

Thermal noise in a boost-invariant matter expansion in relativistic heavy-ion collisions

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(Received 30 November 2017; published 9 May 2018)

We formulate a general theory of thermal fluctuations within causal second-order viscous hydrodynamic evolution of matter formed in relativistic heavy-ion collisions. The fluctuation is treated perturbatively on top of a boost-invariant longitudinal expansion. Numerical simulation of thermal noise is performed for a lattice quantum chromodynamics equation of state and for various second-order dissipative evolution equations. Phenomenological effects of thermal fluctuations on the two-particle rapidity correlations are studied.

DOI: [10.1103/PhysRevC.97.054902](https://doi.org/10.1103/PhysRevC.97.054902)

I. INTRODUCTION

Relativistic dissipative hydrodynamics has become an important tool to study bulk properties of the near-equilibrium system formed in relativistic heavy-ion collisions [1–5]. The large densities and short mean-free times in the system allow for a coarse graining in hydrodynamics which integrates out the microscopic length and timescales. The effective degrees of freedom are then the average conserved quantities, namely the energy, momentum, electric charge, and baryon number, which are dynamically evolved according to the hydrodynamic equations [6–10]. When low enough densities are reached and the interaction times become longer, the system falls out of equilibrium, which ultimately leads to a kinetic freeze-out.

In spite of this inherent coarse graining [11], hydrodynamics was remarkably successful in explaining several experimental observables pertaining to relativistic heavy-ion collisions, for example, the anisotropic flow v_n [12], that characterizes the final-state momentum anisotropy in the plane transverse to the beam direction. The flow was well understood as the collective hydrodynamic response to the initial collision geometry fluctuating event by event [13–18]. The long-range rapidity structures observed in multiparticle correlation measurements in heavy-ion collisions [19] as well as in high-multiplicity collision events involving small projectiles ($p/d/{}^3\text{He}$) [20,21] can also be related to the hydrodynamic behavior.

It is then instructive to investigate whether the thermal noise or the fast microscopic degrees of freedom that survive coarse graining, have any effect on the experimental observables. The fluctuation-dissipation theorem already forces one to consider fluctuations in systems that are in thermal equilibrium. Further, as the size of the fireball formed is just about 10 fm, and there are only a finite number of particles in each coarse-grained fluid cell, fluctuations may play a crucial role. Thermal noise could be even more important for proper interpretation of observables near the critical point for confinement-deconfinement transition where all fluctuations are large in general. In contrast to the perturbation created in the medium from energy deposition by a propagating jet [22], thermal fluctuations are produced at all space-time points in the fluid cells. These local fluctuations are

propagated and diffused via the fluid dynamic evolution equations. Interestingly, thermal fluctuations can have measurable effects on correlation observables and may be used to pin down the transport coefficients such as the shear viscosity to entropy density ratio η_v/s of the quark-gluon plasma (QGP) formed in heavy-ion collisions [14–18,23].

The theory of hydrodynamic fluctuations or noise in a nonrelativistic fluid [24] was extended into the relativistic regime for Navier-Stokes (first-order) viscous fluid by Kapusta *et al.* [23]. The thermal fluctuation $\Xi^{\mu\nu}$ of the energy-momentum tensor was shown to have a nontrivial autocorrelation $\langle \Xi^{\mu\nu}(x)\Xi^{\alpha\beta}(x') \rangle \sim T\eta_v \delta^4(x-x')$, where T is the temperature [23,25,26]. Because of the occurrence of the Dirac delta function, the energy and momentum density averaged value of this white noise becomes $\sim 1/\sqrt{\Delta V \Delta t}$. Thus, even for small shear viscosity η_v , the white noise sets a lower limit on the system cell size ΔV that is essentially comparable to the correlation length. Consequently, white noise could lead to large gradients which makes the basic hydrodynamic formulation (based on gradient expansion) questionable. The divergence problem can be circumvented, by treating the white noise as a perturbation (in a linearized hydrodynamic framework) on top of a baseline nonfluctuating hydrodynamic evolution [23,25,27]. Analytic solutions for hydrodynamic fluctuations were obtained in the case of boost-invariant longitudinal expansion without transverse dynamics (Bjorken flow) [23] and with transverse dynamics (Gubser flow) [28]. However, both these calculations were performed in the relativistic Navier-Stokes theory for a conformal fluid.

It is important to recall that the first-order dissipative fluid dynamics or the Navier-Stokes theory, displays acausal behavior that may lead to unphysical effects. On the other hand, the second-order dissipative fluid dynamics, based on the Müller-Israel-Stewart (MIS) framework [6–8,11], gives hyperbolic equations and restores causality. The commonly used MIS formulation was quite successful in explaining the spectra and azimuthal anisotropy of particles produced in heavy-ion collisions. Recently, formally new dissipative equations have been derived from Chapman-Enskog-like iterative expansion of the Boltzmann equation in the relaxation-time

approximation [29,30] and from the modified 14-moment method which was developed by Denicol *et al.* [31].

In this work, we shall present the formulation of thermal fluctuations for these forms of hydrodynamic dissipative evolution equations in the case of a boost-invariant longitudinal expansion. The fluctuation equations so obtained are somewhat general and will be used along with an equation of state (EOS) corresponding to a conformal fluid and then with the lattice QCD EOS. Because analytical solutions for hydrodynamic fluctuations cannot be obtained for the baseline second-order hydrodynamic approaches, we shall perform numerical simulations of thermal noise and its evolution as a perturbation on top of boost-invariant longitudinal expansion of the viscous medium.

The paper is organized as follows. In Sec. II, we formulate hydrodynamic fluctuations in the linearized limit as a perturbation on top of second-order dissipative equations for boost-invariant expansion. In Sec. III, we calculate the phenomenology of particle freeze-out and the effect of fluctuation on two-particle rapidity correlations. In Sec. IV, we present results from numerical simulations for the rapidity correlations with ideal gas and lattice QCD equations of state. Finally in Sec. V, we summarize our results and conclude.

II. THERMAL FLUCTUATIONS IN BOOST-INVARIANT HYDRODYNAMICS

In this section we formulate thermal fluctuations in the boost-invariant longitudinal expansion of matter within second-order viscous hydrodynamics. In the presence of a thermal noise tensor $\Xi^{\mu\nu}$, the total energy-momentum tensor becomes

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + \pi^{\mu\nu} + \Xi^{\mu\nu}. \quad (1)$$

We shall work in the Landau-Lifshitz frame and disregard particle diffusion current, which is a reasonable approximation from very small values of the net baryon number formed at RHIC and LHC; we further ignore bulk viscosity in our calculation. In the above equation, ϵ and p are, respectively, the energy density and pressure in the fluid's local rest frame (LRF), and $\pi^{\mu\nu}$ is the shear pressure tensor. $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator on the three-space orthogonal to the hydrodynamic four-velocity u^μ which is defined by the Landau matching condition $T^{\mu\nu} u_\nu = \epsilon u^\mu$.

We treat hydrodynamic fluctuations perturbatively such that in the linearized limit the total energy-momentum tensor $T^{\mu\nu}$ can be expressed as the sum of an average and noiseless part $T_0^{\mu\nu}$ (represented by subscript "0") and a thermally fluctuating part $\delta T^{\mu\nu}$ (represented by δ). Because we use fixed initial conditions, the noiseless background evolution is identical for all events, which we have referred to as average part. However, in reality because of event-by-event fluctuations in the initial conditions, each initial state provides a unique baseline on top of which the hydrodynamic fluctuations evolve.

In the presence of fluctuations, the energy density (or temperature), flow velocity, and shear pressure tensor can be

written as [23]

$$\begin{aligned} \epsilon &= \epsilon_0 + \delta\epsilon, \\ u^\mu &= u_0^\mu + \delta u^\mu, \\ \pi^{\mu\nu} &= \pi_0^{\mu\nu} + \delta\pi^{\mu\nu}. \end{aligned} \quad (2)$$

In the linearized limit (keeping terms up to first order in fluctuations), the total energy-momentum tensor becomes

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + \pi^{\mu\nu} + \Xi^{\mu\nu}, \\ &= T_0^{\mu\nu} + \delta T_{\text{id}}^{\mu\nu} + \delta\pi^{\mu\nu} + \Xi^{\mu\nu} \equiv T_0^{\mu\nu} + \delta T^{\mu\nu}. \end{aligned} \quad (3)$$

The total fluctuating part $\delta T^{\mu\nu}$ has contributions from the viscous term $\delta\pi^{\mu\nu}$, the noise term $\Xi^{\mu\nu}$, and the ideal energy-momentum tensor term,

$$\begin{aligned} \delta T_{\text{id}}^{\mu\nu} &= \delta\epsilon u_0^\mu u_0^\nu - \delta p \Delta_0^{\mu\nu} \\ &+ (\epsilon_0 + p_0)(u_0^\mu \delta u^\nu + \delta u^\mu u_0^\nu) + \mathcal{O}(\delta^2). \end{aligned} \quad (4)$$

All of these can be determined by the fluctuating variables $(\delta\epsilon, \delta u^\mu, \delta\pi^{\mu\nu})$. The conservation equations for the total energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, along with the usual conservation for the average part, $\partial_\mu T_0^{\mu\nu} = 0$, lead to

$$\partial_\mu (\delta T_{\text{id}}^{\mu\nu} + \delta\pi^{\mu\nu} + \Xi^{\mu\nu}) \equiv \partial_\mu (\delta T^{\mu\nu}) = 0. \quad (5)$$

Though in a single event thermal noise causes $\delta T^{\mu\nu} \neq 0$, the average over many events results in $\langle \delta T^{\mu\nu} \rangle = 0$. However, noise induces a nonvanishing correlator $\langle \delta T^{\mu\nu} \delta T^{\alpha\beta} \rangle$, that contributes to event-by-event distribution of an observable, e.g., two-particle rapidity correlations [23,27]. To solve (numerically) Eq. (5), which involves the evolution of viscous fluctuation and noise, one requires the averaged quantities $(\epsilon_0, u_0^\mu, \pi_0^{\mu\nu})$. We first deal with the background viscous evolution equations and then formulate the evolution of fluctuation on top of this background.

For Bjorken longitudinal expansion, we work in the Milne coordinates (τ, x, y, η) where proper time $\tau = \sqrt{t^2 - z^2}$, space-time rapidity $\eta = \ln[(t+z)/(t-z)]/2$, and four-velocity $u_0^\mu = (1, 0, 0, 0)$. The conservation equation for the average part of the energy-momentum tensor, $\partial_\mu T_0^{\mu\nu} = -\partial_\mu \pi_0^{\mu\nu}$, gives the evolution equation of noiseless ϵ_0 as

$$\frac{d\epsilon_0}{d\tau} = -\frac{1}{\tau}(\epsilon_0 + p_0 - \pi_0), \quad (6)$$

where $\tau^2 \pi_0^{\eta\eta} \equiv -\pi_0$ is taken as the independent component of the shear pressure tensor. For the three independent variables, we need two more equations, namely, the viscous evolution equation and the equation of state. The simplest choice for the dissipative equation would be the relativistic Navier-Stokes theory, where the instantaneous constituent equation for the shear pressure is $\pi^{\mu\nu} = 2\eta_v \nabla^{(\mu} u^{\nu)}$ $\equiv 2\eta_v \sigma^{\mu\nu}$. Using Eq. (2), the average (noiseless) shear part in the Bjorken case becomes

$$\pi_0 = \frac{4\eta_v}{3}\theta_0, \quad (7)$$

where $\eta_v \geq 0$ is the shear viscosity coefficient, and $\nabla^{(\mu} u^{\nu)}$ $= (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - (\nabla \cdot u)\Delta^{\mu\nu}/3$ and $\nabla^\mu = \Delta^{\mu\nu} \partial_\nu$. For boost-invariant case, the local expansion rate and the time derivative in the LRF are $\theta_0 = 1/\tau$. In the following we shall

use the standard notation $A^{(\mu\nu)} \equiv \Delta_{\alpha\beta}^{\mu\nu} A^{\alpha\beta}$ for traceless symmetric projection orthogonal to u^μ , where $\Delta_{\alpha\beta}^{\mu\nu} \equiv (\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu)/2 - (1/3)\Delta^{\mu\nu} \Delta_{\alpha\beta}$.

The most commonly used second-order dissipative hydrodynamic equation derived from positivity of the divergence of entropy four-current is based on the works of Müller-Israel-Stewart (MIS) [6–9]:

$$\dot{\pi}^{(\mu\nu)} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - 2\eta_v \nabla^{(\mu} u^{\nu)}) - \frac{1}{2}\pi^{\mu\nu} \frac{\eta_v T}{\tau_\pi} \partial_\lambda \left(\frac{\tau_\pi}{\eta_v T} u^\lambda \right), \quad (8)$$

where the above equation involves the full hydrodynamic variables, and $T = T_0 + \delta T$ is the total temperature corresponding to ϵ . In contrast to the first-order equation, the above equation restores causality by enforcing the shear pressure to relax to its first-order value via the relaxation time $\tau_\pi = 2\eta_v \beta_2$, where β_2 is the second-order transport coefficient. In the boost-invariant scaling expansion, the noiseless part of the dissipative Eq. (8) reduces to

$$\frac{d\pi_0}{d\tau} + \frac{\pi_0}{\tau} = \frac{4\eta_v}{3\tau_\pi} \theta_0 - \lambda_\pi \theta_0 \pi_0, \quad (9)$$

where terms up to second order in the velocity gradients are kept in the expansion of the last term in Eq. (8). The resulting expansion coefficient is

$$\lambda_\pi = \frac{1}{2} \left[1 + \frac{\epsilon_0 + p_0}{T_0} \frac{dT_0}{d\epsilon_0} \left(1 - \frac{T_0}{\beta_2} \frac{d\beta_2}{dT_0} \right) \right], \quad (10)$$

which for an ultrarelativistic EOS reduces to $\lambda_\pi = 4/3$. The relaxation time τ_π depends on the underlying microscopic theory namely, weakly coupled QCD, lattice QCD, and $\mathcal{N} = 4$ SYM [11]. For all these theories, one can express $\tau_\pi = \chi \eta_v / (sT_0)$, where the coefficient $2 \lesssim \chi \lesssim 6$. In the present study we consider $\tau_\pi = 2\eta_v \beta_2 = 5\eta_v / (sT_0)$. Hereafter η_v/s is kept fixed, where $s = s_0 + \delta s$ is the total entropy density in the linearized limit with s_0 being the average entropy density.

Alternatively, dissipative evolution equations can be derived from Chapman-Enskog-like (CE) gradient expansion of the nonequilibrium distribution function about the local value, using the Knudsen number as a small expansion parameter [29,30,32]. The relativistic Boltzmann equation, in the relaxation-time approximation for the collision term, can be solved iteratively to yield

$$\dot{\pi}^{(\mu\nu)} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \frac{\sigma^{\mu\nu}}{\beta_2} + 2\pi_\gamma^{(\mu} \omega^{\nu)\gamma} - \frac{10}{7}\pi_\gamma^{(\mu} \sigma^{\nu)\gamma} - \frac{4}{3}\pi^{\mu\nu} \theta, \quad (11)$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity tensor. In the boost-invariant case, the noiseless part of Eq. (11) gives

$$\frac{d\pi_0}{d\tau} + \frac{\pi_0}{\tau} = \frac{4\eta_v}{3\tau_\pi} \theta_0 - \lambda_\pi \theta_0 \pi_0. \quad (12)$$

In the Chapman-Enskog-like approach, the relaxation time naturally comes out to be $\tau_\pi = 2\eta_v \beta_2 = 5\eta_v / (sT_0)$ and $\lambda_\pi = 38/21$ [29]. In this limit the CE equation is equivalent to that obtained by Denicol *et al.* [31] where the expansion is controlled by the Knudsen number and the inverse Reynolds number. We shall explore the effects of the above viscous

equations on the thermal noise correlators and the two-particle rapidity correlations.

For the equation of state (EOS), we have employed the conformal QGP fluid with the thermodynamic pressure $p = \epsilon/3$, and the s95p-PCE EOS [33] which is obtained from fits to lattice data for crossover transition and matches a realistic hadron resonance gas model at low temperatures T , with partial chemical equilibrium (PCE) of the hadrons at temperatures below $T_{\text{PCE}} \approx 165$ MeV. The EOS influences the longitudinal expansion of the fluid and the two-particle correlations.

To obtain the evolution equations for thermal fluctuations, we use the normalization $u^\mu u_\mu = (u_0^\mu + \delta u^\mu)(u_{0\mu} + \delta u_\mu) = 1$, orthogonality $\pi^{\mu\nu} u_\nu = 0$, and tracelessness $\pi^\mu_\mu = 0$. It is important to note that noise, being local in space time, breaks all the symmetries of Bjorken flow, namely, longitudinal boost invariance, and rotational and translational symmetries in the transverse plane. In the present work we explore the effects of longitudinal fluctuations on two-particle rapidity correlations. We follow a similar approach to that used in Refs. [23,34,35], where the effects of transverse dynamics are considered on average by integrating over the transverse dimensions of the fluid. This reduces the complicated (3+1)D linearization problem to an effective (1+1)D set of equations. In fact, the longitudinal correlation pattern was shown to remain unaltered when transverse perturbations were included in the study of propagation of “ripples” in a simplified nonviscous fluid [36].

The transverse-averaged fluctuating quantities depend explicitly on both space-time rapidity and proper time. Thus the three independent variables are $\delta\epsilon \equiv \delta\epsilon(\tau, \eta)$, $\delta u^\eta \equiv \delta u^\eta(\tau, \eta)$, $\delta\pi^{\eta\eta} \equiv \delta\pi^{\eta\eta}(\tau, \eta) = -(\delta\pi^{xx} + \delta\pi^{yy})/\tau^2$. Further, because $\Xi^{\mu\nu}$ satisfies the same constraints as $\pi^{\mu\nu}$, viz. $u_\mu \Xi^{\mu\nu} = 0$ and $\Xi^\mu_\mu = 0$, this results in one independent component, which we take as $\Xi^{\eta\eta}$. The fluctuating part of the energy-momentum conservation equation (5) can then be written as

$$\frac{\partial}{\partial\tau}(\tau\delta\epsilon) + \frac{\partial}{\partial\eta}(\tau\mathcal{U}_0\delta u^\eta) = -\delta\mathcal{V}, \quad (13)$$

$$\frac{\partial}{\partial\tau}(\tau\mathcal{U}_0\delta u^\eta) + \frac{\partial}{\partial\eta} \left(\frac{\delta V}{\tau} \right) = -2\mathcal{U}_0\delta u^\eta, \quad (14)$$

where $\mathcal{U}_0(\tau) \equiv \epsilon_0 + p_0 - \pi_0 = w_0 - \pi_0$ depends on the background variables that are functions of proper time only; w_0 is the enthalpy of the fluid. On the other hand, $\delta V(\eta, \tau) \equiv \delta p + \tau^2 \delta\pi^{\eta\eta}$ consists of the fluctuating variables which depend on the space-time rapidity as well. We have introduced a stochastic variable,

$$\delta\pi^{\eta\eta} = \delta\pi^{\eta\eta} + \Xi^{\eta\eta} \equiv -\delta\pi'/\tau^2, \quad (15)$$

whose evolution will be derived below.

The stochastic part of the dissipative equation corresponding to MIS or CE, can be obtained from Eq. (8) or (11) by using the linearization Eq. (2). For Bjorken expansion, the evolution equation for the independent fluctuating component $\delta\pi'$ reads

$$\frac{\partial\delta\pi'}{\partial\tau} + \frac{\delta\pi'}{\tau} = \frac{1}{\tau_\pi} \left[\tau^2 \xi^{\eta\eta} + \frac{4\eta_v}{3s} (s_0 \delta\theta + \delta s \theta_0) \right] - \lambda_\pi (\theta_0 \delta\pi' + \delta\theta \pi_0) - \frac{\delta\tau_\pi}{\tau_\pi} \left(\lambda_\pi \theta_0 \pi_0 + \frac{d\pi_0}{d\tau} \right). \quad (16)$$

The new noise term $\xi^{\eta\eta}$ defines the equation of motion of $\Xi^{\eta\eta}$, which for the MIS equation is

$$\dot{\Xi}^{\langle\eta\eta\rangle} = -\frac{1}{\tau_\pi}(\Xi^{\eta\eta} - \xi^{\eta\eta}) - \lambda_\pi \Xi^{\eta\eta}\theta, \quad (17)$$

and for the CE equation is

$$\dot{\Xi}^{\langle\eta\eta\rangle} = -\frac{1}{\tau_\pi}(\Xi^{\eta\eta} - \xi^{\eta\eta}) - \frac{10}{7}\Xi_\gamma^{\langle\eta\sigma\rangle\eta\gamma} - \lambda_\pi \Xi^{\eta\eta}\theta. \quad (18)$$

We recall that in the derivation of Eq. (16), the ratio of the shear viscosity and the total entropy density η_v/s is kept fixed during the entire evolution. The local expansion rate of the fluid from velocity fluctuation is of the form $\delta\theta \equiv \partial \cdot \delta u^\eta = \partial_\eta \delta u^\eta$. The variation of the relaxation time τ_π from thermal fluctuation is

$$\delta\tau_\pi = \delta(2\eta_v\beta_2) = -\tau_\pi \frac{\delta T}{T_0}. \quad (19)$$

Equation (16) involves the noise term $\xi^{\eta\eta}$ that generates the fluctuations, which in turn, evolve via the fluctuating hydrodynamic equations.

The equations are closed once the noise $\xi^{\eta\eta}$ (or equivalently $\Xi^{\eta\eta}$) is specified. The autocorrelation of $\Xi^{\mu\nu}$ can be obtained using Eq. (5):

$$\langle \partial_\mu \Xi^{\mu\nu}(x) \partial_\alpha \Xi^{\alpha\beta}(x') \rangle = \langle \partial_\mu (-\delta T_{\text{id}}^{\mu\nu} - \delta\pi^{\mu\nu})(x) \times \partial_\alpha (-\delta T_{\text{id}}^{\alpha\beta} - \delta\pi^{\alpha\beta})(x') \rangle, \quad (20)$$

along with the use of modes in the dissipative hydrodynamic equations and also employing the fluctuation-dissipation theorem [25].

Alternatively, the autocorrelations can also be derived from the nonequilibrium entropy four-current and using the fluctuation-dissipation theorem [23,37]. In the theory of quasistationary fluctuations [24], the rate of change of total entropy can be expressed as $dS/dt = -\sum_a \dot{x}_a X_a$, where the generalized forces $X_a = -\partial S/\partial x_a$ are conjugate to the set of variables x_a . For a system close to equilibrium, the evolution of x_a may be approximated as

$$\dot{x}_a = -\sum_b \gamma_{ab} X_b + y_a, \quad (21)$$

where γ_{ab} are the Onsager coefficients. The random fluctuations y_a then satisfy the autocorrelations $\langle y_a(t) y_b(t') \rangle = (\gamma_{ab} + \gamma_{ba})\delta(t-t')$. In terms of the nonequilibrium part of total energy-momentum tensor, $\pi'^{\mu\nu} = \pi^{\mu\nu} + \Xi^{\mu\nu}$, the entropy four-current (up to second order) in the MIS and CE theories can be written as [32]

$$S^\mu = s u^\mu - \frac{\beta_2}{2T} u^\mu \pi'^{\alpha\beta} \pi'_{\alpha\beta}, \quad (22)$$

where the equilibrium entropy density $s = (\epsilon + p)/T$. From the total (average plus noise) conservation $\partial_\mu T^{\mu\nu} = 0$, one obtains from Eq. (22),

$$\frac{dS}{dt} = \int d^3x \frac{\pi'^{\mu\nu}}{T} [\nabla_\mu u_\nu - \beta_2 \dot{\pi}'_{\mu\nu} - \beta_2 \lambda_\pi \theta \pi'_{\mu\nu}]. \quad (23)$$

Identifying

$$\begin{aligned} \dot{x}_a &\rightarrow \pi'^{\mu\nu}, \\ X_b &\rightarrow -\frac{1}{T} [\nabla_\mu u_\nu - \beta_2 \dot{\pi}'_{\mu\nu} - \beta_2 \lambda_\pi \theta \pi'_{\mu\nu}] \equiv X_{\mu\nu}, \end{aligned}$$

and in analogy with Eq. (21), one can write

$$\pi'^{\mu\nu} = -\gamma^{\mu\nu\alpha\beta} X_{\alpha\beta} + \xi^{\mu\nu}. \quad (24)$$

Owing to symmetries of $\pi'^{\mu\nu}$, one gets $\gamma^{\mu\nu\alpha\beta} = \gamma^{\nu\mu\alpha\beta}$, $\gamma_\mu^{\mu\alpha\beta} = 0$, and $\gamma^{\mu\nu\alpha\beta} u_\mu = 0$. Note that the identification of $X_{\mu\nu}$ is not unique as the transformation $X_{\mu\nu} \rightarrow X_{\mu\nu} + H_{\mu\nu}$, keeps dS/dt invariant, where $H_{\mu\nu}$ is any tensor orthogonal to $\pi'^{\mu\nu}$. To obtain an autocorrelation which is insensitive to such transformations, the above constraints for $\gamma^{\mu\nu\alpha\beta}$ with respect to μ, ν indices should also follow for the α, β indices.

In the MIS theory, we obtain $\gamma^{\mu\nu\alpha\beta}$ by comparing Eq. (8) (using $\pi^{\mu\nu} = \pi'^{\mu\nu} - \Xi^{\mu\nu}$) and Eq. (24):

$$\gamma^{\mu\nu\alpha\beta} = 2\eta_v T \Delta^{\mu\nu\alpha\beta}. \quad (25)$$

From Eqs. (21) and (24) and using the above expression of $\gamma^{\mu\nu\alpha\beta}$, the autocorrelation in the MIS theory can be written as

$$\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle = 4\eta_v T \Delta^{\mu\nu\alpha\beta} \delta^4(x-x'). \quad (26)$$

Similarly, to obtain the autocorrelation in the Chapman-Enskog (CE) theory we compare Eq. (24) with Eq. (11) for $\pi'^{\mu\nu}$. The $\gamma^{\mu\nu\alpha\beta}$, that is consistent with the constraints as stated above, is found to be

$$\begin{aligned} \gamma^{\mu\nu\alpha\beta} &= 2\eta_v T \left(\Delta^{\mu\nu\alpha\beta} - \frac{10}{7} \beta_2 \Delta_{\zeta\kappa}^{\mu\nu} \pi_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\alpha\beta} \right. \\ &\quad \left. + 2\tau_\pi \Delta_{\zeta\kappa}^{\mu\nu} \omega_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\alpha\beta} \right). \end{aligned} \quad (27)$$

Note that the second and third terms in the above equation reproduce $\pi_\gamma^{\langle\mu\sigma\rangle\nu\gamma}$ and $(\omega_\gamma^{\mu\nu} \pi^{\nu\gamma})$, respectively, in the CE theory. Moreover, these terms also give (via contraction with $X_{\alpha\beta}$) additional higher order terms which can be neglected in our second-order formalism. Correspondingly one obtains the noise autocorrelation,

$$\begin{aligned} \langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle &= 4\eta_v T \left(\Delta^{\mu\nu\alpha\beta} - \frac{5}{7} \beta_2 \Delta_{\zeta\kappa}^{\mu\nu} \pi_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\alpha\beta} \right. \\ &\quad - \frac{5}{7} \beta_2 \Delta_{\zeta\kappa}^{\alpha\beta} \pi_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\mu\nu} + \tau_\pi \Delta_{\zeta\kappa}^{\mu\nu} \omega_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\alpha\beta} \\ &\quad \left. + \tau_\pi \Delta_{\zeta\kappa}^{\alpha\beta} \omega_\gamma^{\zeta\kappa} \Delta^{\kappa\gamma\mu\nu} \right) \delta^4(x-x'). \end{aligned} \quad (28)$$

In the boost-invariant case, the autocorrelation for the independent component $\xi^{\eta\eta}$ in the MIS [Eq. (26)] and CE [Eq. (28)] dissipative formalisms reduce to

$$\begin{aligned} \langle \xi^{\eta\eta}(\eta, \tau) \xi^{\eta\eta}(\eta', \tau') \rangle &= \frac{8\eta_v(\tau) T_0(\tau)}{3A_\perp \tau^5} [1 - \mathcal{A} \beta_2 \pi_0] \\ &\quad \times \delta(\tau - \tau') \delta(\eta - \eta'). \end{aligned} \quad (29)$$

The transverse averaging of the noise tensor of Eq. (28) allows us to replace the delta function in the transverse direction $\delta(x-x')\delta(y-y')$ with the inverse of the transverse area A_\perp of the overlapping zone of the colliding nuclei. Note that the

autocorrelation depends only on the background quantities as we have treated the noise as a perturbation on top of background evolution. The coefficient $\mathcal{A} = 0$ in the MIS theory, and $\mathcal{A} = 5/7$ in the CE formalism. The random variable $\xi^{\eta\eta}(\eta, \tau)$ is the stochastic source that obeys the energy-momentum conservation, and propagates each fluctuation $\delta T^{\mu\nu}$ up to later times to their thermal expectation values. In the Navier-Stokes limit, one can show [23] that the autocorrelation for $\Xi^{\eta\eta}$ has an identical form of Eq. (29) with $\mathcal{A} = 0$. It may be mentioned that the autocorrelation has nonvanishing values in the transverse directions, $\langle \Xi^{ij}(\eta, \tau) \Xi^{ij}(\eta', \tau') \rangle (i, j \equiv x, y)$. As mentioned above, we have considered on average such transverse dynamics. The hydrodynamic fluctuation Eqs. (13)–(16) are solved perturbatively in the τ - η coordinates using the MacCormack (a predictor-corrector type) method.

III. FREEZE-OUT AND TWO-PARTICLE RAPIDITY CORRELATIONS

The freeze-out of a near-thermalized fluid to free-streaming (noninteracting) particles can be obtained via the standard Cooper-Frye prescription [38]. For a boost-invariant scenario without fluctuations, freeze-out on a hypersurface of constant temperature would be equivalent to freeze-out at a constant proper time. Inclusion of fluctuation breaks the boost invariance of the system. In an event, the total temperature would be the sum of constant background temperature and the fluctuating temperature which varies for different cells. We shall consider freeze-out at a constant background temperature T_f so that for any hydrodynamic variable, $X(\tau_f, \eta) = X_0(\tau_f) + \delta X(\eta, \tau_f)$, the fluctuating field $\delta X(\eta, \tau_f)$ varies on the hypersurface; τ_f is the freeze-out time corresponding to T_f . For such isothermal (and isochronous) freeze-out at a constant background T_f , the particle spectrum can be obtained from

$$E \frac{dN}{d^3p} = \frac{g}{(2\pi)^3} \int_{\Sigma} d\Sigma_{\mu} p^{\mu} f(x, p), \quad (30)$$

where p^{μ} is the four-momentum of the particle with degeneracy g and $d\Sigma^{\mu}$ is the outward-directed normal vector on an infinitesimal element of the hypersurface $\Sigma(x)$.

In the present (τ, x, y, η) coordinate system, the three-dimensional volume element at freeze-out is

$$d\Sigma_{\mu} = \tau_f (\cosh \eta, \mathbf{0}, -\sinh \eta) d\eta d\mathbf{x}_{\perp}. \quad (31)$$

The four-momentum of the particles is

$$p^{\mu} = (m_T \cosh y, \mathbf{p}_T, m_T \sinh y), \quad (32)$$

where $m_T = \sqrt{p_T^2 + m^2}$ is the transverse mass of the particle with transverse momentum p_T and kinetic rapidity $y = \tanh^{-1}(p_z/p_0)$. The integration measure at the constant temperature freeze-out hypersurface $\Sigma(x)$ is then $p^{\mu} d\Sigma_{\mu} = m_T \cosh(y - \eta) \tau_f d\eta d\mathbf{x}_{\perp}$.

The phase-space distribution function at freeze-out, $f(x, p) = f_{\text{eq}}(x, p) + f_{\text{vis}}(x, p)$ contains the equilibrium contribution,

$$f_{\text{eq}} = [\exp\{(p \cdot u - \mu)/T\} \pm 1]^{-1} \approx \exp(-p \cdot u/T). \quad (33)$$

The last expression in Eq. (33) is valid for classical statistics and for vanishing chemical potential. In this limit, the nonequilibrium viscous correction in the MIS theory has the form based on Grad's 14-moment approximation [39]:

$$f_{\text{vis}} \approx f_{\text{eq}} \frac{p^{\mu} p^{\nu} \pi'_{\mu\nu}}{2(\epsilon + p)T^2}. \quad (34)$$

The total values (noiseless plus noise) for the hydrodynamic variables $X \equiv X(\tau_f, \eta)$ are evaluated at the freeze-out hypersurface coordinates. In the linearized limit, the total distribution function $f(x, p)$ can be expanded as

$$f(x, p) = f_0(x, p) + \delta f(x, p). \quad (35)$$

The noiseless part of the distribution function $f_0(x, p)$ has contributions from ideal and viscous fluctuations:

$$f_0 = (f_{\text{eq}})_0 (1 + K_{0\mu\nu} \pi_0^{\mu\nu}), \quad (36)$$

where $K_0^{\mu\nu} = p^{\mu} p^{\nu} [2(\epsilon_0 + p_0)T_0^2]^{-1}$, and the total temperature $T = T_0 + \delta T$. We recall that $T_0 \equiv T_f$ for freeze-out at a constant background temperature. Similarly, the noise part $\delta f(x, p)$ can be written as the sum of ideal and viscous fluctuations. Using the linearization Eq. (2), this becomes

$$\delta f = \delta f_{\text{eq}} + K_{0\mu\nu} \left[\delta f_{\text{eq}} \pi_0^{\mu\nu} + (f_{\text{eq}})_0 \delta \pi^{\mu\nu} - (f_{\text{eq}})_0 \pi_0^{\mu\nu} \left(2 \frac{\delta T}{T_0} + \frac{\delta \epsilon + \delta p}{\epsilon_0 + p_0} \right) \right], \quad (37)$$

where $(f_{\text{eq}})_0 = \exp(-u_0^{\mu} p_{\mu}/T_0)$ and $\delta f_{\text{eq}} = (f_{\text{eq}})_0 (\delta T u_0^{\mu} p_{\mu}/T_0^2 - \delta u^{\mu} p_{\mu}/T_0)$ are, respectively, the noiseless and the noise parts of the ideal distribution function, and the terms inside the square brackets in Eq. (37) stem from viscous fluctuations. Note that δf , as obtained above, satisfies the Landau-Lifshitz conditions (to linear order in fluctuations), namely $T^{\mu\nu} u_{\nu} = \epsilon u^{\mu}$ and $\epsilon = \epsilon_{\text{eq}}$, where ϵ_{eq} is the energy density contribution from the equilibrium distribution function f_{eq} .

The rapidity distribution of the particle, corresponding to Eq. (30), then reduces to

$$\begin{aligned} \frac{dN}{dy} &= \frac{g \tau_f A_{\perp}}{(2\pi)^3} \int d\eta \cosh(y - \eta) \\ &\times \int dp_x dp_y m_T [f_0(x, p) + \delta f(x, p)] \\ &\equiv (dN/dy)_0 + \delta(dN/dy). \end{aligned} \quad (38)$$

Here $A_{\perp} = \int d\mathbf{x}_{\perp}$ is the usual transverse area of Eq. (29). For the boost-invariant longitudinal flow, the particle rapidity distribution of the average part can be written as

$$\begin{aligned} \left(\frac{dN}{dy} \right)_0 &= \frac{g \tau_f T_0^3 A_{\perp}}{(2\pi)^2} \int \frac{d\eta}{\cosh^2(y - \eta)} \left[\Gamma_3(y - \eta) \right. \\ &\left. + \frac{\pi_0}{4w_0} \left(\mathcal{C}(y - \eta) \Gamma_5(y - \eta) - \frac{m^2}{T_0^2} \Gamma_3(y - \eta) \right) \right]. \end{aligned} \quad (39)$$

Here $\Gamma_k(x) \equiv \Gamma(k, m \cosh x/T_0)$ denotes the incomplete Gamma function of the k th kind [40] and $\mathcal{C}(x) = 3\text{sech}^2 x - 2$.

The second term within the brackets corresponds to viscous corrections. The fluctuating parts can be expressed as

$$\delta \frac{dN}{dy} = \frac{g\tau_f T_0^3 A_\perp}{(2\pi)^2} \int d\eta \left[\mathcal{F}_T(y-\eta) \frac{\delta T(\eta)}{T_0} + \mathcal{F}_u(y-\eta) \tau_f \delta u^\eta(\eta) + \mathcal{F}_\pi(y-\eta) \frac{\delta \pi'(\eta)}{w_0} \right]. \quad (40)$$

Here $\mathcal{F}_{T,u,\pi}$ are the coefficients of the fluctuations, $(\delta T, \delta u^\eta, \delta \pi')$, that are obtained by performing the momentum integrals. In the MIS theory these are given by

$$\mathcal{F}_T \cosh^2 x = \Gamma_4(x) - \frac{\pi_0}{4w_0} \left[\frac{m^2}{T_0^2} (\Gamma_4(x) - \kappa \Gamma_3(x)) - \mathcal{C}(x) (\Gamma_6(x) - \kappa \Gamma_5(x)) \right], \quad (41)$$

$$\mathcal{F}_u \cosh^2 x = \tanh x \Gamma_4(x) - \frac{\pi_0}{4w_0} \tanh x \left[\frac{m^2}{T_0^2} \Gamma_4(x) - \mathcal{C}(x) \Gamma_6(x) - 4\Gamma_5(x) \right], \quad (42)$$

$$\mathcal{F}_\pi \cosh^2 x = \frac{1}{4} \left[\mathcal{C}(x) \Gamma_5(x) - \frac{m^2}{T_0^2} \Gamma_3(x) \right], \quad (43)$$

where $\kappa = 2 + (T_0/w_0)\partial w_0/\partial T_0$. The two-particle rapidity correlator from fluctuations can then be written as

$$\left\langle \delta \frac{dN}{dy_1} \delta \frac{dN}{dy_2} \right\rangle = \left[\frac{g\tau_f T_0^3 A_\perp}{(2\pi)^2} \right]^2 \int d\eta_1 \int d\eta_2 \times \sum_{X,Y} \mathcal{F}_X(y_1 - \eta_1) \mathcal{F}_Y(y_2 - \eta_2) \times \langle X(\eta_1) Y(\eta_2) \rangle. \quad (44)$$

Here $(X, Y) \equiv (\delta T, \delta u^\eta, \delta \pi')$ and $\langle X(\eta_1) Y(\eta_2) \rangle$ are the two-point correlators between the fluctuating variables calculated at the freeze-out hypersurface.

In the Chapman-Enskog-like approach of iteratively solving Boltzmann equation, the viscous correction in the nonequilibrium distribution function has the form [29,30],

$$f_{vis} \approx f_{eq} \frac{5p^\mu p^\nu \pi'_{\mu\nu}}{8pT(u \cdot p)}, \quad (45)$$

with a total flow velocity $u^\mu \equiv u^\mu(\tau_f, \eta)$. Following a similar procedure as done in the MIS theory, the two-particle rapidity correlations in the CE formalism give the same form as in Eq. (44) but with modified coefficients $\mathcal{F}_{T,u,\pi}$.

IV. RESULTS AND DISCUSSIONS

We shall explore two-particle rapidity correlations induced by thermal fluctuations in the Bjorken expansion. A clear understanding of this can be achieved by calculating the time evolution of the correlations among the hydrodynamic variables. The fluctuations employed in our study generates singularities in the correlators at zero separation in rapidity and at the sound horizons that corresponds to maximum distance propagated by the sound wave along rapidity. In the Navier-Stokes theory, the singular and regular parts of the

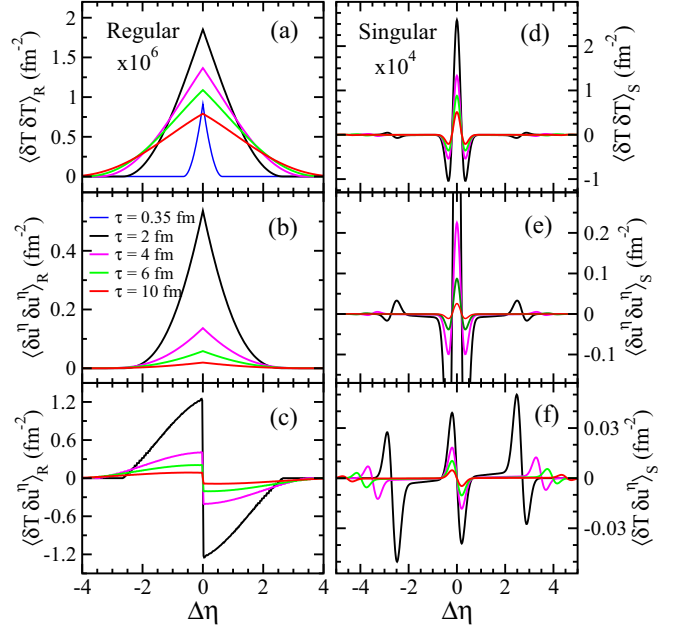


FIG. 1. Regular and singular parts of the equal time temperature-temperature, velocity-velocity, and temperature-velocity correlators computed as a function of space-time rapidity difference $\Delta\eta$ at various proper times in the Navier-Stokes theory. The results are for ultrarelativistic gas EOS ($p = \epsilon/3$) with initial temperature of $T_0 = 550$ MeV, proper time $\tau_0 = 0.2$ fm/c. The shear viscosity to entropy density ratio of $\eta_v/s = 1/4\pi$ is accounted only in the noise correlator. All the correlators in the left and right panels are scaled by 10^6 and 10^4 , respectively.

correlators can be obtained analytically [23]; see Appendix A. Figure 1 shows the rapidity dependence of these equal-time correlators for inviscid fluid $\langle X(\tau, \eta_1) Y(\tau, \eta_2) \rangle$ with $(X, Y) \equiv (\delta T, \delta u^\eta)$. Shear viscosity is neglected in the evolution but accounted for in the noise correlator $\Xi^{\eta\eta}$ of Eq. (29). These correlators represent a wake of the medium behind the shock front associated with the noise propagation. The regular part of the temperature-temperature correlator $\langle \delta T \delta T \rangle$ peaks at zero separation because of short-range correlation which builds up rapidly with increasing proper time. At later times, the peak value decreases and the correlator spreads farther in rapidity because of expansion of the fluid. The time evolution of these structures in the Bjorken expansion is to be contrasted with that of a uniform static system; see Appendix B. The equal-time long-range correlations in the static fluid vanish and correlations at later times are from propagation of sound waves in opposite directions. This clearly underscores the importance of underlying background flow that influences the propagation of fluctuations.

In Fig. 1, we also show the regular part of the velocity-velocity correlator $\langle \delta u^\eta \delta u^\eta \rangle$. It exhibits a similar rapidity dependence as seen in $\langle \delta T \delta T \rangle$, however, with a much smaller magnitude. In contrast, the regular part of the temperature-velocity correlator $\langle \delta T \delta u^\eta \rangle$ is an odd function in $\Delta\eta$. As a consequence this correlator vanishes at $\Delta\eta = 0$ and turns negative (positive) for positive (negative) values of

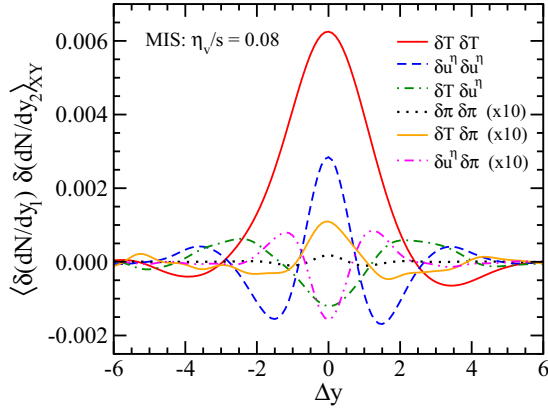


FIG. 2. Two-particle rapidity correlations from various fluctuations calculated at freeze-out for charged pions in the Müller-Israel-Stewart (MIS) dissipative hydrodynamics. The results are for ideal gas EOS ($p = \epsilon/3$) with initial temperature $T_0 = 550$ MeV, proper time $\tau_0 = 0.2$ fm/c, freeze-out temperature $T_f = 150$ MeV, and shear viscosity to entropy density ratio $\eta_v/s = 1/4\pi$.

rapidity separation. Note that the “cross” correlators follow $\langle \delta T \delta u^\eta \rangle = -\langle \delta u^\eta \delta T \rangle$.

Analytic results for the singular part of the correlator can be obtained in the Navier-Stokes theory for an ultrarelativistic gas EOS; see Appendix A. Figure 1 shows the singular part of the equal-time correlators wherein the theta function and its higher-order derivatives have been smeared using a normalized Gaussian distribution of width $\sigma_\eta = 0.2$. In all these correlators, the singularities at $\Delta\eta = 0$ arise from self-correlations, and those at large rapidity separations are induced by sound horizons at $\Delta\eta = \pm 2c_s \log(\tau/\tau_0)$. We note that inclusion of viscosity would dampen the singularities and thereby smear the structures in the longitudinal correlations.

As analytic solutions for thermal fluctuations in second-order dissipative hydrodynamics do not exist, the singular and regular parts of the correlators cannot be separated. However, in the numerical simulation of noise in second-order hydrodynamics, the smearing functions \mathcal{F}_X in Eq. (44) smoothen out all the singularities in the total correlation function $\langle X(\tau, \eta_1)Y(\tau, \eta_2) \rangle$. Hence the computed two-particle rapidity correlation at freeze-out would show a clear structure.

In the second-order Müller-Israel-Stewart viscous hydrodynamics, we present in Fig. 2 the various components, $X, Y \equiv \delta T, \delta u^\eta, \delta \pi$, of the rapidity correlators $\langle (\delta dN/dy_1)(\delta dN/dy_2) \rangle_{X,Y}$ [of Eq. (44)] for charged pions as a function of kinematic rapidity separation $\Delta y = y_1 - y_2$. This was obtained by convoluting the two-point correlators $\langle X(\tau, y_1)Y(\tau, y_2) \rangle$, at freeze-out with the respective smearing functions $\mathcal{F}_{X,Y}$. The calculations are for initial values of temperature $T_0 = 550$ MeV, proper time $\tau_0 = 0.2$ fm/c, and the freeze-out temperature is taken as $T_f = 150$ MeV. A constant $\eta_v/s = 0.08$ is used in both the average and noise parts of the evolution equations. The two-particle correlation functions are essentially manifestations of the sum of their regular and singular parts; see Fig. 1 for the correlators in the Navier-Stokes case. The smearing functions, namely, $F_{\delta T}$ (which is Gaussian about $\delta\eta = 0$) and $F_{\delta u^\eta}$ (which peaks at $\Delta\eta \simeq \pm 1.5$

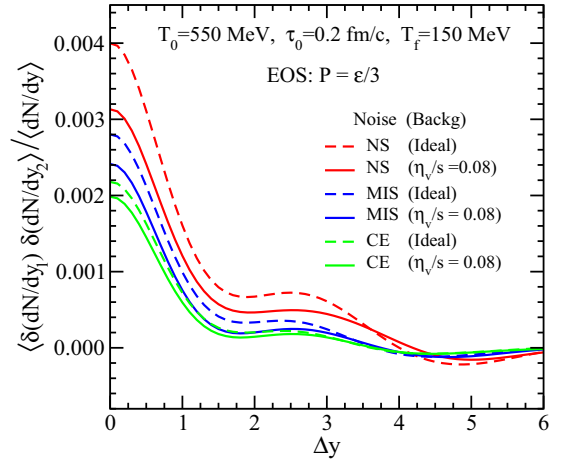


FIG. 3. Correlation function for charged pions normalized with the single-particle rapidity distribution in the Navier-Stokes (NS), Müller-Israel-Stewart (MIS), and Chapman-Enskog (CE) formalisms for the evolution of thermal noise and compared with the ideal background (noiseless) hydrodynamic evolution. The initial and freeze-out conditions are same as in Fig. 2.

and vanishes at $\Delta\eta = 0$) broadens these correlators when convoluted. While the peak at $\delta\eta = 0$ is dominated by the temperature-temperature correlation function, the structures seen at $\Delta\eta \simeq 2 - 4$ for the $\delta T \delta T$, $\delta u^\eta \delta u^\eta$, and $\delta T \delta u^\eta$ correlations are similar in magnitude but have distinct rapidity dependence. The contributions to the correlation functions involving the viscous stress tensor $\delta\pi$ are found to be much smaller.

In Fig. 3 we compare the total two-particle rapidity correlation for charged pions for the Navier-Stokes, Müller-Israel-Stewart, and Chapman-Enskog viscous evolutions for an ultrarelativistic gas EOS ($p = \epsilon/3$). The initial and freeze-out conditions are the same as in Fig. 2. We shall first consider the case of ideal hydrodynamics for the background evolution and explore various dissipative equations for the evolution of thermal fluctuations. The fluctuation in the Navier-Stokes theory gives rise to a larger peak at small rapidity separations as compared to that in the second-order viscous evolutions. This is mainly because of faster build-up of all the correlators and in particular the dominant temperature-temperature correlations in the first-order viscous evolution. In the Chapman-Enskog case, the correlation strength at $\Delta y \approx 0$ is smallest because of the larger coefficient $\lambda_\pi = 38/21$, that results in a slower approach of the viscous fluctuations towards the Navier-Stokes limit.

The inclusion of viscosity in the background evolution damps the correlation peak for all the cases studied. As expected, the maximum reduction in correlation strength occurs in the first-order theory. It may be mentioned that previous studies of rapidity correlations ignored the variation of the relaxation time $\delta\tau_\pi$ of Eq. (19) [23,25,27]. We have found that such an assumption is justified as the rapidity correlation remains practically unaltered when thermal fluctuation of τ_π was not considered.

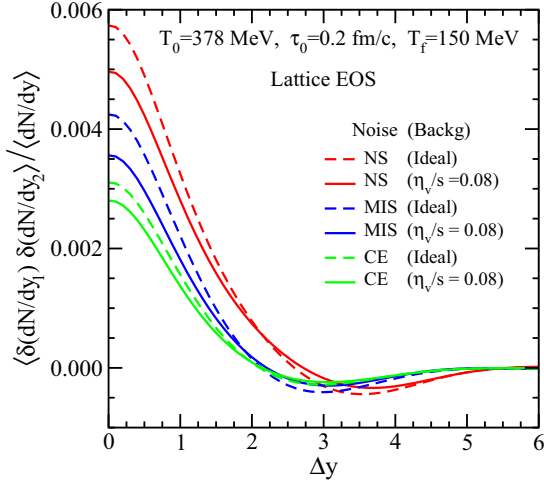


FIG. 4. Similar to Fig. 3 but with lattice QCD EOS; see text for details.

Figure 4 shows numerical results for two-particle rapidity correlation of charged pions in the NS, MIS, and CE formalisms for the lattice QCD EOS that incorporates the transition to a hadron resonance gas at $T_{\text{PCE}} \approx 165 \text{ MeV}$ [33]. It may be mentioned that analytical results for the correlations cannot be obtained for the lattice EOS even in the Navier-Stokes limit. While the initial time and freeze-out temperature are considered the same as used for ideal gas EOS, the initial temperature is set at $T_0 = 378 \text{ MeV}$. This choice stems from the consideration that the event-averaged single-particle rapidity distribution for direct charged pions, $\langle dN/dy \rangle$, in this case is practically identical to that in the ideal gas EOS. Moreover, the freeze-out times for the lattice and conformal equation of states are found similar for each of the dissipative theories. We find that the magnitudes of the correlation between the particles are enhanced for all the cases in the lattice EOS as compared to that for ideal gas EOS. This can be understood as from smaller sound velocity of the fluid near the critical temperature T_c which slows the propagation of perturbations. Consequently, the correlation is solely from the short-range temperature-temperature correlator and the structures associated with the velocity and shear pressure correlators are largely damped and do not spread in rapidity.

It may be mentioned that we have disregarded the effects of bulk viscosity on the hydrodynamic evolution and on the viscous correction term f_{vis} at freeze-out. Heavy-ion collision studies that include bulk viscosity have demonstrated its influence on the results for transverse momentum spectra of hadrons and flow harmonics [41–43]. However, these results are sensitive to (i) the matching of the lattice QCD equation of state for QGP with the EOS for the hadron resonance gas, (ii) the parametrization of bulk relaxation time τ_{Π} [41,44], and (iii) the uncertainties in the form of bulk viscous corrections to f_{vis} at freeze-out [43,45,46]. A detailed study of these effects of bulk viscosity on thermal fluctuations in both the MIS and CE theories will be presented in the future.

V. SUMMARY AND CONCLUSIONS

We have studied the evolution of thermal fluctuations within relativistic second-order dissipative hydrodynamics. The fluctuations were treated in the linearized hydrodynamic framework as a perturbation on top of boost-invariant longitudinal expansion of matter. The analytic form of the auto-correlation function was found to be identical for the acausal Navier-Stokes and the causal Müller-Israel-Stewart theories. However, for the Chapman-Enskog-like dissipative equations, the correlation has an explicit dependence on the shear stress tensor. Within the analytically solvable Navier-Stokes limit in the Bjorken scenario, we demonstrated that the two-particle rapidity correlation at small rapidity separation, $\Delta y \lesssim 2$, is mostly from temperature-temperature correlations and structures seen in the correlations at $\Delta y \approx 2-4$, are caused by varying contributions involving fluid velocity and shear pressure tensor correlations. In general, the two-particle rapidity correlations produced from thermal fluctuations were found to spread to large distance in rapidity separation with magnitude (and pattern) that can be well measured in relativistic heavy-ion collisions. While viscous damping of the correlation is at most $\sim 20\%$, there is further significant damping at small Δy , if one goes from the first-order Navier-Stokes theory to a second-order dissipative hydrodynamic formulation. As compared to the conformal equation of state, the softer lattice QCD EOS, with smaller sound velocity, causes reduced propagation of the fluctuations but leads to a pronounced peak in the rapidity correlations.

ACKNOWLEDGMENTS

We thank A. Mishra for useful discussions during the initial stages of the work. We also thank Jean-Yves Ollitrault for illuminating discussions.

APPENDIX A: SINGULAR PART OF THE CORRELATORS IN NAVIER-STOKES THEORY

The correlation functions for the fluctuating quantities $(X, Y) \equiv (\delta\epsilon, \delta u^\eta)$, which are linear functionals of the noise Ξ^η , can be written as

$$\langle X(\eta, \tau) Y(\eta', \tau) \rangle = \frac{2}{A_\perp} \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'^3} \frac{4\eta_v}{3s w_0(\tau')} G_{XY}(\eta - \eta'; \tau, \tau'). \quad (\text{A1})$$

These Green functions $G_{XY}(\eta - \eta'; \tau, \tau')$ have singular and regular parts which are obtained from the fluctuation evolution equations. The fluctuating component $\delta\pi$ of Eq. (8), in the Navier-Stokes limit reduces to

$$\delta\pi = \frac{4\eta_v}{3s} (s_0 \delta\theta + \delta s \theta_0). \quad (\text{A2})$$

Using this $\delta\pi$ along with the noiseless $\pi_0 = 4\eta_v \theta_0 / 3$ for the Navier-Stokes case, the fluctuating quantities $\delta\epsilon, \delta u^\eta$ are found from the linearized evolution Eqs. (13) and (14) with coefficients $\mathcal{U}_0 = w_0 - (4\eta_v/3s)s_0/\tau$ and $\delta\mathcal{V} = \delta p - \tau^2 \Xi^\eta - (4\eta_v/3s)(s_0 \delta\theta + \delta s \theta_0)$. In the conformal case, these linearized equations have been solved by Fourier transform of $\delta\epsilon, \delta u^\eta$ and finding the corresponding Green functions $G_{\delta\epsilon}(k; \tau, \tau')$

and $G_{\delta u^n}(k; \tau, \tau')$ [23],

$$G_{XY}(\eta - \eta'; \tau, \tau') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(\eta - \eta')} \tilde{G}_{XY}(k; \tau, \tau'), \quad (\text{A3})$$

where $\tilde{G}_{XY}(k; \tau, \tau') \equiv \tilde{G}_X(k; \tau, \tau') \tilde{G}_Y(-k; \tau, \tau')$. The terms which give rise to singular behavior of $G_{XY}(\eta - \eta'; \tau, \tau')$ can be obtained analytically by Laurent series expansion of $\tilde{G}_{XY}(k; \tau, \tau')$ in powers of $1/k$.

Denoting $\rho \equiv 3\delta e/(4e_0)$ and $\omega \equiv \tau \delta u^n$, the expression for the singular part of $G_{\rho\rho}(\eta - \eta'; \tau, \tau')$ stems from

$$\tilde{G}_{\rho\rho}^{\text{sing}}(k; \tau, \tau') = (a_1 k^2 + b_1) + (a_2 k^2 + b_2) \cos(2c_s \gamma k) + \frac{a_3 k^2 + b_3}{k} \sin(2c_s \gamma k). \quad (\text{A4})$$

The coefficients are found to be

$$\begin{aligned} a_1 &= \frac{\beta}{2c_s^2}, \quad a_2 = -a_1, \quad a_3 = \beta \left(\frac{1}{c_s} - \frac{\gamma \delta}{2c_s} \right), \quad b_1 = \beta \frac{c_s^2 + 2\alpha + \delta}{2c_s^2}, \quad b_2 = \beta \left(\frac{1}{2} - \frac{2\alpha + \delta}{2c_s^2} + \frac{\gamma^2 \delta^2}{4} - \gamma \delta \right), \\ b_3 &= \beta \left[\frac{\gamma \delta}{2c_s} (c_s^2 - 2\alpha - \delta) - \frac{\gamma \delta^2}{8c_s} - \frac{c_s \gamma^2 \delta^2}{2} + \frac{c_s \gamma^3 \delta^3}{12} + \frac{2\alpha + \delta}{2c_s} \right], \end{aligned} \quad (\text{A5})$$

where $\alpha = (1 - c_s^2)/2$, $\beta = (\tau'/\tau)^{2\alpha}$, $\gamma \equiv \log(\tau/\tau')$, and $\delta = \alpha^2/c_s^2$. The singular behavior of $G_{\rho\omega}(\eta - \eta'; \tau, \tau')$,

$$\tilde{G}_{\rho\omega}^{\text{sing}}(k; \tau, \tau') = d_1 k + d_2 k \cos(2c_s \gamma k) + (d_3 k^2 + d_4) \sin(2c_s \gamma k), \quad (\text{A6})$$

where the corresponding coefficients are

$$\begin{aligned} d_1 &= -\frac{i}{2} \beta \left(1 - \frac{\alpha + c_s^2}{c_s^2} \right), \quad d_2 = \frac{i}{2} \beta \left(\gamma \delta - 1 - \frac{\alpha + c_s^2}{c_s^2} \right), \quad d_3 = -\frac{i}{2} \beta \frac{1}{c_s}, \\ d_4 &= -\frac{i}{2} \beta \left[-\frac{\alpha + c_s^2}{c_s} \left(1 - \frac{\gamma \alpha^2}{c_s^2} \right) + \frac{\alpha}{c_s} \left(1 + \frac{\alpha}{2c_s^2} \right) + \frac{\gamma \alpha^2}{c_s} \left(1 - \frac{\gamma \alpha^2}{2c_s^2} \right) \right]. \end{aligned} \quad (\text{A7})$$

Finally, the singular behavior of $G_{\omega\omega}(\eta - \eta'; \tau, \tau')$ originates from

$$\tilde{G}_{\omega\omega}^{\text{sing}}(k; \tau, \tau') = (w_1 k^2 + w_2) + (w_3 k^2 + w_4) \cos(2c_s \gamma k) + \left(\frac{w_5 k^2 + w_6}{k} \right) \sin(2c_s \gamma k), \quad (\text{A8})$$

with coefficients

$$\begin{aligned} w_1 &= \frac{1}{2} \beta, \quad w_2 = \frac{1}{2} \beta \left(\frac{\alpha + c_s^2}{c_s} \right)^2, \quad w_3 = \frac{1}{2} \beta, \quad w_4 = -\beta \left[\frac{\gamma \alpha^2}{c_s} \left(\frac{\gamma \alpha^2}{4c_s} - \frac{\alpha + c_s^2}{c_s} \right) + \frac{1}{2} \left(\frac{\alpha + c_s^2}{c_s} \right)^2 \right], \\ w_5 &= \beta \left[\frac{\gamma \alpha^2}{2c_s} - \frac{\alpha + c_s^2}{c_s} \right], \quad w_6 = \beta \left[\frac{\gamma \alpha^2}{2c_s} \frac{\alpha + c_s^2}{c_s} \left(\frac{\gamma \alpha^2}{c_s} - \frac{\alpha + c_s^2}{c_s} \right) + \frac{\gamma \alpha^4}{8c_s^3} \left(1 - \frac{2\gamma^2 \alpha^2}{3} \right) - \frac{\alpha + c_s^2}{c_s} \frac{\delta}{2} \right]. \end{aligned} \quad (\text{A9})$$

APPENDIX B: THERMAL FLUCTUATIONS IN A STATIC FLUID

Consider a static uniform fluid in Cartesian coordinates (t, z) with fluctuations that depend on z and are independent of the transverse (x, y) directions. In the Navier-Stokes theory for a conformal equation of state, the evolution equation for longitudinal fluctuations $(\delta\epsilon, \delta u^z)$ can be written as

$$\begin{aligned} \frac{\partial \delta\epsilon}{\partial t} + w_0 \frac{\partial \delta u^z}{\partial z} &= 0, \\ w_0 \frac{\partial \delta u^z}{\partial t} + c_s^2 \frac{\partial \delta\epsilon}{\partial z} - \frac{4}{3} \eta_v \frac{\partial^2 \delta u^z}{\partial z^2} + \frac{\partial \Xi^{zz}}{\partial z} &= 0, \end{aligned} \quad (\text{B1})$$

where the constant enthalpy of the background is denoted by $w_0 = \epsilon_0 + p_0$. The noise correlator takes the form,

$$\langle \Xi^{zz}(t, z) \Xi^{zz}(t', z') \rangle = \frac{8\eta_v T_0}{3A_{\perp}} \delta(t - t') \delta(z - z'). \quad (\text{B2})$$

Equation (B1) can be solved by taking the Fourier transform,

$$\delta X(t, z) = \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t} e^{-ikz} \delta \tilde{X}(\omega, k), \quad (\text{B3})$$

where the fluctuations are denoted by $X \equiv (\delta\epsilon, \delta u^z)$. The two-point energy correlator becomes

$$\begin{aligned} \langle \delta\epsilon(t, z) \delta\epsilon(t', z') \rangle &= \frac{8T_0 \eta_v}{3A_{\perp}} \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega(t-t')} e^{-ik(z-z')} \\ &\times \frac{k^4}{(\omega^2 - c_s^2 k^2)^2 + \alpha^2 k^4 \omega^2}, \end{aligned} \quad (\text{B4})$$

where $\alpha = 4\eta_v/(3w_0)$. For equal times, the energy correlation becomes $\langle \delta\epsilon(t, z) \delta\epsilon(t, z') \rangle = w_0 T_0 \delta(z - z')/(c_s^2 A_{\perp})$. Thus in a static fluid noise produces only local correlations and does not induce any long-range structures. For unequal times, the correlators in Eq. (B4) admit analytic solutions

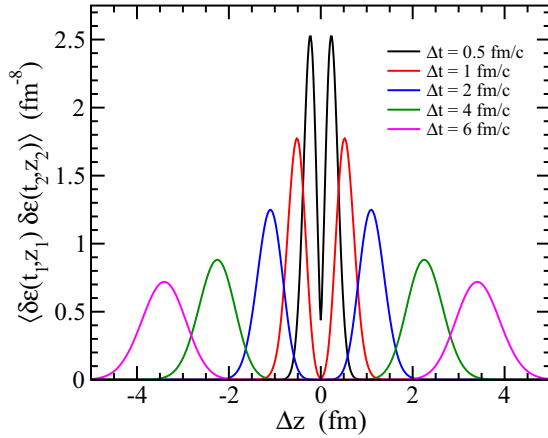


FIG. 5. Spatial dependence of energy-energy correlations at various time intervals Δt from thermal fluctuations created in a static medium. The results are for an ultrarelativistic gas EOS with a temperature of $T_0 = 550$ MeV and shear viscosity to entropy density ratio $\eta_v/s = 1/4\pi$.

only in the limit of $\eta_v \rightarrow 0$:

$$\langle \delta\epsilon(t, z) \delta\epsilon(t', z') \rangle = \frac{w_0 T_0}{2c_s^2 A_\perp} \left[\delta(\Delta z - c_s \Delta t) + \delta(\Delta z + c_s \Delta t) \right], \quad (\text{B5})$$

where $\Delta t = (t - t')$ and $\Delta z = (z - z')$. Thus one finds that in a static fluid, when shear viscosity is neglected in the evolution of fluctuations, the correlations are produced solely by sound waves of velocity $c_s^2 = \partial p / \partial e$ which propagate without attenuation. In the presence of viscosity, the energy-energy correlator of Eq. (B4) has a singular part given by $w_0 T_0 / (c_s^2 A_\perp) \exp(-c_s^2 \Delta t / \alpha) \delta(\Delta z)$. On the other hand, the regular part has to be computed numerically. The time dependence of the regular part of this correlation is shown in Fig. 5 for a constant background temperature of $T_0 = 550$ MeV and $\eta_v/s = 1/4\pi$. With increasing time Δt , viscosity is seen to reduce the amplitude of the two peaks formed at $\Delta z = \pm c_s \Delta t$ as well as broaden the correlations.

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