Two-body contributions to the effective mass in nuclear effective interactions

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Starting from general expressions of well-chosen symmetric nuclear matter quantities derived for both zero- and finite-range effective theories, we derive some universal relations between them. We first show that, independently of the range, the two-body contribution is enough to describe correctly the saturation mechanism but gives an effective mass value around $m^*/m \simeq 0.4$ when the other properties of the saturation point are set near their generally accepted values. Then, we show that a more elaborated interaction (for instance, an effective two-body density-dependent term on top of the pure two-body term) is needed to reach the accepted value $m^*/m \simeq 0.7$–0.8.

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I. INTRODUCTION

The penalty function used in the process of fitting a nuclear effective interaction (either zero- or finite-range) is often built as a mixture of experimental data on nuclei and some empirical values concerning the equation of state of pure neutron matter and symmetric nuclear matter (SNM) in the vicinity of the saturation point. In particular, for SNM, one usually includes the Fermi momentum k_F, the energy per particle E/A at saturation, the compression modulus K_∞, and the effective mass at the Fermi surface m^*/m.

For both zero- and finite-range interactions, it is possible to derive simple expressions for these SNM properties as functions of the interaction parameters. By combining these functions in an appropriate way, we can eliminate some of the parameters and get general relations that are convenient for the fitting process. These simple analytical relations also show that some infinite nuclear matter properties might not be independent and, therefore, some constraints might be conflicting. For example, in the case of standard Skyrme interactions (i.e., zero-range interactions with momentum-dependent terms up to second order and one density-dependent contact term), the manifest correlation among the effective mass, the incompressibility, and the power α of the density in the density-dependent term has been already discussed in Ref. [1].

In the present article, we investigate the origin of such a correlation, and other similar ones, using different families of nonrelativistic effective interactions. An essential point is to discern whether such a correlation, or other similar ones, is general or is an artefact due to the specific form of the adopted interaction.

By inspecting the scientific literature, we observe that when an effective interaction contains only two-body terms (i.e., no explicit three-body, four-body, or density-dependent terms) the effective mass in SNM is inevitably close to $m^*/m = 0.4$. This value is obtained, for example, with the SV [2] and SHZ2 [3] Skyrme interactions, but also with the finite-range interaction B1 [4] and the more recent class of regularized pseudopotentials [5].

The mechanism leading to a low effective mass for pure two-body interactions was already identified years ago by Weisskopf [6] (and more recently discussed by Nakatsukasa *et al.* [7]) and proved to be unavoidable for any interaction that gives a mean-field at most quadratic in momentum in SNM. Our aim is thus to use this evidence as a starting point and to explore the possibility of finding some general relations that can explain this obtained value for generic two-body interactions, including finite-range ones.

The article is organized as follows. In Sec. II we focus on pure two-body terms and obtain some general relations in which we isolate the effective mass and give some numerical values from standard interactions. In Sec. III we extend the analysis by including an explicit two-body density-dependent term. Finally, we give our conclusions in Sec. IV.

II. TWO-BODY INTERACTION

We begin this section by defining some important quantities that will be useful in the following. The starting point for our reasoning is the energy per particle E/A in infinite symmetric nuclear matter. This is a crucial quantity, because all the other relevant thermodynamical properties are related to E/A via simple derivative operations. We define the pressure as

$$P = \rho^2 \frac{\partial E/A}{\partial \rho} \tag{1}$$

and the isothermal compressibility as

$$\frac{1}{\kappa_T} = \frac{\partial P}{\partial \rho}, \tag{2}$$

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where ρ is the nucleon (scalar-isoscalar) density in SNM. For historical reasons, instead of Eq. (2), it is traditional in nuclear physics to consider the compression modulus at saturation defined as

$$K_\infty = 9\left(\rho^2 \frac{\partial^2 E/A}{\partial \rho^2}\right)_{\rho=\rho_0}. \tag{3}$$

Using Eq. (1) this can be equivalently written as

$$K_\infty = 9\left(\frac{1}{\kappa_T} - 2\frac{P}{\rho}\right)_{\rho=\rho_0}. \tag{4}$$

The consistency with Eq. (3) is ensured by the fact that at the saturation density ρ_0 one has $P = 0$.

Notice that any contribution to E/A linear in the density ρ_0 gives a quadratic contribution to P, with the same coefficient. Therefore, these two contributions cancel out exactly in the difference entering the right-hand side of Eq. (4). This applies for instance to the t_0 term of the Skyrme interaction, and this is why the usual expression for K_∞ found in the literature does not contain an explicit dependence on t_0. In the case of a finite-range momentum-independent interaction, because the contribution of the direct term to E/A is also linear in ρ_0, the same conclusion applies. Based on this observation, we adopt for the energy per particle at saturation the following expression:

$$\mathcal{E}_0 = \left(\frac{E}{A} - \frac{1}{\rho}P\right)_{\rho=\rho_0}. \tag{5}$$

As discussed previously, such an expression does not dependent on t_0 in the case of a Skyrme interaction or on the direct term in the case of a finite-range momentum-independent two-body interaction.

A. Weisskopf's relation

Years ago, Weisskopf [6] showed that the mean-field in nuclear matter should be momentum dependent. Assuming a quadratic dependence, he got a relation between the effective mass, the binding energy per particle, and the Fermi momentum. For the sake of clarity, we derive here this relation with a modified notation.

Within the hypothesis of a quadratic momentum dependence, the mean field U_i for a state i with momentum p_i can be written as

$$U_i = U_0 + \frac{p_i^2}{p_F^2} U_1, \tag{6}$$

where U_0 and U_1 are some constants and $p_F = \hbar k_F$ is the Fermi momentum. The effective mass, defined through the relation

$$\frac{p_i^2}{2m} + U_i \equiv \frac{p_i^2}{2m^*} + U_0 \tag{7}$$

can therefore be written as

$$\frac{m}{m^*} = 1 + \frac{U_1}{\varepsilon_F}, \tag{8}$$

where $\varepsilon_F = \hbar^2 k_F^2/2m$ is the Fermi energy.

As an illustration, we can mention that the above hypothesis embraces the standard Skyrme interaction: dropping here the density-dependent term (see Sec. III for a detailed discussion of this term), it is straightforward to show that the mean field can be written as

$$U(k) = \tfrac{3}{4}t_0\rho + \tfrac{3}{80}C_1^{(2)}\rho k_F^2 + \tfrac{1}{16}C_1^{(2)}\rho k^2, \tag{9}$$

where $C_1^{(2)} = 3t_1 + (5 + 4x_2)t_2$. Comparing with Eq. (6) we identify

$$U_1 = \tfrac{1}{16}C_1^{(2)}\rho k_F^2, \tag{10}$$

so that Eq. (8) leads to the following familiar expression for the effective mass,

$$\frac{m^*}{m} = \left[1 + \frac{1}{8}\frac{m}{\hbar^2}C_1^{(2)}\rho\right]^{-1}, \tag{11}$$

for Skyrme interactions.

Weisskopf established an interesting relation between m^*/m, E/A, and ε_F. To rederive it, we start from the energy per particle written as (the brackets indicate an average over the particles states)

$$\begin{aligned} E/A &\equiv \langle T\rangle + \tfrac{1}{2}\langle U\rangle \\ &= \tfrac{3}{5}\varepsilon_F + \tfrac{1}{2}U_0 + \tfrac{3}{10}U_1. \end{aligned} \tag{12}$$

The separation energy of a particle at the Fermi surface is given by

$$S_F = -(\varepsilon_F + U_F) = -(\varepsilon_F + U_0 + U_1), \tag{13}$$

and it is related to the energy per particle via the Hugenholtz–Van Hove theorem [8] with $S_F = -\mathcal{E}_0$, so that one immediately gets[1]

$$U_1 = \tfrac{1}{2}(-5\mathcal{E}_0 + \varepsilon_F). \tag{14}$$

Finally, the effective mass can therefore be written as

$$\frac{m}{m^*} = \frac{3}{2} - \frac{5}{2}\frac{\mathcal{E}_0}{\varepsilon_F}. \tag{15}$$

For the commonly accepted values of $\mathcal{E}_0$ and k_F (i.e., -16 MeV and 1.33 fm^{-1}, respectively), the above equation leads to an effective mass of $m^*/m \simeq 0.4$. Due to the assumed quadratic momentum dependence of the mean field, this result can only be used for a standard Skyrme interaction with no density-dependent term. To compare with a practical case, we thus consider the SV interaction [2], which is one of the very few Skyrme interactions with no density dependence. The relevant SNM properties of the SV interaction leads to the values $\mathcal{E}_0 = -16.05$ MeV, $k_F = 1.32$ fm^{-1}, and $m^*/m = 0.38$. Consistently, the use of Eq. (15) gives $m^*/m = 0.38$.

Finally, let us mention that, as discussed by Weisskopf, the mean-field potential may have a momentum dependence beyond the quadratic one. Such a dependence is examined below.

[1]We indicate in passing that Ref. [6] contains two misprints in the sign of U_1.

B. Zero-range N3LO Skyrme interaction

The central part of the Skyrme N3LO pseudopotential [9,10] with no additional density-dependent term reads

$$
\begin{aligned}
V^{c}_{\mathrm{N3LO}} &= t_0(1+x_0P_\sigma) + \tfrac{1}{2}t_1^{(2)}\big(1+x_1^{(2)}P_\sigma\big)(\mathbf{k}^2+\mathbf{k}'^2)\\
&\quad + t_2^{(2)}\big(1+x_2^{(2)}P_\sigma\big)(\mathbf{k}\cdot\mathbf{k}')\\
&\quad + \tfrac{1}{4}t_1^{(4)}\big(1+x_1^{(4)}P_\sigma\big)[(\mathbf{k}^2+\mathbf{k}'^2)^2+4(\mathbf{k}'\cdot\mathbf{k})^2]\\
&\quad + t_2^{(4)}\big(1+x_2^{(4)}P_\sigma\big)(\mathbf{k}'\cdot\mathbf{k})(\mathbf{k}^2+\mathbf{k}'^2)\\
&\quad + \tfrac{1}{2}t_1^{(6)}\big(1+x_1^{(6)}P_\sigma\big)(\mathbf{k}^2+\mathbf{k}'^2)\\
&\quad \times [(\mathbf{k}^2+\mathbf{k}'^2)^2+12(\mathbf{k}'\cdot\mathbf{k})^2]\\
&\quad + t_2^{(6)}\big(1+x_2^{(6)}P_\sigma\big)(\mathbf{k}'\cdot\mathbf{k})\\
&\quad \times [3(\mathbf{k}^2+\mathbf{k}'^2)^2+4(\mathbf{k}'\cdot\mathbf{k})^2], \qquad (16)
\end{aligned}
$$

where a $\delta(\mathbf{r}_1-\mathbf{r}_2)$ function factorizing all terms is to be understood, but has been omitted for the sake of clarity. From this pseudopotential, we compute the energy per particle as

$$
\begin{aligned}
\frac{E}{A} &= \frac{3}{10}\frac{\hbar^2}{m}c_s\rho^{2/3} + \frac{3}{8}t_0\rho + \frac{3}{80}C_1^{(2)}c_s\rho^{5/3}\\
&\quad + \frac{9}{280}C_1^{(4)}c_s^2\rho^{7/3} + \frac{2}{15}C_1^{(6)}c_s^3\rho^{9/3}, \qquad (17)
\end{aligned}
$$

where we have defined $c_s = (3\pi^2/2)^{2/3}$. The three quantities we are interested in are

$$\mathcal{E}_0 = \tfrac{1}{5}\varepsilon_F - \tfrac{1}{40}C_1^{(2)}\rho_0k_F^2 - \tfrac{3}{70}C_1^{(4)}\rho_0k_F^4 - \tfrac{4}{15}C_1^{(6)}\rho_0k_F^6, \qquad (18)$$

$$K_\infty = -\tfrac{6}{5}\varepsilon_F + \tfrac{3}{8}C_1^{(2)}\rho_0k_F^2 + \tfrac{9}{10}C_1^{(4)}\rho_0k_F^4 + \tfrac{36}{5}C_1^{(6)}\rho_0k_F^6, \qquad (19)$$

and

$$\frac{\hbar^2}{2m^*}k_F^2 = \varepsilon_F + \frac{1}{16}C_1^{(2)}\rho_0k_F^2 + \frac{1}{8}C_1^{(4)}\rho_0k_F^4 + \frac{9}{10}C_1^{(6)}\rho_0k_F^6. \qquad (20)$$

The coupling constants entering these expressions are related to the parameters of the pseudopotential in Eq. (16) as $C_1^{(n)} = 3t_1^{(n)} + (5+4x_2^{(n)})\,t_2^{(n)}$.

In the case of a standard Skyrme interaction (corresponding to N1LO), the three above expressions depend only on the coupling constant labeled $C_1^{(2)}$. This means that these quantities are correlated by pairs. Besides Eq. (15) we can obtain two more relations:

$$15\mathcal{E}_0 + K_\infty = \frac{9}{5}\varepsilon_F, \qquad (21)$$

$$m/m^* = \frac{6}{5} + \frac{K_\infty}{6\varepsilon_F}. \qquad (22)$$

Such relations are clearly specifically related to the form of the interaction (i.e., only valid for SV-like interactions) and do not hold in any other cases.

Using Eqs. (18)–(20) we derive an expression for the effective mass where we exactly cancel out both coefficients $C_1^{(2)}$ and $C_1^{(4)}$ by taking the following combination:

$$\frac{35}{24}\mathcal{E}_0 - \frac{5}{72}K_\infty + \frac{\hbar^2}{2m^*}k_F^2. \qquad (23)$$

It leads to the following relation:

$$\frac{m}{m^*} = \frac{11}{8} + \frac{5}{72}\frac{K_\infty - 21\mathcal{E}_0}{\varepsilon_F} + \frac{1}{90}\frac{1}{\varepsilon_F}C_1^{(6)}\rho_0k_F^6. \qquad (24)$$

This is an interesting result: keeping in mind that the coefficients $C_1^{(n)}$ are related to terms simulating finite-range effects in the N3LO Skyrme interaction [11], this equation states that finite-range effects only appear at third order (N3LO) and thus constitutes an exact result for N1LO and N2LO interactions.

To have a quantitative insight of the effect beyond N2LO, we have displayed in Table I some numerical results for different Skyrme interactions up to the N3LO level (as well as for others discussed later in the article). Within this article we have used the value $\hbar^2/2m = 20.735$ MeV fm^2. Notice that this value may not exactly be the one used by the authors of the various effective interactions mentioned here, but the possible differences are irrelevant for the present discussion. For some forces we have checked that these differences are less than $5\times10^{-3}\%$ and in all cases do not affect the numbers given in Table I.

The selected interactions are the following:

(i) at the N1LO level, the two parametrizations SV [2] and SHZ2 [3], which do not contain any density-dependent terms;
(ii) still at the N1LO level, a panel of various parametrizations which differ mainly for their density-dependent term and for their values of the compression modulus, namely SIII [2], SkM* [12], SLy5 [1,13], BSk1 [14], and SLy5* [15];
(iii) the NℓLO parametrization (ℓ = 2,3) obtained [10] using the Landau parameters derived from the finite-range interactions D1MT [16,17] and M3Y-P2 [18], and hereafter called D1MT-NℓLO and M3Y-P2-NℓLO; and
(iv) the recent SN2LO1 [19] parametrization built up via a complete minimization of the penalty function based on both SNM properties and finite-nuclei observables.

Table I gives the isoscalar bulk properties of each parametrization (first four columns). The second part of Table I (next three columns) gives the two-body contribution to the isoscalar effective mass as given by Eq. (24)—the $C_1^{(6)}$ term is denoted as Δ_{FR} for reasons that will be clarified below—as well as the total isoscalar effective mass for the two-body part only of the interactions. Keep in mind that the partial contributions concern m/m^*, i.e., the inverse of the effective mass.

The five first interactions listed on Table I are pure two-body interactions. Therefore, the values for their effective masses given by Eq. (24) match the full ones given by Eq. (40). These values are slightly different from 0.4 because of the unusual saturation densities and compression moduli predicted by some of these interactions. For all other cases listed in Table I, the contribution to the effective mass from the two-body part of the interaction as given by Eq. (24) is close to 0.4 and therefore in agreement with Weisskopf's estimate. In the case of N3LO interactions, one can see that the contribution of the $C_1^{(6)}$ term is actually very small. To get an estimate of the relative importance of such terms entering Eq. (24), we take as an example the case of the M3Y-P2-N3LO interaction: we

TABLE I. Properties of the two-body interactions used in this study at saturation density. The various contributions to the isoscalar effective mass are given (see text for details). All the finite-range contributions given in the $\Delta_{\rm FR}$ column are from finite-range interactions, i.e., Gogny or Nakada interactions, except for the zero-range Skyrme-like DM1T and M3Y interactions where this column gives the N3LO contribution. All the density dependencies are zero-range, i.e., the t_3 term, except for the D2 Gogny force which uses a finite-range density dependence with a Gaussian form factor. Values given in the column denoted by "t_3" represent the contribution to the effective mass from the last term in Eq. (41). Missing entries are zero.

	$\rho_{\rm sat}$ (fm^{-3})	k_F (fm^{-1})	$\mathcal{E}_0$ (MeV)	K_∞ (MeV)	$\frac{5}{72}(K_\infty - 21\mathcal{E}_0)/\varepsilon_F$	$\Delta_{\rm FR}$	m^*/m Eq. (24)	α	t_3	m^*/m Eq. (41)
SV [2]	0.155	1.319	–16.05	306	1.237		0.383			0.383
SHZ2 [3]	0.157	1.326	–16.27	310	1.241		0.382			0.382
B1 [4]	0.205	1.448	–15.69	183	0.819	0.0225	0.451			0.451
C1 [4]	0.206	1.451	–15.83	218	0.876	0.0133	0.442			0.442
L3 [4]	0.277	1.601	–15.75	216	0.714	0.0090	0.477			0.477
SIII [2]	0.145	1.291	–15.85	355	1.383		0.363	1	–1.448	0.763
SkM* [12]	0.160	1.334	–15.77	217	1.031		0.416	$\frac{1}{6}$	–1.138	0.789
BSk1 [14]	0.157	1.325	–15.80	231	1.074		0.408	$\frac{1}{3}$	–1.496	1.050
SLy5 [1,13]	0.160	1.334	–15.98	230	1.0644		0.410	$\frac{1}{6}$	–1.005	0.697
SLy5* [15]	0.161	1.334	–16.02	230	1.065		0.410	$\frac{1}{6}$	–1.013	0.701
M3Y-P2-N1LO [10]	0.162	1.338	–12.35	217	0.890		0.441	$\frac{1}{3}$	–0.730	0.652
SN2LO1 [19]	0.162	1.339	–15.95	222	1.041		0.414	$\frac{1}{6}$	–1.005	0.709
DM1T-N2LO [10]	0.143	1.284	–9.92	154	0.736		0.474	$\frac{1}{3}$	–0.773	0.748
M3Y-P2-N2LO [10]	0.158	1.327	–15.25	206	1.001		0.421	$\frac{1}{3}$	–0.839	0.651
DM1T-N3LO [10]	0.164	1.345	–15.35	215	0.996	0.0028	0.422	$\frac{1}{3}$	–1.044	0.776
M3Y-P2-N3LO [10]	0.161	1.337	–15.96	217	1.035	0.0126	0.413	$\frac{1}{3}$	–0.886	0.651
D1 [20]	0.166	1.351	–16.30	229	1.050	0.0047	0.412	$\frac{1}{3}$	–0.936	0.670
D1S [21]	0.163	1.342	–16.01	203	1.002	0.0029	0.419	$\frac{1}{3}$	–0.952	0.697
D1M [16]	0.165	1.346	–16.02	225	1.038	0.0030	0.414	$\frac{1}{3}$	–1.076	0.746
D2 [22]	0.161	1.337	–15.82	207	1.011	0.0170	0.416	$\frac{1}{3}$	–1.047	0.738
P2 [18]	0.163	1.340	–16.14	220	1.043	0.0167	0.411	$\frac{1}{3}$	–0.901	0.652
P6 [23]	0.163	1.340	–16.24	240	1.083	0.0161	0.404	$\frac{1}{3}$	–0.797	0.596
P7 [23]	0.163	1.340	–16.23	255	1.111	0.0158	0.400	$\frac{1}{3}$	–0.803	0.589

get $1.375 + 1.0353 + 0.0126$, which leads to an the effective mass 0.4127. Dropping the last term, one gets 0.4149 instead. In conclusion, neglecting the $C_1^{(6)}$ term in Eq. (24) results in an overestimate of m^*/m by less than 0.5%. Keeping this result in mind, we now see how these results are modified with an explicit finite-range interaction.

C. Finite-range interaction

To be as general as possible, we consider a finite-range two-body interaction written as

$$V = \sum_n (W_n + B_n P_\sigma - H_n P_\tau - M_n P_{\sigma\tau}) V(r/\mu_n). \quad (25)$$

The radial form factor is characterized by a set of ranges denoted μ_n. For the sake of simplicity, the index n is omitted in the following; a sum over it is to be understood in every term containing a range μ. From this interaction, we derive the energy per particle as

$$\frac{E}{A} = \frac{3}{5}\frac{\hbar^2}{2m}k_F^2 + 2\pi\rho C_D \int dr r^2 V(r/\mu) - \frac{12}{\pi} C_E k_F^3 \int dr r^2 V(r/\mu) \left[\frac{j_1(k_F r)}{k_F r}\right]^2, \quad (26)$$

where

$$C_D = W + \tfrac{1}{2}B - \tfrac{1}{2}H - \tfrac{1}{4}M, \quad (27)$$

$$C_E = \tfrac{1}{4}W + \tfrac{1}{2}B - \tfrac{1}{2}H - M \quad (28)$$

are respectively the combinations related to the direct and exchange contributions. As already mentioned, we see from the above equation that the direct contribution is linear in ρ_0.

As for the zero-range case, we determine the quantities of interest. The starting expressions are

$$\mathcal{E}_0 = \frac{1}{5}\varepsilon_F - \frac{12}{\pi}C_E k_F^3 \int dr r^2 V(r/\mu)\mathcal{F}^{\mathcal{E}}(k_F r), \quad (29)$$

$$K_\infty = -\frac{6}{5}\varepsilon_F - \frac{12}{\pi}C_E k_F^3 \int dr r^2 V(r/\mu)\mathcal{F}^{K}(k_F r), \quad (30)$$

$$\frac{\hbar^2}{2m^*}k_F^2 = \varepsilon_F + \frac{12}{\pi}C_E k_F^3 \int dr r^2 V(r/\mu)\mathcal{F}^{m}(k_F r), \quad (31)$$

where we have defined the following functions

$$\mathcal{F}^{\mathcal{E}}(x) = \frac{2}{x^2}j_1^2(x) - \frac{2}{3x}j_0(x)j_1(x), \quad (32)$$

$$\mathcal{F}^{K}(x) = 2j_0^2(x) - \frac{12}{x}j_0(x)j_1(x) + \left(\frac{18}{x^2}-2\right)j_1^2(x), \quad (33)$$

$$\mathcal{F}^{m}(x) = \frac{1}{3}j_1^2(x). \quad (34)$$

Taking the linear combination given by Eq. (23), we deduce the following relation (with $x = k_F r$):

$$\frac{m}{m^*} = \frac{11}{8} + \frac{5}{72}\frac{K_\infty - 21\mathcal{E}_0}{\varepsilon_F} + \frac{12}{\pi}\frac{C_E}{\varepsilon_F}\int dx x^2 V\left(\frac{x}{k_F\mu}\right) \times \left\{\mathcal{F}^m(x) + \frac{5}{72}\mathcal{F}^K(x) - \frac{105}{72}\mathcal{F}^{\mathcal{E}}(x)\right\}. \quad (35)$$

As expected, there is no contribution from the direct term C_D. On the contrary, the term containing the C_E coefficient contains all finite-range effects, which go beyond the $C_1^{(6)}$ term of the N3LO equivalent relation given in Eq. (24). As long as the form factor $V(x)$ is short-ranged, it seems reasonable to take the following power expansion:

$$\mathcal{F}^m(x) + \tfrac{5}{72}\mathcal{F}^K(x) - \tfrac{105}{72}\mathcal{F}^{\mathcal{E}}(x) \simeq \tfrac{1}{127\,575}x^6 - \tfrac{8}{9\,823\,275}x^8 + \tfrac{1}{25\,540\,515}x^{10} + \cdots .$$

We see that x^2 and x^4 contributions are exactly canceled out, in agreement with our previous discussion. The first x^6 contribution is thus related to N3LO and the next contributions x^n $(n = 8,\ldots)$ are related to higher orders.

The contribution of the exchange term C_E entering Eq. (35) is denoted as $\Delta_{\rm FR}$. It contains the explicit finite-range contribution to m/m^*.

In Table I are displayed results for several finite-range interactions. We have selected the following:

(i) the Brink-Boeker [4] interactions B1, C1, and L3 which use Gaussian form factors and do not contain any density dependencies;

(ii) the standard D1 [20] and D1S [21] Gogny interactions, which use two Gaussian form factors plus a zero-range density dependence; the D1M parametrization [16] used in a first attempt to reproduce more than 2100 measured masses through a Gogny-Hartree-Fock-Bogoliubov nuclear mass model; and the recent D2 [22] parametrization which uses a finite-range density dependence with a Gaussian form factor; and

(iii) three parametrizations based on three Yukawa form factors [18,23], which also include explicit density-dependent terms.

One can see that $\Delta_{\rm FR}$ is very small. To get an estimate of the relative importance of the terms entering Eq. (35), we give precisely the numerical values obtained for the D1 interaction: $1.375 + 1.0495 + 0.00469$, which results in the effective mass 0.4116. Dropping the last term, one gets 0.4125 instead, which gives an overestimate less than $\approx 0.5\%$. For the Nakada's series the overestimate is larger, i.e., around 1.6%, and this is due to the longer range of the form factors used.

D. Summary concerning the two-body contributions

Either for zero- or finite-range effective interactions, one can write

$$\left.\frac{m}{m^*}\right|_{2B} = \frac{11}{8} + \frac{5}{72}\frac{1}{\varepsilon_F}(K_\infty - 21\mathcal{E}_0) + \Delta_{\rm FR}, \quad (36)$$

where $\Delta_{\rm FR}$ is a short notation for the last terms entering Eqs. (24) and (35) and it takes into account finite-range effects beyond N2LO. In all cases examined here, it turns out that $\Delta_{\rm FR}$ is very small. The value $m^*/m \simeq 0.4$ is obtained for *any* two-body interaction giving reasonable $\mathcal{E}_0$, K_∞, and k_F values. The same value is obtained when using only the pure two-body part of any of the effective interactions considered. This generalizes the older result obtained by Weisskopf.

Because the empirical effective mass in the bulk is expected to be around 0.7 [24,25], other contributions beyond the two-body terms are needed and mainly justify the use of a density dependence to simulate such effects. The next part is devoted to that topic.

III. EFFECTIVE TWO-BODY DENSITY-DEPENDENT INTERACTION

Because an effective interaction limited to a purely two-body term cannot lead to a reasonable value of the effective mass, the inclusion of three- or even four-body terms seems to be unavoidable. However, to the best of our knowledge, even if some pioneering work has been done in the recent past [5,26–28], decisive improvements remain necessary to use such interactions for practical applications. For these reasons, and because we are mainly interested in the properties of infinite nuclear matter at the mean-field level in this article, in this section we limit ourselves to the usual effective two-body density-dependent term that is currently used either in zero- or finite-range interactions. Explicitly, this term reads

$$V_{\rm DD} = \tfrac{1}{6}t_3(1 + x_3 P_\sigma)\rho^\alpha. \quad (37)$$

Let us remark that the factor 1/6 reflects the historical three-body origin of this term [29]. This coefficient is usually used for the zero-range Skyrme interaction, but is disregarded for the finite-range one.

For zero-range interactions, this term does not directly contribute to the effective mass, while its contribution to the

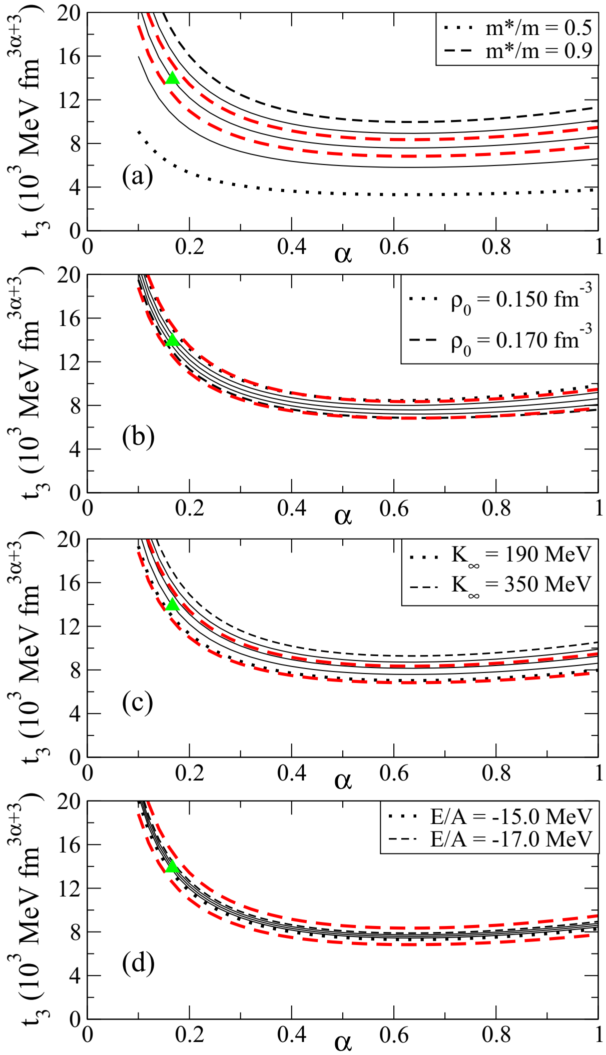

FIG. 1. The parameter t_3 plotted as a function of α for various sets of isoscalar bulk properties. In each case, one of the four bulk properties K_∞, m^*/m, E/A, or $\rho_{\rm sat}$ is varied, keeping constant the three others (see text). The green triangle marks the SLy5* parametrization [15].

other quantities entering Eq. (23) is

$$\mathcal{E}_0|_{\rm DD} = -\tfrac{3}{8}\tfrac{1}{6}t_3\rho_0^{\alpha+1}\alpha, \tag{38}$$

$$K_\infty|_{\rm DD} = \tfrac{27}{8}\tfrac{1}{6}t_3\rho_0^{\alpha+1}\alpha(\alpha+1)\,. \tag{39}$$

It is worth mentioning at this stage that, in the N1LO case, the parameter t_3 is sometimes eliminated with an appropriate combination of $\mathcal{E}_0$ and K_∞, namely

$$9(\alpha+1)\mathcal{E}_0 + K_\infty = \tfrac{3}{5}(3\alpha+1)\varepsilon_F - \tfrac{3}{40}(3\alpha-2)C_1^{(2)}\rho_0 k_F^2. \tag{40}$$

The value $\alpha = 2/3$ leads to a singular situation specific to the standard Skyrme interaction for which K_∞ is determined by $\mathcal{E}_0$ and k_F and is not related to the effective mass. However, because we are interested in getting general relations valid for any effective interaction, either zero- or finite-range, we consider in the following the complete equations, that is, beyond N1LO. Therefore, plugging the density-dependent contributions (38) and (39) into Eq. (23) we get

$$\frac{m}{m^*} = \frac{11}{8} + \frac{5}{72}\frac{K_\infty - 21\mathcal{E}_0}{\varepsilon_F} + \Delta_{\rm FR} - \frac{5}{384}\alpha(10+3\alpha)\frac{t_3\rho_0^{\alpha+1}}{\varepsilon_F}. \tag{41}$$

In Table I are collected the contributions of all the terms entering Eq. (41), but the constant 11/8.

The conclusions of this analysis are clear.

(i) Any pure two-body interaction leads to an effective mass value that is solely determined by the values of $\mathcal{E}_0$, K_∞, k_F. When the accepted values of these parameters are used, one gets $m^*/m \simeq 0.4$, within less than 1% accuracy.

(ii) The density-dependent term substantially modifies that value, increasing it to typically $m^*/m \simeq 0.7$.

An important remark concerning Eq. (41) is that it actually provides an unexpected relation between the parameters t_3 and α. For (reasonable) fixed values of the inputs $\mathcal{E}_0$, K_∞, and k_F, these parameters are not independent. This is reflected in Fig. 1 for several sets of these quantities. Actually, we have dropped the $\Delta_{\rm FR}$ term in these figures. We start from the set $K_\infty = 230 \pm 20$ MeV, $m^*/m = 0.70 \pm 0.02$, $\mathcal{E}_0 = -16.0 \pm 0.5$ MeV, and $\rho_0 = 0.160 \pm 0.005$ fm^{-3}. The two thick dashed lines in all panels represent the limits of t_3 as a function of α taking into account, in a schematic way, the statistical errors related to the uncertainties of the inputs as well as the neglected small contribution from $\Delta_{\rm FR}$. The remaining curves correspond to the central values of three of these inputs, for varying values of the fourth one. In Fig. 1(a) the effective mass is varied in steps of 0.1, in Fig. 1(b) the density is varied in steps of 0.05 fm^{-3}, in Fig. 1(c) the compression modulus is varied in steps of 40 MeV, and in Fig. 1(d) the energy per particle is varied in steps of 0.5 MeV. One can see that the relation between t_3 and α is not very sensitive to the inputs for ρ_0 and $\mathcal{E}_0$, but is sensitive to the (extreme) values for the effective mass.

For values of α in the interval between 0.4 and 0.9, the coefficient t_3 has an almost constant value (around 8000, in the right units), with a minimum for $\alpha \simeq 0.64$. All the curves show a divergence of the t_3 parameter when α decreases showing some dangerous range of values smaller than 1/6. Here, we see the main reason why it is dangerous to include the α parameter in the fitting procedure.

IV. CONCLUSIONS

In this article, we have performed a systematic study on the properties of the effective mass for general two-body interactions, both zero- and finite-range. In particular, starting from the previous work of Weisskopf [6], we have shown that the two-body part of *any* effective interaction induces an effective mass at most of 0.4, irrespectively of the range of the interaction, as long as other infinite matter properties are kept to reasonable values. The result is exact for N1LO (Skyrme) and

N2LO interactions, while there is a minor correction of $\approx$1–2% for the higher-order N3LO pseudopotential and/or finite-range interactions.

To increase the value of the effective mass to higher values without spoiling other infinite matter properties such as saturation density and incompressibility, it is thus necessary to add either an explicit three-body term or equivalently, but more phenomenological, a two-body density-dependent term. The latter is the common strategy used by the vast majority of effective interactions available nowadays.

An interesting result of our analysis is the strong built-in correlation found between the intensity of the density-dependent interaction t_3 and the exponent of the density α. This strong correlation is only marginally affected by the explicit presence of a finite range or equivalently higher-order gradient terms. This correlation reflects a lack of flexibility in our models and in particular in the way three-body terms are treated.

One should keep in mind that the standard density-dependent term was originally generated [29] from a simple zero-range three-body interaction. Including a more general three-body interaction seems to be the proper way to go beyond a pure two-body interaction. Among the several attempts in this direction, we would like to mention two of them: a zero-range three-body interaction including gradient terms [26,28] and a semicontact three-body interaction [30]. In both cases we have verified that one gets the expected result, namely, the pure two-body part gives a contribution to the effective mass of about 0.4, while the three-body part increases this value to the accepted one. It is possible to get for these interactions a relation similar to Eq. (41), in which the density-dependent contribution is replaced with a three-body contribution. Interestingly, this equation provides a correlation between three-body parameters, analogous to that between t_3 and α, which could be usefully utilized in the process of determining the parameters.

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