Number of spin-J states and odd-even staggering for identical particles in a single-j shell

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In this work, new recursion relations for the number of spin-J states for identical particles in a single-j shell are presented. Such relations are obtained using the generating-function technique, which enables one to exhibit an odd-even staggering in the spin distribution of an even number of fermions in a single-j shell: the number of states with an even value of J is larger than the number of states with an odd value of J. An analytical expression of the excess of states with an even value of J is provided, and its asymptotic behavior for large values of j is discussed.

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I. INTRODUCTION

The single-j shell plays an important role in nuclear physics [1]. Investigations concern in particular symmetries [2], isospin relations, or the J-pairing interaction (see for instance Refs. [3–7]). The single-j shell was also successfully modeled using two-body random Hamiltonians [8,9]. The enumeration of the number $N(J,j,n)$ of spin-J states for n identical particles in a single-j shell, first addressed by Bethe [10], is a fundamental issue of nuclear-structure theory. Such a number can be obtained as [11]

$$N(J,j,n) = \sum_{M=J}^{J+1}(-1)^{J-M}D(M,j,n) = D(J,j,n) - D(J+1,j,n), \qquad (1)$$

where $D(M,j,n)$ represents the distribution of the angular-momentum projection M, i.e., the number of states of a given value of M (I use the notations of Talmi's paper [12]). There have been many efforts devoted to the determination of an algebraic expression for $N(J,j,n)$. For instance, Ginocchio and Haxton obtained, in a work on the quantum Hall effect [13], a simple formula for $N(0,j,4)$, which is also equal to $N(j,j,3)$. As pointed out by Talmi [12], such results are interesting, since it was shown that a necessary and sufficient condition for a two-body interaction to be diagonal in the seniority scheme is to have vanishing matrix elements between $\nu = 1$, $J = j$ state (ν being the seniority) and all $\nu = 3$, $J = j$ states of the j^3 configuration [11], and that an equivalent condition is to have vanishing matrix elements between the $\nu = 0$, $J = 0$ state and all $\nu = 4$, $J = 0$ states of the j^4 configuration [14]. Zhao and Arima found empirical formulas of $N(J,j,n)$ for three, four, and five particles [15]. Zamick and Escuderos revisited the Ginocchio-Haxton formula by a combinatorial approach for $J = j$ with $n = 3$ [16] and Talmi derived a recursion relation for $N(J,j,n)$ of n fermions in a j orbit in terms of n, $n-1$, $n-2$, etc., fermions in a $(j-1)$ orbit [12]. In Refs. [17–19], the studies for $n = 3$ and $n = 4$ were extended to the number of states with given spin and isospin T. In Ref. [20], Talmi's recursion formula [12] was further generalized to boson systems and applied to prove the empirical formula for $n = 5$ bosons given in Ref. [15]. The number of states of a given spin was found to be closely related to sum rules of many six-j and nine-j symbols, and coefficients of fractional parentage [17,21–29]. In Ref. [30], it was proven that the number of spin-J states for n fermions in a single-j shell or bosons with spin ℓ equals the number of states of another "boson" system with spin $n/2$, the boson number being equal either to $2j+1-n$ (for n fermions in a j shell) or to 2ℓ (if one considers n spin-ℓ bosons). Jiang *et al.* published analytical formulas for the number of states of a given spin value for three identical particles, in a unified form for both fermions and bosons, by using n virtual bosons with spin $3/2$ (n being equal to $2j-2$ for fermions in a single-j shell or to 2ℓ for bosons with spin ℓ [31]). Recently, Bao *et al.* derived recursive formulas by induction with respect to n and j and applied them to systems of two, three, and five identical particles [32].

In the present work, I propose new recursion relations for $D(M,j,n)$ obtained using generating functions [33–38]. The formalism as well as the new relations are described in Sec. II. The present recurrence relations are different from the one published by Talmi [12], but I show in the Appendix that the latter can be also easily obtained within the present formalism. An approximate statistical modeling of $D(M,j,n)$ is provided in Sec. III and compared to exact results. Finally, in Sec. IV, I investigate, still using generating functions, the J excess, which is the difference between the number of states with an even value of J and the number of states with an odd value of J. An odd-even staggering for the single-j shell with an even number of fermions is observed: the J-excess is always positive and is given by a simple binomial coefficient.

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II. GENERATING FUNCTION AND RECURSION RELATIONS

A. Determination of the generating function

Let us consider a system of n identical fermions in a single-j shell (of degeneracy $g = 2j + 1$) subject to the constraints

$$n = n_1 + \cdots + n_g = \sum_{i=1}^{g} n_i \tag{2}$$

and

$$M = n_1 m_1 + \cdots + n_g m_g = \sum_{i=1}^{g} n_i m_i, \tag{3}$$

m_i being the angular momentum projection of state i and $n_i = 0$ or 1 $\forall\, i$. For a configuration j^n, one has $J_{\min} = [1 - (-1)^n]/4$,

$$M_{\max} = J_{\max} = \sum_{m=j-n+1}^{j} m = \frac{(2j+1-n)n}{2} \tag{4}$$

and $M_{\min} = -M_{\max}$. The corresponding generating function reads

$$f_j(x,z) = \sum_{n=0}^{\infty} \sum_{M=-\infty}^{\infty} z^n x^M \times \sum_{\{n_1,\ldots,n_g\}} \delta_{n,n_1+\cdots+n_g} . \delta_{M,n_1 m_1+\cdots+n_g m_g}, \tag{5}$$

or

$$f_j(x,z) = \sum_{\{n_1,\ldots,n_g\}} z^{n_1+\cdots+n_g} . x^{n_1 m_1+\cdots+n_g m_g}. \tag{6}$$

Since the quantities n_i are independent, it is possible to write

$$f_j(x,z) = \sum_{n_1=0}^{1} z^{n_1} x^{n_1 m_1} \cdots \sum_{n_g=0}^{1} z^{n_g} x^{n_g m_g}, \tag{7}$$

which yields

$$f_j(x,z) = (1 + z x^{m_1}) \times \cdots \times (1 + z x^{m_g}) = \prod_{i=1}^{g} (1 + z x^{m_i}). \tag{8}$$

In that case, $D(M,j,n)$ is related to $f_j(x,z)$ by

$$f_j(x,z) = \sum_{n=0}^{\infty} \sum_{M=-\infty}^{\infty} z^n x^M D(M,j,n), \tag{9}$$

leading to

$$D(M,j,n) = \frac{1}{(2i\pi)^2} \oint \oint \frac{dz_1}{z_1^{n+1}} \frac{dz_2}{z_2^{M+1}} f_j(z_1,z_2). \tag{10}$$

B. New recurrence relations

The generating function (8) can be expanded in powers of z:

$$f_j(x,z) = \sum_{n=0}^{g} z^n f_{j,n}(x), \tag{11}$$

with

$$f_{j,n}(x) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} f_j(x,z)\bigg|_{z=0}. \tag{12}$$

1. First relation

Equation (12) can be rewritten as

$$f_{j,n}(x) = \frac{1}{n!} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[\left(\prod_{k=1}^{g-1} (1 + z x^{m_k}) \right) (1 + z x^{m_g}) \right] \bigg|_{z=0}, \tag{13}$$

and using the Leibniz formula for the multiple derivative of a product of two functions, one obtains

$$f_{j,n}(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\partial^{n-k}}{\partial z^{n-k}} \prod_{k=1}^{g-1} (1 + z x^{m_k}) \right] \bigg|_{z=0} \times \frac{\partial^k}{\partial z^k} (1 + z x^{m_g}) \bigg|_{z=0}, \tag{14}$$

where $\binom{n}{k} = n!/k!/(n-k)!$ is the binomial coefficient. Equation (14) yields

$$f_{j,n}(x) = \frac{1}{n!} \left[\frac{\partial^n}{\partial z^n} \prod_{k=1}^{g-1} (1 + z x^{m_k}) + x^{m_g} \frac{\partial^{n-1}}{\partial z^{n-1}} \prod_{k=1}^{g-1} (1 + z x^{m_k}) \right] \bigg|_{z=0}. \tag{15}$$

The number of states having angular momentum J is given by relation (1) and $D(M,j,n)$ is the coefficient of x^M in

$$f_{j,n}(x) = \sum_{M=M_{\min}}^{M_{\max}} D(M,j,n) x^M, \tag{16}$$

which yields

$$D_g(M,j,n) = D_{g-1}(M,j,n) + D_{g-1}(M - m_g, j, n-1), \tag{17}$$

where $D_g(M,j,n)$ represents the number of states with n fermions (protons or neutrons) in g one-fermion states. In a more general way, one can write the recursion relation (17) as

$$D_k(M,j,n) = D_{k-1}(M,j,n) + D_{k-1}(M - m_k, j, n-1),$$
$$D_k(0,j,n) = \delta(M) \quad \forall\ k.$$

2. Second relation

After a first derivation of $f_j(x,z)$, one gets

$$f_{j,n}(x) = \frac{1}{n!} \sum_{i=1}^{g} x^{m_i} \frac{\partial^{n-1}}{\partial z^{n-1}} \prod_{k=1,k\neq i}^{g} (1 + z x^{m_k}) \bigg|_{z=0}, \tag{18}$$

which is equivalent to

$$f_{j,n}(x) = \frac{1}{n!} \sum_{i=1}^{g} x^{m_i} \frac{\partial^{n-1}}{\partial z^{n-1}} \frac{\prod_{k=1}^{g} (1 + z x^{m_k})}{(1 + z x^{m_i})} \bigg|_{z=0}. \tag{19}$$

Using the Leibniz formula, one obtains

$$f_{j,n}(x) = \frac{1}{n!}\sum_{i=1}^{g} x^{m_i} \sum_{k=0}^{n-1} \binom{n-1}{k} \left[\frac{\partial^k}{\partial z^k}\frac{1}{(1+zx^{m_i})}\right]\Bigg|_{z=0} \times \frac{\partial^{n-1-k}}{\partial z^{n-1-k}} \prod_{k=1}^{g}(1+zx^{m_k})\Bigg|_{z=0}. \quad (20)$$

Since

$$\frac{\partial^k}{\partial z^k}\frac{1}{(1+az)} = \frac{k!(-1)^k a^k}{(1+az)^{k+1}}, \quad (21)$$

one finds

$$f_{j,n}(x) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\left[\sum_{i=1}^{g} x^{k.m_i}\right] f_{j,n-k}(x), \quad (22)$$

which yields, in virtue of Eq. (16),

$$D(M,j,n) = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{g}(-1)^{k-1}D(M-km_i,j,n-k). \quad (23)$$

Such a formalism can be extended to include additional constraints [34,35]. The number of loops required for the three-nested recursion relations (17) and (23) is roughly $n(2j+1)(2M_{\max}+1)$, i.e., (number of fermions)×(number of states)×(number of values of M). The numerical cost is maximum for a half-filled shell, but the recursion relations are much more efficient than the usual combinatorial approach since their cost is polynomial with j and n.

III. STATISTICAL MODELING OF $D(M,j,n)$

The distribution $D(M,j,n)$ having a bell shape, it can be modeled as

$$D(M,j,n) = \frac{G(j^n)}{\sqrt{2\pi v(j^n)}} \exp\left[-\frac{M^2}{2v(j^n)}\right], \quad (24)$$

where $G(j^n) = \binom{2j+1}{n}$ represents the degeneracy of j^n, and $v(j^n)$ its variance:

$$v(j^n) = \sum_{M=-J_{\max}}^{J_{\max}} M^2 = \frac{n(2j+1-n)(j+1)}{6}. \quad (25)$$

One can see in Figs. 1–3, in the cases of shells $(7/2)^4$, $(11/2)^5$, and $(15/2)^6$, respectively, that the statistical modeling of $D(M,j,n)$ is in fairly good agreement with the exact distribution. The results can be improved taking into account the fourth-order moment (kurtosis), and a generalized Gaussian (or hyper-Gaussian) distribution. A first-order Taylor expansion of $J \to D(J,j,n)$ and $J \to D(J+1,j,n)$ at $J+1/2$ gives

$$N(J,j,n) = D(J,j,n) - D(J+1,j,n) \approx -\frac{dD}{dM}\Bigg|_{M=J+1/2}, \quad (26)$$

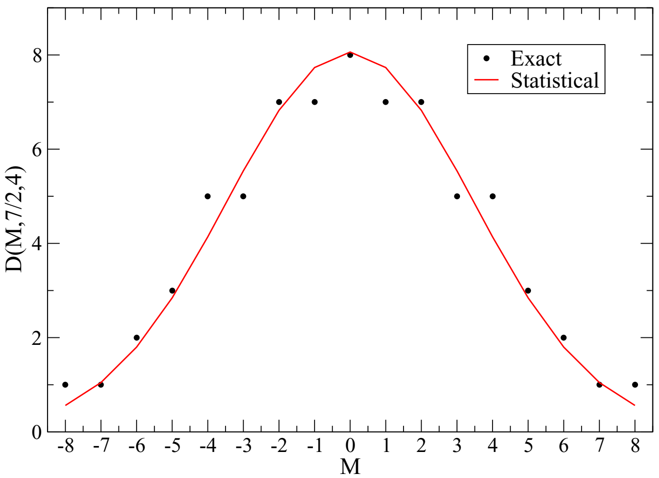

FIG. 1. Distribution $D(M,j,n)$ for $j=7/2$ and $n=4$: exact calculation [relation (23)] and statistical modeling [Eq. (24)].

and one gets, using Eq. (24),

$$N(J,j,n) \approx \frac{G(j^n)}{\sqrt{2\pi}}\frac{(J+1/2)}{[v(j^n)]^{3/2}} \exp\left[-\frac{1}{2v(j^n)}\left(J+\frac{1}{2}\right)^2\right]. \quad (27)$$

It is interesting to compare the latter expression with the Ginocchio-Haxton formula

$$N(j,j,3) = \left[\frac{2j+3}{6}\right], \quad (28)$$

where $[x]$ is the largest integer not exceeding x. One can see in Fig. 4 that the results are rather close to the exact ones. The estimates can be improved either by performing the Taylor-series expansion up to a higher order in Eq. (26), or using the expression

$$N(J,j,n) = \int_{J-1/2}^{J+1/2} D(M,j,n)dM - \int_{J+1/2}^{J+3/2} D(M,j,n)dM. \quad (29)$$

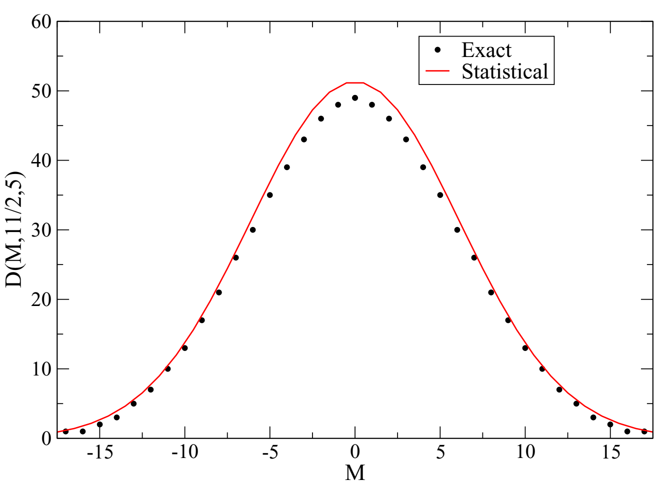

FIG. 2. Distribution $D(M,j,n)$ for $j=11/2$ and $n=5$: exact calculation [relation (23)] and statistical modeling [Eq. (24)].

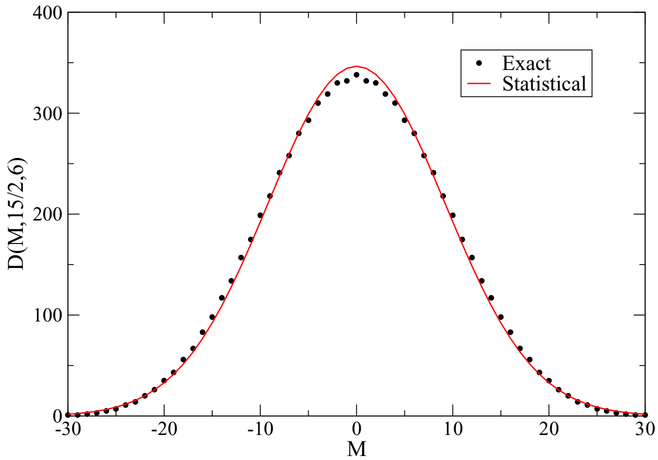

FIG. 3. Distribution $D(M,j,n)$ for $j = 15/2$ and $n = 6$: exact calculation [relation (23)] and statistical modeling [Eq. (24)].

The statistical modeling is of course not as accurate as the recurrence relations, but it can be helpful to better understand the characteristics of the distribution of states and to derive, for instance, asymptotic expressions.

IV. EXCESS OF *J* VALUES

Generating functions can also be of great interest for the determination of the excess of J values, i.e., the difference between the number of even values of J and the number of odd values of J. For a configuration j^n with $n = 2k$, k being a positive integer, the excess of J values is equal to the excess of M values. Since

$$f_{j,n}(x) = \sum_{M=M_{\min}}^{M_{\max}} D(M,j,n)x^M, \tag{30}$$

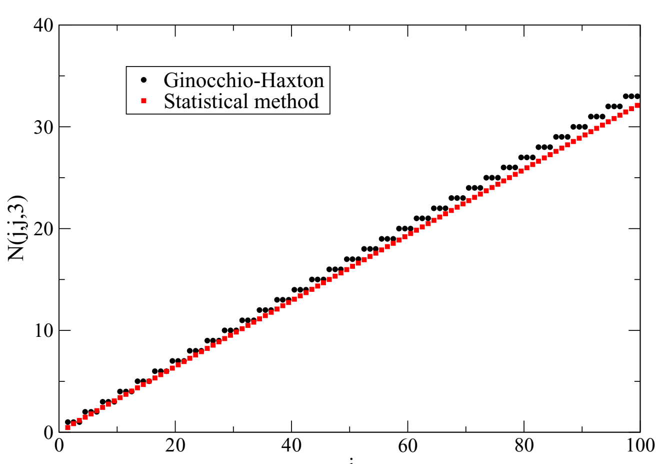

FIG. 4. Comparison between the exact Ginocchio-Haxton formula [relation (28)] and statistical modeling [Eq. (27)] for $J = j$ and $n = 3$.

TABLE I. Excess of even-J states for different j^n shells.

j^n	Excess
$(1/2)^2$	1
$(5/2)^4$	3
$(7/2)^4$	6
$(7/2)^6$	4
$(9/2)^4$	10
$(15/2)^6$	56

the excess E for a configuration $j^{n=2k}$ is equal to

$$E(j^{2k}) = \sum_{M=M_{\min}}^{M_{\max}} (-1)^M D(M,j,n)$$
$$= f_{j,n}(-1) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial z^{2k}} f_j(-1,z)\bigg|_{z=0}. \tag{31}$$

and the function $f_j(-1,z)$ is given by

$$f_j(-1,z) = \prod_{m=-j}^{j}[1+(-1)^m z] = (1+z^2)^{j+1/2}, \tag{32}$$

which implies

$$E(j^{2k}) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial z^{2k}}\left[\sum_{p=0}^{j+1/2}\binom{j+1/2}{p} z^{2p}\right]\Bigg|_{z=0}$$
$$= \binom{j+1/2}{k}. \tag{33}$$

For two fermions (as well as for $n = 2j+1-2 = 2j-1$), the number of odd-J states is zero, since J is necessarily even due to the Pauli exclusion principle (antisymmetric states). The values of the excess for different j^n shells, relativistic or not, are displayed in Table I.

In the case of $(7/2)^4$, there are seven even-J states and one odd-J state, corresponding to $J = 5$ (see Table II). The results can be checked with the tables published by Bayman and Lande [39]. The numbers of even- and odd-J states for the shell $(j = 11/2)^n$ for different values of the number of fermions ($n = 4$ and 6) are provided in Table III, and the number of states for all values of J in Tables IV (for $n = 4$) and V (for $n = 6$). There is of course only one state with spin $J_{\max}$ [the expression of $J_{\max}$ is provided in Eq. (4)]. It is worth mentioning that

TABLE II. Number of even- and odd-J states for the shell $(j = 7/2)^4$.

Even J	Number of states	Odd J	Number of states
0	1	1	0
2	2	3	0
4	2	5	1
6	1	7	0
8	1		
Total (even):	7	Total (odd):	1

TABLE III. Number of even- and odd-J states for the shell $(j = 11/2)^n$ for $n = 4$ and 6.

n	Number of even-J states	Number of odd-J states	Excess
4	24	9	15
6	39	19	20

Talmi derived a recursion relation (which one recovers using the generating-function formalism in the Appendix) and found interesting peculiarities in the distributions of spin-J states: for instance, the states with spins $J_{\max} - 2$ and $J_{\max} - 3$ are unique in a j^n configuration and there is no state with spin $J_{\max} - 1$ [12].

It is interesting to evaluate, for $2 < n < 2j - 1$, the ratio between the excess E and the total number of fixed-spin states N_{tot} for specific configuration j^{2k}:

$$r(j^{2k}) = \frac{E(j^{2k})}{N_{\text{tot}}(j^{2k})}, \tag{34}$$

where N_{tot} reads

$$N_{\text{tot}} = \sum_{J=J_{\min}}^{J_{\max}} N(J,j,n). \tag{35}$$

The latter quantity can be approximated by

$$N_{\text{tot}} \approx \frac{G(j^{2k})}{\sqrt{2\pi\, v(j^{2k})}}, \tag{36}$$

where $G(j^{2k}) = \binom{2j+1}{2k}$ is the degeneracy of configuration j^{2k} and $v(j^{2k})$ the variance [see Eq. (25)]

$$v(j^{2k}) = \frac{k(2j+1-2k)(j+1)}{3}. \tag{37}$$

One has therefore

$$r(j^{2k}) \approx \frac{\binom{j+1/2}{k}}{\binom{2j+1}{2k}} \sqrt{\frac{2\pi k(2j+1-2k)(j+1)}{3}}. \tag{38}$$

TABLE IV. Number of even- and odd-J states for the shell $(j = 11/2)^4$. The excess is equal to 15.

Even J	Number of states	Odd J	Number of states
0	2	1	0
2	3	3	1
4	4	5	2
6	4	7	2
8	4	9	2
10	3	11	1
12	2	13	1
14	1	15	0
16	1		
Total (even):	24	Total (odd):	9

TABLE V. Number of even- and odd-J states for the shell $(j = 11/2)^6$. The excess is equal to 20.

Even J	Number of states	Odd J	Number of states
0	3	1	0
2	4	3	3
4	6	5	3
6	7	7	4
8	6	9	4
10	5	11	2
12	4	13	2
14	2	15	1
16	1	17	0
18	1		
Total (even):	39	Total (odd):	19

Such a quantity reaches its minimum

$$r_{\min} = \frac{\sqrt{j(1+4j(j+2))-3}\,\Gamma\left(\frac{j}{2}+\frac{1}{4}\right)\Gamma\left(\frac{j}{2}+\frac{5}{4}\right)}{2\sqrt{3}\,j!} \tag{39}$$

for $k = j/2 - 1/4$ if j is of the form $j = 2p + 1/2$ with p a positive integer (Γ is the usual gamma function) and

$$r_{\min} = \frac{(2j+1)\sqrt{(j+1)}\,\Gamma\left(\frac{j}{2}+\frac{3}{4}\right)^2}{2\sqrt{3}\,j!} \tag{40}$$

for $k = j/2 + 1/4$ if j is of the kind $j = 2p + 3/2$. Figure 5 represents the quantity $r(j^{2k})$ for $j = 9/2$, $11/2$, and $13/2$.

Using the Stirling formula, one finds that the degeneracy of j^n at the maximum complexity ($n = j + 1/2$) varies as

$$G(j^{j+1/2}) \approx 2^{2j+1} \tag{41}$$

and the following asymptotic form for $j \to \infty$ is obtained:

$$r_{\min} \approx \sqrt{\frac{2\pi}{3}}\,\frac{j^{3/2}}{2^j}. \tag{42}$$

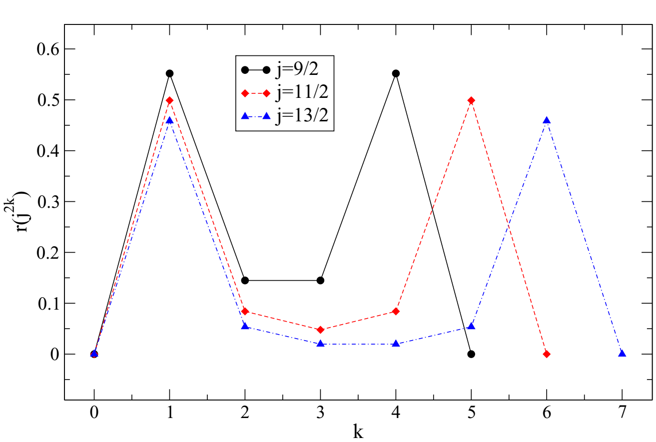

FIG. 5. Values of $r(j^{2k})$ as a function of k for $j = 9/2$, $11/2$, and $13/2$.

V. CONCLUSION

Using the generating-function formalism, new recursion relations were derived for the number of antisymmetric states with a given value of J due to the coupling of n identical fermions in the j orbit. Still using the generating function, an odd-even staggering was found in the spin distribution of a single-j shell with an even number of fermions. The excess of the number of states with an even value of J was calculated and its asymptotic behavior for large values of j was investigated using statistical modeling of the number of states with angular-momentum projection M.

APPENDIX: TALMI'S RECURSION RELATION

The generating-function formalism enables one to derive another recursion relation. Indeed, according to Eq. (11), one has

$$f_{j+1}(x,z) = \prod_{m=-j-1}^{j+1}(1+x^m z) = \sum_{n=0}^{2j+3} z^n f_{j+1,n}(x) = f_j(x,z)(1+zx^{-j-1})(1+zx^{j+1}) \quad \text{(A1)}$$

and therefore

$$f_{j+1,n}(x) = \frac{1}{n!}\frac{\partial^n}{\partial z^n} f_{j+1}(x,z)\Big|_{z=0} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^{n-k}}{\partial z^{n-k}} f_j(x,z)\,\frac{\partial^k}{\partial z^k}\mathcal{P}_j(x,z)\Big|_{z=0}, \quad \text{(A2)}$$

where

$$\mathcal{P}_j(x,z) = (1+zx^{-j-1})(1+zx^{j+1}) = 1+z^2+z(x^{-j-1}+x^{j+1}). \quad \text{(A3)}$$

Since expression (A3) is a second-order polynomial, the only derivatives which are nonzero correspond to

$$k=0:\quad \frac{\partial^0}{\partial z^0}\mathcal{P}_j(x,z)\Big|_{z=0} = 1,$$
$$k=1:\quad \frac{\partial^1}{\partial z^1}\mathcal{P}_j(x,z)\Big|_{z=0} = x^{-j-1}+x^{j+1}, \quad \text{(A4)}$$
$$k=2:\quad \frac{\partial^2}{\partial z^2}\mathcal{P}_j(x,z)\Big|_{z=0} = 2,$$

which leads to

$$D(M,j+1,n) = D(M,j,n) + D(M-j-1,j,n-1) + D(M+j+1,j,n-1) + D(M,j,n-2), \quad \text{(A5)}$$

from which a recursion relation can be deduced for N:

$$N(J,j+1,n) = N(J,j,n) + N(J-j-1,j,n-1) + N(J+j+1,j,n-1) + N(J,j,n-2), \quad \text{(A6)}$$

i.e., in a compact form

$$N(J,j+1,n) = \sum_{i,k=0}^{1} N(J+(i-k)(j+1),j,n-i-k), \quad \text{(A7)}$$

which is the relation (5) derived by Talmi in Ref. [12] for $J \geqslant j$.

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