

Differential correlation measurements with the identity method

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We present an extension of the identity method initially introduced for particle yield fluctuation studies towards measurements of differential correlations. The extension is developed and illustrated in the context of measurements of the normalized two-particle cumulant R_2 but is adaptable to any correlation measurements, including differential flow measurements. The identity method is also extended to account for an arbitrary number of particle identification devices and signals.

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I. INTRODUCTION

Studies of integral and differential correlation functions of elementary particles produced in high-energy nucleus-nucleus collisions provide invaluable information on the particle production dynamics and the collision system evolution, and might also enable the determination of fundamental properties of the quark matter produced in these collisions [1–3]. Such measurements have been carried out for different collision systems, several beam energies, and a host of particle combinations [4–7]. Semiexclusive correlation functions measured for specific particle species (e.g., pions, kaons, protons, etc.) are of particular interest as they probe the influence of specific particle production processes determined by quantum number conservation laws. For instance, extensive measurements of general balance functions should provide detailed probes of the formation, evolution, and hadronization of the quark matter produced in relativistic heavy-ion collisions [8–10]. The difficulty arises, however, in that such measurements of correlation functions require large datasets, and severe particle rejection may be experimentally incurred to achieve high species purity and low contamination. Indeed, traditional methods of selecting the species of interest and rejecting contaminating species are based on cuts on particle identification signals and typically end up throwing away a significant fraction of the measured particles, or severely limiting the kinematic range of the measurement, or both. However, the identity method [11] provides a technique to essentially recover the full statistics and extend the kinematic range of measurements while providing reliable disambiguation of particle species. The technique was first proposed for measurements of the first and second moments of particle multiplicities (integral correlation functions) with two particle species but was successively extended to handle an arbitrary number of species, higher moments [12,13], and measurements of moments in the presence of transverse momentum-dependent efficiency losses [14]. The method is extended here to measurements of differential correlation functions, more specifically measurements of the normalized two-particle cumulants, R_2 . However, the method can be extended to other types of two-particle correlators or to multiparticle

correlation functions. The method is developed for an arbitrary number of particle species and accounts for particle losses due to finite detector efficiency. It is also extended to account for two or more particle identification signals.

This paper is divided as follows. Section II defines the normalized two-particle differential cumulant $R_2^{(p,q)}$, for particle species p and q , and summarizes a technique, introduced elsewhere [15], to discretize and correct measurements for particle losses. Section III builds on the identity method described in Refs. [11–14] and its extension involving an explicit dependence on detection efficiencies, towards measurements of multiplicity moments as a function of relative rapidity and differences in azimuthal angle. Section IV discusses an extension of the identity method for measurements involving more than one source of particle identification, e.g., studies involving joint measurements of energy loss and time of flight. This work is summarized in Sec. V.

II. R_2 DEFINITION AND EVALUATION TECHNIQUE

Measurements of normalized two-particle cumulants $R_2^{(p,q)}(\Delta\eta, \Delta\phi)$ [where p and q represent particle species in specific kinematic ranges while $\Delta\eta$ and $\Delta\phi$ represent rapidity (or pseudorapidity) and azimuthal angle differences], triggered correlation functions $\frac{1}{N_{\text{trig}}} \frac{d^2 N_{\text{pairs}}}{d\Delta\eta d\Delta\phi}$, and balance functions $B(\Delta\eta, \Delta\phi)$ have been carried out in various shapes or forms for a wide range of collision systems and beam energies [16–21]. Physical properties and several measurement techniques of R_2 were reported in [15]. The correlator $R_2^{(p,q)}$ is commonly measured as a function of the relative rapidity (or pseudorapidity), the difference of azimuthal angles of produced particles, or both. However, in this paper, following Method 2 of Ref. [15], one defines $R_2^{(p,q)}$ in four dimensions in terms of single- and two-particle densities, denoted $\rho_1^{(p)}(y, \phi)$ and $\rho_2^{(p,q)}(y_1, \phi_1, y_2, \phi_2)$, respectively, according to

$$R_2^{(p,q)}(y_1, \phi_1, y_2, \phi_2) = \frac{\rho_2^{(p,q)}(y_1, \phi_1, y_2, \phi_2)}{\rho_1^{(p)}(y_1, \phi_1)\rho_1^{(q)}(y_2, \phi_2)} - 1, \quad (1)$$

where y_i and ϕ_i (for $i = 1, 2$) are the rapidity (or pseudorapidity) and azimuthal angle of measured particles. The correlation function is readily reduced to a function of the relative rapidity $\Delta y = y_1 - y_2$ and the azimuthal angle difference $\Delta\phi = \phi_1 - \phi_2$ by averaging across the measurement acceptance:

$$R_2^{(p,q)}(\Delta y, \Delta\phi) = \frac{1}{\Omega(\Delta y)} \int_{\Omega} R_2^{(p,q)}(y_1, \phi_1, y_2, \phi_2) \times \delta(\Delta y - y_1 + y_2) \delta(\Delta\phi - \phi_1 + \phi_2) \times dy_1 d\phi_1 dy_2 d\phi_2, \quad (2)$$

where $\Omega(\Delta y)$ represents the width of the acceptance in $\bar{y} = \frac{1}{2}(y_1 + y_2)$ for a given value of Δy , and the relative angle $\Delta\phi$ is calculated modulo 2π .

The above expression holds for continuous density functions. In practice, the experimental evaluation of $R_2^{(p,q)}$ is based on histograms with finite size bins, and the evaluation of the above integral is formulated as a discrete sum [15] of single and pair yields measured as a function of rapidity (or pseudorapidity), azimuthal angle, and transverse momentum (p_{\perp}). In general, the measurement may be carried out with arbitrarily many bins in all three dimensions for both single particles and pairs of particles, and for species p and q . It is thus convenient to define three-dimensional histograms $H_1^{(p)}(\vec{\alpha})$ and six-dimensional histograms $H_2^{(p,q)}(\vec{\alpha}, \vec{\beta})$ for measurements of single and pair densities, respectively. The three-dimensional vectors $\vec{\alpha} = (\alpha_y, \alpha_{\phi}, \alpha_{p_{\perp}})$ and $\vec{\beta} = (\beta_y, \beta_{\phi}, \beta_{p_{\perp}})$ represent bin indices in rapidity (pseudorapidity), azimuthal angle, and transverse momentum. The number of bins along each axis, denoted by m_y, m_{ϕ} , and $m_{p_{\perp}}$, and range of the variables are to be chosen considering the physics of interest, the available statistics, and the dependence of the detection efficiency on these variables. Considering a selected data sample consisting of N_{ev} events, the analysis involves processing all events and counting numbers of single particles and pairs into single and pair histograms according to their respective momentum vectors, $\vec{\alpha}$ and $\vec{\beta}$. In the absence of (or neglecting) particle losses, statistical estimators of the single and pair densities are obtained according to

$$\hat{\rho}_1^{(p)}(\vec{\alpha}) \equiv \frac{\langle N_p(\vec{\alpha}) \rangle}{\delta y \delta \phi \delta p_{\perp}}, \quad (3)$$

$$\hat{\rho}_2^{(p,q)}(\vec{\alpha}, \vec{\beta}) \equiv \frac{\langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle}{\delta y^2 \delta \phi^2 \delta p_{\perp}^2}, \quad (4)$$

where quantities $\langle O \rangle$ are event ensemble averages of the (single or pair) yields in momentum bins $\vec{\alpha}$ ($\vec{\beta}$) of width $\delta y, \delta\phi$, and δp_{\perp} in rapidity, azimuthal angle, and transverse momentum, respectively. The hats (e.g., $\hat{\rho}$) denote the fact that the above quantities are statistical estimators of the single and pair densities, towards which they converge in the large statistics limit and for infinitesimal bin widths.

In order to obtain measurements of two-particle correlation functions in terms of the particle separation in rapidity Δy and azimuth $\Delta\phi$, one first sums over the p_{\perp} indices $\alpha_{p_{\perp}}$ and $\beta_{p_{\perp}}$ to obtain densities that are functions of rapidity and angle

exclusively:

$$\hat{\rho}_1^{(p)}(\vec{\alpha}^{(2)}) = \sum_{\alpha_{p_{\perp}}=1}^{m_{p_{\perp}}} \hat{\rho}_1^{(p)}(\vec{\alpha}),$$

$$\hat{\rho}_2^{(p,q)}(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)}) = \sum_{\alpha_{p_{\perp}}, \beta_{p_{\perp}}=1}^{m_{p_{\perp}}} \hat{\rho}_2^{(p,q)}(\vec{\alpha}, \vec{\beta}), \quad (5)$$

where $\vec{\alpha}^{(2)} = (\alpha_y, \alpha_{\phi})$ and $\vec{\beta}^{(2)} = (\beta_y, \beta_{\phi})$. The four-dimensional normalized cumulant $R_2^{(p,q)}(y_1, \phi_1, y_2, \phi_2)$ is then evaluated according to

$$R_2^{(p,q)}(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)}) = \frac{\hat{\rho}_2^{(p,q)}(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)})}{\hat{\rho}_1^{(p)}(\vec{\alpha}^{(2)}) \hat{\rho}_1^{(q)}(\vec{\beta}^{(2)})} - 1. \quad (6)$$

Finally, $R_2^{(p,q)}$ is obtained in terms of rapidity and azimuthal angle differences according to

$$R_2^{(p,q)}(\Delta\vec{\alpha}) = \frac{1}{\Omega(\Delta\alpha_y)} \sum_{\alpha_y, \alpha_{\phi}, \beta_y, \beta_{\phi}} R_2^{(p,q)}(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)}) \times \delta(\Delta\alpha_y - \alpha_y + \beta_y) \delta(\Delta\alpha_{\phi} - \alpha_{\phi} + \beta_{\phi}), \quad (7)$$

where the index $\Delta\alpha_y$ corresponds to rapidity difference bins, Δy , in the range $y_{\min} \leq y < y_{\max}$ and the index $\Delta\alpha_{\phi}$ corresponds to azimuthal difference bins, $\Delta\phi$, in the range $0 \leq \phi < 2\pi$, while $\Omega(\Delta\alpha_y)$ is a normalization constant that accounts for the width of the experimental acceptance in $\bar{y} = \frac{1}{2}(y_1 + y_2)$ at a given Δy . The sums are taken over all rapidity and azimuthal bins and the delta functions ensure that the differences of rapidity (angle) bins are properly matched to the Δy ($\Delta\phi$) bins represented by $\Delta\alpha$. Note that the above integer arithmetic yields some bin sharing (often termed aliasing). This bin sharing can be modeled and corrected for or suppressed by oversampling. The bin sharing has modest effects as long as the cumulant changes slowly with Δy and $\Delta\phi$.

Equations (3) and (4) express unbiased estimators of the densities $\rho_1^{(p)}$ and $\rho_2^{(p,q)}$ in the absence of particle losses and contamination from secondary particles or feed-down decays. The strength of the background associated with secondary particles may be evaluated with various track quality criteria, e.g., by applying a selection criterion on the distance of closest approach of charged tracks to the collision primary vertex, while contributions from feed-down may require modeling of such decays. In the context of the extension of the identity method to measurements of differential correlation functions presented in this work, the focus is on the effects of particle losses. To this end, one must first describe the calculation of the moments of the multiplicities in bins $\vec{\alpha}$ and $\vec{\beta}$ in the presence of fluctuations associated with particle losses.

Proceeding similarly as in Ref. [14], one describes fluctuations in the particle production according to a hypothetical (true) joint probability distribution $P_T(N_1, N_2, \dots, N_K)$, in which N_1, N_2, \dots, N_K represent vectors of the (produced) multiplicity of particles of species $p = 1, \dots, K$ in momentum-space bins $\vec{\alpha} \equiv (\alpha_y, \alpha_{\phi}, \alpha_{p_{\perp}})$, where $\alpha_y = 1, \dots, m_y, \alpha_{\phi} = 1, \dots, m_{\phi}$, and $\alpha_{p_{\perp}} = 1, \dots, m_{p_{\perp}}$. It is also convenient to define vectors \vec{n}_p and $\vec{\epsilon}_p$ corresponding to vectors of measured multiplicities and detection efficiencies (defined

later in this section). One can then write

$$\vec{n}_p = (n_p(1, 1, 1), n_p(1, 1, 2), \dots, n_p(m_y, m_\phi, m_{p_\perp})), \quad (8)$$

$$\vec{N}_p = (N_p(1, 1, 1), N_p(1, 1, 2), \dots, N_p(m_y, m_\phi, m_{p_\perp})), \quad (9)$$

$$\vec{\varepsilon}_p = (\varepsilon_p(1, 1, 1), \varepsilon_p(1, 1, 2), \dots, \varepsilon_p(m_y, m_\phi, m_{p_\perp})). \quad (10)$$

Moments of the multiplicities $N_p(\vec{\alpha})$ are calculated according to

$$\langle N_p(\vec{\alpha}) \rangle = \sum_{\vec{N}} N_p(\vec{\alpha}) P_T(\vec{N}_1, \vec{N}_2, \dots, \vec{N}_K), \quad (11)$$

$$\begin{aligned} \langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \\ = \sum_{\vec{N}} N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \\ \times P_T(\vec{N}_1, \vec{N}_2, \dots, \vec{N}_K), \end{aligned} \quad (12)$$

where the shorthand notation $\sum_{\vec{N}}$ is defined according to

$$\begin{aligned} \sum_{\vec{N}} = \sum_{N_1(1,1,1)=0}^{\infty} \dots \sum_{N_1(m_y, m_\phi, m_{p_\perp})=0}^{\infty} \sum_{N_2(1,1,1)=0}^{\infty} \dots \\ \sum_{N_2(m_y, m_\phi, m_{p_\perp})=0}^{\infty} \dots \sum_{N_K(1,1,1)=0}^{\infty} \dots \sum_{N_K(m_y, m_\phi, m_{p_\perp})=0}^{\infty}. \end{aligned} \quad (13)$$

Experimentally, measurements of particle production are subjected to random losses of particles. Assuming the detection of the N particles amounts to N independent processes, i.e., provided that the probability of detecting the N particles jointly is equal to the product of the probabilities of detecting each of the particles independently, one models the particle detection process in bin $\vec{\alpha}$ according to a binomial distribution $B(n_p(\vec{\alpha})|N_p(\vec{\alpha}), \varepsilon_p(\vec{\alpha}))$ defined according to

$$B(n|N, \varepsilon) = \frac{N!}{n!(N-n)!} \varepsilon^n (1-\varepsilon)^{N-n}, \quad (14)$$

where $\varepsilon_p(\vec{\alpha})$ represents the detection efficiency of particle species p in phase-space bin $\vec{\alpha}$, while $n_p(\vec{\alpha})$ and $N_p(\vec{\alpha})$ are the measured and true particle multiplicities in that bin. In general, detection efficiencies differ for species $p = 1, \dots, K$ and may also feature dependences on y , ϕ , and p_\perp , represented here as discretized functions $\varepsilon_p(\vec{\alpha})$.

The joint probability of measuring multiplicities $n_p(\vec{\alpha})$ in bin $\vec{\alpha}$ is represented with a joint probability distribution, $P_M(\vec{n}_1, \dots, \vec{n}_K)$, defined similarly as the true distribution $P_T(\vec{N}_1, \dots, \vec{N}_K)$. For binomial efficiency sampling, $P_M(\vec{n}_1, \dots, \vec{n}_K)$ can be expressed in terms of the true joint

probability distribution $P_T(\vec{N}_1, \dots, \vec{N}_K)$ according to

$$\begin{aligned} P_M(\vec{n}_1, \dots, \vec{n}_K) = \sum_{\vec{N}_p} P_T(\vec{N}_1, \dots, \vec{N}_K) \\ \times \prod_{\vec{\alpha}_1} B(n_1(\vec{\alpha}_1)|N_1(\vec{\alpha}_1), \varepsilon_1(\vec{\alpha}_1)) \\ \times \prod_{\vec{\alpha}_2} B(n_2(\vec{\alpha}_2)|N_2(\vec{\alpha}_2), \varepsilon_2(\vec{\alpha}_2)) \dots \\ \prod_{\vec{\alpha}_K} B(n_K(\vec{\alpha}_K)|N_K(\vec{\alpha}_K), \varepsilon_K(\vec{\alpha}_K)), \end{aligned} \quad (15)$$

where the shorthand notation $\prod_{\vec{\alpha}}$ is defined as

$$\prod_{\vec{\alpha}} = \prod_{\alpha_y=1}^{m_y} \prod_{\alpha_\phi=1}^{m_\phi} \prod_{\alpha_{p_\perp}=1}^{m_{p_\perp}}. \quad (16)$$

The first- and second-order moments of n_p are calculated according to

$$\begin{aligned} \langle n_p(\vec{\alpha}) \rangle = \sum_{\vec{n}} n_p(\vec{\alpha}) P_M(\vec{n}_1, \dots, \vec{n}_K), \quad (17) \\ \langle n_p(\vec{\alpha}) [n_q(\vec{\alpha}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \\ = \sum_{\vec{n}} n_p(\vec{\alpha}) [n_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] P_M(\vec{n}_1, \dots, \vec{n}_K), \end{aligned} \quad (18)$$

where $\sum_{\vec{n}}$ represents sums over all particle species and all kinematic bins $\vec{\alpha}$ and $\vec{\beta}$, and expressions $\langle O \rangle$ once again refer to event ensemble averages of the single and pair multiplicities observed event by event in the distinct kinematic bins $\vec{\alpha}$ and $\vec{\beta}$.

For narrow bins (but wide enough to neglect smearing and bin sharing) and binomial (efficiency) sampling, one readily verifies that the measured single particle and pair multiplicities satisfy

$$\langle n_p(\vec{\alpha}) \rangle = \varepsilon_p(\vec{\alpha}) \langle N_p(\vec{\alpha}) \rangle, \quad (19)$$

$$\begin{aligned} \langle n_p(\vec{\alpha}) [n_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \\ = \varepsilon_p(\vec{\alpha}) \varepsilon_q(\vec{\beta}) \langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle, \end{aligned} \quad (20)$$

Evidently, if the joint detection of particles in bins $(\vec{\alpha})$ and $(\vec{\beta})$ is correlated, one must replace the products $\varepsilon_p(\vec{\alpha}) \varepsilon_q(\vec{\beta})$ by true pair efficiencies $\varepsilon_{pq}(\vec{\alpha}, \vec{\beta})$. In general, however, one finds pair efficiencies factorize to a good approximation, and measurements of the $R_2^{(p,q)}$ correlation function in six dimensions are thus in principle inherently robust against single-particle losses associated with detector or track reconstruction algorithm artifacts [22] given, for instance,

$$\begin{aligned} R_2^M(\vec{\alpha}, \vec{\beta}) = \frac{\langle n_p(\vec{\alpha}) n_q(\vec{\beta}) \rangle}{\langle n_p(\vec{\alpha}) \rangle \langle n_q(\vec{\beta}) \rangle} - 1 \\ = \frac{\varepsilon_p(\vec{\alpha}) \varepsilon_q(\vec{\beta}) \langle N_p(\vec{\alpha}) N_q(\vec{\beta}) \rangle}{\varepsilon_p(\vec{\alpha}) \langle N_p(\vec{\alpha}) \rangle \varepsilon_q(\vec{\beta}) \langle N_q(\vec{\beta}) \rangle} - 1, \end{aligned} \quad (21)$$

$$= \frac{\langle N_p(\vec{\alpha}) N_q(\vec{\beta}) \rangle}{\langle N_p(\vec{\alpha}) \rangle \langle N_q(\vec{\beta}) \rangle} - 1 \equiv R_2^T(\vec{\alpha}, \vec{\beta}), \quad (22)$$

where $R_2^M(\vec{\alpha}, \vec{\beta})$ and $R_2^T(\vec{\alpha}, \vec{\beta})$ represent the measured and true normalized cumulants, respectively. In practice, however, a measurement in six dimensions is challenging because at high transverse momentum the number of particles observed in a given bin $\vec{\alpha}$ may be too small to enable a meaningful evaluation of R_2 with the above expression. Rather than calculating the ratio in six dimensions, it is more practical and common to first integrate the single and pair densities in transverse momentum to obtain a measurement of R_2 in four dimensions, as in Eq. (6), with subsequent averaging over the acceptance to obtain a measurement as a function of Δy and $\Delta\phi$, as in Eq. (7).

Using Eqs. (19) and (20), one writes

$$\langle N_p(\vec{\alpha}^{(2)}) \rangle = \sum_{\alpha_{p\perp}=1}^{m_{p\perp}} \frac{\langle n_p(\vec{\alpha}) \rangle}{\varepsilon_p(\vec{\alpha})}, \quad (23)$$

$$\begin{aligned} & \langle N_p(\vec{\alpha}^{(2)}) [N_p(\vec{\beta}^{(2)}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \\ &= \sum_{\alpha_{p\perp}, \beta_{p\perp}=1}^{m_{p\perp}} \frac{\langle n_p(\alpha_y, \alpha_\phi, \alpha_{p\perp}) [n_p(\beta_y, \beta_\phi, \beta_{p\perp}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle}{\varepsilon_p(\alpha_y, \alpha_\phi, \alpha_{p\perp}) \varepsilon_p(\beta_y, \beta_\phi, \beta_{p\perp})}, \end{aligned} \quad (24)$$

where $\vec{\alpha}^{(2)} = (\alpha_y, \alpha_\phi)$, $\vec{\beta}^{(2)} = (\beta_y, \beta_\phi)$. Division by efficiencies nominally corrects for nonuniform particle losses across the detector acceptance. Note that it is here assumed that pair efficiencies factorize into products of single efficiencies. This may not be appropriate if the momentum bins are very narrow, thereby corresponding to detection configurations in which tracks may nearly or fully overlap (e.g., in a time projection chamber) or share many common detection units (e.g., in segmented tracking chambers). For measurements of pair correlations within such narrow bins, it is then more appropriate to divide by a pair efficiency that accounts for pair losses due to partial or full track overlaps. Either way, the measured normalized cumulants, corrected for efficiencies, become

$$R_2^{M(p,q)}(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)}) = \frac{\langle N_p(\vec{\alpha}^{(2)}) (N_q(\vec{\beta}^{(2)}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}) \rangle}{\langle N_p(\vec{\alpha}^{(2)}) \rangle \langle N_q(\vec{\beta}^{(2)}) \rangle} - 1, \quad (25)$$

where $p, q = 1, \dots, K$ and $\vec{\alpha}^{(2)}$ and $\vec{\beta}^{(2)}$ represent arbitrary kinematic bins in the acceptance of the measurement.

Correlated losses and efficiency dependences on detection geometry (e.g., dependence of the efficiency on the collision vertex position), accelerator luminosity, detector occupancy, etc., can be handled with vertex position or luminosity dependent weights [15]. Such effects are neglected in the discussion that follows but are relatively straightforward to implement. It should be noted in closing this section that the dimensionality reduction achieved in Eqs. (23) and (24) can be trivially extended to yield correlation functions that are functions of Δy or $\Delta\phi$ only, or even integral correlations yielding measures of multiplicity fluctuations such as those discussed in Ref. [14].

III. DIFFERENTIAL R_2 MEASUREMENTS WITH THE IDENTITY METHOD

Studies of R_2 (and similar observables) have been conducted for a variety of collision systems and beam energies, various momentum ranges, and for a wide range of particle pair types ranging from inclusive charged particles to specific charge combinations, and even specific particle species. Measurements of $R_2^{(p,q)}$ for specific particle species p and q , e.g., pions (π^\pm), kaons (K^\pm), or protons (p or \bar{p}), are of particular interest as they provide more detailed information about the particle production process than semiexclusive single particle measurements or inclusive correlation measurements. They may also be combined to obtain charge dependent correlations, balance functions, and general balance functions that may further our understanding of particle production dynamics in nuclear collisions. In the context of traditional measurements, charged particle species are identified with cuts on particle identification (PID) signals from a time projection chamber (TPC), a time-of-flight system (TOF), etc. Unfortunately, with such techniques, the necessity to properly disambiguate particle species typically implies the measured kinematic ranges must be limited to regions of good PID separation thereby leading to potentially substantial particle losses. Using the identity method, however, one can recover most of the statistics lost with conventional cut methods and significantly extend the kinematic range of an analysis. The method was introduced in Ref. [11] for two particle species, $p = 1, 2$, extended in Ref. [12,13] for $K > 2$ species, i.e., for $p, q = 1, \dots, K$, and the determination of higher moments, and further extended in Ref. [14] to explicitly account for p_\perp -dependent detection efficiencies. In this and the next section, one shows that the efficiency-dependent identity method [14] can be further extended to differential correlation functions, such as R_2 , provided one discretizes single and pair densities according to Eqs. (3) and (4). The method presented in this section relies on a single PID variable, e.g., energy loss in a time projection chamber. It is extended to measurements involving two or more PID signals in the following section.

Within the identity method, rather than attempting to unambiguously identify the species of measured particles event by event, one relies on a probabilistic evaluation of the moments $\langle n_k \rangle$ and $\langle n_k(n_k - 1) \rangle$. Specifically, instead of summing integer counts (1 for an identified particle, 0 otherwise), one accounts for ambiguities by summing weights $\omega_k(m)$ for each PID hypothesis. The weights are determined particle by particle for each hypothesis $k = 1, \dots, K$, according to the relative frequency of particles of type k for a PID signal of amplitude m (the ‘‘mass’’ signal) defined by

$$\omega_k(m) \equiv \frac{\rho_k(m)}{\rho(m)}, \quad (26)$$

with

$$\rho(m) \equiv \sum_{k=1}^K \rho_k(m), \quad \int \rho_k(m) dm = \langle N_k \rangle, \quad (27)$$

where $\rho_k(m)$ represents the number density of the PID signal m for particles of type k and $\rho(m)$ is the ensemble averaged PID signal density. The weight $\omega_k(m)$ expresses the

probability a PID signal of amplitude m is generated by a particle of species k .

The goal of this work is to formulate differential correlations as functions of particle pair separation in rapidity and azimuthal angle using the identity method. It is important to first establish that the experimentally measured signal line shape can be meaningfully used to determine the relative probability of particle species on an event-by-event basis. As an example, one considers the energy loss signal dE/dx produced by charged particles in a TPC. The momentum space is discretized in m_y rapidity bins, m_ϕ azimuthal angle bins, and m_{p_\perp} transverse momentum bins. The detector response is thus expressed in terms of the discretized momentum vectors $\vec{\alpha}$ and $\vec{\beta}$ as defined in the previous section.

Let $P(k, \vec{\alpha})$ represent the probability of a particle of type $k = 1, \dots, K$ being produced in momentum bin $\vec{\alpha}$. Further define $P(d|k, \vec{\alpha}) \equiv \varepsilon_k(\vec{\alpha})$ as the conditional probability of the predicate d stating that a particle of type k and momentum $\vec{\alpha}$ is detected in the TPC, and $P(m|d, k, \vec{\alpha})$, the conditional probability density that this particle, being detected, produces a PID signal of amplitude m . The joint probability of having a particle of type k being detected in the TPC and producing a signal of amplitude m is thus

$$P(m, d, k, \vec{\alpha}) = P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha}), \quad (28)$$

where we substituted $\varepsilon_k(\vec{\alpha})$ for $P(d|k, \vec{\alpha})$. We use $P(m, d, k, \vec{\alpha})$ to calculate the probability that a signal of amplitude m corresponds to a particle of type k :

$$P(m, d, k, \vec{\alpha}) = P(k|m, d, \vec{\alpha})P(m|d, \vec{\alpha})P(d, \vec{\alpha}), \quad (29)$$

where $P(k|m, d, \vec{\alpha})$ represents the conditional probability that a track detected in the TPC with a PID signal of amplitude m and momentum-space coordinate bin $\vec{\alpha}$ corresponds to a particle of type k , $P(m|d, \vec{\alpha})$ represents the conditional probability density that a PID signal of amplitude m will be produced when a particle within the momentum-space bin $\vec{\alpha}$ is detected in the TPC, and $P(d, \vec{\alpha})$ represents the joint probability a particle of momentum $\vec{\alpha}$ be observed in the TPC. Using Eqs. (28) and (29), one writes (Bayes' theorem)

$$P(k|m, d, \vec{\alpha}) = \frac{P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})}{P(m|d, \vec{\alpha})P(d, \vec{\alpha})}. \quad (30)$$

The quantity $\varepsilon_k(\vec{\alpha})$ represents the detection efficiency of particles of type k at momentum $\vec{\alpha}$ and can be determined by Monte Carlo simulations of the detector performance or by embedding techniques. $P(m|d, k, \vec{\alpha})$ represents the line shape of the PID signal m associated with a detected particle of type k [it corresponds to $\omega_k(m)$ in Eq. (26)], whereas $P(k, \vec{\alpha}) = P(k|\vec{\alpha})P(\vec{\alpha})$ corresponds to the joint probability, determined statistically from the event ensemble average, that a produced particle of momentum $\vec{\alpha}$ and type k are detected. The quantity $P(m|d, \vec{\alpha})$ represents the probability that a PID signal m is observed when a particle at momentum α is detected, while $P(d, \vec{\alpha})$ represents the joint probability that a particle be detected in the TPC at a momentum $\vec{\alpha}$. $P(m|d, \vec{\alpha})$ is obtained by summing the probability densities of PID signal

m associated with all species:

$$P(m|d, \vec{\alpha})P(d, \vec{\alpha}) = \sum_{k=1}^K P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha}) \quad (31)$$

and

$$P(d, \vec{\alpha}) = \sum_{k=1}^K P(d|k, \vec{\alpha})P(k, \vec{\alpha}) = \sum_{k=1}^K \varepsilon_k(\vec{\alpha})P(k, \vec{\alpha}). \quad (32)$$

The overall line shape $P(m|d, \vec{\alpha}) = \rho(m)/\langle N \rangle$ is given by

$$P(m|d, \vec{\alpha}) = \frac{\sum_{k=1}^K P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})}{\sum_{k=1}^K \varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})}. \quad (33)$$

The conditional probability $P(k|m, d, \vec{\alpha})$ can then be expressed by

$$P(k|m, d, \vec{\alpha}) = \frac{P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})}{\sum_{k'=1}^K P(m|d, k', \vec{\alpha})\varepsilon_{k'}(\vec{\alpha})P(k', \vec{\alpha})}. \quad (34)$$

One finally obtains the line shape $\rho_k(m|\vec{\alpha})$ for particles of type k in the momentum bin $\vec{\alpha}$:

$$\rho_k(m|\vec{\alpha}) = P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})\langle N(\vec{\alpha}) \rangle, \quad (35)$$

where $\langle N(\vec{\alpha}) \rangle = \sum_k \langle N_k(\vec{\alpha}) \rangle$. One thus finds that, indeed, Eq. (34) is equivalent to Eq. (26), and $\omega_k(m)$ corresponds to the probability of species k given a PID signal of amplitude m at a specific momentum $\vec{\alpha}$, which one thus denotes

$$\omega_k(m|\vec{\alpha}) = \frac{P(m|d, k, \vec{\alpha})\varepsilon_k(\vec{\alpha})P(k, \vec{\alpha})}{\sum_{k'=1}^K P(m|d, k', \vec{\alpha})\varepsilon_{k'}(\vec{\alpha})P(k', \vec{\alpha})}. \quad (36)$$

The weights $\omega_k(m|\vec{\alpha})$ provide the correct probability of a particle being of species k given m only if they are evaluated as a function of the momentum vector $\vec{\alpha}$. Indeed, the relative probability of species $k = 1, \dots, K$ may be a function of rapidity, azimuthal angle, and transverse momentum. In practice, it may be unnecessary to use the same level of granularity for the determination of the weights $\omega_k(m|\vec{\alpha})$ and the study of the particle densities. This is particularly important in the context of experiments where efficiencies depend on the collision centrality (or event multiplicity), the collision vertex position, or any other additional variables.

Following the original identity method, one defines an event-by-event quantity W_p , hereafter called an event-wise identity variable, for species $p = 1, \dots, K$, as the sum of the weights $\omega_p(m|\vec{\alpha})$ over all M particles in an event which satisfy the kinematic and quality criteria used in the analysis:

$$W_p(\vec{\alpha}) \equiv \sum_{i=1}^M \omega_p(m_i|\vec{\alpha}). \quad (37)$$

The identity method involves calculating the moments of $W_p(\vec{\alpha})$, and we shall verify that they are linear combinations of the moments of $N_p(\vec{\alpha})$. For measurements of $R_2^{(p,q)}$, one only needs to consider the two lowest moments

$$\langle W_p(\vec{\alpha}) \rangle = \frac{1}{N_{\text{events}}} \sum_{i=1}^{N_{\text{events}}} W_p^{(i)}(\vec{\alpha}), \quad (38)$$

$$\langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle = \frac{1}{N_{\text{events}}} \sum_{i=1}^{N_{\text{events}}} W_p^{(i)}(\vec{\alpha})W_q^{(i)}(\vec{\beta}) \quad (39)$$

in which $W_p^{(i)}(\vec{\alpha})$ and $W_q^{(i)}(\vec{\beta})$ are event-wise identity variables for species p and q in events $i = 1, \dots, N_{\text{events}}$, measured in kinematic bins $\vec{\alpha}$ and $\vec{\beta}$, respectively.

Theoretically, calculations of the expectation values of the moments $\langle W_p(\vec{\alpha}) \rangle$ and $\langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle$ with particle losses proceed similarly as in Ref. [14], but one must properly average over all species, all bins $\vec{\alpha}$, and all particles in those bins. The resulting mathematical expressions are rather large and cumbersome; it is thus convenient to develop some additional shorthand notations. Given that one must account for binomial sampling in each bin $\vec{\alpha}$, for each species p , let us introduce

$$\mathbb{B}(\vec{n}_p, \vec{N}_p, \vec{\varepsilon}_p) = \prod_{\alpha_y=1}^{m_y} \prod_{\alpha_\phi=1}^{m_\phi} \prod_{\alpha_{p\perp}=1}^{m_{p\perp}} B(n_p(\vec{\alpha})|N_p(\vec{\alpha}), \varepsilon_p(\vec{\alpha})), \quad (40)$$

where $\vec{n}_p, \vec{N}_p, \vec{\varepsilon}_p$ represent vectors of values in all bins $\vec{\alpha} = (\alpha_y, \alpha_\phi, \alpha_{p\perp})$ introduced in Eq. (8).

One must also average over all possible values of PID signals, for all species, in all bins $\vec{\alpha}$. To that end, one defines functionals

$$\mathbb{P}_p(n_p(\vec{\alpha})) = \prod_{i=1}^{n_p(\vec{\alpha})} \int P(m_i|d, p, \vec{\alpha}) dm_i, \quad (41)$$

where $n_p(\vec{\alpha})$ is the number of particles of species p detected in bin $\vec{\alpha}$, m_i is the amplitude of the PID signal of the i th particle of type p in that bin, and $P(m_i|d, p, \vec{\alpha})$ is the probability density of such signals. In order to average over all bins $\vec{\alpha}$, one introduces the functionals

$$\mathbb{S}_p(\vec{n}_p) = \prod_{\alpha_y=1}^{m_y} \prod_{\alpha_\phi=1}^{m_\phi} \prod_{\alpha_{p\perp}=1}^{m_{p\perp}} \mathbb{P}_p(n_p(\vec{\alpha})). \quad (42)$$

The integrals within the functionals $\mathbb{P}_p(n_p(\vec{\alpha}))$ and $\mathbb{S}_p(\vec{n}_p)$ are to be evaluated when multiplied on the right by W_p . The expectation value of $W_p(\vec{\alpha})$ may then be written

$$\begin{aligned} \langle W_p(\vec{\alpha}) \rangle &= \sum_{\vec{N}} \sum_{\vec{n}} P_T(\vec{N}) \prod_{k=1}^K \mathbb{B}(\vec{n}_k, \vec{N}_k, \vec{\varepsilon}_k) \mathbb{S}_p(\vec{n}_p) \\ &\times \sum_{k'=1}^K \sum_{i_{k'}=1}^{n_{k'}(\vec{\alpha})} \omega_p(m_{i_{k'}}^{(k')}|\vec{\alpha}). \end{aligned} \quad (43)$$

This expression involves products of several integrals whose evaluation seems daunting. However, note that most of the integrals are of the form $\int P(m) dm = 1$ and thus do not contribute to $\langle W_p(\vec{\alpha}) \rangle$. Only integrals of the form $\int \omega_p(m) P(m|d, q, \vec{\alpha}) dm$ yield nonunitary values and must thus be accounted for. Similarly as in Ref. [14], it is convenient to introduce response coefficients

$$r_{pq}(\vec{\alpha}) = \int \omega_p(m|\vec{\alpha}) P(m|d, q, \vec{\alpha}) dm. \quad (44)$$

Equation (43) may then be written

$$\begin{aligned} \langle W_p(\vec{\alpha}) \rangle &= \sum_{\vec{N}} \sum_{\vec{n}} P_T(\vec{N}) \prod_{k=1}^K \mathbb{B}_k(\vec{n}_p, \vec{N}_p, \vec{\varepsilon}_p) \\ &\times \sum_{k'=1}^K r_{pk'}(\vec{\alpha}) n_{k'}(\vec{\alpha}). \end{aligned} \quad (45)$$

Sequential evaluation of the sums $\sum_{\vec{n}}$ and $\sum_{\vec{N}}$ yields

$$\begin{aligned} \langle W_p(\vec{\alpha}) \rangle &= \sum_{\vec{N}} P_T(\vec{N}) \sum_{k=1}^K r_{pk}(\vec{\alpha}) N_k(\vec{\alpha}) \varepsilon_k(\vec{\alpha}) \\ &= \sum_{k=1}^K r_{pk}(\vec{\alpha}) \langle N_k(\vec{\alpha}) \rangle \varepsilon_k(\vec{\alpha}). \end{aligned} \quad (46)$$

As in Ref. [14], it is convenient to absorb the efficiencies into the moments and write

$$\langle W_p(\vec{\alpha}) \rangle = \sum_{k=1}^K r_{pk}(\vec{\alpha}) \langle n_k(\vec{\alpha}) \rangle, \quad (47)$$

where, by definition, $\langle n_k(\vec{\alpha}) \rangle = \langle N_k(\vec{\alpha}) \rangle \varepsilon_k(\vec{\alpha})$. For a given bin $\vec{\alpha}$, the above equation expresses the averages $\langle W_p(\vec{\alpha}) \rangle$ as a linear combination of the average multiplicities $\langle n_k(\vec{\alpha}) \rangle$ determined by the coefficients $r_{pk}(\vec{\alpha})$. One then introduces vectors

$$\vec{\mathbb{W}}(\vec{\alpha}) \equiv (\langle W_1(\vec{\alpha}) \rangle, \langle W_2(\vec{\alpha}) \rangle, \dots, \langle W_K(\vec{\alpha}) \rangle), \quad (48)$$

$$\vec{\mathbb{N}}(\vec{\alpha}) \equiv (\langle n_1(\vec{\alpha}) \rangle, \langle n_2(\vec{\alpha}) \rangle, \dots, \langle n_K(\vec{\alpha}) \rangle) \quad (49)$$

and the response matrices

$$\mathbb{R}(\vec{\alpha}) = \begin{pmatrix} r_{11}(\vec{\alpha}) & \cdots & r_{1K}(\vec{\alpha}) \\ \vdots & \ddots & \vdots \\ r_{K1}(\vec{\alpha}) & \cdots & r_{KK}(\vec{\alpha}) \end{pmatrix}. \quad (50)$$

The K equations in (47) may then be written

$$\vec{\mathbb{W}}(\vec{\alpha}) = \mathbb{R}(\vec{\alpha}) \vec{\mathbb{N}}(\vec{\alpha}). \quad (51)$$

The average multiplicities $\vec{\mathbb{N}}(\vec{\alpha})$ are thus obtained by inversion of $\mathbb{R}(\vec{\alpha})$:

$$\vec{\mathbb{N}}(\vec{\alpha}) = [\mathbb{R}(\vec{\alpha})]^{-1} \vec{\mathbb{W}}(\vec{\alpha}), \quad (52)$$

and average multiplicities corrected for efficiency losses, $\langle N_p(\vec{\alpha}) \rangle$, are then calculated for each species $p = 1, \dots, K$ according to

$$\langle N_p(\vec{\alpha}) \rangle = \frac{\langle n_p(\vec{\alpha}) \rangle}{\varepsilon_p(\vec{\alpha})}. \quad (53)$$

Note that there are $m_y \times m_\phi \times m_{p\perp}$ independent matrix inversions to carry out, one for each momentum bin $\vec{\alpha}$. If momentum smearing was an important effect, one would have to invoke smearing response functions and all these matrix inversions would be coupled.

Evaluation of the second-order moments proceeds similarly. However, one must consider separately the four cases corresponding to Eq. (39): $\langle W_p(\vec{\alpha})^2 \rangle$, $\langle W_p(\vec{\alpha})W_p(\vec{\beta}) \rangle$,

$\langle W_p(\vec{\alpha})W_q(\vec{\alpha}) \rangle$, and $\langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle$, with $p \neq q$ and $\vec{\alpha} \neq \vec{\beta}$. Toward that end, it is convenient to define

$$r_{pqk}(\vec{\alpha}) = \int \omega_p(m|\vec{\alpha})\omega_q(m|\vec{\alpha})P(m|d, k, \vec{\alpha})dm. \quad (54)$$

The second and cross moments, $\langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle$, are calculated according to

$$\begin{aligned} \langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle &= \sum_{\vec{N}} \sum_{\vec{n}} P_T(\vec{N}) \prod_{k=1}^K \mathbb{B}(\vec{n}_k, \vec{N}_k, \vec{\varepsilon}_k) \mathbb{S}_k(\vec{n}_k) \\ &\times \left[\sum_{k'=1}^K \sum_{i_{k'}=1}^{n_{k'}} \omega_p(m_{i_{k'}}^{(k')}|\vec{\alpha}) \right]^2, \quad (55) \\ &= \sum_{k=1}^K r_{pqk}(\vec{\alpha}) \langle N_k(\vec{\alpha}) \rangle \varepsilon_k(\vec{\alpha}) \delta_{\vec{\alpha}, \vec{\beta}} \\ &+ \sum_{k, k'=1}^K r_{pk}(\vec{\alpha}) r_{qk'}(\vec{\beta}) \langle N_k(\vec{\alpha}) \rangle \langle N_{k'}(\vec{\beta}) \rangle \\ &- \delta_{k, k'} \delta_{\vec{\alpha}, \vec{\beta}} \varepsilon_k(\vec{\alpha}) \varepsilon_{k'}(\vec{\beta}). \quad (56) \end{aligned}$$

The efficiencies can be reabsorbed within the average multiplicities and number of pairs. The above expression simplifies to

$$\begin{aligned} \langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle &= \sum_{k=1}^K r_{pqk}(\vec{\alpha}) \langle n_k(\vec{\alpha}) \rangle \delta_{\vec{\alpha}, \vec{\beta}} \\ &+ \sum_{k, k'=1}^K r_{pk}(\vec{\alpha}) \langle n_k(\vec{\alpha}) \rangle \langle n_{k'}(\vec{\beta}) \rangle - \delta_{k, k'} \delta_{\vec{\alpha}, \vec{\beta}} r_{qk'}(\vec{\beta}), \quad (57) \end{aligned}$$

where $p, q = 1, \dots, K$, while $\vec{\alpha}$ and $\vec{\beta}$ represent arbitrary kinematic bins. It is useful to define the matrices

$$\mathbb{N}(\vec{\alpha}, \vec{\beta}) = \begin{bmatrix} N_{11}(\vec{\alpha}, \vec{\beta}) & N_{12}(\vec{\alpha}, \vec{\beta}) & \cdots & N_{1K}(\vec{\alpha}, \vec{\beta}) \\ N_{21}(\vec{\alpha}, \vec{\beta}) & N_{22}(\vec{\alpha}, \vec{\beta}) & \cdots & N_{2K}(\vec{\alpha}, \vec{\beta}) \\ \vdots & \vdots & \ddots & \vdots \\ N_{K1}(\vec{\alpha}, \vec{\beta}) & N_{K2}(\vec{\alpha}, \vec{\beta}) & \cdots & N_{KK}(\vec{\alpha}, \vec{\beta}) \end{bmatrix} \quad (58)$$

with elements

$$N_{pq}(\vec{\alpha}, \vec{\beta}) = \langle n_p(\vec{\alpha}) \rangle \langle n_q(\vec{\beta}) \rangle - \delta_{p, q} \delta_{\vec{\alpha}, \vec{\beta}} \quad (59)$$

and

$$\mathbb{V}(\vec{\alpha}, \vec{\beta}) = \begin{bmatrix} V_{11}(\vec{\alpha}, \vec{\beta}) & V_{12}(\vec{\alpha}, \vec{\beta}) & \cdots & V_{1K}(\vec{\alpha}, \vec{\beta}) \\ V_{21}(\vec{\alpha}, \vec{\beta}) & V_{22}(\vec{\alpha}, \vec{\beta}) & \cdots & V_{2K}(\vec{\alpha}, \vec{\beta}) \\ \vdots & \vdots & \ddots & \vdots \\ V_{K1}(\vec{\alpha}, \vec{\beta}) & V_{K2}(\vec{\alpha}, \vec{\beta}) & \cdots & V_{KK}(\vec{\alpha}, \vec{\beta}) \end{bmatrix} \quad (60)$$

with elements

$$V_{pq}(\vec{\alpha}, \vec{\beta}) = \langle W_p(\vec{\alpha})W_q(\vec{\beta}) \rangle - \sum_{k=1}^K r_{pqk}(\vec{\alpha}) \langle n_k(\vec{\alpha}) \rangle \delta_{\vec{\alpha}, \vec{\beta}}. \quad (61)$$

Equation (57) can then be written in matrix form,

$$\mathbb{V}(\vec{\alpha}, \vec{\beta}) = \mathbb{R}(\vec{\alpha}) \mathbb{N}(\vec{\alpha}, \vec{\beta}) \mathbb{R}(\vec{\beta})^T. \quad (62)$$

Multiplying on the left and on the right by the inverses of matrices $\mathbb{R}(\vec{\alpha})$ and $\mathbb{R}(\vec{\beta})^T$, one gets

$$\mathbb{N}(\vec{\alpha}, \vec{\beta}) = \mathbb{R}(\vec{\alpha})^{-1} \mathbb{V}(\vec{\alpha}, \vec{\beta}) (\mathbb{R}(\vec{\beta})^T)^{-1}. \quad (63)$$

This expression corresponds to a set of $(m_y \times m_\phi \times m_{p_\perp})^2$ independent equations, one for each pair of bins $\vec{\alpha}$ and $\vec{\beta}$. The matrices $\mathbb{N}(\vec{\alpha}, \vec{\beta})$ can thus be calculated independently for each pair $\vec{\alpha}, \vec{\beta}$. The elements of these matrices then yield the second and cross moments of the multiplicities, $\langle n_p(\vec{\alpha}) [n_q(\vec{\beta}) - \delta_{\vec{\alpha}, \vec{\beta}} \delta_{p, q}] \rangle$. The above formulation in terms of matrices \mathbb{V} involves a significant and convenient simplification of the inversion problem as it was first presented in Ref. [14].

Estimates of the true second moments, corrected for efficiency losses, are finally obtained according to

$$\langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{\vec{\alpha}, \vec{\beta}} \delta_{p, q}] \rangle = \frac{\langle n_p(\vec{\alpha}) [n_q(\vec{\beta}) - \delta_{\vec{\alpha}, \vec{\beta}} \delta_{p, q}] \rangle}{\varepsilon_p(\vec{\alpha}) \varepsilon_q(\vec{\beta})}. \quad (64)$$

Again, in this case, if two-particle efficiencies do not properly factorize into products of single-particle efficiencies, estimates of two-particle efficiencies can be used in the above in lieu of the products of singles.

The matrix inversion technique outlined above provides second moments of particle multiplicities corrected for efficiency across the fiducial acceptance of the experiment. It is worth noting, however, that while in the above formulation the matrices are small (determined by the number of species), there can be many of them to invert. For instance, for an analysis involving a rapidity acceptance $-1 \leq y \leq 1$ in 20 bins, full azimuthal acceptance in 72 bins, and 20 bins in p_\perp , one would need 28 800 matrices. This is evidently not an issue from a computational standpoint with modern computers, but it does have two practical implications. First, the available statistics will be distributed across many bins and it is conceivable that the number of entries in a given bin and the corresponding statistical uncertainty may yield numerically unstable results. Additionally, since the coefficients r_{pq} are based on global fits of the line shapes in each kinematic bin α (although, to reiterate, the granularity required for such fits can likely be coarse), it might be necessary to manually inspect all the fits and make sure they are not subject to idiosyncrasies of the analysis or the detector performance. Differential analyses with the identity method thus clearly have high computing and storage costs.

Finally, note that once the moments $\langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{\vec{\alpha}, \vec{\beta}} \delta_{p, q}] \rangle$ are obtained for some nominal range of transverse momentum, i.e., given m_{p_\perp} bins for α_{p_\perp} and β_{p_\perp} , one can readily obtain sums $\langle N_p(\vec{\alpha}^{(2)}) [N_q(\vec{\alpha}^{(2)}) - \delta_{\vec{\alpha}, \vec{\beta}} \delta_{p, q}] \rangle$ including all m_{p_\perp} bins or only a restricted range of p_\perp using Eq. (24). It is thus possible to compare results obtained with the identity method described here with those obtained with traditional cut methods (applicable only over a limited range of transverse momentum) by selecting appropriate p_\perp sum ranges for each species of interest.

IV. IDENTITY METHOD WITH TWO OR MORE IDENTITY SIGNALS

Large collider experiments commonly feature partially redundant and complementary techniques of particle identification. For instance, the STAR and ALICE experiments both include particle identification devices based on specific energy loss (dE/dx) and time-of-flight (TOF) measurements. The ALICE detector additionally features transition radiation detectors geared towards the identification of electrons. Bayesian identification techniques based on cuts have already been developed that exploit the joint information from several PID detectors on a track-by-track basis. While such techniques maximize the use of information from the multiple components of a detector, such as the ALICE detector [23–28], they nonetheless suffer statistical losses associated with the use of PID selection criteria. This section describes an extension of the identity method applied to detectors featuring several PID signal types available for each track.

As a preamble to the discussion, note that PID detector components suffer efficiency losses, and tracking algorithms may fail to associate a given PID detector signal to a track. In particular, there are kinematic regions in which usable dE/dx and TOF signals may not be obtainable. One may thus end up reconstructing tracks that feature no usable dE/dx signal but a reliable TOF signal, no TOF signal but a reliable dE/dx signal, or no useful PID signal at all. Since the point of the identity method is to utilize all of the available information, one needs to devise a technique to statistically include all tracks featuring PID signals, even though the information may be incomplete. One must thus first consider the combination of probabilistic statements about the PID of particles.

In this context, one once again uses the many probability functions [e.g., $P(p, \vec{\alpha})$, $P(d|p, \vec{\alpha})$] that were introduced in Sec. III. However, one must also introduce a few additional definitions and probability functions. Assume there are N_D detector components potentially producing PID signals that may be associated to a track. Let D_j , for $j = 1, \dots, N_D$, represent the predicate “the track is detected (or matched to a signal) in device j ,” where one arbitrarily assigns $j = 1$, for instance, to a TPC, $j = 2$ to a TOF detector, and so on. Additionally, let E_j , for $j = 1, \dots, N_D$, represent the predicate “the PID info of device j is usable.” Finally, let m_j represent the PID signals produced by devices $j = 1, \dots, N_D$. For a given track, these can be conveniently expressed as $\vec{m} = (m_1, m_2, \dots, m_{N_D})$. Assuming detector topologies similar to those of STAR and ALICE, consideration of the PID information provided by a detector component $j \geq 2$ is only meaningful if a track is first detected in device $j = 1$ (e.g., a TPC track). Indeed, in the context of these experiments, the detection of a hit in the TOF detector is not useful unless it can be matched to a track from the TPC. One consequently requires that D_1 be true. However, E_1 is not necessarily required so long as one of the other devices produces a usable PID signal, i.e., if there exists one E_j .

Let us first consider the predicate logic for a detection system involving two components. One shall see how it can be generalized to more than two components later in this paragraph. In the following, one indicates a true predicate by its name: D_2 means that a given particle is detected

or matched in device $j = 2$, while a barred predicate, \bar{D}_2 , indicates the track is not detected or matched in device $j = 2$. Using commas to denote logical conjunctions, for a two-component detection system, only the predicate combinations (D_1, E_1, \bar{D}_2) , (D_1, E_1, D_2, E_2) , $(D_1, E_1, D_2, \bar{E}_2)$, and $(D_1, \bar{E}_1, D_2, E_2)$ provide conditions with useful PID information. For instance (D_1, E_1, \bar{D}_2) means a track was detected (D_1) in device $j = 1$, produced a usable PID signal (E_1) in that device, but was not detected in device $j = 2$ (\bar{D}_2). Clearly, usable PID information from detector 2 (E_2) can only be present if there is a signal in detector 2 (D_2). The alternative, \bar{E}_2 , encompasses the case in which there is a signal in detector 2 (D_2) but no usable PID information as well as the cases where there is no signal in detector 2 (\bar{D}_2). Therefore the information about D_2 and \bar{D}_2 is absorbed into E_2 and \bar{E}_2 . If additional PID devices are available, one needs to consider all permutations deemed appropriate. For instance, with the addition of a third device, one might have (D_1, E_1, E_2, E_3) , $(D_1, \bar{E}_1, E_2, E_3)$, $(D_1, E_1, \bar{E}_2, E_3)$, $(D_1, E_1, E_2, \bar{E}_3)$, $(D_1, \bar{E}_1, \bar{E}_2, E_3)$, $(D_1, \bar{E}_1, E_2, \bar{E}_3)$, and $(D_1, E_1, \bar{E}_2, \bar{E}_3)$. For the sake of simplicity in the remainder of this work, the discussion is limited to two PID devices only, but extensions to $N_D > 2$ are relatively straightforward.

The momentum and species of the particles must also be accounted for. As in Sec. III, let k, p, q , with $k, p, q = 1, \dots, K$, denote species indices (assuming K distinct possibilities) and let $\vec{\alpha}$ and $\vec{\beta}$ represent momentum bin index vectors. The probability of detecting a track produced by a particle of species p in momentum bin $\vec{\alpha}$ (i.e., the efficiency) is denoted $d_1 \equiv P(D_1|p, \alpha)$.

The probabilities that a particle produces a meaningful PID signal in both detectors (E_1, E_2), in detector 1 but not in detector 2 (E_1, \bar{E}_2), in detector 2 but not in detector 1 (\bar{E}_1, E_2), or neither detector (\bar{E}_1, \bar{E}_2) are given by $\varepsilon_{12} \equiv e_1 e_2 d_1$, $\varepsilon_{11} \equiv e_1(1 - e_2)d_1$, $\varepsilon_{22} \equiv e_2(1 - e_1)d_1$, and $\varepsilon_0 \equiv (1 - e_1)(1 - e_2)d_1 + 1 - d_1$, respectively. Here, e_1 denotes the probability of having a usable PID signal in detector 1, and e_2 denotes the product of the probabilities of detecting, matching, and having a useful signal in detector 2. Given an event with $N(p, \alpha)$ particles of type p within the momentum bin $\vec{\alpha}$, the number of tracks detected with conditions (E_1, E_2) , (E_1, \bar{E}_2) , (\bar{E}_1, E_2) , are hereafter denoted $n_{12}(p, \vec{\alpha})$, $n_{11}(p, \vec{\alpha})$, $n_{22}(p, \vec{\alpha})$, and the number of undetected tracks (i.e., tracks not detected or those detected without a usable PID signal) is n_0 . These numbers shall evidently fluctuate event by event. The probability of a given combination of the numbers is given by a multinomial probability distribution $M(n_{12}(p, \vec{\alpha}), n_{11}(p, \vec{\alpha}), n_{22}(p, \vec{\alpha})|N, \varepsilon_{12}(p, \vec{\alpha}), \varepsilon_{11}(p, \vec{\alpha}), \varepsilon_{22}(p, \vec{\alpha}), \varepsilon_0(p, \vec{\alpha}))$ defined according to

$$M(n_{12}, n_{11}, n_{22}|N, \varepsilon_{12}, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_0) = \frac{N!}{n_{12}!n_{11}!n_{22}!n_0!} \varepsilon_{12}^{n_{12}} \varepsilon_{11}^{n_{11}} \varepsilon_{22}^{n_{22}} \varepsilon_0^{n_0}, \quad (65)$$

where the labels p and $\vec{\alpha}$ were omitted for the sake of simplicity, and $n_0 = N - n_{12} - n_{11} - n_{22}$.

One must next consider the probability density distributions of signals m_1 and m_2 . Assuming the generation of PID signals m_1 and m_2 are statistically independent, let

$P(m_1|E_1, D_1, p, \alpha)$ and $P(m_2|E_2, D_1, p, \alpha)$ respectively represent the probability densities of signals m_1 and m_2 , with normalization $\int P(m_i|E_i, D_1, p, \alpha) dm_i = 1$, for $i = 1, 2$. Probability distribution functions expressing the probability that a measured particle is of type p given PID signals of amplitude m_i , $i = 1, 2$ are obtained with Bayes' theorem:

$$P(p|m_1, m_2, E_1, E_2, D_1, \alpha) = \frac{P(m_1|E_1, D_1, p, \alpha)P(m_2|E_2, D_1, p, \alpha)P(E_1, E_2, D_1, p, \alpha)}{\sum_q P(m_1, m_2, E_1, E_2, D_1, q, \alpha)}, \quad (66)$$

$$P(p|m_1, E_1, \bar{E}_2, D_1, \alpha) = \frac{P(m_1|E_1, D_1, p, \alpha)P(E_1, \bar{E}_2, D_1, p, \alpha)}{\sum_q P(m_1, E_1, \bar{E}_2, D_1, q, \alpha)}, \quad (67)$$

$$P(p|m_2, \bar{E}_1, E_2, D_1, \alpha) = \frac{P(m_2|E_2, D_1, p, \alpha)P(\bar{E}_1, E_2, D_1, p, \alpha)}{\sum_q P(m_1, \bar{E}_1, E_2, D_1, q, \alpha)}, \quad (68)$$

It is convenient to use the shorthand notation $\vec{m} = (m_1, m_2)$ to define weights according to

$$\omega_p^{(12)}(\vec{m}|\vec{\alpha}) = P(p|m_1, m_2, E_1, E_2, D_1, \vec{\alpha}), \quad (69)$$

$$\omega_p^{(1)}(\vec{m}|\vec{\alpha}) = P(p|m_1, E_1, \bar{E}_2, D_1, \vec{\alpha})\delta(m_2), \quad (70)$$

$$\omega_p^{(2)}(\vec{m}|\vec{\alpha}) = P(p|m_2, \bar{E}_1, E_2, D_1, \vec{\alpha})\delta(m_1), \quad (71)$$

where both signals m_1 and m_2 are included in all three cases for notational convenience in the following. The delta function factors $\delta(m_1)$ and $\delta(m_2)$ are included to signify explicitly that the signals m_1 and m_2 are not relevant for weights $\omega_p^{(2)}(\vec{m}|\vec{\alpha})$ and $\omega_p^{(1)}(\vec{m}|\vec{\alpha})$, respectively. The weights $\omega_p^{(12)}(\vec{m}|\vec{\alpha})$, $\omega_p^{(1)}(\vec{m}|\vec{\alpha})$, and $\omega_p^{(2)}(\vec{m}|\vec{\alpha})$, can thus be represented as $\omega_p^{(T)}(\vec{m}|\vec{\alpha})$ with types $T = (12), (1)$, and (2) , respectively.

The event-wise identity variable W_p is defined according to

$$W_p(\vec{\alpha}) = \sum_{i=1}^{n_{12}} \omega_p^{(12)}(\vec{m}_i|\vec{\alpha}) + \sum_{i=1}^{n_1} \omega_p^{(1)}(\vec{m}_i|\vec{\alpha}) + \sum_{i=1}^{n_2} \omega_p^{(2)}(\vec{m}_i|\vec{\alpha}), \quad (72)$$

$$= \sum_T \sum_{i_T=1}^{n_T} \omega_p^{(T)}(\vec{m}_{i_T}|\vec{\alpha}), \quad (73)$$

where in the first line the three sums account for tracks satisfying (E_1, E_2) , (E_1, \bar{E}_2) , and (\bar{E}_1, E_2) , respectively, while in the second line they were replaced with the sum $\sum_{(T)}$, which represents a sum (of sums) for cases (12) , (1) , and (2) .

One next proceeds to calculate the expectation value of the moments $W_p(\vec{\alpha})$ and $W_p(\vec{\alpha})W_q(\vec{\beta})$. To this end, one defines coefficients $r_{pj}^{(T)}(\vec{\alpha})$ and $r_{pqj}^{(T)}(\vec{\alpha})$ with $(T) = (12), (1), (2)$, which are analogs of coefficients defined by Eqs. (44) and (54), according to

$$r_{pj}^{(T)}(\vec{\alpha}) = \int \omega_p^{(T)}(\vec{m}|\vec{\alpha})P(\vec{m}|T, j, \vec{\alpha})dm_1dm_2, \quad (74)$$

$$r_{pqj}^{(T)}(\vec{\alpha}) = \int \omega_p^{(T)}(\vec{m}|\vec{\alpha})\omega_q^{(T)}(\vec{m}|\vec{\alpha})P(\vec{m}|T, j, \vec{\alpha})dm_1dm_2, \quad (75)$$

where, for convenience, we also used the shorthand T within the probabilities $P(\vec{m}|T, j, \vec{\alpha})$ to represent the permutations (E_1, E_2, D_1) , (E_1, \bar{E}_2, D_1) , (\bar{E}_1, E_2, D_1) . In order to carry out sums on the measured particles, one needs to insert multinomial distributions in each kinematic bin. One must also average over all possible multiplicity configurations in moment space spanned by $\vec{\alpha}$. We thus define the notation

$$\mathbb{M}(\vec{n}_p^{(12)}, \vec{n}_p^{(1)}, \vec{n}_p^{(2)}, \vec{N}_p, \vec{\varepsilon}_p^{(12)}, \vec{\varepsilon}_p^{(1)}, \vec{\varepsilon}_p^{(2)}) = \prod_{\alpha_y=1}^{m_y} \prod_{\alpha_\phi=1}^{m_\phi} \prod_{\alpha_{p_\perp}=1}^{m_{p_\perp}} M(n_p^{(12)}(\vec{\alpha}), n_p^{(1)}(\vec{\alpha}), n_p^{(2)}(\vec{\alpha})|N_p(\vec{\alpha}), \vec{\varepsilon}_p^{(12)}, \vec{\varepsilon}_p^{(1)}, \vec{\varepsilon}_p^{(2)}), \quad (76)$$

where $\vec{n}_p^{(12)}$, $\vec{n}_p^{(1)}$, $\vec{n}_p^{(2)}$ represent vectors of values of the number of particles detected with (E_1, E_2) , (E_1, \bar{E}_2) , and (\bar{E}_1, E_2) , respectively, in all bins $\vec{\alpha} = (\alpha_y, \alpha_\phi, \alpha_{p_\perp})$. \mathbb{M} expresses the probability of measurement outcomes for a given species p over the full space $\vec{\alpha}$. One must also average over all possible values of PID signals, for all species, and for all bins $\vec{\alpha}$. To that end, one define functionals

$$\mathbb{P}_p^{(T)}(n_p^{(T)}(\vec{\alpha})) = \prod_{k=1}^{n_p^{(T)}(\vec{\alpha})} \int P(\vec{m}_k|T, p, \vec{\alpha})d\vec{m} \quad (77)$$

and

$$\mathbb{S}_p(\vec{n}_p^{(12)}, \vec{n}_p^{(1)}, \vec{n}_p^{(2)}) = \prod_{\alpha_y=1}^{m_y} \prod_{\alpha_\phi=1}^{m_\phi} \prod_{\alpha_{p_\perp}=1}^{m_{p_\perp}} \mathbb{P}_p^{(12)}(n_p^{(12)}(\vec{\alpha})) \mathbb{P}_p^{(1)}(n_p^{(1)}(\vec{\alpha})) \mathbb{P}_p^{(2)}(n_p^{(2)}(\vec{\alpha})). \quad (78)$$

The integrals within the functionals $\mathbb{P}_p(n_p(\vec{\alpha}))$ and $\mathbb{S}_p(\vec{n}_p^{(12)}, \vec{n}_p^{(1)}, \vec{n}_p^{(2)})$ are to be evaluated when multiplied on the right by $W_p(\vec{\alpha})$. The expectation value of $W_p(\vec{\alpha})$ may then be written

$$\langle W_p(\vec{\alpha}) \rangle = \sum_{\vec{N}} \sum_{\vec{n}} P_T(\vec{N}) \prod_{j'=1}^K \mathbb{M}_{j'}(\vec{n}_{j'}^{(12)}, \vec{n}_{j'}^{(1)}, \vec{n}_{j'}^{(2)}, \vec{N}_{j'}, \vec{\varepsilon}_{j'}) \mathbb{S}_{j'}(\vec{n}_{j'}^{(12)}, \vec{n}_{j'}^{(1)}, \vec{n}_{j'}^{(2)}) \sum_{j=1}^K \left(\sum_T \sum_{i_j=1}^{n_j^{(T)}(\vec{\alpha})} \omega_p^{(T)}(m_{1,i_j}, m_{2,j}|\vec{\alpha}) \right), \quad (79)$$

$$= \sum_{j=1}^K \left[\sum_T \varepsilon_j^{(T)}(\vec{\alpha}) r_{pj}^{(T)}(\vec{\alpha}) \right] \langle N_j(\vec{\alpha}) \rangle, \quad (80)$$

where, in the last line, only the relevant integrals are kept and included in the form of the coefficients $r_{pj}^{(T)}(\vec{\alpha})$ defined in Eq. (74). Note that it is not possible, in this case, to reabsorb the efficiencies into the multiplicities as in the previous section because these are now associated with different response coefficients $r_{pj}^{(T)}(\vec{\alpha})$. One next defines the matrices $\mathbb{R}(\vec{\alpha})$ with elements $R_{pj} = \sum_T \varepsilon_j^{(T)}(\vec{\alpha}) r_{pj}^{(T)}(\vec{\alpha})$. The first moments $\vec{N}(\vec{\alpha})$ are thus given by the linear equations

$$\vec{N}(\vec{\alpha}) = [\mathbb{R}(\vec{\alpha})]^{-1} \mathbb{W}(\vec{\alpha}). \quad (81)$$

The evaluation of the second moments and cross-moments of $W_p(\vec{\alpha})$ proceeds in a similar fashion:

$$\begin{aligned} \langle W_p(\vec{\alpha}) W_q(\vec{\beta}) \rangle &= \sum_{k=1}^K \left[\sum_T r_{pk}^{(T)}(\vec{\alpha}) \varepsilon_k^{(T)}(\vec{\alpha}) \right] \langle N_k(\vec{\alpha}) \rangle \delta_{\vec{\alpha}, \vec{\beta}} + \sum_{k,k'=1}^K \left[\sum_{T,T'} r_{pk}^{(T)}(\vec{\alpha}) r_{qk'}^{(T')}(\vec{\beta}) \varepsilon_k^{(T)}(\vec{\alpha}) \varepsilon_{k'}^{(T')}(\vec{\beta}) \right] \\ &\quad \times \langle N_k(\vec{\alpha}) [N_{k'}(\vec{\beta}) - \delta_{k,k'} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle. \end{aligned} \quad (82)$$

As in the previous section, one next defines the matrices $\mathbb{V}(\vec{\alpha}, \vec{\beta})$ with elements $V_{pq}(\vec{\alpha}, \vec{\beta})$ calculated according to

$$V_{pq}(\vec{\alpha}, \vec{\beta}) = \langle W_p(\vec{\alpha}) W_q(\vec{\beta}) \rangle - \sum_{k=1}^K \left[\sum_T r_{pk}^{(T)}(\vec{\alpha}) \varepsilon_k^{(T)}(\vec{\alpha}) \right] \langle N_k(\vec{\alpha}) \rangle \delta_{\vec{\alpha}, \vec{\beta}}. \quad (83)$$

Equations (82) are then rewritten

$$V_{pq}(\vec{\alpha}, \vec{\beta}) = \sum_{k,k'=1}^K \left[\sum_T r_{pk}^{(T)}(\vec{\alpha}) \varepsilon_k^{(T)}(\vec{\alpha}) \right] \langle N_k(\vec{\alpha}) [N_{k'}(\vec{\beta}) - \delta_{k,k'} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \left[\sum_{T'} r_{qk'}^{(T')}(\vec{\beta}) \varepsilon_{k'}^{(T')}(\vec{\beta}) \right] \quad (84)$$

Redefining the elements of the matrices \mathbb{N} and \mathbb{R} according to

$$N_{pq}(\vec{\alpha}, \vec{\beta}) = \langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle \quad (85)$$

and

$$R_{pk}(\vec{\alpha}) = \sum_T r_{pk}^{(T)}(\vec{\alpha}) \varepsilon_k^{(T)}(\vec{\alpha}), \quad (86)$$

one gets matrix equations

$$\mathbb{V}(\vec{\alpha}, \vec{\beta}) = \mathbb{R}(\vec{\alpha}, \vec{\beta}) \mathbb{N}(\vec{\alpha}, \vec{\beta}) [\mathbb{R}(\vec{\alpha}, \vec{\beta})]^T, \quad (87)$$

which are solved by multiplying on the left and right by inverses of the matrices $\mathbb{R}(\vec{\alpha})$ and $\mathbb{R}(\vec{\beta})^T$, thereby yielding expressions of the form of Eqs. (63) that provide the moments $\langle N_p(\vec{\alpha}) [N_q(\vec{\beta}) - \delta_{p,q} \delta_{\vec{\alpha}, \vec{\beta}}] \rangle$. One thus concludes that, in the context of analyses involving several PID signals, the determination of multiplicity moments proceeds essentially as in the case of a single type of PID signal. However, it is not possible, in general, to reabsorb the efficiencies in the moments because they enter in linear combinations within the

coefficients r_{pk} . Inversion of the matrix equations thus requires both the knowledge of the functions $r_{pk}^{(T)}$ as well as that of the efficiencies $\varepsilon_k^{(T)}(\vec{\alpha})$.

The above formalism was derived assuming a particular PID scheme. However, it can be adapted to other PID requirements with little to no change to the equations. Additionally, one could also adapt the equations so that different PID schemes are used in different p_\perp ranges, e.g., TPC PID at low p_\perp , TOF

PID at high p_{\perp} , and Cherenkov or transition radiation detectors in between.

V. SUMMARY

A binning technique to discretize six-dimensional two-particle correlation functions R_2^{pq} was first introduced to evaluate two particle correlations as functions of rapidity, azimuthal angle, and transverse momentum, and project them onto two-dimensional correlators that are functions of the particles rapidity and azimuthal angle differences. Such discretized functions were next shown to be amenable to measurements with the identity method first in the context of experiments

with a single PID device and finally for experiments featuring two PID devices. The method is also applicable to multiple-particle correlations and for measurement devices featuring more than two PID techniques.

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