Resonance properties including asymptotic normalization coefficients deduced from phase-shift data without the effective-range function

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Recently, a new Δ method for the calculation of asymptotic normalization coefficients (ANC) from phase-shift data has been formulated, proved, and used for bound states. This method differs from the conventional one by fitting only the nuclear part of the effective-range function which includes a partial phase shift. It should be applied to large-charge nuclei when the conventional effective-range expansion or the Padé approximations using the effective-range function $K_l(k^2)$ fitting do not work. A typical example is the nucleus vertex $\alpha + {}^{12}\mathrm{C} \longleftrightarrow {}^{16}\mathrm{O}$. Here we apply the Δ method, which totally excludes the effective-range function, to isolated resonance states. In fact, we return to the initial renormalized scattering amplitude with a denominator which defines the well-known pole condition. Concrete calculations are made for the resonances observed in the ${}^{3}\mathrm{He}$-${}^{4}\mathrm{He}$, α-α, and α-${}^{12}\mathrm{C}$ collisions. We use the experimental phase-shift and resonant energy data including their uncertainties and find the ANC variations for the states considered. The corresponding results are in a good agreement with those for the S-matrix pole method which uses the differing formalism. The simple formula for narrow resonances given in the literature is used to check the deduced results. The related ANC function clearly depends on the resonance energy (E_0) and width (Γ), which is used to find the ANC uncertainty (ΔANC) through the energy (ΔE_0) and the width ($\Delta\Gamma$) uncertainties.

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I. INTRODUCTION

It is known that many reactions in supernovae explosions proceed through subthreshold bound states and low-energy resonance states. To calculate the rate of such reactions, one needs to find the asymptotic normalization coefficient (ANC) of the radial wave function for bound and resonance states, which can be used to calculate radiative capture cross sections at low energies. The radiative capture process is one of the main sources of new element creation.

In our recent paper [1], we validate a new algorithm for the bound states ANC calculation when the input data include a phase-shift energy dependence at low-energy region and bound-state pole position. The related form of the renormalized scattering amplitude is proposed (without proof) in Ref. [2] (see also Ref. [3]). We call this algorithm the Δ method. This new method allows us to avoid the problems arising when the charges of colliding particles increase. In Ref. [1], we note that the effective-range expansion (ERE) and Padé approximations for finding the ANC are especially limited by the values of the colliding particle charges. These approaches do not work for large charges when the nuclear term of the effective-range function (ERF) is too small compared with the Coulomb term. The $\alpha{}^{12}\mathrm{C}$ is just a proper example of such a situation. This problem is revealed in our work [4] where we see that the ERF for $\alpha{}^{12}\mathrm{C}$ have a similar behavior in the different ${}^{16}\mathrm{O}$ states because the Coulomb term in the ERF does not depend on nuclear state.

In the present paper, we apply the Δ method to resonance states. The first attempt to use the ERF approach for resonances is made in Ref. [5]. In Ref. [6] the problem of calculating resonance pole properties is solved using the S-matrix pole approach in the frame of a potential model.

In the S-matrix pole method (SMP) (see Ref. [7]), an analytical continuation to the resonance pole is accomplished for the so-called potential (or no resonant) phase shift in the complex k plane. Having found the ANC, we now know the asymptotic part of the Gamow wave function. This allows us to normalize it correctly if we choose a nuclear potential of the interaction between the two nuclei considered. In Refs. [6,7] there is a more detailed discussion about the Gamow states and its normalization.

The ANC method has been explored as an indirect experimental method for determining the cross sections of peripheral reactions at low energy [8]. There are several methods of deriving the bound state ANCs from experimental data (see Refs. [9,10] and references therein). Recently, the ERE method has been developed to find the ANC for bound states from the elastic-scattering phase-shift δ_l analysis (see Refs. [11,12] and references therein). A renormalized scattering amplitude, taking into account the Coulomb interaction, is derived in Ref. [13] to enable an analytic continuation of this amplitude

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to negative energies. It is shown in Ref. [13] that the ERF is a real analytic function with the possible exception of single poles. This means that the ERF can be presented by the ERE or Padé approximations, whose coefficients can be found by fitting the experimental phase shifts. The same is valid for the Δ method as long as the Δ_l function is included in one of the two terms defining the ERF.

An important step in calculating a bound state ANC using the ERF was first taken by Iwinski *et al.* [14], who discuss a radiative capture process ^{3}He(^{4}He, γ)^{7}Be and calculate ANCs for the bound P states with total angular momentum J and parity π ($J^\pi = 3/2^-$ and $1/2^-$) of the ^{7}Be nucleus applying the Padé approximant. In the past, the results of the scattering phase-shift analyses were often presented in the form of the ERF. Consequently, from 1984 onwards, the ERF has been used to deduce the ANC. It is much simpler to use the few ERF parameters instead of the δ_l tables. A first example is the nucleon-nucleon scattering when the expansion coefficients are considered as independent characteristics of the NN interaction. The scattering length a_l, effective range r_l, and the shape parameter P_l for the orbital momentum l were introduced, although ERE should be convergent and, consequently, may include an unlimited number of terms. The denomination of the coefficients in the polynomial $K_l(k^2) = -1/a_l + (r_l/2)k^2 - P_l r_l^3 k^4$ is due to the fact that one can truncate the series in the low-energy region. Another approximation for $K_l(k^2)$ with a limited number of fitting parameters is the Padé approximant when the ERF has poles.

In fact, it is necessary to fit only the nuclear part. Excluding the ERF leads to the original renormalized scattering amplitude form which does not include the Coulomb part of the ERF. Simple algebra gives an inverse transformation from the amplitude including the ERF to this original form. When charges are large enough, the original renormalized scattering amplitude form should be used to deduce resonance properties, including ANC, from the experimental phase shifts. This can be used for smaller charges as well.

Below we apply the Δ method to the concrete systems ^{3}Heα, $\alpha\alpha$, and α^{12}C. Processes such as the scattering of α particles, triple-α reaction, and radiative α capture play a major role in stellar nucleosynthesis. In particular, α capture on carbon determines the ratio of carbon to oxygen during helium burning and affects the subsequent carbon, neon, oxygen, and silicon burning stages. The authors of a recent paper [15] describe an *ab initio* calculation of α-α scattering that uses lattice Monte Carlo simulations to a two-cluster system.

The article is organized as follows. In Sec. II we present the main formulas of the Δ method for resonances and show that the original renormalized scattering amplitude should be used, which can be analytically continued to a resonant pole. It is important that this amplitude does not include the ERF and its Coulomb part. The ratio of Coulomb to nuclear parts increases quickly with the growth of the product of the colliding nuclei charges. In our paper [1], one can see from Eqs. (9) and (12) that the nuclear term includes $\exp(2\pi\eta)$ in the denominator. That is why the relatively small variation of η (or charges product) leads to a strong reduction of the nuclear component compared with the Coulomb term in square brackets in Eq. (9) in Ref. [1]. It is notable that $\eta = 1/(a_B k)$ is the only argument of the functions responsible for the Coulomb effects. Here a_B is the Bohr radius. η has a scaling property: a decrease of a_B is equivalent to the same decrease of the relative momentum k of the colliding nuclei and the corresponding decrease of the energy when the role of the Coulomb barrier increases. As a result, the nuclear part, including the phase shift, is a small addition to the Coulomb part $h(\eta)$ which can be ignored with reasonable precision. The ratio of the Coulomb to nuclear parts is about 10^3 for the α-^{12}C system due to a relatively large value of the Sommerfeld parameter η. As a result, the corresponding phase shift is unreproducible from the experimental ERF fit which leads to an incorrectly calculated ANC. By definition, the Δ_l-function fit reproduces the input phase shift. That is why we named the corresponding algorithm the Δ method. So the final equation for the renormalized scattering amplitude can be applied to calculate the nuclear vertex constant (NVC) or $\tilde{G}_l$, the residue W_l, and the ANC (C_l). The relationships between these observables are well known in the literature. The simple analytic ANC formula for narrow resonances is written, borrowed from Ref. [16]. This simple formula clearly depends on the resonance energy and width. The ANC uncertainty equation is due to the uncertainties in the E_0 and Γ and is derived from this ANC expression.

In Sec. III we present the main SMP-method equations which describe a different formalism compared with the Δ method. The only common elements in both approaches are the one-channel approximation and model which does not take into account the internal structure of colliding nuclei. The SMP-method results are published in Ref. [7] for the resonance states of ^{5}He, ^{5}Li, and ^{16}O.

In Sec. IV the results are given for the Δ-method calculations for the resonance levels of ^{7}Be, ^{8}Be, and ^{16}O. Tables I–III for the three nuclear systems studied here include the experimental and calculated resonant energies E_0 and the widths Γ for the different methods. The resulting ANCs are compared with those calculated by the SMP method and with those using the simple formula for narrow resonances. These tables show a good agreement between the results obtained by both of these methods. The results for narrow resonances serve to check our Δ-method calculation results. We give the absolute values $|C_l|$ because the Schrödinger equation is uniform, so the phase multiplier can be omitted.

The effects in the calculated ANCs of the experimental uncertainties in the phase shift and resonant energy are investigated. The conclusions following from the analysis of the results of all the tables are formulated. We stress the stability of the different results found for the ^{8}Be ground state, which plays a special role in astrophysics.

In Sec. V we summarize the main points of the present paper. In the following we use the unit system $\hbar = c = 1$.

II. Δ METHOD FOR RESONANT STATES WITHOUT USING THE EFFECTIVE-RANGE FUNCTION

The partial amplitude of the nuclear scattering in the presence of the Coulomb interaction is

$$f_l(k) = \exp(2i\sigma_l)[\exp(2i\delta_l) - 1]/2ik, \quad (1)$$

where

$$\exp(2i\sigma_l) = \Gamma(l+1+i\eta)/\Gamma(l+1-i\eta). \quad (2)$$

Here δ_l is the nuclear phase shift modified by the Coulomb interaction, and $\eta = \xi/k$ is the Sommerfeld parameter; $\xi = Z_1 Z_2 \mu\alpha$; $k = \sqrt{2\mu E}$ is the relative momentum; μ and E are the reduced mass and the center-of-mass energy of the colliding nuclei with the charge numbers Z_1 and Z_2, respectively; and α is the fine-structure constant.

The amplitude (1) has a complicated analytical property in the complex momentum plane due to the Coulomb factor. According to Ref. [13], we renormalize the partial amplitude of the elastic-scattering multiplying it by the function [the Coulomb correcting or renormalizing factor $CF_l(k)$]

$$CF_l(k) = \frac{(l!)^2 e^{\pi\eta}}{[\Gamma(l+1+i\eta)]^2}. \quad (3)$$

The general pole condition $\cot\delta_l - i = 0$ follows from the expression for the renormalized amplitude of the elastic scattering (see, for example, Ref. [1])

$$\tilde{f}_l(k) = \frac{1}{k(\cot\delta_l - i)\rho_l(k)}, \quad (4)$$

where the function ρ_l is defined by the equation

$$\rho_l(k) = \frac{2\pi\eta}{e^{2\pi\eta}-1}\prod_{n=1}^{l}\left(1+\frac{\eta^2}{n^2}\right). \quad (5)$$

Writing the expression $\cot\delta_l$ in a nonphysical energy region in Eq. (4) and elsewhere, we mean its analytical continuation, since the phase shift is defined only in the positive energy region. The renormalized scattering amplitude of the conventional method is written as

$$\tilde{f}_l(k) = \frac{k^{2l}}{K_l(k^2) - 2\xi D_l(k^2)h(\eta)} \quad (6)$$

(see, for example, Ref. [11] and definitions below), where the effective-range function $K_l(k^2)$ borrowed from Ref. [17] has the form:

$$K_l(k^2) = 2\xi D_l(k^2)\left[C_0^2(\eta)(\cot\delta_l - i) + h(\eta)\right], \quad (7)$$

$$h(\eta) \equiv \Psi(i\eta) + (2i\eta)^{-1} - \ln(i\eta), \quad (8)$$

where $\Psi(i\eta)$ is the digamma function.

It is easy to derive (4), substituting (7) into the denominator of (6). Using simple algebra we obtain the expression

$$\tilde{f}_l(k) = \frac{k^{2l}}{2\xi D_l(k^2)C_0^2(\eta)(\cot\delta_l - i)}, \quad (9)$$

where the function $h(\eta)$ (8) is absent. In Eqs. (6)–(9) we use the following notations:

$$C_0^2(\eta) = \frac{\pi}{\exp(2\pi\eta)-1}, \quad (10)$$

$$D_l(k^2) = \prod_{n=1}^{l}\left(k^2 + \frac{\xi^2}{n^2}\right), \quad D_0(k^2) = 1. \quad (11)$$

We define the $\Delta_l(k^2)$ function as in Ref. [1]

$$\Delta_l(k^2) = C_0^2(\eta)\cot\delta_l \quad (12)$$

in the positive energy semiaxis. Using (12) we can recast (9) as

$$\tilde{f}_l(k) = \frac{k^{2l}}{2\xi D_l(k^2)\left[\Delta_l(k^2) - iC_0^2(\eta)\right]}. \quad (13)$$

We note that $C_0^2(\eta) \to k/2\xi$ and $D_l(k^2) \to k^{2l}$ if $\eta = \xi/k \to 0$. Therefore $\tilde{f}_l(k) \to 1/k(\cot\delta_l - \mathrm{i})$ and $K_l(k^2) \to k^{2l+1}\cot\delta_l$, as it should be when there is no Coulomb interaction. The factor $C_0^2(\eta)$ secures a regularity of the Δ_l function at point $E = 0$. The physical meaning of the function $C_0^2(\eta)$ is its role as the compensating factor, excluding the essential phase-shift singularity in the function δ_l. Moreover, this is a multiplier in the Coulomb penetration factor squared [see Eq. (5)]. Separating this factor from the total partial amplitude leads to the renormalized amplitude (or "effective amplitude" as it is called in Ref. [13]). This has analytic properties similar to amplitude properties for a short-range potential.

It is easy to show that the expression (13) is equivalent to the original formula (4), although this is obvious because (13) is derived from (4) and from the expression (7) for $K_l(k^2)$. To prove this, we express the function $k\rho(k)$ in terms of $C_0^2(\eta)$ and $D_l(k^2)$ as

$$k\rho(k) = 2\xi C_0^2(\eta)D_l(k^2)/k^{2l}$$

and include it in Eq. (4). The function $C_0^2(\eta)$, having the analytical form (10), does not need fitting. This function clearly depends on the momentum k through $\eta(k)$ which leads to the square root cut of the renormalized amplitude in the E plane.

The results of the $\Delta_l(k^2)$ fitting using the experimental phase shifts can be applied to resonances, taking into account that the resonance energy position is defined by the condition

$$\Delta_l(k^2) - iC_0^2(\eta) = 0. \quad (14)$$

Next we write down the expression for the residue of the $\tilde{f}_l(k)$ at the resonance pole. This residue of the renormalized amplitude can be written as

$$W_l\left(k_r^2\right) = \frac{(k_r)^{2l}}{2\xi D_l\left(k_r^2\right)\lim_{k\to k_r}\left\{\frac{d}{dk}\left[\Delta_l(k^2) - iC_0^2(\eta)\right]\right\}}. \quad (15)$$

Here $E_r = E_0 - i\Gamma/2$, $k_r = \sqrt{2\mu E_r}$, $k_r = k_0 - ik_i$.

According to the known relations between the NVC ($\tilde{G}_l$), ANC (C_l), and the residue W_l we can write

$$\tilde{G}_l^2 = -\frac{2\pi k_r}{\mu^2}W_l, \quad (16)$$

$$C_l = \frac{i^{-l}\mu}{\sqrt{\pi}}\frac{\Gamma(l+1+i\eta_r)}{l!}e^{-\frac{\pi\eta_r}{2}}\tilde{G}_l, \quad (17)$$

where $\eta_r = \xi/k_r$. A simple relation for the ANC derived in Ref. [16] for narrow resonances, which we call the Dolinsky-Mukhamedzhanov (DM) method,

$$\left|C_l^a\right| = \sqrt{\frac{\mu\Gamma}{k_0}}, \quad (18)$$

is used to check our calculations.

The uncertainty of the absolute value of the ANC which follows from Eq. (18) is obtained by deducing a differential of

the right-hand side of Eq. (18) which is the function of the two arguments E_0 and Γ:

$$\Delta C_l = |C_l^a|\left(\frac{\Delta\Gamma}{2\Gamma} + \frac{\Delta E_0}{4E_0}\right), \tag{19}$$

where we put the increment signs instead of the differentials, which should not be confused with the already-used signs for the Δ method and the $\Delta_l(k)$ function. One can see from the last equation that the uncertainty in the width ($\Delta\Gamma$) contributes twice as much compared to that (ΔE_0) in the resonance energy into the relative ANC uncertainty $\Delta C_l/|C_l^a|$.

III. THE ANC FROM THE ELASTIC-SCATTERING AMPLITUDE BASED ON THE ANALYTIC PROPERTIES OF THE *S* MATRIX (SMP METHOD)

Near an isolated resonance the partial S-matrix element can be represented as in Ref. [18]

$$S_l(k) = e^{2i\nu_l(k)}\frac{(k+k_r)(k-k_r^*)}{(k-k_r)(k+k_r^*)}, \tag{20}$$

where $k_r = k_0 - ik_i$ is the complex wave number of a resonance [$k_0 > k_i > 0$, and the symbol (*) means the complex conjugate operation]. Here $k_0 > k_i$ because we do not consider subthreshold resonances. Energy E_0 of this resonance and its width Γ are

$$E_0 = \frac{k_0^2 - k_i^2}{2\mu}, \quad \Gamma = \frac{2k_0k_i}{\mu}. \tag{21}$$

The partial scattering nonresonant phase shift $\nu_l(k)$ is a smooth function near the pole of the S-matrix element, corresponding to the resonance. The S-matrix element defined by Eq. (20) fulfills the conditions of analyticity, unitarity, and symmetry. It is possible to recast Eq. (20) in the form

$$S_l(k) = e^{2i(\nu_l+\delta_r+\delta_a)}, \tag{22}$$

where

$$\delta_r = -\arctan\frac{k_i}{k-k_0}$$

represents the resonance phase shift, while

$$\delta_a = -\arctan\frac{k_i}{k+k_0}$$

is the additional phase shift which contributes to the whole scattering phase shift. Thus the total phase shift is

$$\delta_l = \nu_l + \delta_r + \delta_a. \tag{23}$$

After simplification and replacing $\exp(2i\delta_l)$ by $S_l(k)$, we get

$$\tilde{f}_l(k) = \frac{S_l(k)-1}{2ik\rho_l(k)}, \tag{24}$$

This renormalized amplitude $\tilde{f}_l(k)$ can be analytically continued like the partial scattering amplitude, corresponding to the short-range interaction and has its pole at the point k_r according to Eq. (20). In the vicinity of pole k_r, the partial scattering amplitude (24) can be represented as

$$\tilde{f}_l(k) = \frac{W_l}{k-k_r} + \tilde{f}_{l,\mathrm{nonres}}(k), \tag{25}$$

where the function $\tilde{f}_{l,\mathrm{nonres}}(k)$ is regular at this point. The simple derivation of the residue W_l leads to the expression

$$W_l = \operatorname{res}\tilde{f}_l = \lim_{k\to k_r}[(k-k_r)\tilde{f}_l(k)] = -\frac{k_i e^{i2\nu_l(k_r)}}{k_0\rho_l(k_r)}. \tag{26}$$

Using the relationship between the NVC ($\tilde{G}_l$) and ANC (C_l) (17), we obtain

$$\begin{aligned} C_l &= \frac{i^{-l}\mu}{\sqrt{\pi}}\frac{\Gamma(l+1+i\eta_r)}{l!}e^{-\frac{\pi\eta_r}{2}}\tilde{G}_l \\ &= i^{-l}\sqrt{\frac{\mu\Gamma}{k_0}}e^{-\frac{\pi\eta_r}{2}}\frac{\Gamma(l+1+i\eta_r)}{l!} \\ &\quad\times e^{i\nu_l(k_r)}\sqrt{(1-ik_i/k_0)/\rho_l(k_r)}. \end{aligned} \tag{27}$$

The derived equations are valid for both narrow and broad resonances. For narrow resonances, when $\Gamma \ll E_0$ ($k_i \ll k_0$), one can simplify Eq. (27) for the ANC replacing k_r by k_0 and using the equality

$$e^{-\frac{\pi\eta}{2}}\frac{\Gamma(l+1+i\eta)}{l!\sqrt{\rho_l(k_0)}} = e^{i\sigma_l} \tag{28}$$

to obtain

$$C_l^a = \sqrt{\frac{\mu\Gamma}{k_0}}e^{i(\nu_l(k_0)+\sigma_l(k_0)-\pi l/2)}, \tag{29}$$

which coincides with the result obtained in Ref. [16].

The nonresonant phase shift $\nu_l(k)$ is the analytical function, excluding the origin. In Ref. [19], the authors present the behavior of $\nu_l(k)$ near the origin as

$$\nu_l(k) = -\frac{2\pi}{(l!)^2}k^{2l+1}\eta^{2l+1}a_l e^{-2\pi\eta}, \tag{30}$$

where a_l is the scattering length for colliding nuclei. We see that $k=0$ is the point of the essential singularity of the scattering phase shift. However, as a function of the momentum k, it has normal analytical properties near the point corresponding to the resonance. Therefore we can expand $\nu_l(k)$ to a series

$$\nu_l(k) = \sum_{n=0}^{\infty} c_n(k-k_s)^n \tag{31}$$

in the vicinity of the pole corresponding to the resonance. If we wish to determine the value of the phase shift $\nu_l(k)$ by applying Eq. (31) at a point in the complex plane close to the centered point k_s, then only the first few terms of the convergent series may be taken into account with reasonable precision. The expansion coefficients c_n of Eq. (31) as well as k_0 and k_i are determined by fitting the experimental values of the elastic-scattering phase shifts δ_l given by Eq. (23).

IV. RESONANCE ANC CALCULATED BY THE Δ METHOD AND ITS COMPARISON WITH SMP-METHOD RESULTS FOR LOW-ENERGY ^{7}Be, ^{8}Be, and ^{16}O LEVELS

As mentioned in the Introduction we apply the Δ method described above to the ^{7}Be, ^{8}Be, and ^{16}O resonances using a model with the configurations $^{3}\mathrm{He}+\alpha$, $\alpha+\alpha$, and $\alpha+{}^{12}\mathrm{C}$. For the α-^{12}C collision the absolute value of the nuclear part of the ERF is very small compared with those for the Coulomb part (see Ref. [1]) due to the "large" product of the colliding particle charges. As explained above, this is due to $\exp(2\pi\eta)$ in the denominator in Eq. (10) for C_0^2. Therefore, the ERF approach is not valid for this nucleus.

In Ref. [1] we find that the results of the fitting are quite sensitive to the selection of the energy region. For bound states the low-energy area is especially important, while for resonances it is necessary to secure a proper description of Δ_l in the vicinity of resonance energy E_0.

We need to satisfy this demand in our calculations while choosing a fitting model. In Ref. [1] we use different models for the orbital momenta $l=0, 1, 2$ of the bound states.

Here we change the model only for ^{16}O states to secure a proper agreement of the resonance energy defined from Eq. (14) with the experimental values of E_0 and Γ.

The figures for the Δ_l fit are given for the experimental phase-shift values as well as for E_0 and Γ, written in the corresponding captions for the different nuclear systems and states.

The results in the tables below are given for the different methods and take into account the uncertainties in the resonance energy and in the experimental phase shifts when known.

A simple model of the Δ_l fit is used for ^{7}Be ($7/2^-$, $5/2^-$) and ^{8}Be (0^+, 2^+):

$$\Delta_l(E)=a_0+a_1E+a_2E^2. \tag{32}$$

For ^{16}O ($^{4}\mathrm{He}+{}^{12}\mathrm{C}$) we use the following more complex fitting models:

(1) $J^\pi=0^+$. There is a narrow resonance in this state (in the R-matrix fit in Ref. [20] Γ = 3 keV). The input phase shift δ_0 is zero or π at $E_z=4.8823$ MeV. So the following fitting is used:

$$\Delta_0=\frac{a_0+a_1(E-E_z)+a_2(E-E_z)^2+a_3(E-E_z)^3}{1-E/E_z}. \tag{33}$$

(2) $J^\pi=1^-$. Due to the near-threshold bound state at $E=-\epsilon_1=-0.045$ MeV ($\Delta=0$ at a bound pole [1]) we choose

$$\Delta_1=(1+E/\epsilon_1)(a_0+a_1E+a_2E^2). \tag{34}$$

(3) $J^\pi=2^+$. There are two resonances which are observed in the energy interval 2.5–5.0 MeV. The bound state pole is situated at $E=-\epsilon_2=-0.245$ MeV. The input phase shift δ_2 is zero or π at $E_{z1}=2.680$ MeV where $\cot\delta_2$ goes to infinity. Consequently, the fitting model in the region of the lowest resonance may be taken as

$$\Delta_2=\frac{(1+E/\epsilon_2)(a_0+a_1E+a_2E^2)}{1-E/E_{z1}}. \tag{35}$$

The second energy value where $\sin\delta_2=0$ is $E_{z2}=3.970$ MeV. Therefore, the fitting model in the region of the second resonance can be taken as

$$\Delta_2=\frac{a_0+a_1(E-E_{z2})+a_2(E-E_{z2})^2}{1-E/E_{z2}}, \tag{36}$$

where we take E_{z2} as the centered point.

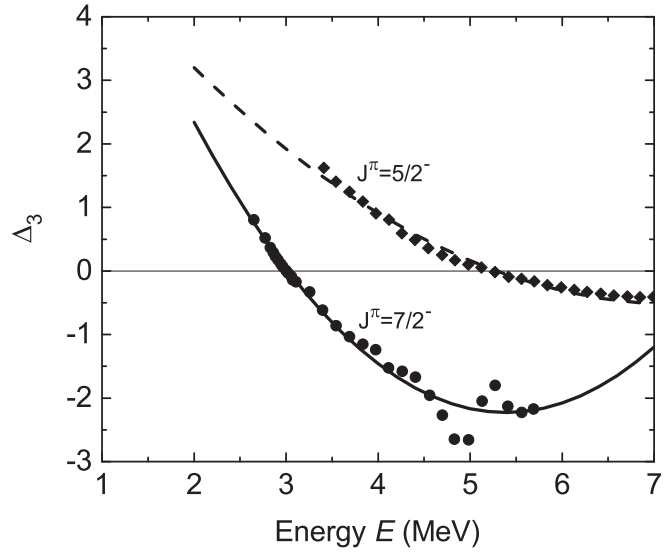

FIG. 1. Dependence of the fitted Δ_l functions [Eq. (32)] vs. the center-of-mass energy E of the ^{3}He-α collision. Solid and dashed lines are for $J^\pi=7/2^-$ and $5/2^-$, respectively. The experimental data (dots) correspond to the phase shifts taken from Ref. [21]. Results of the extracted resonance energy are $E_0=3.017$ MeV, Γ = 177 keV for $J^\pi=7/2^-$ and $E_0=5.106$ MeV, Γ = 1.212 MeV for $J^\pi=5/2^-$.

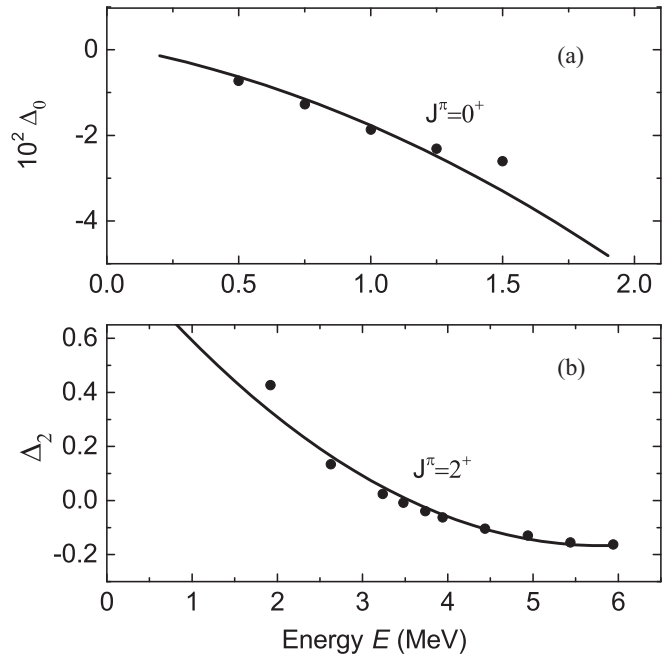

FIG. 2. As for Fig. 1 for the α-α collision for $J^\pi=0^+$ (a) and 2^+ (b). The experimental data (dots) correspond to the experimental phase shifts taken from Ref. [22]. The extracted resonance energy and width are $E_0=0.093$ MeV, Γ = 0.0055 keV for $J^\pi=0^+$ and $E_0=3.096$ MeV, Γ = 1.512 MeV for $J^\pi=2^+$.

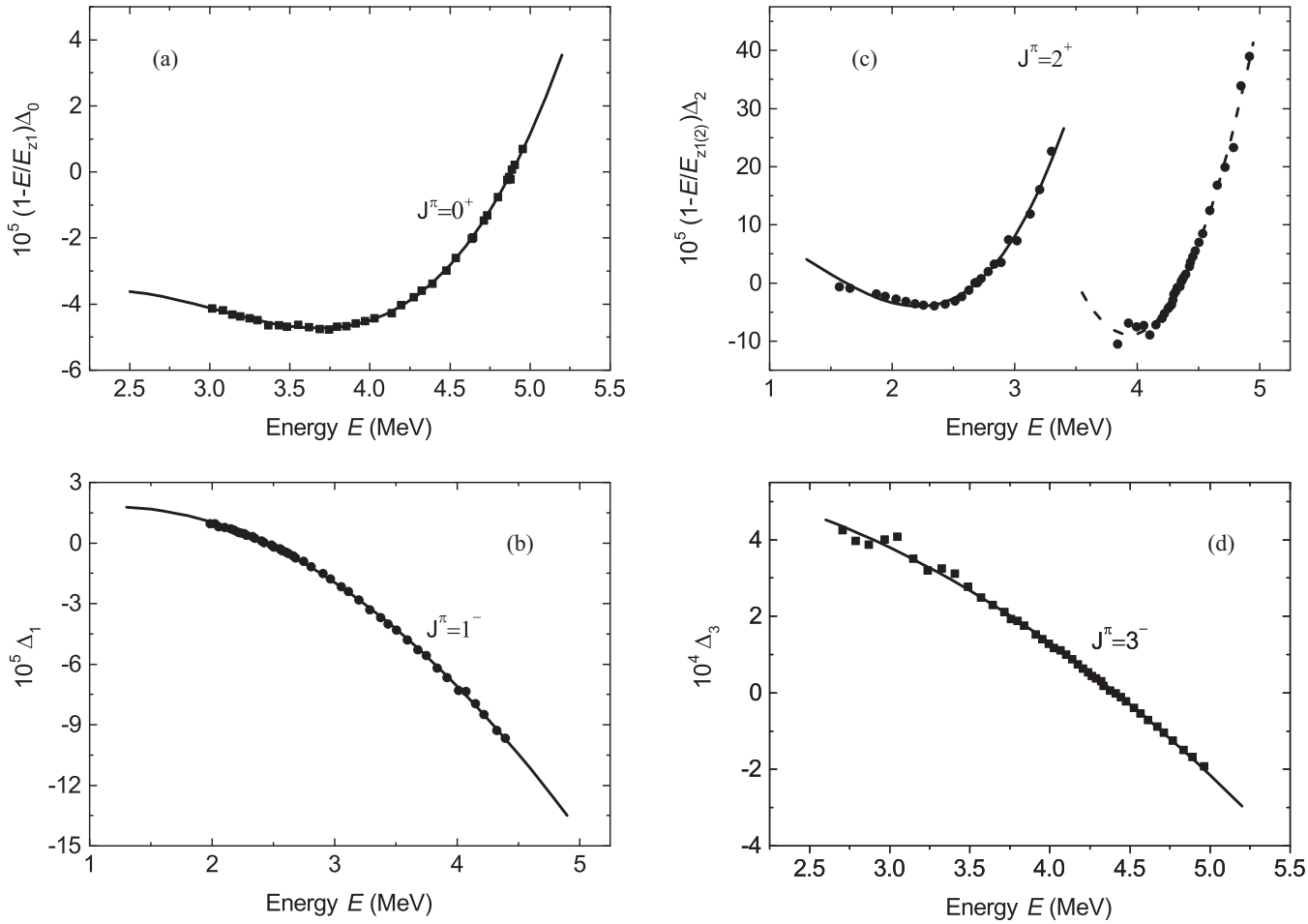

FIG. 3. As for Fig. 1 for the α-^{12}C collision. The fitting models are given in Eqs. (33)–(37). Solid lines are for $J^\pi = 0^+$ (a), 1^- (b), 2^+ (c), left curve, and for 3^- (d). Dashed line is for $J^\pi = 2^+$ (c), right curve. The state 2^+ includes two resonances. The experimental data (dots) correspond to the R-matrix fit phase shifts taken from Ref. [20]. The extracted resonance energy and width are $E_0 = 4.887$ MeV, $\Gamma = 3.52$ keV (0^+); $E_0 = 2.364$ MeV, $\Gamma = 0.333$ MeV (1^-); $E_0 = 2.693$ MeV, $\Gamma = 0.597$ keV (2^+); $E_0 = 4.359$ MeV, $\Gamma = 78$ keV (2^+); $E_0 = 4.228$ MeV, $\Gamma = 0.817$ MeV (3^-).

(4) $J^\pi = 3^-$. For the fitting we use an expansion with the centered point at $E_z = 4.32$ MeV in the vicinity of the resonance pole:

$$\Delta_3 = a_0 + a_1(E - E_z) + a_2(E - E_z)^2. \qquad (37)$$

In Figs. 1–3, we show a comparison between the experimental Δ_l-function values and the fitting curves for the models given in Eqs. (32)–(37). The phase-shift experimental data are taken from papers [21] for ^{7}Be, [22] for ^{8}Be and [20] for the R-matrix phase-shift fit for ^{16}O. There is a fairly good agreement between the fitting curves and the experimental data in the energy intervals considered.

In the Tables I–III all the experimental data for the energies and widths of the resonances are taken from Ref. [23]. The left column "Method" in all tables includes the following designations. $\Delta(1)$ means the calculation by the Δ method, using the models given above with the experimental phase-shift values. Δ(up) and Δ(low) mean the calculation by the Δ method, using the upper and lower experimental phase-shift values taken from Ref. [22] (the α-α scattering phase shifts) and Ref. [24] (the α-^{12}C scattering phase shifts).

The rows DM(+) and DM(−) show the results of the calculations by the Eq. (18) for narrow resonances, where "+" and "−" are related to the maximal and minimal values of the experimental energy and width of the resonance in accordance with the uncertainty defined by the Eq. (19), respectively. In Tables I and II, the row denoted by SMP shows our new calculations for ^{7}Be and ^{8}Be using the SMP method. In Table III, all the SMP-method results are taken from our paper [7]. In Table II, for the ground 0^+ state the results noted ERE(1) and ERE(2) are calculated by the conventional method using the G_l^2 results found in Ref. [25] for differing resonant energies. The corresponding results for the 2^+ state are noted by the ERE(1, 2, 3). These ERF-method results also show the effects of the resonant energy and width variations. One can compare these ERF results with those using other methods.

Analysis of the calculation results in all the tables leads to the following conclusions:

(1) For narrow resonances, different accepted methods lead to good agreement between the $|C_l|$ and $|C_l^a|$. Such agreement can be considered as a criterion of a

resonance narrowness. This is slightly better for $\Delta(1)$ compared with the SMP-method results.

(2) For wider resonances, as a rule there is an inequality $|C_l| < |C_l^a|$, but the differences are not very big for the nuclear systems considered. We include the results of $|C_l^a|$ for broad resonances to find out how large the differences are between $|C_l^a|$ and $|C_l|$.

(3) There is very good agreement between $|C_l|$ and $|C_l^a|$ for both DM(+) and DM(−) in Tables I and II with one exception in Table III for $J^\pi = 0^+$. This is due to the high sensitivity of $|C_l|$ to the Γ value. Significantly bigger variations of the resonant width are visible in this case, which lead to the variation $|C_0| = 0.0094$–0.0174 fm$^{-1/2}$ although the difference between $|C_0|$ for $\Delta(1)$ model (0.0174 fm$^{-1/2}$) and SMP method (0.0160 fm$^{-1/2}$) is rather small. In Table I for $J^\pi = 5/2^- E_0$ changes while Γ is mostly stable. This lead to the variation $|C_2| = 0.260$–0.277 fm$^{-1/2}$. For the narrow resonance in the $7/2^-$ state which is most pronounced experimentally, the variation is smaller $|C_3| = 0.118$–0.123 fm$^{-1/2}$. We note that in Ref. [26] the resonance energy $E_0(2^+) = 3.03 \pm 0.01$ MeV, $\Gamma = 1.49 \pm 0.02$ MeV. These $E_0(2^+)$ and Γ are slightly smaller compared with the experimental data [23] as are some values given in Table II.

(4) Table II shows for the α-α system good agreement ($|C_0| = 0.0016$–0.0017 fm$^{-1/2}$) for all the models, including the ERF for the $J^\pi = 0^+$ state, which is especially important in astrophysics. For the 2^+ state, the experimental uncertainties in the resonant energy and the phase shift lead to the variation $|C_2| = 0.32$–0.35 fm$^{-1/2}$.

(5) Table III for the $\alpha + {}^{12}$C system is the most important in the present paper because the ERF method is invalid. It contains much more information than the other tables, including the effects of uncertainties of the phase shifts for all the states considered except the 0^+ and first 2^+ resonance states. In the second state 2^+ where we find a stability of E_0 but also an essential Γ variation which significantly affects the value of the $|C_l|$. Nevertheless, there is quite good agreement between the results for the $\Delta(1)$ and the SMP methods.

TABLE I. ^{7}Be $\leftrightarrow \alpha + {}^3$H. Calculation method, J^π, resonance energy E_0 and its width Γ, corresponding values of ANCs $|C_l|$ and $|C_l^a|$ calculated by Eq. (18) for narrow resonances. Energy values given in the center-of-mass system. Experimental data [23]: $E_0(5/2^-) = 5.143 \pm 0.1$ MeV, $\Gamma(5/2^-) = 1.2$ MeV; $E_0(7/2^-) = 2.983 \pm 0.05$ MeV, $\Gamma(7/2^-) = 175 \pm 7$ keV.

Method	J^π	E_0(MeV)	Γ(keV)	$\lvert C_l\rvert$(fm$^{-1/2}$)	$\lvert C_l^a\rvert$(fm$^{-1/2}$)
$\Delta(1)$	$5/2^-$	5.106	1212	0.260	0.277
DM(+)		5.243	1200	0.276	0.274
DM(−)		5.043	1200	0.274	0.276
SMP		4.983	1275	0.264	0.286
$\Delta(1)$	$7/2^-$	3.017	177	0.120	0.121
DM(+)		3.033	182	0.123	0.122
DM(−)		2.933	168	0.118	0.119
SMP		2.987	182	0.122	0.123

TABLE II. ^{8}Be $\leftrightarrow \alpha + \alpha$. The definitions of the method, state, and the calculated results are the same as in Table I. Experimental data [23]: $E_0(0^+) = 91.84$ keV, $\Gamma(0^+) = 5.57 \pm 0.25$ eV; $E_0(2^+) = 3.122 \pm 0.010$ MeV, $\Gamma(2^+) = 1.513 \pm 0.015$ MeV.

Method	J^π	E_0(MeV)	Γ(keV)	$\lvert C_l\rvert$(fm$^{-1/2}$)	$\lvert C_l^a\rvert$(fm$^{-1/2}$)
$\Delta(1)$	0^+	0.093	0.0055	0.0016	0.00170
ERE(1)		0.0918	0.0058	0.00169	0.00172
ERE(2)		0.0918	0.0053	0.00165	0.00165
DM(+)		0.0918	0.0058	0.00172	0.00172
DM(−)		0.0918	0.0053	0.00165	0.00165
SMP		0.0093	0.0056	0.0016	0.00170
$\Delta(1)$	2^+	3.096	1512	0.321	0.363
Δ(up)		2.925	1456	0.329	0.362
Δ(low)		2.899	1669	0.348	0.387
ERE(1)		2.87	1310	0.348	0.345
ERE(2)		2.91	1370	0.361	0.351
ERE(3)		3.04	1510	0.387	0.365
DM(+)		3.132	1528	0.366	0.365
DM(−)		3.112	1498	0.362	0.362
SMP		3.122	1513	0.291	0.362

TABLE III. ^{16}O $\leftrightarrow \alpha + {}^{12}$C. The definitions of the method, state, and the calculated results are the same as in Table I. Experimental data [23]: $E_0(0^+) = 4.887 \pm 0.002$ MeV, $\Gamma(0^+) = 1.5 \pm 0.5$ keV; $E_0(1^-) = 2.423 \pm 0.011$ MeV, $\Gamma(1^-) = 0.420 \pm 0.020$ MeV; $E_0(2^+) = 2.683$ MeV$\pm$ 0.5 keV, $\Gamma(2^+) = 0.625 \pm 0.100$ keV; $E_0(2^+) = 4.358$ MeV ± 4 keV, $\Gamma(2^+) = 71 \pm 3$ keV; $E_0(3^-) = 4.438$ MeV ± 20 keV, $\Gamma(3^-) = 0.800 \pm 0.1$ MeV.

Method	J^π	E_0(MeV)	Γ(keV)	$\lvert C_l\rvert$(fm$^{-1/2}$)	$\lvert C_l^a\rvert$(fm$^{-1/2}$)
$\Delta(1)$	0^+	4.887	3.52	0.0174	0.0174
DM(+)		4.889	2.0	0.0132	0.0131
DM(−)		4.885	1.0	0.0094	0.0093
SMP		4.887	3.0	0.0160	0.0160
$\Delta(1)$	1^-	2.364	333	0.179	0.185
Δ(up)		2.213	319	0.178	0.202
Δ(low)		2.327	323	0.177	0.200
SMP		2.364	356	0.185	0.209
$\Delta(1)$	2^+	2.693	0.597	0.0083	0.0085
DM(+)		2.683	0.725	0.0092	0.0092
DM(−)		2.682	0.525	0.0078	0.0078
SMP		2.364	0.760	0.0094	0.0094
$\Delta(1)$	2^+	4.359	78	0.0835	0.0842
Δ(up)		4.380	80.35	0.0846	0.0853
Δ(low)		4.386	73.84	0.0810	0.0817
SMP		4.350	79.1	0.0840	0.0847
$\Delta(1)$	3^-	4.228	817	0.236	0.274
Δ(up)		4.266	809	0.240	0.272
Δ(low)		4.257	825	0.238	0.275
SMP		4.350	79.1	0.230	0.273

V. CONCLUSION

In the present paper we apply the Δ method to resonances. We use the original form (4) of the renormalized scattering amplitude where there is no expression for the ERF. We emphasize that the renormalized scattering amplitude does not include the Coulomb term $h(\eta)$ (8) which is part of the ERF in Eq. (7). The function $h(\eta)$ forms a background for the nuclear term and obviously leads to a wrong ANC for "large" charges of colliding nuclei. The ^{16}O states with the configuration $\alpha + {}^{12}$C is an example. We show that the Δ or SMP methods should be applied when the calculations using the ERF fitting are invalid. In Table III, the calculation results denoted by SMP are taken from Ref. [7].

We include some uncertainties of the experimental data in our ANC calculations. The formula for narrow resonances is used to derive a simple expression for the increment ΔC_l related to both the uncertainties of the resonance energy (ΔE_0) and width ($\Delta\Gamma$). Some experimental uncertainties of the phase-shift data are also included and their effects in ANCs are analyzed. The system $\alpha + {}^{12}$C is studied in more detail using the Δ method as the conventional ERF method is not valid for this system or for those with larger charge product. We also study the lighter systems ^{3}He^{4}He, and $\alpha\alpha$. The ground state of ^{8}Be is especially important in astrophysics for the creation of the organic elements and life itself on Earth [27]. While considering the different pole conditions for bound and resonance state, we stress the role of the square root cut on the complex energy plane of the partial scattering amplitude. In addition, the renormalized amplitude is real on the imaginary momentum axis. This is due to the similarity of its analytical properties to those of the amplitude for a short-range potential (see, for example, Ref. [28]).

The reasonable agreement between the resonant energies and the ANC results obtained by both the Δ and SMP methods, as well as that between the ANC results and the ANCs calculated for narrow resonances mean that these results are credible for the nuclear systems considered within the found limits of the variations. They can be used in nuclear astrophysics and in the nuclear reactions theory based on Feynman diagrams.

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