Effects of causality on the fluidity and viscous horizon of quark-gluon plasma

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The second-order Israel-Stewart-Müller relativistic hydrodynamics was applied to study the effects of causality on the acoustic oscillation in relativistic fluid. Causal dispersion relations have been derived with nonvanishing shear viscosity, bulk viscosity, and thermal conductivity at nonzero temperature and baryonic chemical potential. These relations have been used to investigate the fluidity of quark-gluon plasma (QGP) at finite temperature (T). Results of the first-order dissipative hydrodynamics have been obtained as a limiting case of the second-order theory. The effects of the causality on the fluidity near the transition point and on the viscous horizon are found to be significant. We observe that the inclusion of causality increases the value of fluidity measure of QGP near T_c and hence makes the flow strenuous. It was also shown that the inclusion of the large magnetic field in the causal hydrodynamics alters the fluidity of QGP.

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I. INTRODUCTION

The collision of heavy ions at relativistic energies create matter in a new state called quark-gluon plasma (QGP) [1,2]. The QGP can be created with different temperatures (T) and net baryonic chemical potential (μ) by altering the energy of the colliding beams [3]. For example, the system formed in the nuclear collisions at Large Hadron Collider (LHC) as well as at the highest Relativistic Heavy Ion Collider (RHIC) energy will have very small μ but large T. On the other hand the matter created at Facility for Antiproton and Ion Research (GSI-FAIR) energy, Nuclotron-based Ion Collider fAcility (JINR-NICA), and at lower energy run of RHIC will have larger μ but smaller T. Nature of the transition from QGP to hadrons depends on the values of T and μ [4]. It is expected that at high μ and low T the phase transition is first order but at high T low μ it is a continuous transition from QGP to hadrons [5–8]. When the QGP reverts to hot hadrons from cooling caused by expansion, the system may encounter the critical point in the QCD phase diagram during the transition from QGP to hadrons. The characterization of the fluid at the critical point is one of the most crucial problems in heavy ion collision at relativistic energies.

Lattice QCD simulations at zero μ indicate that strongly interacting nuclear matter undergoes a rapid transition from a chirally broken confined hadronic phase to a chirally symmetric, deconfined QGP around $T_c \sim 155$ MeV [8]. The QGP expands very fast because of internal pressure and its evolution in space time can be studied by using relativistic viscous hydrodynamics. In general, the presence of nonzero transport coefficient, like shear and bulk viscosities and thermal conductivity make the evolution and characterization of QGP very challenging and complex. The Navier-Stokes equation is not suitable to describe relativistic fluid as it suffers from severe flaws, e.g., it violates causality and leads to unstable solutions [9]. These unphysical behaviors were resolved by Müller [10] using Grad's 14 moment method [11] and its relativistic covariant form is from Israel and Stewart [12]. These theories are based on extended irreversible thermodynamics known as second-order theories. The first-order and second-order hydrodynamical descriptions stem from the definition of entropy four-current. The conservations of energy momentum and conserved charge (e.g., net baryon number) along with the second law of thermodynamics lead to the dynamical transport equations which are hyperbolic in nature and respect causality.

The transport coefficients such as shear viscosity, bulk viscosity, thermal conductivity, etc., are taken as input in first-order hydrodynamics. In addition to these standard transport coefficients, the causal or second-order theory contains a few more thermodynamic functions which are known as second-order coefficients. These coefficients along with the standard transport coefficients, correspond to different relaxation times and relaxation lengths for various dissipative fluxes which are absent in acausal theory. The results of acausal theory can be obtained by setting these extra coefficients to zero in causal theory. In this work we use the relativistic causal hydrodynamics to investigate propagation of acoustic wave through dissipative fluid with nonzero net (baryonic) charge, shear viscosity, bulk viscosity, and thermal conductivity following the procedure outlined in Ref. [18]. In the present work we investigate the effects of causality on the fluidity of QGP in contrast to earlier work where the fluidity of QGP was studied [13] within the scope of first-order theory which is flawed because of causality violation in the relativistic domain. The aim of this work is to estimate the shift on the fluidity of relativistic fluid by using second-order hydrodynamics which respects causality. Maartens *et al.* [14] has used causal hydrodynamics to explore

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the dissipation of acoustic waves in baryon-photon fluid in early universe.

The present article is organized as follows: In Sec. II, we will discuss the formalism used to derive the transverse and longitudinal dispersion relations for sound wave within the framework of causal hydrodynamics. Dispersion relations for sound wave in the dissipative system with the inclusion of magnetic field have been derived in Sec. III. The impact of the causality and external magnetic field on the fluidity was discussed in Sec. IV. Section V was devoted to present results and finally Sec. VI was dedicated to summary and discussions. We have used the natural unit, i.e., $c = \hbar = k_B = 1$ here and the Minkowski metric is set as $g^{\lambda\mu} = \text{diag}(-,+,+,+)$.

II. FORMALISM: DERIVATION OF CAUSAL DISPERSION RELATIONS

The relativistic energy-momentum tensor ($T^{\lambda\mu}$) in the Israel-Stewart [12] second-order theory is given by

$$T^{\lambda\mu} = \epsilon u^\lambda u^\mu + P\triangle^{\lambda\mu} + 2h^{(\lambda}u^{\mu)} + \tau^{\lambda\mu}, \tag{1}$$

where the dissipative viscous stress tensor $\tau^{\lambda\mu} = \Pi\triangle^{\lambda\mu} + \pi^{\lambda\mu}$ with $\pi^\lambda_\lambda = h^\lambda u_\lambda = \tau^{\lambda\mu}u_\lambda = 0$ where the projection operator is defined by $\triangle^{\lambda\mu} = g^{\lambda\mu} + u^\mu u^\lambda$ with $u^\mu u_\mu = -1$. The heat flux four vector is given by $q^\mu = h^\mu - n^\mu(\epsilon + P)/n$, the particle four flow $N^\mu = nu^\mu + n^\mu$ with $n^\mu u_\mu = 0$, where n is the net number density, Π is the bulk pressure, u^μ is the fluid four velocity, ϵ is the energy density, P is the thermodynamic pressure, and $h(= \epsilon + P)$ is the enthalpy density. The symmetric tensor $h^{(\lambda}u^{\mu)}$ is defined as $h^{(\lambda}u^{\mu)} = \frac{1}{2}(h^\lambda u^\mu + h^\mu u^\lambda)$.

The definition of fluid four velocity in Eq. (1) can be fixed by choosing a proper reference frame attached to the fluid element either from Landau-Lifshitz (LL) or Eckart. The Eckart frame [15] represents a local rest frame for which the net charge dissipation is zero but the net energy dissipation is nonzero. The LL frame [16] represents a local rest frame where the energy dissipation is zero but the net charge dissipation is nonzero. We consider the LL frame here to study a system with net nonzero charge (baryon number).

In the LL frame, $h^\mu = 0$, $n^\mu = -nq^\mu/(\epsilon + P)$ and the different viscous fluxes are given by [12]

$$\begin{aligned}
\Pi &= -\tfrac{1}{3}\zeta\left(u^\mu_{|\mu} + \beta_0 D\Pi - \alpha_0 q^\mu_{|\mu}\right),\\
q^\lambda &= \chi T\triangle^{\lambda\mu}\big[(\partial_\mu\alpha)nT/(\epsilon + P)\\
&\quad - \beta_1 Dq_\mu + \alpha_0\partial_\mu\Pi + \alpha_1\Pi^\nu_{\mu|\nu}\big],\\
\Pi_{\lambda\mu} &= -2\eta[u_{\langle\lambda|\mu\rangle} + \beta_2 D\Pi_{\lambda\mu} - \alpha_1 q_{\langle\lambda|\mu\rangle}],
\end{aligned} \tag{2}$$

where $D \equiv u^\mu\partial_\mu$, is the well-known co-moving derivative or material derivative. In the local rest frame, $D\Pi = \partial_0\Pi \equiv \dot{\Pi}$. The different coefficients appearing in Eq. (2) are as follows: $\alpha = \mu/T$ is known as thermal potential, ζ is the coefficient of bulk viscosity, η is the coefficient of shear viscosity, χ is the coefficient of thermal conductivity, β_0,β_1,β_2 are relaxation coefficients, and α_0 and α_1 are coupling coefficients. The relaxation times for the bulk pressure (τ_Π), the heat flux (τ_q), and the shear tensor (τ_π) are defined as [17]

$$\tau_\Pi = \zeta\beta_0, \quad \tau_q = k_B T\beta_1, \quad \tau_\pi = 2\eta\beta_2. \tag{3}$$

The relaxation lengths which couple the heat flux and bulk pressure ($l_{\Pi q},l_{q\Pi}$), the heat flux, and shear tensor ($l_{q\pi},l_{\pi q}$) are defined as

$$l_{\Pi q} = \zeta\alpha_0, \quad l_{q\Pi} = k_B T\alpha_0, \quad l_{q\pi} = k_B T\alpha_1, \quad l_{\pi q} = 2\eta\alpha_1. \tag{4}$$

The symmetric, trace free part of the spatial projection is defined by $A_{\langle\lambda\mu\rangle} \equiv [\triangle^\alpha_{(\lambda}\triangle^\beta_{\mu)} - \frac{1}{3}\triangle_{\lambda\mu}\triangle^{\alpha\beta}]A_{\alpha\beta}$ and $u^\mu_{|\mu} \equiv \partial_\mu u^\mu$. Because in the energy frame $h^\mu = 0$, then the energy-momentum tensor reduces to

$$T^{\lambda\mu} = \epsilon u^\lambda u^\mu + P\triangle^{\lambda\mu} + \Pi\triangle^{\lambda\mu} + \pi^{\lambda\mu}. \tag{5}$$

We put the explicit forms of Π,q^λ, and $\pi^{\lambda\mu}$ given by Eq. (2) into Eq. (1) to get

$$\begin{aligned}
T^{\lambda\mu} &= \epsilon u^\lambda u^\mu + P\triangle^{\lambda\mu} - \frac{1}{3}\zeta u^\sigma_{|\sigma}\triangle^{\lambda\mu} + \frac{1}{9}\zeta^2\beta_0\dot{u}^\rho_{|\rho}\triangle^{\lambda\mu}\\
&\quad + \frac{nT^2}{P+\epsilon}\frac{\zeta\alpha_0\chi}{3}\partial_\sigma[\triangle^{\sigma\rho}(\partial_\rho\alpha)\triangle^{\lambda\mu}] - 2\eta u^{\langle\lambda|\mu\rangle}\\
&\quad + 4\eta^2\beta_2\dot{u}^{\langle\lambda|\mu\rangle} + \frac{2nT^2}{P+\epsilon}\alpha_1\eta\chi\Big[\triangle^{(\lambda}_\alpha\triangle^{\mu)}_\beta\\
&\quad - \frac{1}{3}\triangle_{\alpha\beta}\triangle^{\lambda\mu}\Big]\partial^\beta\triangle^{\alpha\rho}_\rho\alpha,
\end{aligned} \tag{6}$$

where we have kept terms up to second order in space-time derivatives and neglected all the higher order space-time derivatives. We impart small perturbations P_1,ϵ_1,n_1,T_1 and u^α_1 to P,ϵ,n,T and u^α, respectively, to study the acoustic oscillations set by these perturbations. In this work we consider a nonexpanding fluid with $u^\alpha = (1,0,0,0)$. Then the perturbation u^α_1 will be $u^\alpha_1 = (0,u^i_1)$ to satisfy the constraint $u'^\alpha u'_\alpha = u^\alpha u_\alpha = -1$, where $u'^\alpha = u^\alpha + u^\alpha_1$.

To analyze the fate of the perturbation in the dissipative medium we assume that the space-time dependence of the perturbation is $\sim\exp[-i(kx - \omega t)]$. The perturbations in different components of $T^{\lambda\mu}$ appear as follows (Appendix A1):

$$\begin{aligned}
T^{00}_1 &= \epsilon_1,\\
T^{i0}_1 &= (\epsilon + P)u^i_1 + \frac{\zeta}{3}\frac{nT^2}{\epsilon + P}\chi\alpha_0 u^i_1\nabla^2\alpha + 2\alpha_1\eta\chi\frac{nT^2}{P+\epsilon}\left\{(\vec{u}_1\cdot\vec{\nabla})\partial^i\alpha - \frac{1}{3}u^i_1\nabla^2\alpha\right\},\\
T^{ij}_1 &= P_1 g^{ij} - \frac{1}{3}\zeta\left\{i\vec{k}\cdot\vec{u}_1 - \frac{1}{3}\zeta\beta_0\omega(\vec{k}\cdot\vec{u}_1)\right\}g^{ij} + \frac{1}{3}\zeta\alpha_0\chi\Big[i\{(\vec{k}\cdot\vec{\nabla})\alpha\}\aleph + \aleph\nabla^2\alpha + \frac{nT^2}{P+\epsilon}\{2(\vec{u}_1\cdot\vec{\nabla})\dot{\alpha}\\
&\quad + i\dot{\alpha}(\vec{k}\cdot\vec{u}_1) - i\omega(\vec{u}_1\cdot\vec{\nabla})\alpha\}\Big]g^{ij} - i\eta\left[k^j u^i_1 + k^i u^j_1 - \frac{2}{3}g^{ij}(\vec{k}\cdot\vec{u}_1)\right] + 2\eta^2\beta_2\omega\left\{k^j u^i_1 + k^i u^j_1 - \frac{2}{3}g^{ij}(\vec{k}\cdot\vec{u}_1)\right\}
\end{aligned}$$

$$+i\alpha_1\eta\chi\aleph\left[k^j(\partial^i\alpha)+k^i(\partial^j\alpha)-\frac{2}{3}g^{ij}k_l(\partial^l\alpha)\right]+2\alpha_1\eta\chi\frac{nT^2}{P+\epsilon}\left\{u_1^i(\partial^j\dot{\alpha})+u_1^j(\partial^i\dot{\alpha})-\frac{2}{3}g^{ij}(\vec{u}_1\cdot\vec{\nabla})\dot{\alpha}\right\}$$
$$+i\frac{nT^2}{P+\epsilon}\alpha_1\eta\chi\left\{k^ju_1^i\dot{\alpha}+k^iu_1^j\dot{\alpha}-\frac{2}{3}g^{ij}(\vec{k}\cdot\vec{u}_1)\dot{\alpha}\right\}, \tag{7}$$

where

$$\aleph=\left\{\frac{n_1T^2+2TT_1n}{P+\epsilon}-\frac{nT^2(P_1+\epsilon_1)}{(P+\epsilon)^2}\right\}. \tag{8}$$

The equations of motions (EoMs) of perturbations dictated by the conservations of energy momentum and net charge of the fluid are given by

$$\partial_\mu T^{\mu\lambda}=0,\quad \partial_\mu N^\mu=0. \tag{9}$$

The EoMs in the frequency-wave vector space take the following form:

$$0=\omega T_1^{i0}-k_jT_1^{ij}$$
$$=\omega(\epsilon+P)u_1^i+\frac{1}{3}\frac{nT^2}{\epsilon+P}\zeta\omega\chi\alpha_0u_1^i\nabla^2\alpha+2\alpha_1\eta\chi\omega\frac{nT^2}{P+\epsilon}\left\{(\vec{u}_1\cdot\vec{\nabla})\partial^i\alpha-\frac{1}{3}u_1^i\nabla^2\alpha\right\}-k^iP_1$$
$$+\frac{1}{3}\zeta k^i\left\{i\vec{k}\cdot\vec{u}_1-\frac{1}{3}\zeta\beta_0\omega(\vec{k}\cdot\vec{u}_1)\right\}-\frac{1}{3}\zeta\alpha_0\chi k^i\left[i\aleph(\vec{k}\cdot\vec{\nabla})\alpha+\aleph\nabla^2\alpha+\frac{nT^2}{P+\epsilon}\{2(\vec{u}_1\cdot\vec{\nabla})\dot{\alpha}+i\dot{\alpha}(\vec{k}\cdot\vec{u}_1)-i\omega(\vec{u}_1\cdot\vec{\nabla})\alpha\}\right]$$
$$+i\eta\left[k^2u_1^i+k^i(\vec{k}\cdot\vec{u}_1)-\frac{2}{3}k^i(\vec{k}\cdot\vec{u}_1)\right]-2\eta^2\beta_2\omega\left\{k^2u_1^i+k^i(\vec{k}\cdot\vec{u}_1)-\frac{2}{3}k^i(\vec{k}\cdot\vec{u}_1)\right\}$$
$$-i\alpha_1\eta\chi\aleph\left[k^2(\partial^i\alpha)+k^i(\vec{k}\cdot\vec{\nabla})\alpha-\frac{2}{3}k^i(\vec{k}\cdot\vec{\nabla})\alpha\right]-\frac{nT^2}{P+\epsilon}2\alpha_1\eta\chi\left[u_1^i(\vec{k}\cdot\vec{\nabla})\dot{\alpha}+(\vec{k}\cdot\vec{u}_1)\partial^i\dot{\alpha}-\frac{2}{3}k^i(\vec{u}_1\cdot\vec{\nabla})\dot{\alpha}\right]$$
$$-i\frac{nT^2}{P+\epsilon}\alpha_1\eta\chi\left\{k^2u_1^i\dot{\alpha}+k^i(\vec{k}\cdot\vec{u}_1)\dot{\alpha}-\frac{2}{3}k^i(\vec{k}\cdot\vec{u}_1)\dot{\alpha}\right\}. \tag{10}$$

The other components of the energy momentum tensor satisfies

$$0=\omega T_1^{00}-k_iT_1^{i0}$$
$$=\omega\epsilon_1-(\epsilon+P)(\vec{k}\cdot\vec{u}_1)-\frac{1}{3}\zeta\chi\alpha_0\frac{nT^2}{\epsilon+P}(\vec{k}\cdot\vec{u}_1)\nabla^2\alpha$$
$$-2\alpha_1\eta\chi\frac{nT^2}{P+\epsilon}\left\{(\vec{u}_1\cdot\vec{\nabla})(\vec{k}\cdot\vec{\nabla})\alpha-\frac{1}{3}(\vec{k}\cdot\vec{u}_1)\nabla^2\alpha\right\}, \tag{11}$$

and the number conservation equation gives

$$0=\omega n_1-n(\vec{k}\cdot\vec{u}_1). \tag{12}$$

P_1 and ϵ_1 can be expressed in terms of the independent variables, n_1 and T_1 as follows:

$$\epsilon_1=\left(\frac{\partial\epsilon}{\partial T}\right)_nT_1+\left(\frac{\partial\epsilon}{\partial n}\right)_Tn_1, \tag{13}$$

and

$$P_1=\left(\frac{\partial P}{\partial T}\right)_nT_1+\left(\frac{\partial P}{\partial n}\right)_Tn_1. \tag{14}$$

We decompose the fluid velocity into directions perpendicular and parallel to the direction of wave vector $\vec{k}$ as

$$\vec{u}_1=\vec{u}_{1\perp}+\vec{k}(\vec{k}\cdot\vec{u}_1)/k^2. \tag{15}$$

The modes propagating along the direction of $\vec{k}$ are called longitudinal and those perpendicular to $\vec{k}$ are called transverse modes. Inserting Eq. (15) in the EoMs with the help of Eqs. (13) and (14) and collecting the transverse components, we get the dispersion relation for the transverse mode as

$$\omega^\perp=\frac{-ik^2(\eta-\alpha_1\chi\dot{\alpha})+\frac{nT^2}{(P+\epsilon)}2\alpha_1\eta\chi(\vec{k}\cdot\vec{\nabla})\dot{\alpha}}{\left[P+\epsilon-2\eta^2\beta_2k^2+\frac{nT^2}{3(P+\epsilon)}\chi(\alpha_0\zeta-2\alpha_1\eta)\nabla^2\alpha\right]}. \tag{16}$$

In the acausal limit ($\beta_2=\alpha_1=\alpha_0=0$), Eq. (16) reduces to

$$\omega^\perp=\frac{-ik^2\eta}{P+\epsilon}=i\omega_{\rm Im}^\perp, \tag{17}$$

which is the result obtained in acausal hydrodynamics [18]. We observe that ζ does not appear in the imaginary part of $\omega^\perp$ and it is purely imaginary if $\chi=0$.

The derivation of dispersion relation for the longitudinal component is lengthy and tedious to derive. The details are given in the Appendix A2. For $\chi=0$, the imaginary part of the longitudinal component of the dispersion relation is

$$\omega_{\rm Im}^{\parallel}=\frac{-k^2\left\{\frac{1}{3}\zeta+\frac{4}{3}\eta\right\}}{2\left\{(P+\epsilon)-\frac{1}{9}k^2\zeta^2\beta_0-\frac{8}{3}k^2\eta^2\beta_2\right\}}. \tag{18}$$

In the acausal limit, taking $\beta_0=\beta_2=0$ Eq. (18) reduces to

$$\omega_{\rm Im}^{\parallel}=\frac{-k^2\left\{\frac{1}{3}\zeta+\frac{4}{3}\eta\right\}}{2(P+\epsilon)}, \tag{19}$$

which matches with results of [18] for $\chi=0$. The coefficient of ζ in Eq. (18) differs from that of the one given in [18] because of the different numerical coefficient of ζ in $T^{\lambda\mu}$ used here.

The real part of the dispersion for the longitudinal modes turns out to be

$$\omega_{\rm Re}^{\parallel}=-\Bigg[-\bigg\{k^2\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{1}{3}\zeta+\frac{4}{3}\eta\right)\bigg\}^2-4\bigg\{(P+\epsilon)\left(\frac{\partial\epsilon}{\partial T}\right)_n-\frac{1}{9}k^2\zeta^2\beta_0\left(\frac{\partial\epsilon}{\partial T}\right)_n-\frac{8}{3}k^2\eta^2\beta_2\left(\frac{\partial\epsilon}{\partial T}\right)_n\bigg\}\times\bigg\{k^2n\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T-k^2n\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T-k^2(P+\epsilon)\left(\frac{\partial P}{\partial T}\right)_n\bigg\}\Bigg]^{1/2}\Bigg/\left[2\bigg\{(P+\epsilon)-\frac{1}{9}k^2\zeta^2\beta_0-\frac{8}{3}k^2\eta^2\beta_2\bigg\}\left(\frac{\partial\epsilon}{\partial T}\right)_n\right]. \tag{20}$$

In the acausal limit, considering vanishing net number density(n), Eq. (20) reduces to

$$\omega_{\rm Re}^{\parallel}=-\Bigg[-\bigg\{k^2\left(\frac{\partial\epsilon}{\partial T}\right)\left(\frac{1}{3}\zeta+\frac{4}{3}\eta\right)\bigg\}^2-4(P+\epsilon)\left(\frac{\partial\epsilon}{\partial T}\right)\times\bigg\{-k^2(P+\epsilon)\left(\frac{\partial P}{\partial T}\right)\bigg\}\Bigg]^{1/2}\Bigg/\left[2\bigg\{(P+\epsilon)\left(\frac{\partial\epsilon}{\partial T}\right)\bigg\}\right], \tag{21}$$

which appears as

$$\omega_{Re}=c_s|k|+(\text{constant})k^2, \tag{22}$$

where c_s is the speed of sound wave in the fluid. The acausal limit $\omega_{\rm Re}=c_s|k|$ can be recovered by keeping only the linear term [18]. The causal dispersion relation derived here can reproduce all the known relations existing in the acausal limit.

III. EFFECTS OF MAGNETIC FIELD

It was shown that a ultrahigh but transient magnetic field is generated in the collision of heavy ions at RHIC and LHC energies [19]. Survivability of the magnetic field will depend on the value of the conductivity of the QGP medium formed in these collisions. The presence of the magnetic field will affect the properties of the fluid through its contribution to the energy-momentum tensor. Considering constant magnetic field (B), the magnetic contribution is given by [20]

$$T_m^{\mu\nu}=\frac{B^2}{8\pi}(2u^\mu u^\nu+g^{\mu\nu}-2n'^\mu n'^\nu), \tag{23}$$

where n'^μ is the unit vector in the direction of the magnetic field $n'^\mu=B^\mu/B$ with $n'^\mu n'_\mu=-1$ and $u^\mu n'_\mu=0$ The conservation equation then reads

$$\partial_\mu T_{\rm tot}^{\mu\nu}=\partial_\mu T^{\mu\nu}+\partial_\mu T_m^{\mu\nu}=0, \tag{24}$$

where

$$T_{\rm tot}^{\mu\nu}=T^{\mu\nu}+T_m^{\mu\nu}.$$

For small perturbation, $T_{1m}^{\mu\nu}$ to $T_m^{\mu\nu}$ the first and third term of Eq. (23) will be changed. The changes in different components of $T_{1,m}^{\mu\nu}$ are given in the Appendix A3. After taking Fourier transformation, the equations of motions in the presence of the magnetic field becomes

$$0=\omega T_{1,\rm tot}^{00}-k_iT_{1,\rm tot}^{i0}=F_1-\frac{B^2}{8\pi}\left[2(\vec{k}\cdot\vec{u}_1)-\frac{(\vec{B}\cdot\vec{k})}{B^2}(\vec{B}\cdot\vec{u}_1)\right], \tag{25}$$

$$0=\omega T_{1,\rm tot}^{i0}-k_jT_{1,\rm tot}^{ij}=F_2+\frac{\omega B^2}{8\pi}\left[2u_1^i-\frac{B^i}{B^2}(\vec{B}\cdot\vec{u}_1)\right], \tag{26}$$

where F_1 and F_2 are the expressions given in the right-hand side of Eqs. (10) and (11), respectively. Decomposing the fluid velocity as Eq. (15), Eq. (25) becomes

$$0=F_1'-\frac{B^2}{8\pi}\left[2(\vec{k}\cdot\vec{u}_1)-\frac{(\vec{B}\cdot\vec{k})}{B^2}(\vec{B}\cdot\vec{u}_{1\perp})-\frac{(\vec{B}\cdot\vec{k})^2}{B^2k^2}(\vec{k}\cdot\vec{u}_1)\right]. \tag{27}$$

If we take the constant magnetic field along the direction of wave vector k, then $(\vec{B}\cdot\vec{k})=kB$ and $(\vec{B}\cdot\vec{u}_{1\perp})=0$ as $\vec{u}_1\perp\vec{k}$. In that case Eq. (27) becomes

$$0=F_1'-\frac{B^2}{4\pi}(\vec{k}\cdot\vec{u}_1). \tag{28}$$

Similarly, after decomposition Eq. (26) reads as

$$0=F_2'+\frac{\omega B^2}{8\pi}\left[2u_{1\perp}^i+\frac{2k^i}{k^2}(\vec{k}\cdot\vec{u}_1)-\frac{B^i}{Bk}(\vec{k}\cdot\vec{u}_1)\right], \tag{29}$$

and the number conservation equation remains unchanged. Here F_1' and F_2' represents the right-hand side (RHS) of Eq. (10) and (11), respectively, after decomposition of fluid velocity. The dispersion relation in the transverse direction in the presence of constant B is given by

$$\omega^\perp=\frac{-ik^2(\eta-\alpha_1\chi\dot{\alpha})+\frac{nT^2}{(P+\epsilon)}2\alpha_1\eta\chi(\vec{k}\cdot\vec{\nabla})\dot{\alpha}}{\left[P+\epsilon-\frac{B^2}{4\pi}-2\eta^2\beta_2k^2+\frac{nT^2}{3(P+\epsilon)}\chi(\alpha_0\zeta-2\alpha_1\eta)\nabla^2\alpha\right]}. \tag{30}$$

For $B=0$, Eq. (30) reduces to Eq. (16). The imaginary part of the dispersion relation in the longitudinal direction can be expressed as

$$\omega_{\rm Im}^{\parallel}=\frac{-k^2\left(\frac{1}{3}\zeta+\frac{4}{3}\eta\right)}{2\left[P+\epsilon+\frac{B^2}{8\pi}-\frac{1}{9}k^2\zeta^2\beta_0-\frac{8}{3}k^2\eta^2\beta^2\right]}, \tag{31}$$

and the real part is

$$\omega_{\rm Re}^{\parallel}=\Bigg[-\left\{k^2\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{1}{3}\zeta+\frac{4}{3}\eta\right)\right\}^2-4\left\{(P+\epsilon)+\frac{B^2}{8\pi}-\frac{1}{9}k^2\zeta^2\beta_0-\frac{8}{3}k^2\eta^2\beta_2\right\}\left(\frac{\partial\epsilon}{\partial T}\right)_n\Bigg\{k^2n\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T$$
$$-\frac{k^2B^2}{4\pi}\left(\frac{\partial P}{\partial T}\right)_n-k^2n\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T-k^2(P+\epsilon)\left(\frac{\partial P}{\partial T}\right)_n\Bigg\}\Bigg]^{1/2}\Bigg/$$
$$\left[2\left\{(P+\epsilon)+\frac{B^2}{8\pi}-\frac{1}{9}k^2\zeta^2\beta_0-\frac{8}{3}k^2\eta^2\beta_2\right\}\left(\frac{\partial\epsilon}{\partial T}\right)_n\right], \tag{32}$$

where we have considered $\chi=0$ to keep the expression compact, however, the derivation of the dispersion relation for $\chi\neq 0$ is straightforward.

IV. EFFECTS OF CAUSALITY ON FLUIDITY

What is the difference that it makes to characterize a relativistic fluid by using causal vis-a-vis acausal dispersion relations? In the following we will study this aspect in detail. The fluidity of QGP can be studied [13] by introducing the ratio of two length scales—one of those is related to the wave length of the sound wave propagating through the fluid. The other one is the interparticle distance in the fluid.

A. Viscous horizon

In the following we will provide the threshold value of wave vector k_v above which no sound wave can propagate. The quantity, $R_v\sim k_v^{-1}$, determines the length scale called viscous horizon [22]. The imaginary part of the dispersion relation dictates the attenuation of the sound wave in the fluid. A sound wave damps in time as $\sim\exp(\omega_{\rm Im}t)$ (for $\omega_{\rm Im}<0$) in the viscous medium. This can be expressed in terms of perturbation to $T^{\mu\nu}$ as

$$T_1^{\mu\nu}(t)=T_1^{\mu\nu}(t_i)\exp(\omega_{\rm Im}t), \tag{33}$$

where $T_1^{\mu\nu}(t_i)$ represents the perturbation to $T^{\mu\nu}$ at the initial time t_i. The dispersion relation derived in the previous section may be used to determine the upper limit of wave vector k_v of the sound wave that can propagate in the medium, which can be obtained by setting $|\omega_{\rm Im}|t=1$,

$$k_{v,\rm long}^{\rm causal}\equiv\frac{1}{R_{v,\rm long}^{\rm causal}}=\sqrt{\frac{P+\epsilon}{\frac{t}{2}\left(\frac{\zeta}{3}+\frac{4\eta}{3}\right)+\frac{1}{9}\zeta^2\beta_0+\frac{8}{3}\eta^2\beta_2}}. \tag{34}$$

We note that the viscous horizon scale, $R_v\sim\sqrt{t}$, in contrast to the sound horizon which varies linearly with t. The above condition implies that a longitudinal mode with magnitude of k larger than $k_{v,\rm long}^{\rm causal}\equiv 1/R_{v,\rm long}^{\rm causal}$ will be killed by dissipation and all other longitudinal modes with lower values of k will propagate. The known result [22] in the acausal limit ($\beta_0=\beta_2=0$) can be obtained as

$$k_{v,\rm long}^{\rm acausal}\equiv\frac{1}{R_{v,\rm long}^{\rm acausal}}=\sqrt{\frac{P+\epsilon}{\frac{t}{2}\left(\frac{\zeta}{3}+\frac{4\eta}{3}\right)}}. \tag{35}$$

Similarly for the causal transverse mode we have the upper limit,

$$k_{v,\rm tran}^{\rm causal}\equiv\frac{1}{R_{v,\rm tran}^{\rm causal}}=\sqrt{\frac{P+\epsilon}{\eta(t+2\eta\beta_2)}}, \tag{36}$$

and in the acausal limit the above relation turns out to be

$$k_{v,\rm tran}^{\rm acausal}\equiv\frac{1}{R_{v,\rm tran}^{\rm acausal}}=\sqrt{\frac{P+\epsilon}{\eta t}}. \tag{37}$$

We have already seen in the previous section that the application of the magnetic field changes the dispersion relations. Therefore, the viscous horizon in the presence of the magnetic field should also change to

$$k_{v,{\rm tran},B}^{\rm causal}\equiv\frac{1}{R_{v,{\rm tran},B}^{\rm causal}}=\sqrt{\frac{P+\epsilon-\frac{B^2}{4\pi}}{\eta(t+2\eta\beta_2)}}, \tag{38}$$

$$k_{v,{\rm long},B}^{\rm causal}\equiv\frac{1}{R_{v,{\rm long},B}^{\rm causal}}=\sqrt{\frac{P+\epsilon+\frac{B^2}{8\pi}}{\frac{t}{2}\left(\frac{\zeta}{3}+\frac{4\eta}{3}\right)+\frac{1}{9}\zeta^2\beta_0+\frac{8}{3}\eta^2\beta_2}}. \tag{39}$$

The viscous horizon has an impact on the flow harmonics. It is argued in [23] that the properties related to the ratio of higher order to second order harmonics, i.e., v_n/v_2 with ($n>2$) can be understood in terms of the propagation of sound wave through the dissipative medium and hence such studies will help in estimating the size of the sound horizon and viscous horizon [22].

B. Measure of fluidity

The sound wave in a viscous fluid will stop propagating if its wave length is smaller than some threshold value, $\lambda_{\rm th}=2\pi/k_v$. The value of $\lambda_{\rm th}$ will depend on the values of dissipative coefficients, η,ζ,χ, etc. The fluidity of the system was defined in Refs. [13,21] with the introduction of a new quantity which depends on the intrinsic properties of the fluid and enables one to compare fluids of wide varieties such as nonrelativistic fluid like water and relativistic, extremely dense, and hot fluid like QGP. For example, the temperature of water and QGP differ by a factor $\sim O(10^{10})$. Now if we want to compare their fluidity we may find the dissipation per interparticle separation. In Ref. [13] the linearized first-order dispersion relation of the sound mode was used:

$$\omega = c_s k - \frac{i}{2}k^2 \frac{\frac{4}{3}\eta}{h/c^2}. \tag{40}$$

The imaginary part of the dispersion relation represents the dissipation of the sound wave in the medium. The sound mode with wave vector k will propagate if the imaginary part of frequency is small, i.e.,

$$\left|\frac{\omega_{\rm Im}(k)}{\omega_{\rm Re}(k)}\right| \ll 1. \tag{41}$$

The limiting value can be found by setting $|\omega_{\rm Im}/\omega_{\rm Re}| = 1$, which gives $k = 3hc_s/(2\eta)$ then the resulting threshold for the wavelength of the sound mode becomes

$$\lambda_{\rm th} = \frac{2\pi}{k_v} = \frac{4\pi}{3}\frac{\eta}{hc_s} = \frac{4\pi}{3}L_\eta, \tag{42}$$

where $L_\eta = \eta/(hc_s)$. The L_η gives an estimation for the lowest sound wavelength ($\lambda_{\rm th}$) which can propagate through the viscous fluid. The L_η has the dimension of length and can be used to characterize fluids. However, introduction of a dimensionless scale will enable us to compare fluids with varying densities. Quantities like Reynolds or Knudsen numbers have been used in Refs. [24,25], respectively, to study flow properties. However, both of these quantities involve parameters, like dimension of the system which is not connected with the intrinsic properties of the fluid. The particle number density(ρ) can be used to estimate the interparticle distance, $L_\rho \sim \rho^{-1/3}$, which is related to the intrinsic properties of the fluid. The ratio of L_η to L_ρ may be used to characterize the fluid. For relativistic QGP with vanishing net baryon number density, entropy density (s) can be used to estimate ρ by using $\rho \sim s/4$. The ratio of these two length scales can be used as a measure of fluidity,

$$F \equiv \frac{L_\eta}{L_\rho}. \tag{43}$$

What is the corresponding expression of F for causal fluid dynamics involving other transport coefficients in addition to η? We use dispersion relations derived from causal relativistic hydrodynamics involving shear, bulk viscosities, thermal conductivity, and different relaxation coefficients to estimate the fluidity. We would contrast our results to those obtained with acausal relation [13]. The length scale analogous to L_η for causal fluid dynamics is denoted by L_T and depends on the transport coefficients like ζ, χ, β_0, β_2 in addition to η. L_T for the longitudinal mode is given by

$$\begin{aligned} L_T = \left[\frac{1}{4}\right]\Bigg\{&\left(\frac{\partial\epsilon}{\partial T}\right)_n\zeta^2 + 8\zeta\eta\left(\frac{\partial\epsilon}{\partial T}\right)_n + 16\eta^2\left(\frac{\partial\epsilon}{\partial T}\right)_n + 2(P+\epsilon)\beta_0\zeta^2\left(\frac{\partial P}{\partial T}\right)_n + 2n\beta_0\zeta^2\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T \\ &- 2n\zeta^2\beta_0\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T + 48\eta^2\beta_2(P+\epsilon)\left(\frac{\partial P}{\partial T}\right)_n + 48\eta^2\beta_2 n\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T - 48\eta^2\beta_2 n\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T\Bigg\}^{\frac{1}{2}} \Bigg/ \\ &\left\{(P+\epsilon)^2\left(\frac{\partial P}{\partial T}\right)_n + n(P+\epsilon)\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T - n(P+\epsilon)\left(\frac{\partial\epsilon}{\partial n}\right)_T\left(\frac{\partial P}{\partial T}\right)_n\right\}^{1/2}. \end{aligned} \tag{44}$$

We use $(\partial P/\partial T) = (\partial P/\partial\epsilon)(\partial\epsilon/\partial T)$ to express F as

$$\begin{aligned} F = \left[\frac{\rho^{\frac{1}{3}}}{4}\right]\Bigg\{&\zeta^2 + 8\zeta\eta + 16\eta^2 + 2(P+\epsilon)\beta_0\zeta^2\left(\frac{\partial P}{\partial\epsilon}\right)_n + 2n\beta_0\zeta^2\left(\frac{\partial P}{\partial n}\right)_T - 2n\zeta^2\beta_0\left(\frac{\partial P}{\partial\epsilon}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T + 48\eta^2\beta_2 \\ &\times(P+\epsilon)\left(\frac{\partial P}{\partial\epsilon}\right)_n + 48\eta^2\beta_2 n\left(\frac{\partial P}{\partial n}\right)_T - 48\eta^2\beta_2 n\left(\frac{\partial P}{\partial\epsilon}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T\Bigg\}^{\frac{1}{2}} \Bigg/ \\ &\left\{(P+\epsilon)^2\left(\frac{\partial P}{\partial\epsilon}\right)_n + n(P+\epsilon)\left(\frac{\partial P}{\partial n}\right)_T - n(P+\epsilon)\left(\frac{\partial\epsilon}{\partial n}\right)_T\left(\frac{\partial P}{\partial T}\right)_n\right\}^{1/2}. \end{aligned} \tag{45}$$

This is measure of fluidity of a relativistic fluid for $\chi = 0$.

For a fluid having vanishing net charge density ($n = 0$) the above equation becomes

$$\begin{aligned} F = \Bigg[\frac{\rho^{\frac{1}{3}}}{4}\Bigg\{&\zeta^2 + 8\zeta\eta + 16\eta^2 + 2(P+\epsilon)\beta_0\zeta^2\left(\frac{\partial P}{\partial\epsilon}\right) \\ &+ 48\eta^2\beta_2(P+\epsilon)\left(\frac{\partial P}{\partial\epsilon}\right)\Bigg\}^{\frac{1}{2}}\Bigg] \Bigg/ \left\{(P+\epsilon)^2\left(\frac{\partial P}{\partial\epsilon}\right)\right\}^{1/2}. \end{aligned} \tag{46}$$

It is clear from this result that dispersion relations become more complex if relativistic causal hydrodynamics is used. Two more coefficients β_0 and β_2 enter into the expression for fluidity. In the acausal limit, i.e., for vanishing β_0 and β_2 as well as neglecting nonlinear terms in the real part of ω, the F reads

$$F = \frac{\rho^{1/3}\eta}{hc_s}, \tag{47}$$

which is exactly what is given in Ref. [13]. It may be noted from Eq. (45) that the fluidity measure F of the causal fluid has a complicated functional dependence on various transport coefficients and thermodynamic variables of the fluid. In contrast to the causal case the F has simpler dependence on transport coefficients and thermodynamical variables in an acausal scenario [Eq. (47)].

C. Fluidity in presence of magnetic field

We have already seen that nonzero B affects the real and imaginary part of ω along the longitudinal direction and hence it modifies the fluidity measure also. For vanishing net charge and $\zeta = \chi = 0$ the L_T becomes

$$L_T = \left[\left(\frac{\partial\epsilon}{\partial P}\right)(\zeta + 4\eta)^2 + 2\beta_0\zeta^2\left(\frac{B^2}{4\pi} + P + \epsilon\right) + 48\eta^2\beta_2\left(\frac{B^2}{4\pi} + P + \epsilon\right)\right]^{1/2} \Bigg/ \left[4\left\{2\left(\frac{B^2}{8\pi}\right)^2 + \frac{3B^2}{8\pi}(P+\epsilon) + (P+\epsilon)^2\right\}^{1/2}\right]. \tag{48}$$

For simplicity we kept only η as nonzero. However, it is straightforward to find F with nonzero n,ζ, and χ.

V. RESULTS AND DISCUSSION

In this section we discuss the dispersion relation for the transverse and longitudinal modes for nonexpanding fluid.

A. Transverse mode

To see how causality or causal hydrodynamics affects the damping of the sound wave, first we consider the transverse component of the dispersion relation. For $\chi = 0$, Eq. (16) reads

$$\omega^{\perp}_{\rm Im} = \frac{-k^2\eta}{[P + \epsilon - 2\eta^2\beta_2 k^2]}. \tag{49}$$

It is interesting to note that the bulk viscosity does not appear in the dispersion relation for the transverse mode. The coefficient β_2 appearing in the denominator is the signature of causal hydrodynamics. In the ultrarelativistic limit it has the limiting value [12],

$$\beta_2 = \frac{3}{4P}. \tag{50}$$

We estimate the damping of the sound wave by using the thermodynamic relation for vanishing net charge density (such as baryon free QGP), $P + \epsilon = sT$. In Fig. 1 we display the damping of the transverse mode with k for $\eta/s = 1/4\pi$ at $T = 200$, 300, and 400 MeV. We find that damping is stronger for larger η/s, lower T, and larger wave numbers or smaller wave lengths. The imaginary part of the dispersion relation leads to the variation of amplitude with k as $\sim \exp(-\Gamma_s k^2)$ where the Γ_s, square of the characteristic dissipation length that picks up different values at causal and acausal scenarios resulting in a different damping rate for different k. Although for small k it is not significant but at large $k > 200$ MeV the difference is distinctly visible in the results displayed in Fig. 1. The decay of the perturbation with time is shown in Fig. 2 for $\eta/s = 1/4\pi$ for different k. We observe that at $T = 400$ MeV

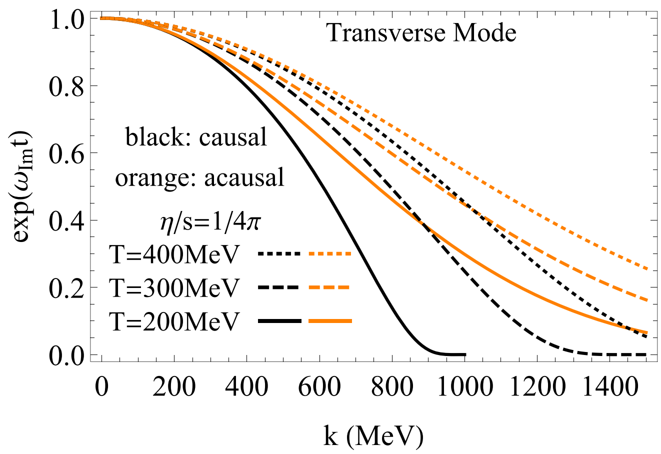

FIG. 1. Damping of transverse perturbative modes in QGP with k at $T = 200, 300$, and 400 MeV for causal and acausal hydrodynamics. We have taken $t = 0.6$ fm/c.

the perturbations decay faster in the causal than in the acausal hydrodynamic as k increases. Stronger damping is observed at $T = 200$ and 300 MeV (not shown in the figure). At large t, the amplitude of the perturbations for causal and acausal scenarios is close because at large t the amplitude decays to a very small value irrespective of the value of $\omega_{\rm Im}$. Similarly at small t the amplitude of the perturbation is also close. The enhanced magnitude of η/s enforces faster decay. All these results represent a physically consistent picture because it is well known that in the acausal (first order) hydrodynamics a nonequilibrium system evolves to the equilibrium instantly. However, in second-order hydrodynamics the nonequilibrium system does not go to the equilibrium state instantaneously but takes some nonzero time. This nonzero time lag is incorporated in the second-order hydrodynamics by means of relaxation coefficients such as β_0,β_1,β_2. In other words the second-order hydrodynamics effectively enhances the dissipation of the system. As any disturbance will dissipate faster in a higher order viscous hydrodynamics than the lower one, the

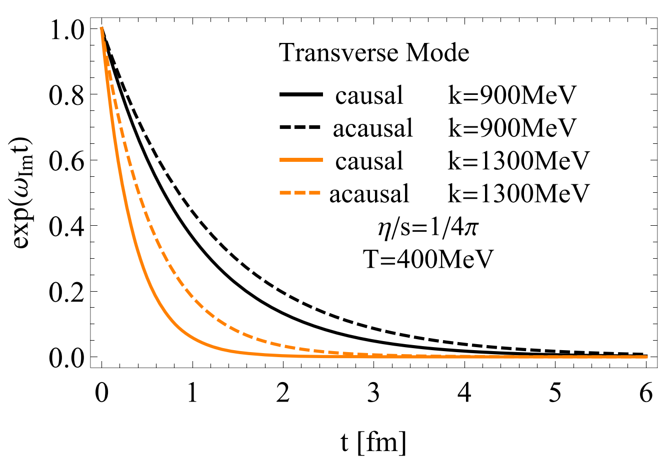

FIG. 2. Damping of the transverse perturbative modes in QGP with time (t) at $T = 400$ MeV for $\eta/s = 1/4\pi$ for causal and acausal hydrodynamics for various k.

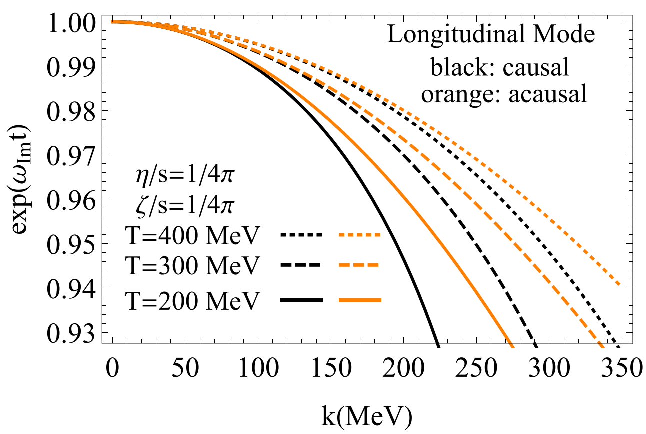

FIG. 3. Damping of perturbations with k in the longitudinal direction for $T = 200,300$, and 400 MeV. We have taken $t = 0.6$ fm/c.

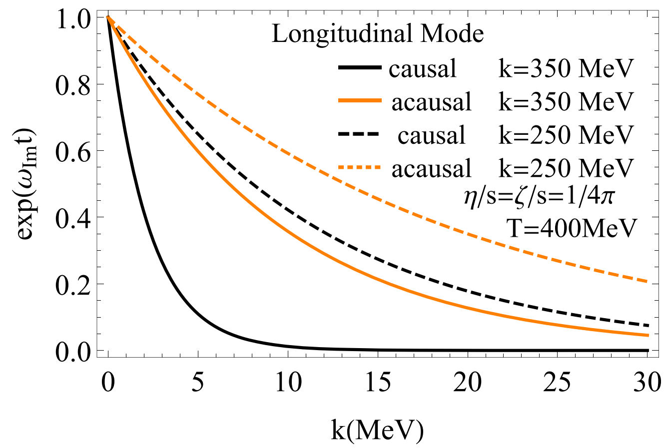

FIG. 4. Damping of the longitudinal mode with time (t) at $T = 400$ MeV for $\eta/s = 1/4\pi$ and $\zeta/s = 1/4\pi$ for various k values.

perturbations in causal disturbances fall faster than the acausal one. We have observed that the amplitude of the sound wave falls faster with an increase in η/s and decrease in T.

B. Longitudinal mode

To study the perturbations in longitudinal direction, we encounter a new relaxation coefficient β_0 that was absent in acausal theory. In the ultrarelativistic limit β_0 is given by [12]

$$\beta_0 = \frac{216}{P\beta^4}, \tag{51}$$

where $\beta = m/T$. We have used thermal mass to estimate β. To study the propagation of the longitudinal modes in the fluid we consider the gluonic fluid. The thermal mass of gluon is given by [26]

$$\frac{m_g}{T} = g\sqrt{\frac{C_A + N_f/2}{6}} \quad \Rightarrow \beta = g\sqrt{\frac{C_A + N_f/2}{6}}, \tag{52}$$

where $g = \sqrt{4\pi\alpha_s}$, $C_A = 3$, and $N_f = 2$ (for two flavors). In the present work we have taken $\alpha_s = 0.2$. We use Eq. (18) with the aid of β_0 to study the dissipation of the longitudinal modes. One major difference with the transverse mode is the appearance of bulk viscosity in the longitudinal mode and it will be seen later that bulk viscosity plays a dominant role in the damping of the perturbations. The nature of variation of the perturbations of the longitudinal mode is similar to that of transverse modes. The perturbation decays faster with k in causal than in acausal hydrodynamics (Fig. 3). At lower T a faster decay is observed. In Fig. 4, we depict the dissipation of the perturbations with time for $\eta/s = 1/4\pi$ for different k values. A faster decay is observed at higher η/s and lower T. Similar to the transverse modes the difference in the decay of longitudinal amplitudes in causal and acausal hydrodynamics is significant. We have discussed before that the longitudinal dispersion relation is controlled not only by shear but by the bulk viscosity as well. The damping of the longitudinal modes from shear and bulk viscous coefficients and the relative importance of these coefficients are investigated. The variation of the damping with k was depicted in Fig. 5. The result indicates a bigger influence of the bulk viscosity on the longitudinal modes than the shear viscosity.

As mentioned in Sec. III the QGP fluid may be subjected to the external magnetic field (B) created from the relativistic motion of the colliding nuclei. The magnitude of the field during evolution of QGP will depend on the rate of decay of the field which is controlled by the value of electrical conductivity of QGP. We assume a nonzero constant magnetic field in the QGP and study its effects on the fluid properties. We find that the energy from the magnetic field appears with opposite sign in the denominators of $\omega^{\perp}$ and $\omega^{\parallel}$ given by Eqs. (30) and (31), respectively. This is reflected in the results displayed Figs. 6 and 7 for the variation of damping with k and t, respectively. The transverse modes decay faster in causal hydrodynamics. An opposite trend is observed for the longitudinal modes.

C. Quantitative changes in the viscous horizon

We would like to estimate the shift in the viscous horizon caused by causal hydrodynamics as compared to the acausal one. The viscous horizon size scales with time as $R_v \sim 1/\sqrt{t}$. Through the relation, $R_v(\text{fm}) \approx 197/k_v(\text{MeV})$, it determines

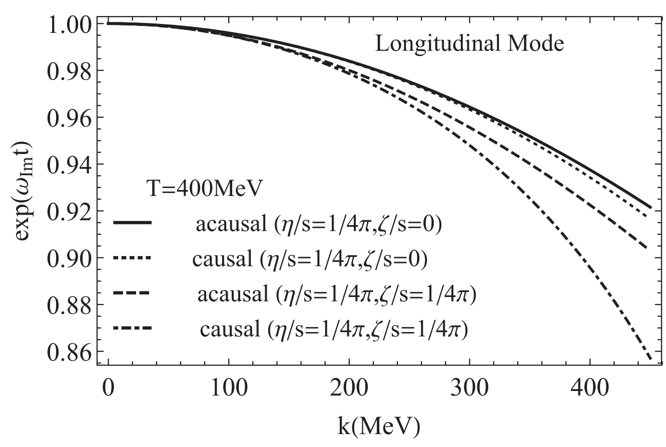

FIG. 5. Damping of the perturbations in QGP for different values of η/s and ζ/s. t is taken as 0.6 fm/c.

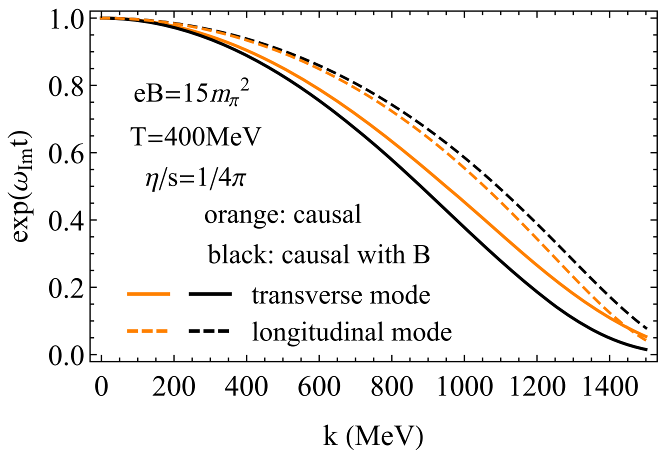

FIG. 6. Damping of perturbations with k in QGP in the presence of B along the transverse direction (solid line) and longitudinal direction (dashed line) at $T = 400$ MeV. The value of t is taken as 0.6 fm/c here.

the wave length that is unable to propagate in the dissipative medium, i.e., if the wave length is less than $2\pi/k_v$ then those waves will dissipate.

Using Eqs. (34)–(37) we can estimate viscous horizon scales at different times. The variation of k_v with t for causal and acausal hydrodynamics was depicted in Fig. 8 at $T = 400$ MeV. It is observed that k_v for the causal scenario approaches the k_v for the acausal scenario at large t. This trend can be understood from the mathematical expressions of Eqs. (34) and (35). However, if the time variation of pressure from hydrodynamic evolution is considered then β_0 and β_2 will also increase with time as evident from Eqs. (50) and (51) and in such a situation the difference between the causal and acausal scenario may survive at large t also.

In Fig. 9, we display the ratio of viscous horizon lengths for causal and acausal hydrodynamics as a function of t for $T = 200$ and 400 MeV. We find that the longitudinal scale in causal hydrodynamics is almost 3 times larger than acausal one

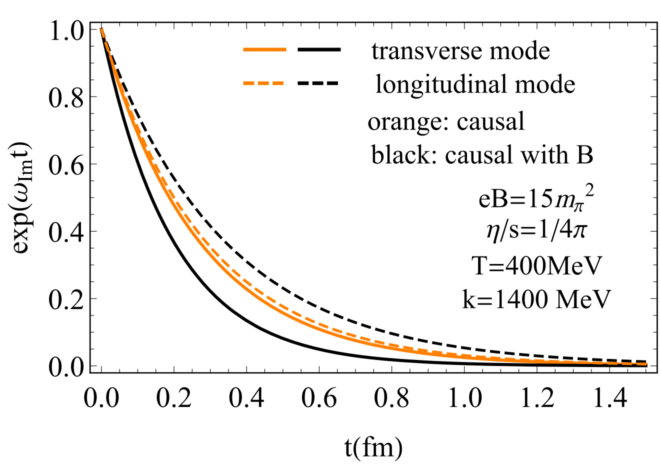

FIG. 7. Time variation of damping of the transverse mode (solid line) and longitudinal mode (dashed line) at T= 400 MeV in the presence of B.

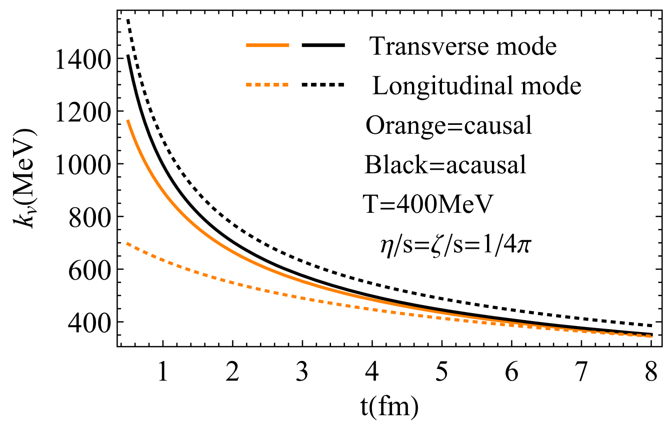

FIG. 8. Variation of k_v with t for causal and acausal hydrodynamics at $T = 400$ MeV.

at $t = 0.6$ fm for $T = 200$ MeV and $\eta/s = \zeta/s = 1/4\pi$. The same ratio becomes 2.07 for $T = 400$ MeV at $t = 0.6$ fm/c. We also note that the difference in the viscous horizon length for transverse modes is smaller than the longitudinal modes.

The viscous damping controls the highest order of flow harmonic (n_v) that will survive against the dissipative effects. The relation between n_v and R_v is given by [23]: $n_v = 2\pi R/R_v$ where R is the size of the fluid system. Therefore, an increase in R_v will reduce the value of n_v resulting in a shift in its value between casual and acausal scenarios. Because the value of n_v depends on η/s, measurement of amplitudes of various harmonics will help in determining the viscosity and consequently characterizating QGP [22].

D. Measure of fluidity

First we consider a system devoid of bulk viscosity. Then the fluidity measure of such a system can be obtained by putting $\zeta = 0$ in Eq. (46) which leads to

$$F = \frac{\rho^{\frac{1}{3}}\left\{16\eta^2 + 48\eta^2\beta_2(P+\epsilon)\left(\frac{\partial P}{\partial \epsilon}\right)\right\}^{1/2}}{\left[4\left\{(P+\epsilon)^2\left(\frac{\partial P}{\partial \epsilon}\right)\right\}^{\frac{1}{2}}\right]}, \tag{53}$$

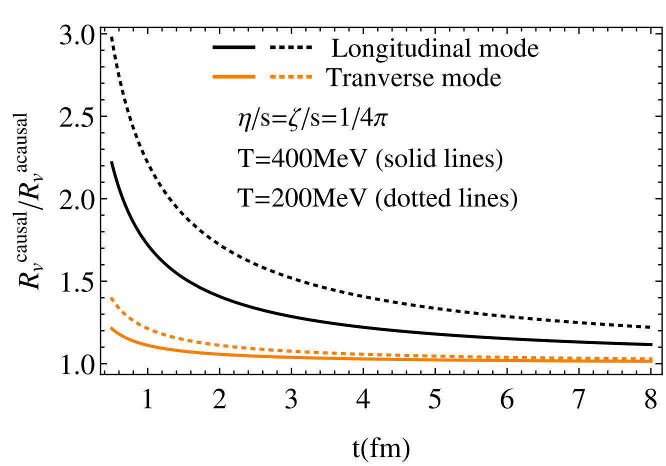

FIG. 9. Variation of the ratio of the viscous horizon length (R_v) with t for causal and acausal hydrodynamics.

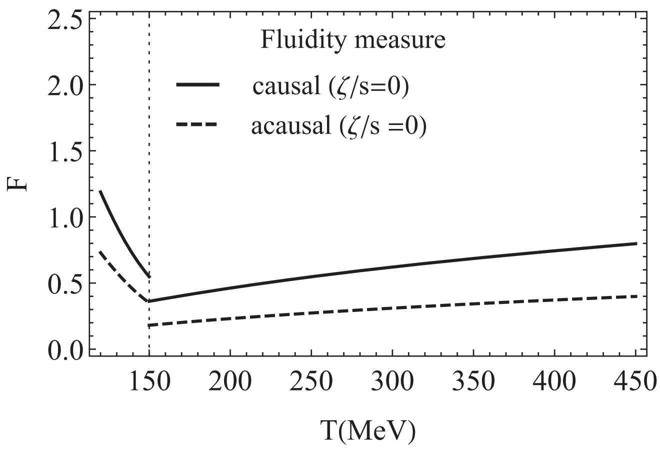

FIG. 10. Temperature variation of F in the QGP ($T > 150$ MeV) and hadronic phase ($T < 150$ MeV). The verticle line represents $T_c = 150$ MeV.

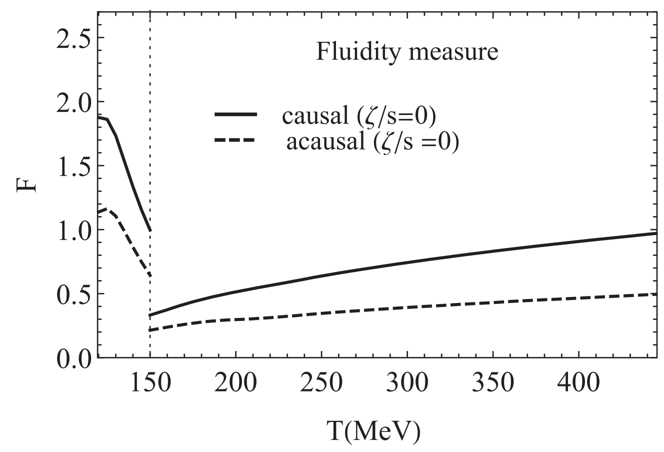

FIG. 11. Same as Fig. 10 with velocity of sound and other thermodynamic quantities taken from lattice QCD calculations (see text).

where $\beta_2 = 3/4P$ in the relativistic limit. We use Eq. (53) to display the variation of F with T for the following inputs. The particle number density (ρ) is estimated from entropy density (s) by using the relation $\rho \sim s/4$. We have used the parametric form of specific viscosity given in Ref. [27] as

$$\frac{\eta(T)}{s(T)} \approx \frac{1}{4\pi}\left(\frac{s_Q}{s_H}\right)\left(\frac{T}{T_c}\right)^{1-\frac{1}{c_s^2}} \text{ for } T < T_c$$
$$\approx \frac{1}{4\pi}\left[1 + W \ln\frac{T}{T_c}\right]^2 \text{ for } T > T_c, \tag{54}$$

where s_Q and s_H are the entropy densities in the QGP and hadrons at the transition temperature ($T_c = 150$ MeV). W is given by

$$\frac{W^2}{4\pi} = \frac{9\beta_0'^2}{\left[80\pi^2 K_{SB} \ln\left\{\frac{4\pi}{g^2(T)}\right\}\right]}, \tag{55}$$

where

$$[g^2(T)]^{-1} = \tfrac{9}{8}\pi^2 \ln(2\pi T/\Lambda) + \tfrac{4}{9}\pi^2 \ln 2[\ln(2\pi T/\Lambda)], \tag{56}$$

and $K_{SB} = 12$, $\Lambda = 190$ MeV, and $\beta_0' = 10$. The value of entropy density (s) and c_s^2 for the hadronic and QGP phases have been estimated from hadronic resonance gas (HRG) model [28] and quasiparticle QGP model. The relevant thermodynamic quantities have been derived from the partition function using standard relations. The F is displayed as a function of T in Fig. 10 for $\eta/s = 1/4\pi$. We observe that the value of F has increased in the causal scenario compared to the acausal dynamics. It is to be also noted that the enhancement is more with larger specific shear viscosity. The F has a nonlinear dependence on the transport coefficients and thermodynamic variables in causal scenario. However, in the acausal case the dependence on the coefficient of viscosity is linear. This is reflected in the results already depicted in Fig. 10 as well as results displayed below. We observe a sharp decrease of F in the hadronic phase with the increase in temperature, i.e., hadrons flow easily with rise in temperature. However, the temperature variation of F in the QGP phase is slower. As F is larger in the causal limit the fluid flow becomes difficult compared to the acausal case.

To study the sensitivity of the results on the velocity of sound we use the value of c_s^2 and other relevant thermodynamic variables, like entropy density, etc., from lattice QCD calculations [7]. The variation of F with T is displayed in Fig. 11. A larger discontinuity in F was seen when T_c and c_s^2 are taken from lattice QCD calculations. The shift of fluidity in second-order hydrodynamics from the first order is about 35% both in the hadronic as well as in the QGP phase near T_c. The same value of η/s was used for second- and first-order hydrodynamics, therefore, the shift in F is from stronger damping in causal hydrodynamics.

Figure 12 shows the dependence of fluidity of QGP on bulk viscosity in a causal dynamical scenario determined by Eq. (46). β_0, β_2, and β are taken as $216/P\beta^4$, $3/4P$, and 0.7, respectively. The bulk viscosity of the QGP phase was taken

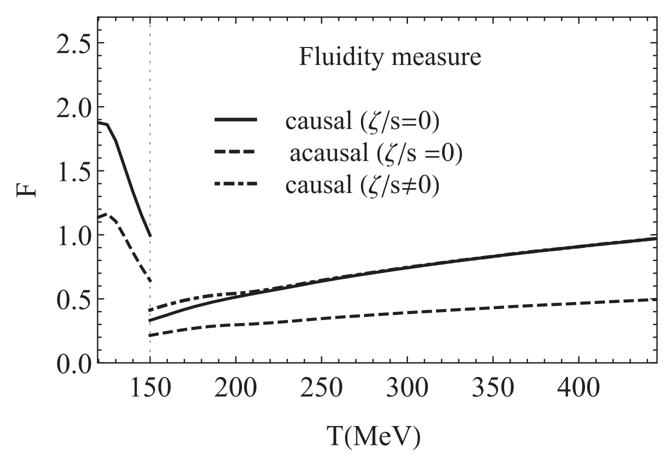

FIG. 12. Same as Fig. 11 in the presence of bulk viscosity (dot-dashed curve).

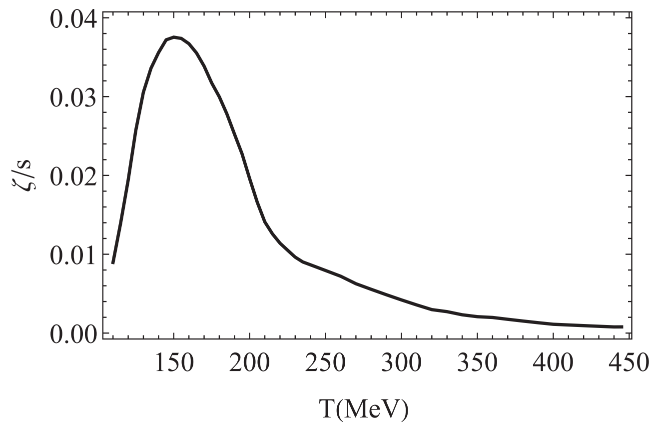

FIG. 13. Temperature variation of ζ/s.

in terms of shear viscosity as [18]

$$\frac{\zeta}{s} \approx 15\frac{\eta}{s}\left(1/3 - c_s^2\right)^2, \tag{57}$$

where the parametric form of η/s is taken from Eq. (54). We find a peak in the value of ζ/s around $T \sim 150$ MeV (Fig. 13). This peak is reflected as a bump in the temperature variation of F just above T_c, because of large conformal breaking $(1/3 - c_s^2)^2$ near T_c. It is also interesting to note that the bulk viscosity hardly plays any role at higher T because of its small numerical value. As T increases, beyond $T = 250$ MeV, conformal invariance restores and that results in almost vanishing ζ/s. However, a constant $\zeta/s = 1/4\pi$ represents a different picture as shown in Fig. 14. It is clear that nonzero value of $\zeta/s(\sim\eta/s)$ will play a crucial role in determining the fluidity of the system.

We have shown before that the magnetic field alters both the transverse and longitudinal modes. Therefore, it will affect the fluidity of the QGP as shown in Fig. 15. The F for the hadronic phase with the magnetic field was not shown, because the magnetic field will decay substantially and hence will have insignificant effects on fluidity of the hadronic phase which

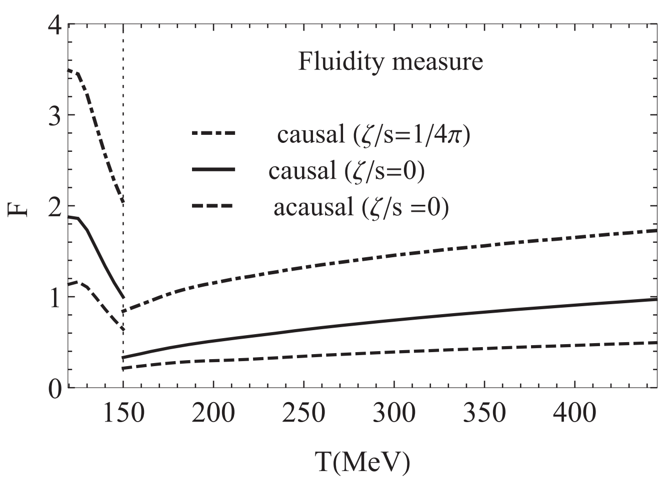

FIG. 14. Same as Fig. 10 in the presence of constant $\zeta/s\,(= 1/4\pi)$.

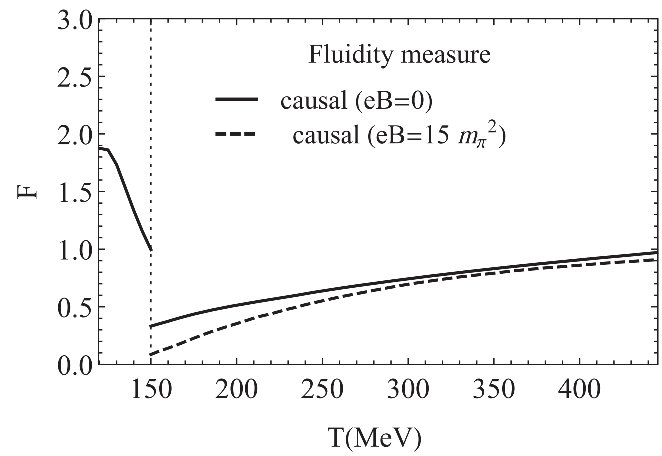

FIG. 15. Same as Fig. 11 in the presence of magnetic field $(eB = 15m_\pi^2)$.

appears late in the evolution history. As we discussed earlier B makes the fluid less dissipative in the QGP phase. Near T_c, F reduces significantly and hence the flow becomes easier near T_c.

For the AdS/CFT system, we have taken the well-known KSS lower bound ($\eta/s = 1/4\pi$) of shear viscosity [29] to show the variation of F with T above T_c (Fig. 16). We have taken $L_\rho = 1/T$ [13] which gives $F \approx 0.2$ in acausal hydrodynamics and $F \approx 0.4$ in its causal counterpart. The fluidity factor F gets enhanced as expected in Israel-Stewart hydrodynamics by a factor of 2 hence makes it harder for the fluid to flow.

In Fig. 17 the variation of the ratio of two length scales, L_T/L_η was plotted as a function of T. We find that the ratio remains above unity for the temperature range considered. It is discussed in Ref. [13] that the applicability of hydrodynamics may be resolved from the the ratio of L_η estimated in acausal hydrodynamics to some external length scale, say, the size of the system R. Because $L_T/L_\eta > 1$, therefore, the applicability of hydrodynamics becomes poorer when causality effects are

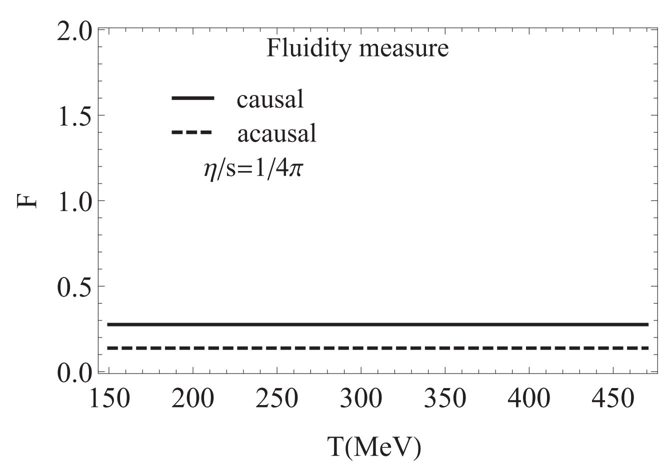

FIG. 16. Same as Fig. 11 for AdS/CFT fluid.

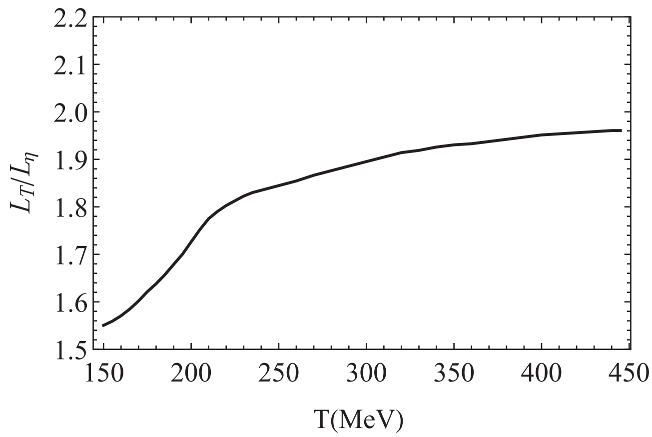

FIG. 17. The ratio of length scales L_T and L_η corresponding to causal and acausal hydrodynamics (see text) as a function of temperature.

included in the fluid dynamics, if all other relevant quantities are kept the same in causal and acausal scenarios.

VI. SUMMARY AND CONCLUSION

In summary, we have derived dispersion relations of relativistic fluid using Israel-Stewart second-order causal viscous hydrodynamics. It is shown that the dispersion relations in acausal hydrodynamics can be obtained from the causal results as a limiting case. The perturbations in viscous fluid damp faster within the scope of causal hydrodynamics than its acausal counterpart. The waves with large k suffer more damping than the waves with short k. In both the longitudinal and transverse dispersion relations the difference between the causal and acausal hydrodynamics is significant. The difference increases with the magnitude of viscosities. It was also noted that the bulk viscosity does not play any role in the dissipation of the transverse modes but it plays a crucial role in the dispersion for the longitudinal modes. The dispersion relations in the presence of magnetic field have also been derived and it is shown that the magnetic field affects the longitudinal and transverse modes oppositely. The magnetic field makes the fluid effectively less dissipative. The dispersion relations derived here have been used to find a viscous measure of the fluid as well as the viscous horizon. We have seen that the use of causal relations enhances the size of the viscous horizon of the longitudinal mode by more than a factor two for the parameter values used here. Inclusion of the causality enhances the F of QGP near T_c. The bulk viscosity affects the fluidity strongly near T_c. However, its role becomes less important at higher temperature with the restoration of conformal symmetry resulting in lower ζ. We also find that the effects of ζ on F is more prominent than η if η and ζ have similar magnitudes. Magnetic field makes a fluid more perfect by compensating the effects of viscosity near T_c. The fluidity is enhanced by a constant factor for AdS/CFT fluid within the causal hydrodynamics.

In Ref. [13] the fluidity was studied in the supercritical domain within the purview of acausal hydrodynamics. We observed a shift in fluidity from causal hydrodynamics as compared to acausal hydrodynamics. It is expected that a similar shift will be seen in the supercritical region too.

In a nutshell the incorporation of causality in relativistic hydrodynamics makes the following changes with respect to acausal hydrodynamics: (i) The fluidity measure F increases and thus flow of the fluid becomes strenuous, (ii) the value of the highest order of flow harmonics (n_v) reduces as the viscous horizon R_v increases, and (iii) applicability of the hydrodynamics becomes poorer because $L_T > L_\eta$ in the temperature range considered. In these conclusions it was tacitly assumed that the relevant quantities, like η/s, etc., are kept the same in both the causal and acausal scenarios.

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APPENDIX

1. Perturbations in $T^{\lambda\mu}$

We evaluate the perturbation in the energy-momentum tensor ($T^{\lambda\mu}$), of the Israel-Stewart hydrodynamics. We denote perturbations in P, ϵ, n, T, and u^α by P_1,ϵ_1,n_1,T_1, and u_1^α, respectively, and decompose $T^{\lambda\mu}$ in Eq. (6) into sum of $A^{\lambda\mu}$, $B^{\lambda\mu}$, $C^{\lambda\mu}$, $D^{\lambda\mu}$, $E^{\lambda\mu}$, and $F^{\lambda\mu}$. We assume the perturbations as $P' = P + P_1, \epsilon' = \epsilon + \epsilon_1, n' = n + n_1, T' = T + T_1$, and $u' = u + u_1$. With perturbation $A^{\lambda\mu}(= \epsilon u^\lambda u^\mu + P\triangle^{\lambda\mu})$ changes to

$$A'^{\lambda\mu} = \epsilon u^\lambda u^\mu + \epsilon_1 u^\lambda u^\mu + \epsilon u_1^\lambda u^\mu + \epsilon u^\lambda u_1^\mu + P\triangle^{\lambda\mu} + P u_1^\lambda u^\mu + P u^\lambda u_1^\mu + P_1\triangle^{\lambda\mu},$$

where we keep only the linear terms in perturbations. Thus, the change in $A^{\lambda\mu}$ reads

$$A_1^{\lambda\mu} = \epsilon_1 u^\lambda u^\mu + \epsilon u_1^\lambda u^\mu + \epsilon u^\lambda u_1^\mu + P u_1^\lambda u^\mu + P u^\lambda u_1^\mu + P_1\triangle^{\lambda\mu}. \quad \text{(A1)}$$

Similarly the change in the term $B^{\lambda\mu}(= -\frac{1}{3}\zeta u^\sigma_{|\sigma}\triangle^{\lambda\mu} + \frac{1}{9}\zeta^2\beta_0\dot{u}^\rho_{|\rho}\triangle^{\lambda\mu})$ arising due from perturbation is

$$B_1^{\lambda\mu} = -\tfrac{1}{3}\zeta\left(\partial_\sigma u_1^\sigma - \tfrac{1}{3}\zeta\beta_0\partial_0\partial_\sigma u_1^\sigma\right)\triangle^{\lambda\mu}. \quad \text{(A2)}$$

Perturbation in $C^{\lambda\mu}(=\frac{nT^2}{P+\epsilon}\frac{\zeta\alpha_0\chi}{3}\partial_\sigma[\triangle^{\sigma\rho}(\partial_\rho\alpha)\triangle^{\lambda\mu}])$ is

$$C_1^{\lambda\mu}=\frac{\zeta}{3}\alpha_0\chi\partial_\sigma\left\{(n_1T^2+2TT_1n)\frac{\triangle^{\sigma\rho}}{P+\epsilon}\partial_\rho\alpha-\frac{nT^2(P_1+\epsilon_1)}{(P+\epsilon)^2}\triangle^{\sigma\rho}\partial_\rho\alpha+\frac{nT^2}{P+\epsilon}\left(u_1^\sigma u^\rho+u^\sigma u_1^\rho\right)\partial_\rho\alpha\right\}$$
$$\times\triangle^{\lambda\mu}+\frac{nT^2}{(P+\epsilon)}\frac{\zeta\alpha_0\chi}{3}\partial_\sigma\{\triangle^{\sigma\rho}\partial_\rho\alpha\}\left(u_1^\lambda u^\mu+u^\lambda u_1^\mu\right).$$

Perturbation in $D^{\lambda\mu}(=-2\eta u^{\langle\lambda|\mu\rangle})$ reads

$$D_1^{\lambda\mu}=-\eta\left(\triangle_\alpha^\lambda\triangle_\beta^\mu+\triangle_\alpha^\mu\triangle_\beta^\lambda-\tfrac{2}{3}\triangle^{\lambda\mu}\triangle_{\alpha\beta}\right)\partial^\beta u_1^\alpha. \tag{A3}$$

Change in the term $E^{\lambda\mu}(=4\eta^2\beta_2\dot{u}^{\langle\lambda|\mu\rangle})$ from perturbation is

$$E_1^{\lambda\mu}=2\eta^2\beta_2\partial_0\left\{\left(\triangle_\alpha^\lambda\triangle_\beta^\mu+\triangle_\alpha^\mu\triangle_\beta^\lambda-\tfrac{2}{3}\triangle^{\lambda\mu}\triangle_{\alpha\beta}\right)\partial^\beta u_1^\alpha\right\}. \tag{A4}$$

The $F^{\lambda\mu}(=\frac{2nT^2}{P+\epsilon}\alpha_1\eta\chi[\triangle_\alpha^{(\lambda}\triangle_\beta^{\mu)}-\frac{1}{3}\triangle_{\alpha\beta}\triangle^{\lambda\mu}]\partial^\beta\triangle^{\alpha\rho}\partial_\rho\alpha)$ is perturbed by the term,

$$F_1^{\lambda\mu}=\alpha_1\chi\eta\left[\triangle_\alpha^\lambda\triangle_\beta^\mu+\triangle_\alpha^\mu\triangle_\beta^\lambda-\frac{2}{3}\triangle^{\lambda\mu}\triangle_{\alpha\beta}\right]\partial^\beta\left[\left\{\frac{n_1T^2+2TT_1n}{P_1+\epsilon_1}-\frac{nT^2(P_1+\epsilon_1)}{(P+\epsilon)^2}\right\}\triangle^{\alpha\rho}\partial_\rho\alpha\right.$$
$$\left.+\frac{nT^2}{P+\epsilon}\left(u_1^\rho u^\alpha+u^\rho u_1^\alpha\right)\partial_\rho\alpha\right]+\frac{nT^2}{P+\epsilon}\alpha_1\chi\eta\left[\left(u_1^\lambda u_\alpha+u^\lambda u_{1\alpha}\right)\triangle_\beta^\mu+\left(u_1^\mu u_\beta+u^\mu u_{1\beta}\right)\triangle_\alpha^\lambda\right.$$
$$\left.+\left(u_1^\mu u_\alpha+u^\mu u_{1\alpha}\right)\triangle_\beta^\lambda+\left(u_1^\lambda u_\beta+u^\lambda u_{1\beta}\right)\triangle_\alpha^\mu-\frac{2}{3}\left\{\left(u_1^\lambda u_\mu+u^\lambda u_{1\mu}\right)\triangle_{\alpha\beta}+(u_{1\alpha}u_\beta+u_\alpha u_{1\beta})\triangle^{\lambda\mu}\right\}\right]\partial^\beta[\triangle^{\alpha\rho}\partial_\rho\alpha]. \tag{A5}$$

The net change in $T^{\lambda\mu}$ from perturbation is the sum of all terms discussed above:

$$T_1^{\lambda\mu}=A_1^{\lambda\mu}+B_1^{\lambda\mu}+C_1^{\lambda\mu}+D_1^{\lambda\mu}+E_1^{\lambda\mu}+F_1^{\lambda\mu}. \tag{A6}$$

2. Dispersion relation for the longitudinal mode

The linearized equation of motion (EoM) of the Israel-Stewart hydrodynamics can be written in terms of the independent variables (perturbations), e.g., $(\vec{k}\cdot\vec{u_1}),T_1$, and n_1. Then the dispersion relation can be obtained by setting the determinant of the coefficients of the linear algebraic equations satisfied by $(\vec{k}\cdot\vec{u_1}),T_1$, and n_1 to zero. Expanding this determinant and solving for ω leads to the dispersion relation for the longitudinal component. The determinant formed by three unknown coefficients in Eqs. (10)–(12) is

$$0=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix},$$

where the values of the different matrix elements are given below:

$$a_{11}=\omega(P+\epsilon)+\frac{nT^2}{3(\epsilon+P)}\zeta\chi\alpha_0\omega\nabla^2\alpha+\frac{2nT^2}{(\epsilon+P)}\alpha_1\eta\omega\chi\left\{\frac{(\vec{k}\cdot\vec{\nabla})(\vec{k}\cdot\vec{\nabla})\alpha}{k^2}-\frac{1}{3}\nabla^2\alpha\right\}+i\frac{\zeta}{3}k^2-\frac{1}{9}\zeta^2\beta_0\omega k^2$$
$$-\frac{nT^2}{3(P+\epsilon)}\zeta\alpha_0\chi[2(\vec{k}\cdot\vec{\nabla})\dot{\alpha}+i\dot{\alpha}k^2-i\omega(\vec{k}\cdot\vec{\nabla})\alpha]+i\frac{4}{3}\eta k^2-\frac{8}{3}\eta^2\beta_2\omega k^2-\frac{8nT^2}{3(P+\epsilon)}\alpha_1\chi\eta(\vec{k}\cdot\vec{\nabla})\dot{\alpha}-i\frac{4\alpha_1\eta\chi}{3}k^2\dot{\alpha},$$

$$a_{12}=-k^2\left(\frac{\partial P}{\partial T}\right)_n-\mathrm{Re}\left[\frac{2nT}{P+\epsilon}-\frac{nT^2}{(P+\epsilon)^2}\left\{\left(\frac{\partial P}{\partial T}\right)_n+\left(\frac{\partial\epsilon}{\partial T}\right)_n\right\}\right],$$

$$a_{13}=-k^2\left(\frac{\partial P}{\partial n}\right)_T-\mathrm{Re}\left[\frac{T^2}{P+\epsilon}-\frac{nT^2}{(P+\epsilon)^2}\left\{\left(\frac{\partial P}{\partial n}\right)_T+\left(\frac{\partial\epsilon}{\partial n}\right)_T\right\}\right],$$

$$a_{21}=-(\epsilon+P)-\frac{\zeta}{3}\frac{nT^2}{(\epsilon+P)}\chi\alpha_0\nabla^2\alpha-\frac{2nT^2}{(\epsilon+P)}\alpha_1\eta\chi\left\{\frac{(\vec{k}\cdot\vec{\nabla})(\vec{k}\cdot\vec{\nabla})\alpha}{k^2}-\frac{1}{3}\nabla^2\alpha\right\},$$

$$a_{22}=\omega\left(\frac{\partial\epsilon}{\partial T}\right)_n,\quad a_{23}=\omega\left(\frac{\partial\epsilon}{\partial n}\right)_T,\quad a_{31}=-n,\quad a_{32}=0,\quad a_{33}=\omega,$$

where

$$\mathrm{Re} \equiv i\frac{\zeta\alpha_0\chi}{3}k^2(\vec{k}\cdot\vec{\nabla})\alpha + \frac{\zeta\chi\alpha_0}{3}k^2\nabla^2\alpha + i\frac{4\alpha_1\eta\chi}{3}k^2(\vec{k}\cdot\vec{\nabla})\alpha.$$

Expanding the above determinant and keeping terms up to second order in $\eta T/h,\zeta T/h$ and their products like $\eta\chi,\eta\zeta,\zeta\chi$, we get an equation of the form,

$$\omega(a\omega^2 + b\omega + c) = 0, \tag{A7}$$

which has a trivial solution $\omega = 0$ and the other two roots can be found by solving the quadratic equation $(a\omega^2 + b\omega + c) = 0$. The coefficients of the quadratic equation is given by

$$a = \bigg[(P+\epsilon) + \frac{nT^2}{3(P+\epsilon)}\zeta\chi\alpha_0\nabla^2\alpha - \frac{1}{9}k^2\zeta^2\beta_0 + \frac{2nT^2\eta\chi\alpha_1}{(P+\epsilon)}\frac{(\vec{k}\cdot\vec{\nabla})(\vec{k}\cdot\vec{\nabla})\alpha}{k^2} - \frac{2nT^2}{3(P+\epsilon)}\eta\chi\alpha_1\nabla^2\alpha$$
$$- \frac{8}{3}k^2\eta^2\beta_2 + i\frac{nT^2}{3(P+\epsilon)}\zeta\chi\alpha_0(\vec{k}\cdot\vec{\nabla})\alpha\bigg]\left(\frac{\partial\epsilon}{\partial T}\right)_n, \tag{A8}$$

$$b = \bigg[-\frac{2}{3}\frac{nT^2}{(P+\epsilon)}\zeta\chi\alpha_0(\vec{k}\cdot\vec{\nabla})\dot{\alpha} - \frac{8}{3}\frac{nT^2\eta\chi\alpha_1}{(P+\epsilon)}(\vec{k}\cdot\vec{\nabla})\dot{\alpha} + i\left\{\frac{1}{3}k^2\zeta + \frac{4}{3}k^2\eta - \frac{1}{3}\frac{nT^2}{(P+\epsilon)}k^2\dot{\alpha}\zeta\chi\alpha_0 - \frac{4}{3}k^2\alpha_1\eta\chi\dot{\alpha}\right\}\bigg]\left(\frac{\partial\epsilon}{\partial T}\right)_n, \tag{A9}$$

$$c = -k^2(P+\epsilon)\left(\frac{\partial P}{\partial T}\right)_n + k^2n\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T - k^2n\left(\frac{\partial P}{\partial n}\right)_T\left(\frac{\partial\epsilon}{\partial T}\right)_n - \frac{2}{3}k^2nT\zeta\chi\alpha_0\nabla^2\alpha + \frac{2n^2T}{(P+\epsilon)}\alpha_0\zeta\chi k^2\left(\frac{\partial\epsilon}{\partial n}\right)_T\nabla^2\alpha$$
$$- \frac{n^2T^2}{3(P+\epsilon)^2}\alpha_0\zeta\chi k^2\left(\frac{\partial\epsilon}{\partial n}\right)_T\left(\frac{\partial P}{\partial T}\right)_n\nabla^2\alpha + \frac{n^2T^2}{3(P+\epsilon)^2}\alpha_0\zeta\chi k^2\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T\nabla^2\alpha + \frac{2nT^2}{3(P+\epsilon)}\alpha_1\eta\chi k^2\left(\frac{\partial P}{\partial T}\right)_n\nabla^2\alpha$$
$$- \frac{2nT^2}{P+\epsilon}\eta\chi\alpha_1\left(\frac{\partial P}{\partial T}\right)_n(\vec{k}\cdot\vec{\nabla})(\vec{k}\cdot\vec{\nabla})\alpha + i\bigg[-\frac{2}{3}k^2nT\zeta\chi\alpha_0 + \frac{nT^2}{P+\epsilon}\zeta\chi\alpha_0k^2\left(\frac{\partial P}{\partial T}\right)_n + \frac{2n^2T}{3(P+\epsilon)}\zeta\chi\alpha_0k^2$$
$$- \frac{n^2T^2}{3(P+\epsilon)^2}\zeta\chi\alpha_0k^2\left(\frac{\partial P}{\partial T}\right)_n\left(\frac{\partial\epsilon}{\partial n}\right)_T + \frac{n^2T^2}{3(P+\epsilon)^2}\zeta\chi\alpha_0k^2\left(\frac{\partial P}{\partial n}\right)_T\left(\frac{\partial\epsilon}{\partial T}\right)_n - \frac{8}{3}k^2nT\eta\chi\alpha_1 + \frac{4nT^2}{3(P+\epsilon)}\eta\chi\alpha_1k^2\left(\frac{\partial P}{\partial T}\right)_n$$
$$+ \frac{8n^2T}{3(P+\epsilon)}\eta\chi\alpha_1k^2\left(\frac{\partial\epsilon}{\partial n}\right)_T - \frac{4n^2T^2}{3(P+\epsilon)^2}\eta\chi\alpha_1k^2\left(\frac{\partial\epsilon}{\partial n}\right)_T\left(\frac{\partial P}{\partial T}\right)_n + \frac{4n^2T^2}{3(P+\epsilon)^2}\eta\chi\alpha_1k^2\left(\frac{\partial\epsilon}{\partial T}\right)_n\left(\frac{\partial P}{\partial n}\right)_T\bigg](\vec{k}\cdot\vec{\nabla})\alpha. \tag{A10}$$

The physical solution of Eq. (A7) gives the general dispersion relation for nonzero η,ζ,χ as well for nonzero (baryonic) conserved charge.

3. Perturbations in $T^{\lambda\mu}$ in the presence of magnetic field

The energy-momentum tensor in the presence of magnetic field (B) is given by Eq. (23) with $n'^{\mu} = B^{\mu}/B$ where $B_{\mu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\nu\alpha}u^{\beta}$ and $F^{\mu\nu} = (E^{\mu}u^{\nu} - E^{\nu}u^{\mu}) + (1/2)\epsilon^{\mu\nu\beta\gamma}(u_{\beta}B_{\gamma} - u_{\gamma}B_{\beta})$. For vanishing electric field (E), the expression for energy-momentum tensor [Eq. (23)] can be written as

$$T_m^{\lambda\mu} = \frac{B^2}{8\pi}\bigg(2u^{\lambda}u^{\mu} + g^{\lambda\mu} - \frac{1}{8}\epsilon^{\lambda\rho\sigma\eta}\epsilon_{\rho\sigma\alpha\beta}\epsilon^{\rho'\sigma'\eta'\mu}\epsilon_{\rho'\sigma'\alpha'\beta'},$$
$$\times(u^{\alpha}B^{\beta} - u^{\beta}B^{\alpha})(u^{\alpha'}B^{\beta'} - u^{\beta'}B^{\alpha'})\bigg)u_{\eta}u_{\eta'}. \tag{A11}$$

Using the following relation satisfied by the Levi-civita tensor,

$$\epsilon^{\rho\sigma\eta\lambda}\epsilon_{\rho\sigma\alpha\beta} = -2\big(g_{\alpha}^{\eta}g_{\beta}^{\lambda} - g_{\beta}^{\eta}g_{\alpha}^{\lambda}\big),$$

Eq. (A11) can be written as

$$T_m^{\lambda\mu} = \frac{B^2}{8\pi}\bigg[2u^{\lambda}u^{\mu} + g^{\lambda\mu} - \frac{1}{2B^2}\big(g_{\alpha}^{\eta}g_{\beta}^{\lambda} - g_{\beta}^{\eta}g_{\alpha}^{\lambda}\big),$$
$$\times\big(g_{\alpha'}^{\eta'}g_{\beta'}^{\mu} - g_{\beta'}^{\eta'}g_{\alpha'}^{\mu}\big)(u^{\alpha}B^{\beta}u^{\alpha'}B^{\beta'} - u^{\alpha}B^{\beta}u^{\alpha'}B^{\beta'}$$
$$- u^{\beta}B^{\alpha}u^{\alpha'}B^{\beta'} + u^{\beta}B^{\alpha}u^{\beta'}B^{\alpha'})u_{\eta}u_{\eta'}\bigg]. \tag{A12}$$

The last term of the RHS of Eq. (A12) can be decomposed into 16 terms, each of which will contain the product of four fluid velocity (u^{μ}). If we write, $u^{\mu} \to u^{\mu} + u_1^{\mu}$ and use $u^0 = 1$, $u^i = 0$, $u_1^0 = 0$, $B^0 = 0$, and $B^{\mu}u_{\mu} = 0$, then the only nonzero terms are $(1/2)u^{\alpha}u^{\alpha'}u_{1\eta}u_{\eta'}B^{\beta}B^{\beta'}g_{\beta}^{\eta}g_{\alpha}^{\lambda}g_{\alpha'}^{\eta'}g_{\beta'}^{\mu}$ and $(1/2)u^{\beta}u^{\beta'}u_{1\eta}u_{\eta'}B^{\alpha}B^{\alpha'}g_{\alpha}^{\eta}g_{\beta}^{\lambda}g_{\beta'}^{\eta'}g_{\alpha'}^{\mu}$. From these two terms the perturbations are estimated. The only nonzero components of the perturbation, $T_{1m}^{\lambda\mu}$ to $T_m^{\lambda\mu}$ exist for $\lambda = i$ and $\mu = 0$. The magnitude of the perturbations are

$$T_{1m}^{00} = 0,$$
$$T_{1m}^{i0} = \frac{B^2}{8\pi}\left(2u_1^i - \frac{B^i}{B^2}(\vec{u}_1\cdot\vec{B})\right), \tag{A13}$$
$$T_{1m}^{ij} = 0.$$

These expressions for the energy momentum tensor from the presence of the magnetic field in the fluid have been used to calculate the dispersion relation for the longitudinal and transverse wave in this work.

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