}$ mesons $(i = A+1)$, $\mathbf{p}_i = -i\hbar d/d\mathbf{r}_i$ is the momentum operator for nucleons with number i $(i = 1, \dots, A)$ or $\pi^{\pm}$ mesons $(i = A+1)$, $V(\mathbf{r}_1 \dots \mathbf{r}_{A+1})$ is the general form of the potential of interactions between nucleons and $\pi^{\pm}$ meson, $\boldsymbol{\sigma}$ are Pauli matrices, $A_i = (\mathbf{A}_i, A_{i,0})$ is the potential of the electromagnetic field formed by moving nucleons with number i $(i = 1, \dots, A)$ or $\pi^{\pm}$ mesons $(i = A+1)$. Let us turn to the center-of-mass frame. Introducing coordinates of the center of mass for the nucleus $\mathbf{R}_A = \sum_{j=1}^{A} m_j\mathbf{r}_{Aj}/m_A$, the coordinates of the center of mass of the complete system $\mathbf{R} = (m_A\mathbf{R}_A + m_\pi\mathbf{r}_\pi)/(m_A + m_\pi)$, the relative coordinates $\boldsymbol{\rho}_{Aj} = \mathbf{r}_j - \mathbf{R}_A$, and $\mathbf{r} = \mathbf{r}_\pi - \mathbf{R}_A$, we obtain new independent variables $\mathbf{R}$, $\mathbf{r}$, and $\boldsymbol{\rho}_{Aj}$ $(j = 1, \dots, A-1)$[4]

$$\mathbf{R} = \frac{1}{m_A + m_\pi}\left\{\sum_{j=1}^{A} m_{Aj}\mathbf{r}_{Aj} + m_\pi\mathbf{r}_\pi\right\},$$
$$\mathbf{r} = \mathbf{r}_\pi - \frac{1}{m_A}\sum_{j=1}^{A} m_{Aj}\mathbf{r}_{Aj}, \tag{A4}$$
$$\boldsymbol{\rho}_{Aj} = \mathbf{r}_{Aj} - \frac{1}{m_A}\sum_{k=1}^{A} m_{Ak}\mathbf{r}_{Ak}$$

and calculate the operators of corresponding momenta,

$$\mathbf{p}_\pi = -i\hbar\frac{d}{d\mathbf{R}_\pi} = \frac{m_\pi}{m_A + m_\pi}\mathbf{P} + \mathbf{p},$$
$$\mathbf{p}_{Aj} = -i\hbar\frac{d}{d\mathbf{R}_{Aj}} = \frac{m_{Aj}}{m_A + m_\pi}\mathbf{P} - \frac{m_{Aj}}{m_A}\mathbf{p} + \frac{m_A - m_{Aj}}{m_A}\tilde{\mathbf{p}}_{Aj} - \frac{m_{Aj}}{m_A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak}, \tag{A5}$$

where $\mathbf{P} = -i\hbar d/d\mathbf{R}$, $\mathbf{p} = -i\hbar d/d\mathbf{r}$, $\tilde{\mathbf{p}}_{Aj} = -i\hbar d/d\boldsymbol{\rho}_{Aj}$, and m_π and m_A are the masses of the scattering $\pi^{\pm}$ and nucleus, respectively.

Let us study the leading emission operator of the system composed of $\pi^{\pm}$ and a nucleus in the laboratory frame. We obtain its view in Eq. (A3) by neglecting terms at $\mathbf{A}_i^2$, $A_{i,0}$, and the spinor term:

$$\hat{H}_\gamma = -\frac{z_\pi e}{m_\pi c}\mathbf{A}(\mathbf{r}_\pi, t)\hat{\mathbf{p}}_\pi - \sum_{j=1}^{A}\frac{z_j e}{m_j c}\mathbf{A}(\mathbf{r}_j, t)\hat{\mathbf{p}}_j. \tag{A6}$$

Here, $\mathbf{A}(\mathbf{r}_s, t)$ describes the emission of a photon caused by a $\pi^{\pm}$ meson or nucleon ($s = \pi$ is for $\pi^{\pm}$, $s = j$ is for the nucleons of the nucleus). By using its presentation in the form (5) of Ref. [30], for the emission operator in the laboratory frame we obtain

$$\hat{H}_\gamma = -e\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}e^{-i\mathbf{k}\cdot\left[\mathbf{R} - \frac{m_\pi}{M+m_\pi}\mathbf{r}\right]}\left\{\frac{1}{M+m_\pi}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}z_\pi + \sum_{j=1}^{A} z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{P} + \left[e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{z_\pi}{m_\pi} - \frac{1}{M}\sum_{j=1}^{A} z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{p} + \sum_{j=1}^{A-1}\frac{z_{Aj}}{m_{Aj}}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\tilde{\mathbf{p}}_{Aj} - \frac{1}{M}\left[\sum_{j=1}^{A} z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\right\}, \tag{A7}$$

[4]Sometimes, for clarity, we add an additional bottom index A to variables for nucleons.

where the star denotes complex conjugation, $\mathbf{e}^{(\alpha)}$ are unit vectors of polarization of the photon emitted ($\mathbf{e}^{(\alpha),*} = \mathbf{e}^{(\alpha)}$), $\mathbf{k}$ is the wave vector of the photon, and $w_{\rm ph} = kc = |\mathbf{k}|c$. Vectors $\mathbf{e}^{(\alpha)}$ are perpendicular to $\mathbf{k}$ in the Coulomb gauge. We have two independent polarizations $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ for the photon with momentum $\mathbf{k}$ ($\alpha = 1, 2$).

APPENDIX B: WAVE FUNCTION OF $\pi^{\pm}$-NUCLEUS SYSTEM

Emission of the bremsstrahlung photons is caused by the relative motion of nucleons and the charged $\pi^{\pm}$ mesons of the full system. However, we assume that the most intensive emission of photons is formed by relative motion of $\pi^{\pm}$ related to the nucleus. Thus, it is sensible to represent the total wave function via coordinates of relative motion of these complicated objects. Following such a logic, we define the wave function of the full nuclear system as

$$\Psi_s = \Phi_s(\mathbf{R})\Phi_{\pi\text{-nucl,s}}(\mathbf{r})\psi_{\rm nucl,s}(\beta_A), \tag{B1}$$

where $s = i$ or f (indexes i and f denote the initial state, i.e., the state before emission of photon, and the final state, i.e., the state after emission of photon), $\mathbf{K}_s$ is the full momentum of the $\pi^{\pm}$-nucleus system (in the laboratory frame), $\Phi_s(\mathbf{R})$ is the wave function describing the motion of the center of mass of the full nuclear system in the laboratory frame, $\Phi_{\pi\text{-nucl,s}}(\mathbf{r})$ is the function describing the relative motion (with tunneling for under-barrier energies) of $\pi^{\pm}$ related to the nucleus (without description of internal relative motions of nucleons inside the nucleus), $\psi_{\rm nucl,s}(\beta)$ is the many-nucleon function describing the internal states of nucleons in the nucleus (it determines space states on the basis of relative distances $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_A$ of nucleons of the nucleus related to its center of mass, and spin-isospin states also), and β_A is a set of numbers $1, \ldots, A$ of nucleons of the nucleus. We assume

$$\Phi_{\bar{s}}(\mathbf{R}) = N_{\bar{s}} e^{-i\mathbf{K}_{\bar{s}}\cdot\mathbf{R}}, \quad \mathbf{K}_i = 0, \tag{B2}$$

where $N_{\bar{s}}$ is a normalization factor that will be defined later.

Motion of nucleons of the nucleus relative to each other does not have a large influence on the states describing the relative motion of $\pi^{\pm}$ related to the nucleus. Therefore, such a representation of the full wave function can be considered as an approximation. However, the relative internal motions of nucleons of the nucleus give its own contributions to the full bremsstrahlung spectrum and they can be estimated. We include the many-nucleon structure in the wave function $\psi_{\rm nucl,s}(\beta_A)$ of the nucleus while we assume that the wave function of the relative motion $\psi_{\pi\text{-nucl,s}}(\mathbf{r})$ is calculated without it but with maximal orientation of the $\pi^{\pm}$-nucleus potential well extracted from experimental data of the $\pi^{\pm}$-nucleus scattering (one can take the many-nucleon corrections into account in the next step). Such a line allows us to accurately keep the wave function of relative motion which gives the leading contribution to the bremsstrahlung spectrum, while the many-nucleon structure should be estimated after as a correction.

APPENDIX C: MATRIX ELEMENT OF EMISSION

We find the matrix element on the basis of the operator of emission (A7) and wave function (B1):

$$\begin{aligned}
\langle\Psi_f|\hat{H}_\gamma|\Psi i\rangle = -N_i N_f e\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\Bigg\{&\langle\Psi_f|e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\frac{1}{m_A+m_\pi}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}z_\pi + \sum_{j=1}^{A} z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{P}|\Psi_i\rangle \\
&+ \langle\Psi_f|e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{z_\pi}{m_\pi} - \sum_{j=1}^{A}\frac{z_{Aj}}{m_A}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{p}|\Psi_i\rangle \\
&+ \langle\Psi_f|e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\left[\sum_{j=1}^{A-1}\frac{z_{Aj}}{m_{Aj}}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\tilde{\mathbf{p}}_{Aj}\right]|\Psi_i\rangle \\
&- \langle\Psi_f|e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{M+m_\pi}}\frac{1}{m_A}\left[\sum_{j=1}^{A} z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\right]|\Psi_i\rangle\Bigg\}.
\end{aligned} \tag{C1}$$

The first term describes emission of photons caused by motion of the full nuclear system in the laboratory frame and its response to the emission of photons. We calculate the spectra in the center-of-mass frame, so we neglect this term. The second term describes emission of photons caused by $\pi^{\pm}$ mesons and each nucleon of the nucleus, at relative motion of $\pi^{\pm}$ with respect to the nucleus. This term is leading and gives the main contribution to the full bremsstrahlung spectrum. The third and fourth terms describe emission of photons caused by each nucleon of the nucleus, in relative motions of nucleons of the nucleus inside its space region (any nuclear deformations during emission can be connected with such terms).

We consider the leading matrix element on the basis of the second term in Eq. (C1). In calculations, we must integrate over all independent space variables given in Eq. (A4):

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1 = -N_i N_f e\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\int e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}d\mathbf{R}\langle\Phi_{\pi\text{-nucl,f}}(\mathbf{r})\psi_{\rm nucl,f}(\beta_A)| \times e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{z_\pi}{m_\pi}-\sum_{j=1}^{A}\frac{z_{Aj}}{m_A}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}\right]\mathbf{p}|\Phi_{\pi\text{-nucl,i}}(\mathbf{r})\psi_{\rm nucl,i}(\beta_A)\rangle. \quad (C2)$$

Introducing the *effective charge of the* $\pi^\pm$*-nucleus system* as

$$Z_{\rm eff}(\mathbf{k},\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}\frac{m_\pi}{m_A+m_\pi}}\left\{\frac{m_A z_\pi}{m_A+m_\pi}-e^{i\mathbf{k}\cdot\mathbf{r}}\frac{m_\pi Z_{\rm A}(\mathbf{k})}{m_A+m_\pi}\right\} \quad (C3)$$

and the *charged form factor of the nucleus* as

$$Z_{\rm A}(\mathbf{k}) = \langle\psi_{\rm nucl,f}(\boldsymbol{\rho}_{A1},\ldots,\boldsymbol{\rho}_{AA})|\sum_{j=1}^{A}z_{Aj}e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}}|\psi_{\rm nucl,i}(\boldsymbol{\rho}_{A1},\ldots,\boldsymbol{\rho}_{AA})\rangle, \quad (C4)$$

we obtain

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1 = -N_i N_f\frac{e}{m}\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}(2\pi)^3\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\delta(\mathbf{K}_f-\mathbf{K}_i-\mathbf{k})\langle\Phi_{\pi\text{-nucl,f}}(\mathbf{r})|Z_{\rm eff}(\mathbf{k},\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}|\Phi_{\pi\text{-nucl,i}}(\mathbf{r})\rangle, \quad (C5)$$

where $m = m_\pi m_A/(m_\pi + m_A)$ and we use integral representation of the δ function,

$$\int e^{i(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k})\cdot\mathbf{R}}d\mathbf{R} = (2\pi)^3\delta(\mathbf{K}_i-\mathbf{K}_f-\mathbf{k}). \quad (C6)$$

We define the normalizing factors N_i and N_f as

$$N_i = N_f = (2\pi)^{-3/2}. \quad (C7)$$

We calculate cross sections of the emitted photons not dependent on momentum $\mathbf{K}_f$ (momentum of the full $\pi^\pm$-nucleus system after emission of a photon in the laboratory frame). Thus, we must integrate the matrix element over momentum $\mathbf{K}_f$. From (C5) we obtain

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1 = -\frac{e}{m}\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\langle\Phi_{\pi\text{-nucl,f}}(\mathbf{r})|Z_{\rm eff}(\mathbf{k},\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}|\Phi_{\pi\text{-nucl,i}}(\mathbf{r})\rangle, \quad \mathbf{K}_i = \mathbf{K}_f+\mathbf{k}. \quad (C8)$$

The effective charge of the system in the first approximation of $\exp(i\mathbf{k}\cdot\mathbf{r}) \to 1$ (i.e., at $\mathbf{k}\cdot\mathbf{r} \to 0$, called a *dipole* concerned with the effective charge) obtains the form

$$Z_{\rm eff}^{(\rm dip)}(\mathbf{k}) = \frac{m_A z_\pi - m_\pi Z_{\rm A}(\mathbf{k})}{m_A+m_\pi}. \quad (C9)$$

One can see the independence of the effective charge with respect to the relative distance between $\pi^\pm$ and the center of mass of the nucleus in such an approximation.

A simple determination of the matrix element can be obtained by neglecting relative displacements of nucleons of the nucleus inside its space region in calculations of the form factor (i.e., in an approximation where the nucleus is considered as point like and we use $e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{Aj}} \to 1$ for each nucleon). Here, the form factor of the nucleus represents the summed electromagnetic charge of nucleons of the nucleus, where the dependence on the characteristics of the emitted photon is lost:

$$Z_{\rm A}(\mathbf{k}) \to \langle\psi_{\rm nucl,f}(\boldsymbol{\rho}_{A1},\ldots,\boldsymbol{\rho}_{AA-1})|\sum_{j=1}^{A}z_{Aj}|\psi_{\rm nucl,i}(\boldsymbol{\rho}_{A1},\ldots,\boldsymbol{\rho}_{AA-1})\rangle = \sum_{j=1}^{A}z_{Aj} = Z_{\rm A}, \quad (C10)$$

as the functions $\psi_{\rm nucl,s}$ are normalized. At such approximations we obtain the matrix element (we added the index "dip"):

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1^{(\rm dip)} = -\frac{e}{m}\sqrt{\frac{2\pi\hbar}{w_{\rm ph}}}Z_{\rm eff}^{(\rm dip,0)}\sum_{\alpha=1,2}\mathbf{e}^{(\alpha),*}\langle\Phi_{\pi\text{-nucl,f}}(\mathbf{r})|e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}|\Phi_{\pi\text{-nucl,i}}(\mathbf{r})\rangle, \quad Z_{\rm eff}^{(\rm dip,0)} = \frac{m_A z_\pi - m_\pi Z_{\rm A}}{m_A+m_\pi}. \quad (C11)$$

Using as the functions $\psi_{\pi\text{-nucl,s}}(\mathbf{r})$, the wave packets (as in the formalism of Refs. [32–41]),

$$\Phi_{\pi\text{-nucl,s}}(\mathbf{r},t) = \int_0^{+\infty} g(k-k_s)\psi_{\pi\text{-nucl,s}}(\mathbf{r})e^{-iw(k)t}dk, \quad (C12)$$

we rewrite the matrix element above [see Eq. (6) in Ref. [30] without spinor terms],

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle_1^{(\mathrm{dip})} = \frac{e}{m}\sqrt{\frac{2\pi\hbar}{w_{\mathrm{ph}}}}\,p_{\mathrm{el}} 2\pi\delta(w_i - w_f - w),\; p_{\mathrm{el}} = -Z_{\mathrm{eff}}^{(\mathrm{dip},0)} \sum_{\alpha=1,2} \mathbf{e}^{(\alpha),*}\langle\psi_{\pi\text{-nucl,f}}(\mathbf{r})|e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}|\psi_{\pi\text{-nucl,i}}(\mathbf{r})\rangle. \quad (\mathrm{C13})$$

We calculate the matrix element in multipolar expansion of wave function of photons

$$p_{\mathrm{el}} = iZ_{\mathrm{eff}}^{(\mathrm{dip},0)}\sqrt{\frac{\pi}{2}}\sum_{l_{\mathrm{ph}}=1}(-i)^{l_{\mathrm{ph}}}\sqrt{2l_{\mathrm{ph}}+1}\big[p_{l_{\mathrm{ph}}}^{M} - i p_{l_{\mathrm{ph}}}^{E}\big], \quad p_{l_{\mathrm{ph}}}^{M} = \sum_{\mu=\pm1} h_\mu \mu p_{l_{\mathrm{ph}}\mu}^{M}, \quad p_{l_{\mathrm{ph}}}^{E} = \sum_{\mu=\pm1} h_\mu p_{l_{\mathrm{ph}}\mu}^{E}, \quad (\mathrm{C14})$$

where

$$p_{l_{\mathrm{ph}}\mu}^{M} = \int \varphi_f^*(\mathbf{r})\bigg(\frac{\partial}{\partial\mathbf{r}}\varphi_i(\mathbf{r})\bigg)\mathbf{A}_{l_{\mathrm{ph}}\mu}^{*}(\mathbf{r}, M)d\mathbf{R}, \quad p_{l_{\mathrm{ph}}\mu}^{E} = \int \varphi_f^*(\mathbf{r})\bigg(\frac{\partial}{\partial\mathbf{r}}\varphi_i(\mathbf{r})\bigg)\mathbf{A}_{l_{\mathrm{ph}}\mu}^{*}(\mathbf{r}, E)d\mathbf{R}. \quad (\mathrm{C15})$$

Here, $\mathbf{A}_{l_{\mathrm{ph}}\mu}(\mathbf{r}, M)$ and $\mathbf{A}_{l_{\mathrm{ph}}\mu}(\mathbf{r}, E)$ are magnetic and electric multipolar terms (j_{ph} is a quantum number characterizing the eigenvalue of the full momentum operator, while $l_{\mathrm{ph}} = j_{\mathrm{ph}} - 1,\, j_{\mathrm{ph}},\, j_{\mathrm{ph}} + 1$ is connected with orbital momentum operator, but it defines eigenvalues of photon parity).

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