Microscopic description of pair transfer between two superfluid Fermi systems: Combining phase-space averaging and combinatorial techniques

David Regnier[*] and Denis Lacroix[†]

Institut de Physique Nucléaire, IN2P3-CNRS, Université Paris-Sud, Université Paris-Saclay, F-91406 Orsay Cedex, France

Guillaume Scamps and Yukio Hashimoto

Center for Computational Sciences, University of Tsukuba, Tsukuba 305-8571, Japan

In a mean-field description of superfluidity, particle number and gauge angle are treated as quasiclassical conjugated variables. This level of description was recently used to describe nuclear reactions around the Coulomb barrier. Important effects of the relative gauge angle between two identical superfluid nuclei (symmetric collisions) on transfer probabilities and fusion barrier have been uncovered. A theory making contact with experiments should at least average over different initial relative gauge-angles. In the present work, we propose a new approach to obtain the multiple pair transfer probabilities between superfluid systems. This method, called phase-space combinatorial (PSC) technique, relies both on phase-space averaging and combinatorial arguments to infer the full pair transfer probability distribution at the cost of multiple mean-field calculations only. After benchmarking this approach in a schematic model, we apply it to the collision $^{20}O + {}^{20}O$ at various energies below the Coulomb barrier. The predictions for one pair transfer are similar to results obtained with an approximated projection method, whereas significant differences are found for two pairs transfer. Finally, we investigated the applicability of the PSC method to the contact between nonidentical superfluid systems. A generalization of the method is proposed and applied to the schematic model showing that the pair transfer probabilities are reasonably reproduced. The applicability of the PSC method to asymmetric nuclear collisions is investigated for the $^{14}O + {}^{20}O$ collision and it turns out that unrealistically small single- and multiple pair transfer probabilities are obtained. This is explained by the fact that relative gauge angle play in this case a minor role in the particle transfer process compared to other mechanisms, such as equilibration of the charge/mass ratio. We conclude that the best ground for probing gauge-angle effects in nuclear reaction and/or for applying the proposed PSC approach on pair transfer is the collisions of identical open-shell spherical nuclei.

I. INTRODUCTION

Although its contribution to the energy is rather small, superfluidity plays a significant role in static properties of atomic nuclei [1,2]. Its influence on time evolution of nuclei is, however, scarcely known. One effect that could be naturally anticipated is the enhancement of simultaneous transfer of two nucleons when the two particles form a pair [3,4]. Another predicted effect is the global increase of pair transfer when coherent oscillations of pairs exist, the so-called pairing vibration [5–7]. While a possible effect of pairing on transfer is rarely contested [3,4], its quantitative influence remains to be clarified. A detailed analysis has been, for instance, made recently in Ref. [8], where different levels of description from a pure mean field to beyond mean field have been considered. In parallel, to understand the competition between transfer and fusion reaction, new highly accurate experimental measurements of transfer probabilities have been achieved giving test-bench for nuclear models [9–13].

In recent years, intensive efforts have been made to include pairing into dynamical microscopic theories [14–19] to simulate both isolated nuclei and/or collisions between nuclei. These approaches offer the possibility to understand the influence of superfluidity on the dynamics from a different point of view than traditional approaches where nuclear structure is treated separately from nuclear reaction. The basic tools to include superfluidity is the Hartree-Fock-Bogoliubov (HFB) or BCS theory where the $U(1)$ symmetry associated to the gauge angle is spontaneously broken to include pairing correlations in a simple way. Similarly to interacting bosonic systems it is quite natural to investigate if the interaction of two nuclei is affected by their relative gauge angles when they enter into contact. Recently, several works have uncovered rather large effects of the relative gauge angle between identical nuclei. This case of symmetric collisions is quite special in the sense that effects such as charge/mass equilibration between collision partners are absent. This context allows the pairing fluctuations to become a major driver of the pair transfer. A first hint on the role of gauge angle in symmetric reactions was given in Ref. [20], where its influence on pair transfer has been addressed for the first time within TDHFB. It was indeed found in particular that particle transfer is sensibly affected

---
[*]regnier@ipno.in2p3.fr
[†]lacroix@ipno.in2p3.fr

by the relative gauge angle between nuclei. The sensitivity to the gauge angle has been further explored in Refs. [21–23], confirming again its importance in the particle transfer process. Besides, these works emphasize two surprising behaviors. First, when fusion does not occur, the kinetic energy of fragments after reseparation is significantly affected by the initial relative phase in gauge space. Second, the relative angle changes the fusion threshold leading to a considerable contribution to the so-called extra-push energy.

The fact that some spontaneous symmetry breaking affects the physics close to the Coulomb barrier is by itself not a surprise. Another typical example is given by the spontaneous breaking of the rotational symmetry at the mean-field level that leads to deformed nuclei in their intrinsic frame. The role of the gauge angle is in this case replaced by the relative orientation between the two main axes of deformation of the collision partners. These collective degrees of freedom lead to fluctuations in the Coulomb barrier properties that can be probed experimentally [24,25]. Note that an extensive discussion of the possible role of spontaneous symmetry breaking (e.g., deformation in real and/or gauge space) on transfer reactions can also be found in Ref. [26]. Mean-field theories like Hartree-Fock and/or Hartree-Fock-Bogoliubov provide a proper framework to describe spontaneous symmetry breaking in the sense that these approaches naturally yield deformed ground states. However, they treat the collective variables classically while, in principle, a fully quantum collective approach would be required. To overcome this difficulty, the mean-field picture requires some mechanism to restore the broken symmetries of the total system before, during, and after the collisions. A natural technique to ensure proper symmetries in state-of-the-art nuclear structure studies is to project the trial wave function on good particle number, parity, angular momentum [27–29]. This technique is, however, much more involved for nuclear reactions where two evolving systems with various exit channels are considered. Up to now, the projection technique has been essentially used after the reaction involving either two normal [30] or only one superfluid nucleus [19].

A first attempt was made recently to use projection technique during the evolution [31,32]. An important advantage is eventually to be able to treat interferences between different mean-field trajectories. The technique proposed in Ref. [31] turns out to be rather involved even if a drastic approximation was made on the mixing of different mean-field trajectories. In addition, two difficulties show up: (i) the results strongly depend on the conventions for the equation of motion used to solve the TDHFB equations; (ii) the final transfer probabilities might have rather important oscillations depending on the phase evolution convention even when the two nuclei are well separated.

Based on the recent investigation on the role of gauge angle on nuclear reactions, there is a number of emerging interrogations: the first one is that the gauge angle itself is a concept that only has a meaning in a symmetry breaking theory while the number of protons and neutrons in nuclei are fixed. Then, are the predicted effects surviving in a particle number conserving theory? If yes, do the huge effect predicted on reactions in Ref. [22] persists once the symmetry is properly restored? Are these effects still important in collisions involving a target

different from the projectile? Behind these questions, one may wonder if the classical mechanics mean-field approach based on gauge angle has a physical observable reality in nuclei. Another difficulty is that contrary to cold atoms, and even if nuclei might have rather large number of nucleons, the true number of nucleons (essentially those close to the Fermi energy) forming pairs is rather small. Then, finite-size corrections to the BCS/HFB approaches are expected to be significant as well as correlations much beyond the mean-field picture.

This article is organized as follows. First, we discuss some of the points raised above and question if a classical picture for the gauge space is meaningful in the context of transfer reactions (Sec. II). In Sec. III we review several existing technics to restore the symmetry associated to the number of particles in the context of nuclear reactions and propose a new method that we call phase-space combinatorial (PSC) technique. This method is benchmarked in a schematic model of collision between two identical systems (symmetric collisions) and at energies below the Coulomb barrier. Section IV highlights its application to the realistic collision ^{20}O + ^{20}O. Finally, we discuss in Sec. V the applicability of the PSC method to the case of asymmetric collisions, i.e., when the two superfluids are different, for which the main driver of particle transfer may not be the residual pairing interaction.

II. DESCRIPTION OF TRANSFER REACTIONS IN A SCHEMATIC MODEL

In this section, we focus on the exchange of pairs of fermions happening when two symmetric and superfluid systems come into contact. We follow Refs. [31,32] and consider a minimal schematic Hamiltonian to describe the transfer between a superfluid system A and a superfluid system B. In this simple model, the systems are assumed to stay at any time in fully paired states. The Hamiltonian is a sum of three terms:

$$H = H_A + H_B + V(t). \tag{1}$$

The operator H_A (respectively, H_B) describes the isolated Fermi system A (respectively, B) and is supposed to take the form of a simple pairing Hamiltonian (see, for instance, Ref. [2]):

$$H_A = \sum_{k>0}^{\Omega_A} \varepsilon_k^A (a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - g_A \sum_{k\neq l>0}^{\Omega_A} a_k^\dagger a_{\bar{k}}^\dagger a_{\bar{l}} a_l,$$

$$H_B = \sum_{k>0}^{\Omega_B} \varepsilon_k^B (b_k^\dagger b_k + b_{\bar{k}}^\dagger b_{\bar{k}}) - g_B \sum_{k\neq l>0}^{\Omega_B} b_k^\dagger b_{\bar{k}}^\dagger b_{\bar{l}} b_l. \tag{2}$$

Here, $\{a_k, a_{\bar{k}}\}$ (respectively, $\{b_k, b_{\bar{k}}\}$) correspond to a set of Ω_A (respectively, Ω_B) pairs of states, where $\bar{k}$ denotes the time-reversed state of k. The term $V(t)$ describes the interaction between the two systems during the contact time. As stated in Ref. [33], two mechanisms may drive the transfer of particles between two colliding nuclei. The first one is the tunneling of single particles coming from the mean field and can be mimicked by operators $a_k^\dagger b_l$, whereas the second one is the direct transfer of pairs due to the residual pairing interaction

and is related to $a_k^\dagger a_{\bar{k}}^\dagger b_{\bar{l}} b_l$. In the present model, the first term is omitted and therefore it implicitly assumes that the transfer of pairs mainly occurs simultaneously. As we will see below, such a model can eventually model the pair transfer during symmetric heavy-ion collisions but not asymmetric reactions. In the present work, we mainly concentrate on symmetric collisions where large gauge-angle effects has been uncovered. Then, neglecting the single-particle tunneling, the transfer process is described using [32]

$$V(t) = v(t) \sum_{k>0}^{\Omega_A} \sum_{l>0}^{\Omega_B} (a_k^\dagger a_{\bar{k}}^\dagger b_{\bar{l}} b_l + b_l^\dagger b_{\bar{l}}^\dagger a_{\bar{k}} a_k). \quad (3)$$

The time dependency of the coupling is assumed to be Gaussian and centered at the collision time $t = 0$:

$$v(t) = v_0 \exp\left[-t^2/\tau_c^2\right], \quad (4)$$

where τ_c is the interaction time. Under simple assumptions, the two parameters of the coupling strength $v(t)$ may be related to a few characteristics of the input channel namely the charges Z_A, Z_B, the reduced mass of the colliding nuclei, the relative kinetic energy and the scattering angle in the center of mass frame [33].

In the following, for the sake of compactness, we denote generically by $(c_k^\dagger, c_{\bar{k}}^\dagger)$ a pair of states belonging to either the system A or B. Then, the total Hamiltonian is simply written as

$$H = \sum_{k>0} \varepsilon_k (c_k^\dagger c_k + c_{\bar{k}}^\dagger c_{\bar{k}}) - \sum_{k \neq l > 0} G_{kl}(t) c_k^\dagger c_{\bar{k}}^\dagger c_{\bar{l}} c_l. \quad (5)$$

With these notations, $\varepsilon_k = \varepsilon_k^A$ (respectively, ε_k^B) if $(k, \bar{k})$ belong to A (respectively, B). The matrix element G_{kl} equals, respectively, to g_A, g_B, and $v(t)$ if the couple of indices (k,l) refers to states that are both in A, both in B, or one in A and one in B.

A direct diagonalization technique applied in every subspace of given seniority gives the complete set of eigenstates of the pairing Hamiltonian [34]. This direct technique is possible as long as the single-particle Hilbert space is not too large. This advantageous feature of the pairing Hamiltonian has been widely leveraged to study the static properties of a variety of small superfluid systems [2,35,36] as well as to test approximate treatment of pairing [37]. The full Hamiltonian Eq. (1) was first proposed to study transfer reaction in Ref. [33] and further analyzed in Refs. [38,39]. It was also used by Broglia *et al.* to discuss the semiclassical nature of the mean-field approximation (see, for instance, Ref. [26]). The exact solution of this schematic Hamiltonian guided us to propose, in this paper, an approximate treatment of the pair transfer between two colliding nuclei. This approximate treatment aims at being applicable to heavy systems where the direct diagonalization of the pairing Hamiltonian becomes impractical. We first discuss below briefly the exact and mean-field solutions.

A. Exact solution

We assume at initial time that each subsystem is in its ground state. In these ground states, all the particles are paired. Since the complete Hamiltonian Eq. (1) does not break pairs during

the evolution, the total wave function of the composite system remains in a subspace where all nucleons are paired during the whole evolution. Calculations can, therefore, be performed in the basis of orthonormal states $\{|n\rangle\}$ defined as

$$|n\rangle = \prod_{k>0}^{\Omega} (c_k^\dagger c_{\bar{k}}^\dagger)^{n_k} |0\rangle, \quad n_k = 0,1, \quad (6)$$

where $\Omega = \Omega_A + \Omega_B$. Due to the invariance of the total particle number respected by the model Hamiltonian, this basis can even be reduced to states verifying the condition

$$\sum_{k>0}^{\Omega} n_k = \frac{1}{2}\left(N_A^0 + N_B^0\right), \quad (7)$$

where N_A^0 and N_B^0 denote the initial numbers of particle in systems A and B, respectively. The string of bits $n_\Omega n_{\Omega-1} \cdots n_1$ totally defines the state $|n\rangle$ and provides a direct mapping between this state and the integer n having $n_\Omega n_{\Omega-1} \cdots n_1$ for binary representation. The initial state of each subsystem is determined by diagonalizing its own Hamiltonian within the subspace spanned by $|n\rangle$ states with appropriate number of particles; i.e., $\sum_{k \in A} n_k = N_A^0/2$ and $\sum_{k \in B} n_k = N_B^0/2$. The time-evolution of the total wave-function $|\Psi\rangle$ is then obtained by solving the coupled-channel equations on the components $c_n(t)$ with

$$|\Psi(t)\rangle = \sum_n c_n(t)|n\rangle. \quad (8)$$

From the coefficients, any quantity related to transfer can be computed. For instance, the probability to have (N_A, N_B) particles at final time t_∞ is given by

$$P(N_A, N_B, t_\infty) = \sum_{n \in \mathcal{E}(N_A, N_B)} |c_n(t_\infty)|^2, \quad (9)$$

where the sum runs over the set of states $\mathcal{E}(N_A, N_B)$ defined by the condition

$$\sum_{k \in A} n_k = N_A/2, \quad \sum_{k \in B} n_k = N_B/2. \quad (10)$$

Since the total particle number N is conserved, the final probability given by Eq. (9) is zero if $N_B \neq N - N_A$. In the following, we introduce the exact pair transfer probability P_{xn}^{exa} that is equal to

$$P_{xn}^{\text{exa}} = P\left(N_A = N_A^0 + x, N_B = N_B^0 - x, t_\infty\right). \quad (11)$$

We follow the standard terminology and call addition (respectively, removal) probabilities, the probability for $x > 0$ (respectively, $x < 0$), implicitly assuming that the addition/removal is defined with respect to the system A.

Using this practical scheme, we compute the multiple pair transfer probabilities during the collision of two identical systems. This example is used as a reference calculation throughout this article. Each system consists of one degenerate shell with single-particle energies set to $\epsilon_k^A = \epsilon_k^B = 0$. The shell degeneracy is set to $\Omega_A = \Omega_B = 6$ and both systems are initialized in their half filling situation, i.e., $N_A^0/2 = N_B^0/2 = 3$ pairs of particles. This simple assumption can be regarded as a minimal description of the transfer of nucleon pairs
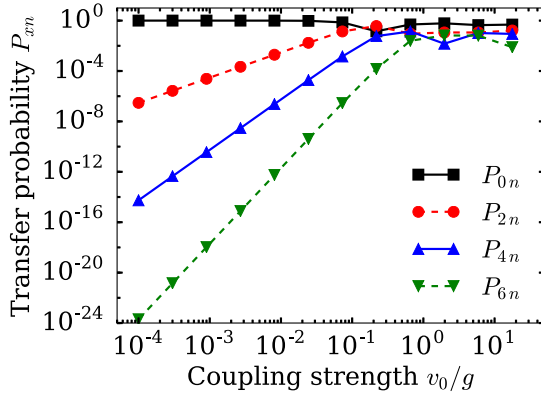
FIG. 1. Exact asymptotic probabilities of multiple pair transfer as a function of the coupling strength v_0/g obtained for the symmetric degenerate case with $\Omega_A = \Omega_B = 6$ and $N_A^0 = N_B^0 = 6$.

from one degenerate shell of nucleus A (respectively, B) to another degenerate shell of nucleus B (respectively, A). The pairing strength is assumed to be the same in the two systems $g = g_A = g_B = 1$ MeV. The characteristic contact time defining the coupling Hamiltonian is set to $\tau_c = 0.28 \, \hbar/g$ and the evolution takes place from $t_0 = -2.28$ to $t_\infty = +2.28$ in the same units. This time interval is wide enough to probe the asymptotic regime both before and after the collision. The initial product state is evolved in time using a Taylor expansion of the exponential propagator up to fourth order with a sufficiently small time step $dt = 10^{-4} \, \hbar/g$. After the collision, the transfer probabilities are recovered using Eq. (9).

Figure 1 shows the results obtained for a wide range of coupling strength v_0/g. Two regions of coupling strength can be identified. For $v_0/g > 2.10^{-2}$, the probabilities to transfer several pairs have the same order of magnitude. This region corresponds to a highly nonperturbative regime where strong quantum interferences between different transfer channels play an important role. On the other hand, for $v_0/g < 2.10^{-2}$, the interaction acts as a small perturbation. The probability to transfer one pair becomes prominent compared to multipair transfer. The perturbative nature of the transfer can be directly inferred from the simple scaling behavior of P_{xn} observed in Fig. 1. Indeed, for small values of the coupling, P_{xn} is proportional to $(v_0)^x$, which is consistent with the scaling deduced from the first nonzero term appearing in time-dependent perturbation theory (see the Appendix). It is finally worth mentioning that the observed probability dependence with v_0/g looks very much the same as the observed evolution of transfer probabilities below the Coulomb barrier when plotted as a function of the minimal distance of approach [9–12]. This underlines that the perturbative regime is certainly the most relevant for these experiments.

B. TDHFB solution

We now consider a mean-field description of transfer. The natural approach to circumvent the combinatorial growth of the exact Hilbert space is to restrain the system to a TDHFB ansatz. In the case of our model Hamiltonian, the HFB trial function reduces to a BCS one, which takes the form

$$|\Psi(t)\rangle = \prod_{k>0}(U_k^*(t) + V_k^*(t)c_k^\dagger c_{\bar{k}}^\dagger)|0\rangle. \quad (12)$$

The single-particle occupation numbers n_k and the anomalous density components κ_k, defined as

$$n_k(t) = \langle c_k^\dagger c_k \rangle = \langle c_{\bar{k}}^\dagger c_{\bar{k}} \rangle = |V_k(t)|^2,$$
$$\kappa_k(t) = \langle c_{\bar{k}} c_k \rangle = U_k(t)V_k^*(t),$$

contain all the information on the system.

The mean-field trajectory fulfills the Ehrenfest theorem $i\hbar\partial_t\langle \hat{O}\rangle = \langle[\hat{O},\tilde{H}]\rangle$ for any operator $\hat{O}$ that are linear combinations of $\{c_k^\dagger c_k, c_{\bar{k}} c_k, c_k^\dagger c_{\bar{k}}^\dagger\}$ operators (here $\tilde{H}$ just means that it might contain or not the constraint on particle number $-\lambda\hat{N}$). This leads to the set of equations of motion:

$$i\hbar\frac{dn_k}{dt} = \Delta_k^*\kappa_k - \kappa_k^*\Delta_k,$$
$$i\hbar\frac{d\kappa_k}{dt} = 2\tilde{\varepsilon}_k\kappa_k + \Delta_k(2n_k - 1). \quad (13)$$

The pairing gap is defined as $\Delta_k = \sum_l G_{kl}\kappa_l$ and the single-particle energies write as $\tilde{\varepsilon}_k = \varepsilon_k - \lambda_k$. While Eqs. (13) can be solved directly, it is common to use instead the equation of motion on the quasiparticle components (U_k, V_k):

$$i\hbar\frac{d}{dt}\begin{pmatrix}U_k(t)\\V_k(t)\end{pmatrix} = \begin{pmatrix}\tilde{\varepsilon}_k - \gamma_k & \Delta_k \\ \Delta_k^* & -\tilde{\varepsilon}_k - \gamma_k\end{pmatrix}\begin{pmatrix}U_k(t)\\V_k(t)\end{pmatrix}. \quad (14)$$

As previously noted [15,40], the above set of equations are not unique and can be solved using an arbitrary factor γ_k. This arbitrary factor brings a change in the evolution of the global phase of the quasiparticle vacuum $|\Psi(t)\rangle$, while conserving the equation of motion Eq. (13). At the mean-field level, any expectation value will be independent of this factor and therefore γ_k may be chosen arbitrarily [15,40]. If one tries to go beyond mean-field (e.g., using theories requiring calculation of overlaps between different quasiparticle vaccua), the results strongly depend on the choice of this global phase. The fact that no specific choice has yet been established on first principle argument renders the treatment of the interaction between two superfluids rather tricky [31].

Within the TDHFB approach, the initial state is a product of two quasiparticle ground states associated with A and B. The breaking of $U(1)$ symmetry for these two states leaves us with an arbitrary relative initial gauge angle θ_{AB}^0 between the two subsystems:

$$\kappa_A(0)\kappa_B^*(0) = |\kappa_A(0)||\kappa_B(0)|e^{i\theta_{AB}^0}. \quad (15)$$

As demonstrated in several works [31,41], the final result of the collision treated within TDHFB depends on the initial relative gauge-angle.

To better grasp this effect, we consider as in Sec. II A the case of two symmetric fully degenerated systems. In this simple model, the gauge angle and relative number of particles play the role of classical conjugated variables that obey simple coupled equations of motion. Due to the degeneracy in each system, only four parameters describe the TDHFB evolution: n_A, n_B,

κ_A, and κ_B. Then, Eq. (13) reduces to

$$i\hbar\frac{dn_A}{dt} = \Omega_B v(t)(\kappa_B^* \kappa_A - \kappa_A^* \kappa_B) = \frac{\Omega_B}{\Omega_A}\frac{dn_B}{dt}, \quad (16)$$

$$i\hbar\frac{d\kappa_A}{dt} = 2\tilde{\varepsilon}_A \kappa_A + (\Delta_A + \Delta_{AB}(t))(2n_A - 1), \quad (17)$$

$$i\hbar\frac{d\kappa_B}{dt} = 2\tilde{\varepsilon}_B \kappa_B + (\Delta_B + \Delta_{BA}(t))(2n_B - 1), \quad (18)$$

with

$$\Delta_A = g_A \Omega_A \kappa_A, \ \ \Delta_B = g_B \Omega_B \kappa_B,$$
$$\Delta_{AB} = v(t)\Omega_B \kappa_B, \ \ \Delta_{BA} = v(t)\Omega_A \kappa_A. \quad (19)$$

At any time t, we may define the time-dependent relative angle $\theta_{AB}(t)$ by $\kappa_A \kappa_B^* = |\kappa_A||\kappa_B|e^{i\theta_{AB}}$. Finally, introducing the number of particles $N_A(t) = 2\Omega_A n_A(t)$ in the subsystem A (respectively, $N_B(t) = 2\Omega_B n_B(t)$ in B) yields the following evolutions for the average particle numbers:

$$\frac{dN_A}{dt} = 4v(t)\frac{\Omega_A \Omega_B}{\hbar}|\kappa_A(t)||\kappa_B(t)|\sin[\theta_{AB}(t)]$$
$$= -\frac{dN_B}{dt}. \quad (20)$$

This evolution is rather complicated as it depends explicitly on the anomalous density of each system. However, in the weak coupling regime, one might neglect the coupling term in Eqs. (17) and (18). Then, the evolution of the anomalous densities in the two subsystems become independent from each other and can be integrated in time:

$$\kappa_{A/B}(t) \simeq \kappa_{A/B}^0 \exp(-2i\omega_{A/B}t), \quad (21)$$

with $\hbar\omega_{A/B} = \epsilon_{A/B} - \lambda + g_{A/B}\Omega_{A/B}(n_{A/B} - 1/2)$. The relative gauge orientation then rotates with a nearly constant frequency $\omega_{AB} = \omega_A - \omega_B$:

$$\theta_{AB}(t) \simeq -2\omega_{AB}t + \theta_{AB}^0. \quad (22)$$

Reporting in the anomalous densities evolutions and integrating in time, we finally obtain an expression of $N_A(t)$:

$$N_A(t) = N_A^0 + S_v(t)\sin\theta_{AB}^0 - A_v(t)\cos\theta_{AB}^0, \quad (23)$$

with

$$S_v(t) = \int_{t_0}^t W(s)\cos(2\omega_{AB}s)ds,$$

$$A_v(t) = \int_{t_0}^t W(s)\sin(2\omega_{AB}s)ds,$$

and where we have introduced the notation

$$W(t) = \frac{4}{\hbar}v(t)\Omega_A \Omega_B |\kappa_A^0||\kappa_B^0|. \quad (24)$$

In the asymptotic regime after the collision, we may choose $t_\infty = -t_0$ so that only the symmetric term will contribute to pair transfer. In the symmetric case, $\omega_{AB} \simeq 0$ and S_v can be integrated explicitly as

$$S_v(t) = 2\left[\frac{v_0\tau_c}{\hbar}\right]v_0\Omega_A\Omega_B|\kappa_A^0||\kappa_B^0|\sqrt{\pi}$$
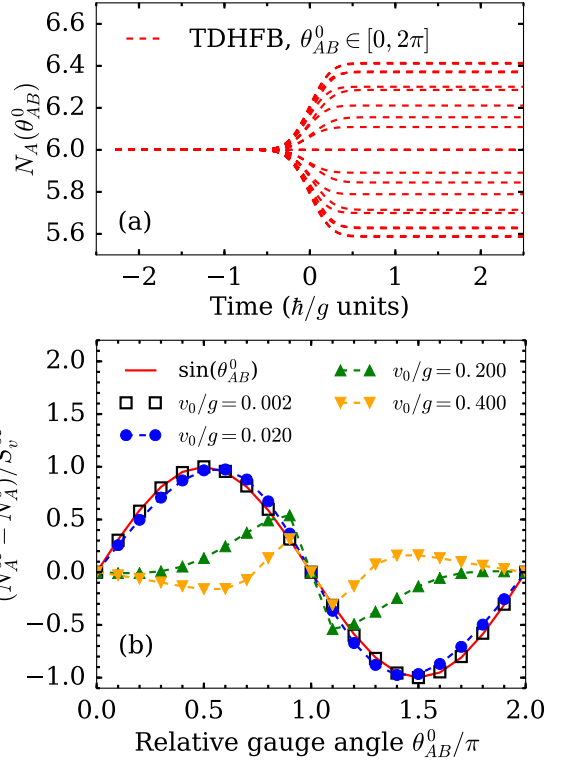$$\times[\text{erf}(t/\tau_c) - \text{erf}(t_0/\tau_c)], \quad (25)$$

FIG. 2. (a) TDHFB evolutions of the particle number in the system A starting from the relative gauge angles $\theta_{AB}^0 = 2k\pi/20$, $0 \leqslant k < 20$ (same as Fig. 1 of Ref. [31] given here for completeness), and for a coupling strength $v_0/g = 0.02$. (b) Asymptotic number of particles transferred to the system A as a function of the initial relative gauge angle for four different coupling strengths. The number of transferred particles is normalized by S_v^∞ and the weak coupling analytical formula, Eq. (26) is shown in red solid line. All TDHFB calculations are performed with a rescaling factor $\alpha = 1.2$ on the interaction (cf. Sec. III B 3).

with the standard definition $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}ds$. As a conclusion, the average number of particle transferred in the weak coupling conditions is found to depend on the sinus of the relative gauge angle with an amplitude proportional to the a-dimensional parameter $(v_0\tau_c/\hbar)$:

$$N_A^\infty \simeq N_A^0 + S_v(t_\infty)\sin\theta_{AB}^0. \quad (26)$$

In Fig. 2, the time evolution of N_A obtained with mean field is illustrated using different initial relative phases for the case considered in Sec. II A ($\Omega_A = \Omega_B = 6$ and starting from the half filling configuration). In the top panel, we see that starting from different θ_{AB}^0 leads to different mean-field trajectories as soon as the two systems start to interact. Similar behavior has been obtained in Ref. [31]. The bottom part of Fig. 2 shows the dependency of the asymptotic number of particles transferred as a function of the initial relative gauge angle. As we decrease the coupling strength, the asymptotic particle number tends toward the sinusoidal dependency given by Eq. (26). Already for $v_0/g = 2.10^{-2}$, the agreement with the analytical formula is very good which is consistent with the beginning of the perturbative regime shown for the exact case in Fig. 1.

III. DESCRIPTION OF PAIR TRANSFER PROBABILITIES BEYOND MEAN FIELD

Mean-field theories that treat superfluidity have inherent limitations. The first one is obviously the breaking of the $U(1)$ symmetry associated to particle number, which makes difficult the extraction of transfer probabilities. Indeed, finite-size effects and quantum fluctuations associated to the eventual symmetry restoration are anticipated to play a significant role, especially if the number of particles involved is small. A second limitation is the quasiclassical nature of the mean field in collective space.

In this section, we first give a nonexhaustive overview of some approaches that can include beyond mean-field effects as well as their possible shortcomings. We then present a new approach named phase-space combinatorial technique to compute pair transfer probabilities based on multiple TDHFB trajectories.

A. Quantum methods to go beyond mean field

A natural extension to include quantum effects beyond mean field would be to use a path integral approach. In the simple degenerated model described here, it is indeed possible to include interferences between trajectories leading to the same final number of particles. From Fig. 2, one indeed realizes that there are always at least two trajectories (weak coupling) or more (strong coupling) leading to the same N_A^∞. Using the stationary phase approximation (SPA), we may estimate the transfer probability as a weighted average over different paths with the proper phase factor. This approach has already been applied with some success to transfer reaction, starting from the classical action $S_{cl}(\theta_{AB}, N_A)$ and making additional efforts to access probabilities in the classically forbidden region [26]. We could have followed the same technique and most probably got reasonable probabilities. However, we anticipate that such a method may not be applicable for realistic nuclear collisions where the collective coordinates associated to the particle number and relative gauge angle are coupled to other degrees of freedom (e.g., the relative distance between nuclei, the deformation,...). A clear fingerprint of such a coupling is the large dependence of the fusion barrier with the gauge-angle empirically observed in Ref. [22]. Treating this feature explicitly when performing the stationary phase approximation leads to an increase of the problem complexity that seems prohibitive.

A second strategy consists in performing a proper quantum mixing of the TDHFB trajectories during the evolution. In such a framework, the symmetry is restored with an appropriate variation after projection (VAP) onto good particle number [27] both for the initialization of the two partners of the reaction but also during the whole evolution. The VAP technique itself is already the state of the art theory in nuclear structure and requires a large amount of computational resources. Besides, a proper formalism for its time-dependent equivalent (time-dependent VAP), where the many-body wave function would be written as a set of evolving quasiparticle many-body vacua is still missing. A first attempt has been made in Refs. [31,32], where the mixing is made approximately by assuming that each

quasiparticle vacua evolves independently from each other. This attempt was the original motivation of the present work. We made extensive tests of this technique and realized that changing the phase convention during the TDHFB evolution significantly affects the results and therefore jeopardizes the reliability of the prediction.

B. Semiclassical phase-space average over initial orientations

A simpler approach toward the symmetry restoration relies on semiclassical averages over the initial gauge angle configurations. The attractive feature of this method is that it keeps the computational costs to the level of computing several mean-field trajectories.

1. Phase-space estimation of the moments of an observable

The method starts with the statement that no orientation in gauge space should be *a priori* privileged in the initial state. The situation is similar to the case of deformed nuclei where semiclassical methods with random orientations of deformation axis between nuclei have been considered followed by a set of classical evolutions to describe barrier fluctuations (see, for instance, Refs. [42,43]).

Since the initial relative gauge angle θ_{AB}^0 is arbitrarily chosen before the two systems interact, one should at least perform a phase-space average of all the possible orientations between 0 and 2π. This is equivalent to assuming a uniform initial probability for the relative gauge angle distribution:

$$P(\theta_{AB}^0) = \frac{1}{2\pi}. \tag{27}$$

A simple observable $\hat{O}$[1] is considered as a classical variable whose evolution is given by its expectation value along the mean-field path. These evolutions are denoted by $\mathcal{O}[\theta_{AB}^0, t]$, since they explicitly depend on the initial relative orientation. In this picture, the quantum fluctuations in the gauge space are mimicked by the fact that $\mathcal{O}[\theta_{AB}^0, t]$ becomes a random variable. The moments of order k of the observable $\hat{O}$ after the collision is estimated through the semiclassical average:

$$\overline{O^k} \equiv \int_0^{2\pi} \mathcal{O}^k[\theta_{AB}^0, t_\infty] P(\theta_{AB}^0) d\theta_{AB}^0, \tag{28}$$

while its associated centered moment μ_k^{sc} reads

$$\mu_k^{sc} = \int_0^{2\pi} \left(\mathcal{O}[\theta_{AB}^0, t_\infty] - \overline{O} \right)^k P(\theta_{AB}^0) d\theta_{AB}^0 \tag{29}$$

This *brute-force* semiclassical treatment, that was already discussed extensively in Ref. [26], is very similar to the stochastic mean-field (SMF) approach [44,45] especially to its superfluid version [46]. It should however be noted that here the initial phase-space is taken to restore in a classical picture the broken symmetry while in Ref. [46] the initial phase-space was chosen to simulate quantum fluctuations of a

---

[1]Here simple observable means that its expectation value can be written as a linear combination of the one-body density matrix and anomalous density matrix elements.
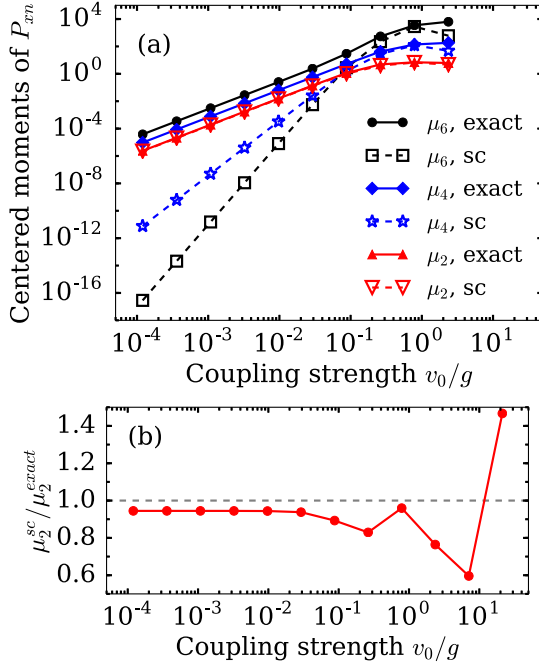
FIG. 3. (a) Centered moments of the probability distribution P_{xn} at final time as a function of the coupling strength v_0/g. The semiclassical results (dashed lines) are compared to the exact results (solid lines). (b) Second centered moment obtained with the semiclassical average divided by its exact counterpart. The results are obtained for the symmetric degenerate case with $\Omega_A = \Omega_B = 6$ and $N_A^0 = N_B^0 = 6$. All TDHFB calculations are performed with a rescaling factor $\alpha = 1.2$ on the interaction.

quasiparticle vacuum or more generally of correlated systems. Besides being simpler technically, such a direct phase-space average has some advantages compared to alternative formulations where the quantum expectation are kept together with the gauge-angle average as in Refs. [47,48]. Indeed, assuming an isolated Fermi superfluid with a fixed number of particles, all centered moments μ_k^{sc} of the total particle number distribution will be 0 in the semiclassical picture while spurious fluctuations will persist if a quantum average is performed.

2. Transfer probabilities from the phase-space approach

In this section, we apply the phase-space averaging to compute the moments of the observable $\hat{X} = \hat{N}_A - N_A^0$ (number of pairs transferred to the system A). For this model case, we can compare the semiclassical results with the exact quantum distribution for the same observable. Indeed, once the exact solution has been evolved in time according to Sec. II A, the exact centered moments for $\hat{X}$ can be evaluated with

$$\mu_k^{exa} = \sum_x (x - \langle \hat{X}(t_\infty) \rangle)^k \, P_{xn}^{exa}, \qquad (30)$$

where Eq. (11) provides the probabilities P_{xn}^{exa} and sum runs over the even integers x.

Figure 3 compares the evolution of the second, fourth, and sixth centered moments of $\hat{X}$ for a wide range of coupling strength. Both moments obtained from the exact case and the semiclassical phase-space approach are represented. In the strong coupling regime, significant differences between exact and semiclassical estimations are present for all three moments. In the low coupling regime, the situation is quite different. An impressive result is that the approximate and exact second moments coincide over a wide range of coupling strength. We see in particular (bottom part of Fig. 3) that the ratio of the two moments is nearly constant and close to one up to $v_0/g \simeq 3.10^{-2}$ in the perturbative regime. In this weak coupling regime, the different centered moments of the number of nucleons in the subsystem A can be expressed analytically by performing the averages of $N_A^k[\theta_{AB}^0, t_\infty]$ over θ_{AB}^0, where $N_A[\theta_{AB}^0, t]$ is given by Eq. (26). For the second moments, it gives, for instance,

$$\mu_2^{sc} = \overline{N_A^2} - \overline{N_A}^2 = \tfrac{1}{2} S_v^2(t_\infty). \qquad (31)$$

The fact that exact and semiclassical second-order moments become proportional in the low coupling regime can in this case be shown analytically as

$$\mu_2^{sc} = \frac{1}{2} S_v^2(t_\infty) \propto \left(\frac{v_0 \tau_c}{\hbar} \right)^2, \quad \mu_2^{ex} \simeq 8 P_{2n} \propto \left(\frac{v_0 \tau_c}{\hbar} \right)^2. \tag{32}$$

The exact estimation of P_{2n} comes from keeping only the leading-order term in the time-dependent perturbative approach developed in the Appendix.

On the contrary, higher moments obtained with the semiclassical average completely fail to reproduce their exact counterparts for all coupling strengths. In particular, the fourth and sixth semiclassical moments fall down much faster than the exact case when v_0 decreases. Using the analytical formula Eq. (25), we can, for example, investigate the asymptotic behavior of the fourth semiclassical moment:

$$\mu_4^{sc} = \frac{3}{8} S_v^4(t_\infty) \propto \left(\frac{v_0 \tau_c}{\hbar} \right)^4. \tag{33}$$

Due to the hierarchy of the probabilities $P_{2kn} \gg P_{2(k+1)n}$ in the perturbative region, the exact solution will necessarily result in a different behavior $\mu_4^{ex} \simeq 4\mu_2^{ex} \propto (\frac{v_0 \tau_c}{\hbar})^2$. This relation between the order two and fourth exact moments clearly appears in Fig. 3 and similar arguments can be used to explain the mismatch between exact and semiclassical moments for the higher orders.

As illustrated from the mismatch between moments of order higher than two, the distribution of the random variable $X[\theta_{AB}^0, t_\infty]$ is a poor approximation of the exact quantum distribution of the observable $\hat{X}$ at t_∞. There are several reasons for the failure of a direct semiclassical phase-space average. First, semiclassical variable $X[\theta_{AB}^0, t_\infty]$ takes continuous real values, whereas only even integers are possible in the exact treatment. In addition, the domain of variation of $X[\theta_{AB}^0, t_\infty]$ may in practice be very different from the range of possible measurements of $\hat{X}$. To illustrate this, we emphasize that in the weak coupling regime one can eventually obtain an analytical expression of the probability P_{xn}^{sc} probability using [26]

$$P_{xn}^{sc} = \frac{1}{\pi} \left(\frac{dX[\theta_{AB}^0, t_\infty]}{d\theta_{AB}^0} \right)^{-1}. \tag{34}$$

This expression[2] accounts for the fact that two initial relative orientations lead to the same final number of particles (see Fig. 2). Inserting Eq. (26), we find that the number of particles is bounded,

$$N_A^0 - S_v(t_\infty) \leqslant N_A^\infty \leqslant N_A^0 + S_v(t_\infty), \qquad (35)$$

which corresponds to the "classically allowed" region. In the exact treatment, particle numbers that are outside the classical region are always populated.

We conclude from this analysis that, even in our simple model case, the direct phase-space approach that consists in averaging over different TDHFB does not provide a precise description of the transfer probability distribution. However, this semiclassical average procedure seems to recover correctly the second moments of the pair transfer distribution by somehow including effects beyond the independent quasiparticle picture.

3. Precision on the comparison between exact and phase-space treatment

When comparing an approximate treatment of the many-body problem with the exact one, we should *a priori* use the same Hamiltonian. In Ref. [31], it has been argued that the interactions strength used in the Hamiltonian for the mean-field-based calculations should be rescaled compared to the exact case, in such a way that the initial total energy is the same. More precisely, for the symmetric case, it was assumed in Ref. [31] that $g'_A = \alpha g_A$, $g'_B = \alpha g_B$, and $v'_0 = \alpha v_0$ in the energy where α is a factor that depends on the specific case under study. For $\Omega_A = \Omega_B = 6$ in the half filling situation we get $\alpha = 1.2$ and the corresponding scaling was applied in the phase-space calculations presented here. This adjustment might appear surprising but it is quite close to what is done nowadays in nuclear structure where the strength of the pairing interaction is adjusted to reproduce either the pairing gap for midshell nuclei or the global trend in the two nucleons separation energies. It should, however, be kept in mind that this adjustment is based on rather empirical arguments. The comparison between the exact solution and approximate methods using this rescaling should be taken with caution and conclusions should only be qualitative.

C. The phase-space combinatorial technique (PSC)

Although the semiclassical technique presented in Sec. III B is not able to describe the complete richness of the transfer probabilities, it provides a very good estimation of the second moment for a wide range of interaction strength v_0. This nontrivial feature of the semiclassical approach for transfer was already emphasized in Ref. [31] and is confirmed on a more systematic basis in this work. Based on this empirical assessment, we propose a method to compute the pair transfer

---

probabilities from the first semiclassical moments in the perturbative regime. The method relies on two major assumptions:

(a) The first and second moments obtained from a semiclassical distribution of many TDHFB evolutions with different relative gauge-angles are realistic. This hypothesis can only be validated *a posteriori* by comparing the result of the approach developed here with experimental observations.

(b) The transfer of interest takes place in the weak coupling regime. In this regime, we do expect a hierarchy in the transfer probabilities.

$$P_{0n} \gg (P_{2n}, P_{-2n}) \gg (P_{4n}, P_{-4n}) \gg \cdots \qquad (36)$$

Such hierarchy is typically observed in reactions below the Coulomb barrier [9–12], which establishes an interesting range of applications.

1. One pair transfer

The one pair transfer is fully determined by the two probabilities P_{2n} and P_{-2n}. For the symmetric case, we have in addition $P_{xn} \simeq P_{-xn}$. We can then use the hypothesis (b) to obtain an approximate expression between the variance of the distribution P_{xn} and the two-particles addition/removal probabilities:

$$\mu_2(t) \simeq 8P_{2n}(t) = 8P_{-2n}(t). \qquad (37)$$

We have checked that this is indeed realized up to 0.4% in the exact calculations as long as we stay in the perturbative regime $v_0/g < 2.10^{-2}$. Consistently with the assumption made above, P_{0n} is automatically obtained from the approximate relation:

$$P_{0n} \simeq 1 - P_{2n} - P_{-2n}.$$

The relation Eq. (37) provides a straightforward way to access the two-particles addition/removal probabilities and avoids the complexity of multiple projections at different times [31]. In addition, it only requires the computation of independent mean-field trajectories. In the weak coupling regime between two symmetric degenerated systems, one can eventually use Eq. (31) to get the analytical form $P_{2n}^{sc} = S_v^2(t)/16$ where $S_v^2(t)$ is given by Eq. (25).

In Fig. 4, the numerical and analytical semiclassical estimates of the two-particle pair transfer probabilities are compared to the exact ones as a function of time for different coupling strengths. At very small coupling, all the probabilities are in close agreement with each other. The simple strategy proposed here reproduces to a good extent the behavior of the exact result in the weak coupling regime. In particular, the time evolution of $P_{2n}^{sc}(t)$ is smooth and does not suffer from the spurious oscillations of asymptotic probabilities observed in Refs. [31,32]. As could be anticipated, the numerical estimate has a wider range of applicability than the analytical one obtained from $S_v^2(t)/16$. Finally, and without surprise, more and more deviation is observed as the coupling strength enters the nonperturbative regime.

2. From one to multiple pair transfer

The success of the above method is an incentive to generalize it to multiple pair transfer. A naive attempt in this

---

[2]Note that this expression is only defined when $\frac{dX}{d\theta_{AB}^0} \neq 0$. According to equation this is not the case for a few discrete values of θ_0^{AB} (e.g., $\theta_0^{AB} = \pi/2$). This is actually not a problem as continuous distribution of probabilities may have non defined values on a support of measure null.
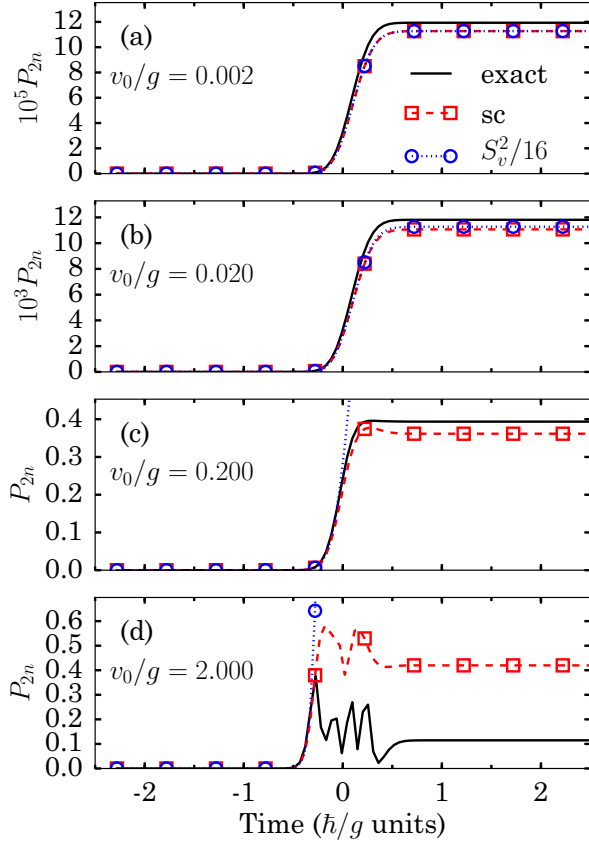
FIG. 4. Two-particle transfer probability as a function of time and for different coupling strength v_0/g. The semiclassical estimation (red dashed line) and its approximate analytical expression in the weak coupling regime (blue dotted line) are plotted along with the exact solution (black solid line). The calculations are performed in the symmetric degenerate case with $\Omega_A = \Omega_B = 6$ and $N_A^0 = N_B^0 = 6$. The rescaling factor on the interaction for TDHFB calculations is $\alpha = 1.2$.

direction would be (i) to compute the higher-order semi-classical moments μ_k^{sc} and (ii) to invert the set of equations $\mu_k^{sc} = \sum_n (2n)^k (P_{2kn} + P_{-2kn})$ to retrieve the probabilities. This technique would actually work if the high-order moments μ_k^{sc} would match their exact counterparts. However, the Fig. 3 clearly shows that the phase-space approach fails to predict the centered moments of order higher than 2.

To circumvent this difficulty, we propose to model the shape of the probability distribution P_{xn} with an analytical formula involving a sufficiently small number of free parameters. Once the generic shape of the distribution is decided, we determine the parameters so to reproduce the first and second moments predicted by the semiclassical average. This gives us eventually the possibility to extrapolate the distribution P_{xn} to multiple pair transfer.

To propose a shape for the P_{xn} distribution, we start from the following simplifying assumption.

(c) In the perturbative regime, the transfer of several pairs from one superfluid system to another can be essentially treated as a sequence of uncorrelated pair transfers.

This hypothesis is guided by the exact resolution that is discussed in the Appendix. In the exact case, the transfer probabilities result from a rather complicated process involving interferences between different channels (see Fig. 12 of the Appendix). In the perturbative regime, the interference between channels can be neglected and we simply end-up with a sequence of transfer that could be depicted for the addition or removal process from A to B, respectively, as

$$N_A \xrightarrow{\mathcal{P}_{+1}} N_A + 2 \xrightarrow{\mathcal{P}_{+2}} N_A + 4 \cdots \qquad (38)$$

and

$$N_A \xrightarrow{\mathcal{P}_{-1}} N_A - 2 \xrightarrow{\mathcal{P}_{-2}} N_A - 4 \cdots \qquad (39)$$

The removal and addition probabilities of k pairs can then be written as a product:

$$P_{2kn} = \mathcal{P}_{+1} \cdots \mathcal{P}_{+k}, \quad P_{-2kn} = \mathcal{P}_{-1} \cdots \mathcal{P}_{-k}. \qquad (40)$$

A simplified expression of the $\mathcal{P}_{+k}$ is illustrated by Eq. (A12). These transfer probabilities are in general rather complex since they contain the information on the internal structure of the system before and after the transfer, as well as the dynamical effects of the time-dependent interaction. However, as shown in the Appendix, in some limiting situation, the product of probabilities can be rewritten as

$$P_{2kn} = W_k p^k, \quad P_{-2kn} = W_{-k} q^k, \qquad (41)$$

where $W_{k/-k}$ are combinatorial factors while p and q can be interpreted as the elementary probability for a pair to be added or removed during the reactions. In the exact case, this probability is governed by the coupling v_0 as well as the contact time, i.e., by the time-dependent interaction between the two systems. The factors W_k and W_{-k} contain here the information on the available number of particles to transfer as well as the number of possible states reachable when accounting for the Pauli principle. It may also contain in an approximate way some information on the intrinsic structure of the initial and final states. In the Appendix, we obtained the expression of $W_{k/-k}$ for two specific cases:

(i) We first consider the case where the transition frequencies at each step of the process Eq. (38) are constant. Then we have

$$W_k = C_{\Omega_A - n_A}^k C_{n_B}^k \frac{(n_A + k)!}{n_A!} \frac{(\Omega_B - n_B + k)!}{(\Omega_B - n_B)!}. \qquad (42)$$

This situation is the one anticipated for the case of degenerate system *in the absence of pairing*, i.e., when both systems are in their normal phase.

(ii) We then consider the case where the transition frequencies at the step k is proportional to k, leading to

$$W_k = C_{\Omega_A - n_A}^k C_{n_B}^k C_{n_A + k}^k C_{\Omega_B - n_B + k}^k. \qquad (43)$$

This second case, is relevant for two degenerate systems *in the presence of superfluidity* starting from half-filling and is then expected to be more realistic for the present study.

In both cases, W_{-k} is deduced from W_k simply by making the replacement $\Omega_A \leftrightarrow \Omega_B$ and $n_A \leftrightarrow n_B$. Parts of the

combinatorial factors appearing in Eqs. (42) and (43) have a simple statistical interpretation. Indeed, $C_{n_B}^k$ counts the number of configurations of k pairs initially in system B that could be transferred to A, while $C_{\Omega_A - n_A}^k$ counts the number of possibilities to put k pairs in $\Omega_A - n_A$ empty spots. Note, finally, that this second estimate of the pair transfer probabilities will lead to lower probabilities, since Eq. (43) can be obtained by dividing Eq. (42) by $(k!)^2$. The approach that uses combinatorial arguments is called hereafter PSC (phase-space combinatorial).

In the following, we will systematically show the result of both prescriptions. Since both are deduced from simple approximate treatment of the internal structure of the systems, the variation of transfer probabilities from one prescription to another gives us an idea on the uncertainty related to the proper treatment of the structure of the two systems.

Our starting point to obtain multiple pair transfer probabilities are the assumed Eq. (41), where we see that the important quantities for pair transfer addition and removal are the elementary probabilities p and q. These parameters can directly be inferred using our previous technique to estimate two-particle transfer probabilities from phase-space average (Sec. III C 1). Focusing first on the symmetric case where $p = q$, we obtain simply

$$p = q = \frac{\mu_2}{8W_1}. \tag{44}$$

From the knowledge of p and q, we can directly calculate the different probabilities when more than one pair is transferred using Eqs. (41). It is worth mentioning that for the symmetric case, we have the recurrence relation,

$$P_{2(k+1)n} = p \frac{W_{k+1}}{W_k} P_{2kn}, \tag{45}$$

as well as a direct connection between the probability to transfer k pairs with the probability to transfer one pair:

$$P_{2kn} = \frac{W_k}{W_1^k}[P_{2n}]^k, \quad P_{-2kn} = \frac{W_{-k}}{W_{-1}^k}[P_{-2n}]^k. \tag{46}$$

The method proposed here provides a straightforward way to obtain the transfer probabilities from a semiclassical average using solely the second moment of the simulated distribution. Let us now study to what extent the forms given by Eqs. (41) are valid for the considered model. To do so, we compute the ratios $p_k = [W_k P_{2(k+1)n}^{ex}]/[W_{(k+1)} P_{2kn}^{ex}]$ obtained from the exact probabilities. The shape of the probability distribution is realistic when these ratios are almost independent of k for a given value of v_0/g [cf. Eq. (45)]. The Fig. 5 highlights these ratios for different k and different initial size of the systems A and B. We used the values of W_k given by Eq. (43). All ratios match to a good extend with each others as long as not too many pairs are transferred. Note that, if the prescription Eq. (42) is used, more deviations between ratios are observed as anticipated.

To further illustrate the accuracy of the PSC approach, we show in Fig. 6 the approximate pair transfer probabilities deduced when using the exact second moment μ_2^{ex} in the combinatorial method presented here and compare it to the exact probabilities. We see that the pair transfer are relatively
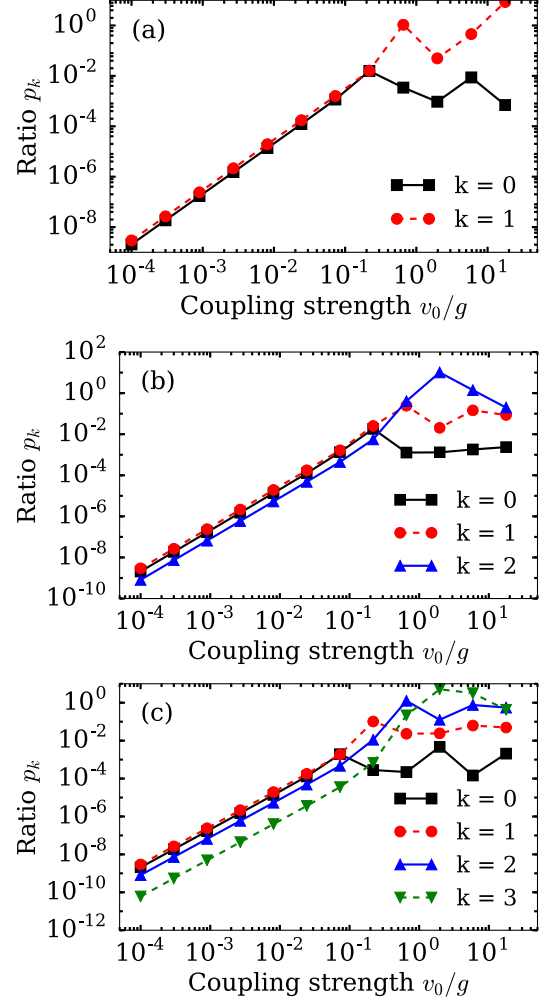

FIG. 5. Ratios $p_k = [W_k P_{2(k+1)n}^{ex}]/[W_{(k+1)} P_{2kn}^{ex}]$ as a function of the coupling strength. We used the W_k coming from Eq. (43). The results are displayed for the symmetric degenerate case with (a) $\Omega_A = \Omega_B = 4$, (b) $\Omega_A = \Omega_B = 6$, and (c) $\Omega_A = \Omega_B = 8$, starting from the half filling configuration for both systems.

well reproduced with more and more deviations as the number of transferred pairs or as the coupling strength increases. Nevertheless, in general, the probability to transfer two pairs and sometimes three pairs are reasonably close to the exact probabilities. Figure 8 presents the systematic calculation of the multiple pair transfer as a function of the coupling strength v_0/g in the perturbative regime. We see that the behavior of the distribution is monotone in this region and the semiclassical result gives a nice estimate of the exact probabilities in this case.

We finally couple the combinatorial approach with the semiclassical estimate of the second moment. Provided that the phase-space averaging is accurate to describe this moment [hypothesis (a)], we compute in a straightforward way the different pair transfer probabilities starting from μ_2^{sc}. Figures 7 and 8 compare again the approximate transfer probabilities using μ_2^{sc} with the exact ones for the symmetric case. The overall evolution of the probability with increasing number of pairs transferred is rather well reproduced as well as its v_0
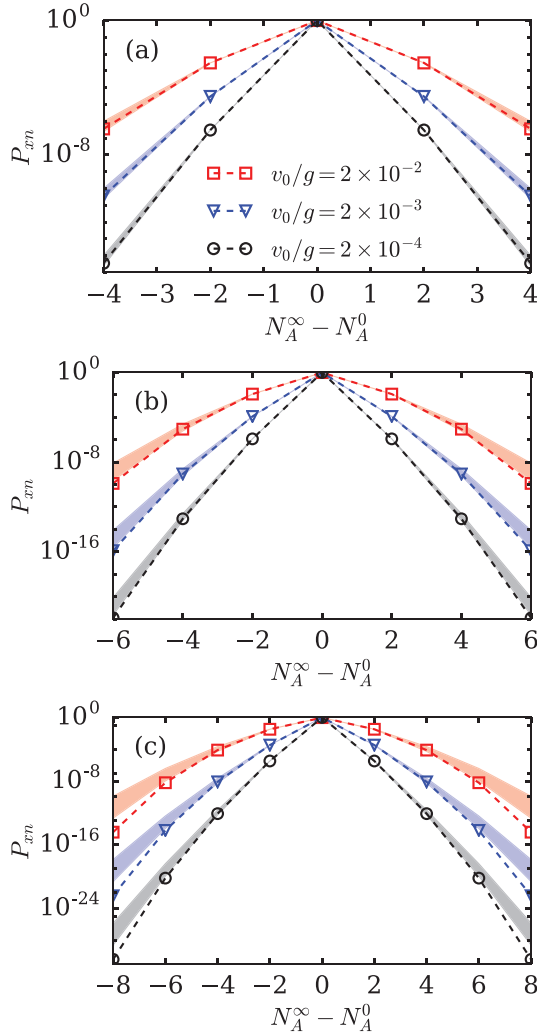
FIG. 6. Asymptotic probabilities to transfer one or several pairs as a function of the quantity $N_A^\infty - N_A^0$ obtained from a symmetric degenerate situation with $\Omega_A = \Omega_B = N_A^0 = N_B^0 = 4$ (a), 6 (b), and 8 (c). The colored bands span the area obtained with the distribution Eq. (41) with the two cases for the W_k factors. The distribution are obtained by using the exact second moment μ_2^{ex} to determine $p = q$. Approximate probabilities are systematically compared with the exact values (dashed line) for three coupling strength v_0/g in the perturbative regime.

FIG. 7. Same as described in the caption of Fig. 6, except that p is determined with the second moment μ_2^{sc} obtained by performing the semiclassical phase-space average.

In summary, we introduced here the new PSC method that works in two steps. First the second moments μ_2^{sc} is estimated from phase-space averaging. This requires computing a set of independent TDHFB trajectories. Then the complete pair transfer distribution is recovered from the combinatorial factors W_k. The main benefit is that we obtain beyond mean-field fluctuations at the cost of several mean-field calculations only. The growth of the computation time with the number of particles is therefore no more than the one associated to the TDHFB calculations themselves.

dependency. As expected, the exact solution is closer to the case of linear transition frequencies given by Eq. (43). Note that in all cases we have $\mu_2^{sc} \sim \mu_2^{ex}$. Therefore, using the exact or semiclassical second moments does not change the resulting probabilities. Part of this matching is a direct consequence of the fact that the interaction has been rescaled in the TDHFB case to match the exact ground state energy (cf. Sec. III B 3). In the different cases considered in Fig. 7, a scaling factor $\alpha = 1.333, \alpha = 1.2$, and $\alpha = 1.144$ has been used for $\Omega_A = 4$, 6, and 8, respectively, leading to a difference between the exact and approximate μ_2 that is at maximum 10%. Again, we would like to insist on the fact that the scaling procedure is irrelevant for realistic applications and one should instead suppose that the hypothesis (a) is valid.
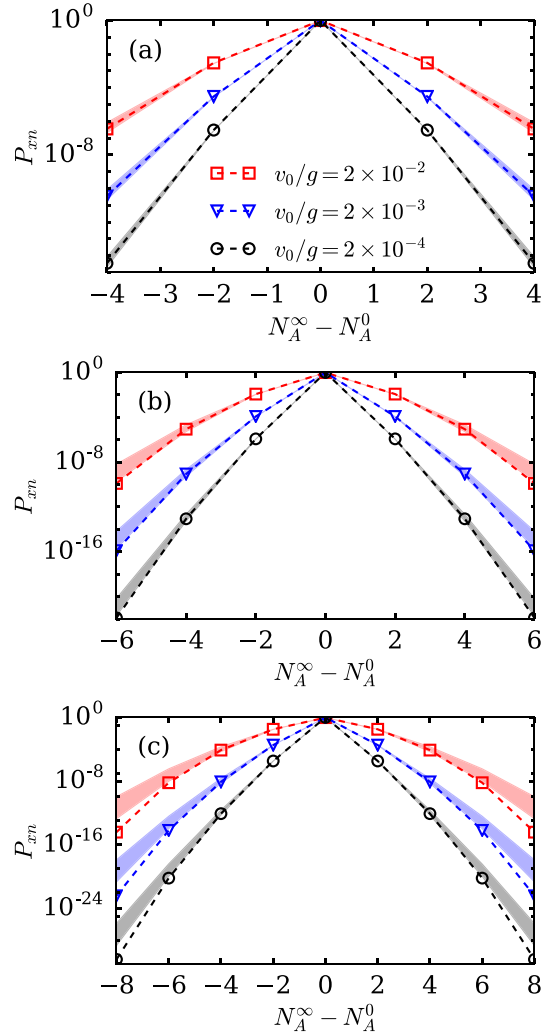
IV. APPLICATION TO ^{20}O + ^{20}O REACTIONS BELOW THE COULOMB BARRIER

To test the applicability the PSC method to a realistic situation, we consider the symmetric reaction ^{20}O + ^{20}O discussed in Ref. [31]. The TDHFB equation using the Gogny interaction [17,18] has been used to simulate the central collision of two ^{20}O superfluid nuclei at various energies and gauge angles below the Coulomb barrier. In Ref. [32], a
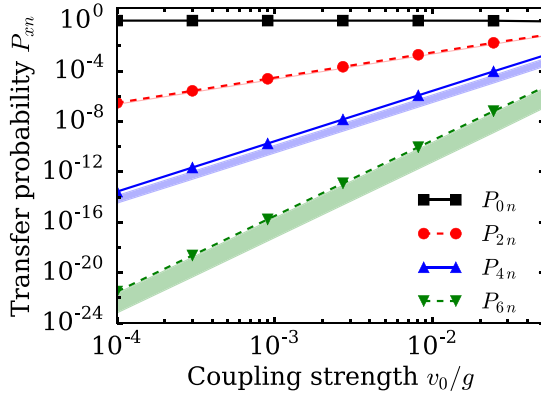
FIG. 8. Asymptotic probabilities to transfer one or several pairs as a function of the coupling strength v_0/g and obtained from a symmetric degenerate situation with $\Omega_A = \Omega_B = N_A^0 = N_B^0 = 6$. The colored bands span the area obtained with the distribution Eq. (41) with the two cases for the W_k factors. The distribution are obtained by using the semiclassical second moment μ_2^{sc} to determine $p = q$. Approximate probabilities are compared with the exact values (dashed line). All TDHFB calculations are performed with the rescaling factor $\alpha = 1.2$.

multiple projection technique was used to extract two particle transfer probabilities. With the same projection technique, the probability to transfer several pairs can also be obtained and can serve as an element of comparison to the PSC approach proposed here. The one pair and two pairs transfer probabilities obtained with projections are represented in Fig. 9 by solid lines. Note that in this case, the error bars stem from the fluctuations of the probabilities after the nucleus do reseparate.

An important and nontrivial ingredient to be able to apply the PSC approach is to figure out the number of pairs contributing to the transfer and the size of the phase-space
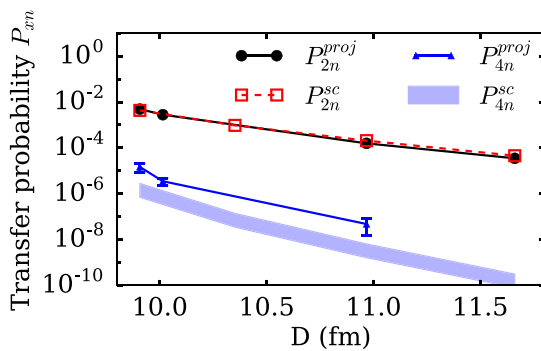

FIG. 9. Probability to transfer one and two pairs in the symmetric central collision $^{20}O + ^{20}O$ as a function of the closest distance of approach D during the reaction. The probabilities P_{xn}^{proj} are computed with the approximate projection method of Ref. [32], whereas P_{xn}^{sc} results from the PSC approach presented in this paper. The error bars associated to P_{xn}^{proj} correspond to the fluctuations of the particle number after reseparation of the collision partners. They are too small to be visible for P_{xn}^{proj}. The colored band for P_{2n}^{sc} spans the area obtained with the two prescriptions for the W_k factors. These factors are computed using $n_A = n_B = 2$ and $\Omega_A = \Omega_B = 6$ (see text).

available for particles to be transmitted to the other nucleus. This number does not impact the probability of one pair exchange, but it drives the predictions for the multiple pairs transfer. In a simple shell-model picture, for the ^{20}O, we expect to have four particles in the last occupied level $1d_{5/2}$, therefore we assumed that the number of pairs equals $n_A = n_B = 2$. The available phase-space after transfer is more difficult to identify, a reasonable assumption is to suppose that all states in the sd shell contribute to the phase-space; i.e., ($\Omega_A = \Omega_B = 6$). We show in Fig. 9 the result of the combinatorial approach compared to the result obtained by projection. It is first remarkable to notice that the probability to transfer two particles are almost identical in the two approaches. This is already a great success of our approach in view of its relative simplicity compared to the method proposed in Ref. [32]. Indeed, the two-particle transfer is obtained in the present work using the simple formula Eq. (37) once the second moment is computed from the phase-space average. Another advantage is the absence of dependence of the result with respect to the phase convention used to solve the TDHFB equations. Note that the agreement of the two particles transfer indirectly validates the projector approach to estimate P_{2n} and P_{-2n}.

The situation is different when more than one pair is transferred. We see that the PSC approach leads systematically to probabilities that are smaller than the projection case. One possible origin of the discrepancy could stem from the size of the available phase-phase that is assumed to perform the PSC calculation. Indeed, the multiple pair transfer increases with degeneracy Ω_A. To obtain an upper limit of P_{4n} with the PSC approach, we simply assumed an infinite number of possible final states, i.e., $\Omega_A = \Omega_B = +\infty$. Using Eqs. (42) and (43) lead to the scaling $P_{4n} = 1/6[P_{2n}]^2$ and $P_{4n} = 1/24[P_{2n}]^2$, respectively. These upper limits are actually within 10% of the values obtained with $\Omega_A = 6$ which tells us that (i) the PSC is quite robust relatively to a change of Ω_A, (ii) having more states active in the reaction does not explain the discrepancy with the projection case where the scaling $P_{4n} \simeq 1/2[P_{2n}]^2$ has been empirically found. Having in mind the success of the approach in the toy model and henceforth supposing that the approach is suited to predict the multiple pair transfer, this discrepancy suggests that the projection technique overestimates the probabilities when more than one pair is transferred.

V. APPLICABILITY TO REACTIONS BETWEEN NONIDENTICAL SUPERFLUID SYSTEMS

The major motivation for this work was the surprisingly large effects related to gauge angle in the collision between two identical superfluids observed in Ref. [22]. In this section, we give some hint on whether these effects are still present in the case of the contact between nonidentical superfluids systems and if the PSC method is still applicable in this case.

A. Schematic model for asymmetry reactions

1. Generalization of the PSC method for nonidentical systems

As a first step, we show how the PSC method modified to properly describe the pair transfer process in the schematic model considered previously when the two initial superfluids

are different, i.e., when they have for instance different particle numbers, degeneracies, single-particle levels, etc.). Two major differences appear in this case: (i) the probability to transfer pairs comes not only from fluctuations but also from an average drift of the mean particle number transferred from one system to the other; (ii) the elementary probability to transfer a pair from A to B or from B to A are *a priori* different, i.e., $p \neq q$ and in general $P_{xn} \neq P_{-xn}$.

Starting from the same hypotheses (a–c), we generalize below the PSC technique to access the pair addition/removal probabilities. In the perturbative regime, we can still assume that only P_{0n}, P_{2n}, and P_{-2n} are dominating. Denoting $\delta n_A = \bar{N}_A^\infty - N_A^0$ the average number of pairs transferred from B to A, and using hypothesis (b), we can express the probabilities P_{2n} and P_{-2n}. Indeed, we now have

$$\delta n_A \simeq 2(P_{2n} - P_{-2n}), \tag{47}$$

while

$$\mu_2 \simeq 4(P_{2n} + P_{-2n}). \tag{48}$$

Note that here we used the fact that $P_{\pm 2n}^2 \ll P_{\pm 2n}$.

Inverting these equations leads to the simple expressions

$$P_{2n} \simeq \frac{\mu_2 + 2\delta n_A}{8} = pW_1,$$

$$P_{-2n} \simeq \frac{\mu_2 - 2\delta n_A}{8} = qW_{-1}, \tag{49}$$

which give a straightforward method to extract the values of p and q. Equations (45) are now extended as

$$P_{2(k+1)n} = p\frac{W_{k+1}}{W_k}P_{2kn},$$

$$P_{-2(k+1)n} = -q\frac{W_{-(k+1)}}{W_k}P_{-2kn}. \tag{50}$$

It is worth mentioning that the above expressions should be used with some care. Indeed, when the absolute value of the drift $|\delta n_A|$ increases and exceed $\mu_2/2$, one of the probabilities becomes negative which is unphysical. This directly stems from the breakdown of the hypothesis (b). Indeed as $|\delta n_A|$ increases, on of the two particle transfer probabilities starts to decrease and becomes comparable to P_{4n} and or P_{-4n}. This constraint on the applicability of Eq. (49) can be further quantified. Let us first assume that $\delta n_A > 0$, which means that the transfer from B to A is dominant and $P_{2n} > P_{-2n}$. Then, the condition $P_{-2n} \gg P_{4n}$ gives the condition

$$\left(\frac{8W_1^2}{W_2}\right)\frac{[\mu_2 - 2\delta n_A]}{[\mu_2 + 2\delta n_A]^2} \gg 1. \tag{51}$$

Equivalently for the case $\delta n_A < 0$, which corresponds to the case where the transfer from A to B dominates, the condition becomes

$$\left(\frac{8W_{-1}^2}{W_{-2}}\right)\frac{[\mu_2 + 2\delta n_A]}{[\mu_2 - 2\delta n_A]^2} \gg 1. \tag{52}$$

If these conditions are not met, i.e., if the drift becomes too high compared to the second moment, one of the two-particle transfer probabilities dominates the other. Then, Eqs. (47) and

(48) simplify and we have

$$\mu_2 \simeq 2|\delta n_A| \simeq 4P_{\eta 2n}, \tag{53}$$

where η is the sign of the δn_A. The moments of order one and two contain a redundant information, which characterizes only the dominant branch (addition or removal). In practice, the probability p (or q) associated to this branch may still be obtained by matching the first moment with $2P_{\eta 2n}$. It is finally interesting to mention, that for large drift we obtain $\mu_2 \propto |\delta n_A|$ that is similar to the results obtained in the nucleon exchange model where the transfer is driven by randomness [49–51].

2. Benchmark with the exact second moment

To benchmark this approach, we follow the same methodology as for the symmetric case and first compute the probability distribution P_{xn} from the exact average drift and second moment. In the present study, we consider three different types of asymmetric reactions: (I) the case of degenerate system with a different initial number of particles in each system $N_A^0 = 4$, $N_B^0 = 8$; (II) the same situation with an additional asymmetry coming from a shift in the single-particle energy between the two systems, namely, $\Delta e = \epsilon_B - \epsilon_A = g$; (III) the case of two nondegenerated systems with equidistant single-particle level spacing $\Delta \varepsilon/g = 1$ in each system and different initial number of particles in A and B. In all cases, we assume $\Omega_A = \Omega_B = 6$. Note that we also tested situations where $\Omega_A \neq \Omega_B$, leading essentially to the same conclusions.

In Fig. 10 the pair transfer probabilities obtained for the three asymmetric reactions using the PSC approach with the exact δn_A and second moment. These probabilities are compared to the exact ones. We end up essentially to the same conclusions as in the symmetric case. We systematically see that the probability to transfer one or two pairs are reasonably well reproduced, while as the number of transferred pair increases, more and more deviation appears with respect to the exact distribution. Overall, the shape of the distribution still matches correctly the exact one.

3. Critical discussion on the semiclassical moments in collisions between nonidentical systems

Note also that for the case of asymmetric collisions that is considered here, there is no more strong argument to apply the same scaling for g_A, g_B, and $v(t)$ to compare with the exact solution. For the sake of simplicity, we kept the same procedure as for the symmetric case.

For the three asymmetric cases considered, we compute the semiclassical moments $\delta n_A^{\rm sc}$ and $\mu_2^{\rm sc}$ from the average over different TDHFB trajectories. When going from symmetric case to asymmetric case, we did not found any systematic arguments to obtain the scaling on the coupling constant discussed in Sec. III B 3. For the sake of simplicity, we therefore adopted the same scaling procedure as in the symmetric case and found $\alpha = 1.2$ for cases (I) and (II) and $\alpha = 1.289$ for case (III).

Figure 11 compares the semiclassical estimation of the two first moments with the exact one. In the three cases, this procedure yields moments that are proportional to their exact counterparts in the perturbative region. The semiclassical drift
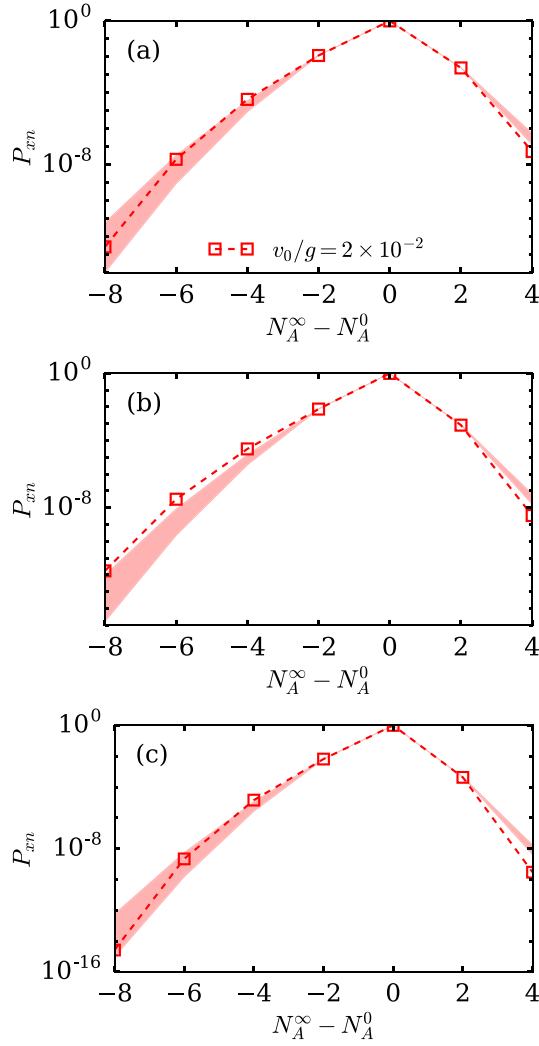
FIG. 10. Same as Fig. 6 for three cases of asymmetric reactions. The exact probabilities are shown with dashed lines and systematically compared to the combinatorial results using the exact values of $\bar{n}$ and μ_2 (colored bands). The different panels (a)–(c) correspond, respectively, to the asymmetric case (I), (II), and (III).
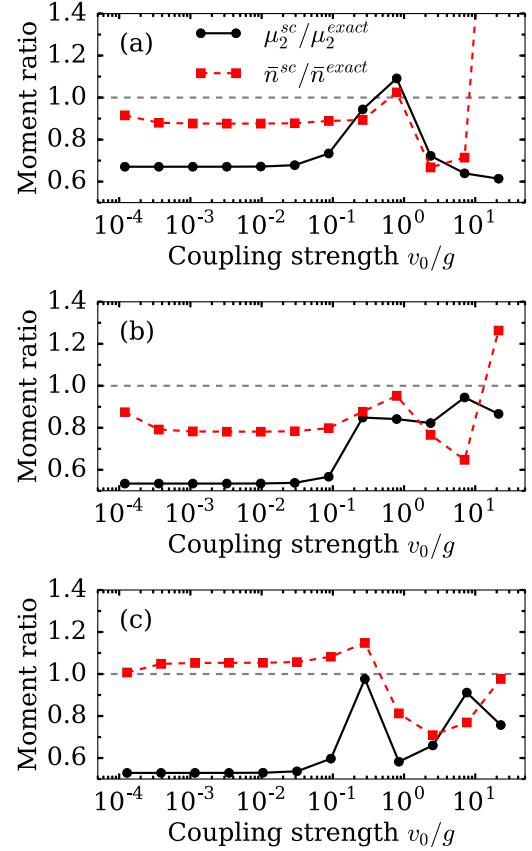

FIG. 11. First- and second-order moments obtained with the semiclassical average normalized to their exact counterparts as a function of the coupling strength. The different panels (a)–(c) correspond, respectively, to the asymmetric case (I), (II), and (III).

lies within 20% of the exact drift in this regime, whereas the second moment is systematically underestimated, $\mu_2^{sc} \sim 0.5$–$0.7\mu_2^{ex}$. The absence of clear procedure to generalize the scaling technique strongly bias the comparison between the exact and the semiclassical approach combined with the combinatorial technique. From these results, it is not clear whether the underestimation of the second order moment comes essentially from the arbitrary rescaling procedure or an intrinsic feature of the semiclassical phase-space method. This question is crucial as such an underestimation in a realistic case would severely jeopardize the method proposed for asymmetric reactions. For example, in the model cases (II) and (III) we find that whatever the values of p and q, no distribution represented by Eq. (A12) could reproduce the moments estimated from the semiclassical average.

To conclude, a generalization of the PSC method to the case of nonidentical superfluid systems is technically possible. Its

application to our toy model shows that in the case where the transfer is dominated by an average drift of pairs only a part of the distribution could be recovered. In addition, it is not clear from this study whether the low order moments of the distribution can be correctly estimated within the phase-space averaging procedure. To answer this remaining question, an application of the PSC method to a realistic collision between nonidentical systems is required.

B. Application to ^{14}O + ^{20}O reaction below the Coulomb barrier

Using the semiclassical average over the initial relative gauge angle, we computed the semiclassical drift and second order moment of the pair transfer distribution for the asymmetric reaction ^{14}O + ^{20}O. The calculations were repeated at three different energies below the Coulomb barrier and the Table I summarizes the results. For the sake of completeness, we also give the results obtained by the projection method proposed in Ref. [31]. In the case of nonidentical nuclei, almost no influence of the initial gauge-angle on the particle transfer process is found leading to a small, almost zero, value of μ_2^{sc} from which we deduce very small values for the pair transfer using Eq. (53). This is at variance with both the toy model for a symmetric reaction and the ^{20}O + ^{20}O application. This is also different from the toy model used previously for interacting

TABLE I. Average drift along with fluctuation of the number of transferred particles (and pair transfer probabilities P_{2n}^{sc}) obtained within a phase-space averaging for the $^{14}O + ^{20}O$ reaction at different energies in the center of mass (E_{cm}). The probabilities to transfer one P_{1n}^{proj} and two P_{2n}^{proj} particles obtained from an approximated projection technique are also provided for comparison.

E_{cm} (MeV)	δn_A	$P_{2n}^{sc} = \mu_2^{sc}/4$	P_{1n}^{proj}	P_{2n}^{proj}
7.903	0.28×10^{-3}	0.04×10^{-9}	0.28×10^{-3}	0.05×10^{-3}
8.903	7.93×10^{-3}	0.14×10^{-9}	3.31×10^{-3}	1.23×10^{-3}
9.403	40.11×10^{-3}	0.48×10^{-9}	13.04×10^{-3}	7.11×10^{-3}

nonidentical superfluids. The difference with the toy model can be directly traced back to the fact that one-body components treating single-particle tunneling has not been considered in the toy model. This process seems to dominate for the asymmetric collisions considered here washing out any significant effects of gauge-angle.

As a matter of fact, this is not surprising because the transfer process in asymmetric systems is known to be mainly dominated by the fast N/Z equilibration, due to a fast equilibration of the chemical potentials. This process is already accounted for by the mean-field Hamiltonian and is not connected to superfluidity. It is worth mentioning that mean field alone (with or without pairing) can describe the mean drift but not the fluctuations around the mean drift.

The absence of gauge-angle influence obviously leads to a failure of the PSC approach. Indeed, a prerequisite of the success of the approach presented here is that the $U(1)$ symmetry breaking dominates the physical process under interest.

Our conclusions are twofold. First, the PSC method applied to nuclear reactions can only be successful for symmetric collisions between superfluid nuclei to avoid at maximum the pollution from pure one-body effects. Second, for the same reason, experiments involving symmetric reactions between midshell spherical nuclei is the best test-bench to probe any significant gauge-angle effects, if any.

VI. CONCLUSION

In this work, we analyze in detail the pair transfer between two identical superfluid systems. We use a simple model where the two systems governed by a pairing Hamiltonian only interact with each other for a short finite time. This model mimics the transfer of pairs during a heavy-ion reaction in the presence of pairing and can be solved exactly for small systems. The possibility to describe accurately the pair transfer by combining the TDHFB framework with some average over the relative gauge angle is discussed. We show in particular that a brute force semiclassical average over a distribution of gauge angles can only partially describe the final distribution of particles in each subsystem. While the second moment of this distribution is reasonably described, higher-order moments significantly differ from the exact results.

Based on this observation, a method is proposed to obtain the probability to transfer multiple pairs in the perturbative regime. This approach supplements the phase-space average with a combinatorial technique to infer the probabilities to transfer more than one pair. The PSC method is benchmarked with respect to the schematic model and then successfully

applied to the head-on collision of two ^{20}O. For this realistic case, the results are systematically compared with the projection method proposed in Ref. [32]. Despite the fact that the PSC method is technically much simpler than the projection method, both techniques lead to similar two particle transfer probabilities. The PSC that was shown to be effective in the schematic model to reproduce multiparticle transfer leads in general to multiple pairs transfer probabilities that are much smaller than the projection approach.

Finally, the applicability of the PSC method to the case of asymmetric collision is discussed. We show in particular that in its current state, the method fails to describe the transfer in the asymmetric reaction $^{14}O + ^{20}O$. The main reason is that the fluctuations associated with the relative gauge angle in this reactions are not the main driver of the transfer.

In the future, it would be interesting to compare the different approaches to describe multiple pair transfer to high-precision experiments. In recent years, several experiments have been performed for energies well below the Coulomb barrier [9–12] where the perturbative regime is relevant. However, these experiments involve targets different from the projectile where many effects other than pairing play a role in the transfer. We believe that the best situation to compare theory and experience and to unambiguously probe relative gauge-angle effects in multiple particles transfer would be to consider symmetric collisions between medium mass spherical nuclei, like $^{108-120}Sn$, where the last occupied level is close to half-filling.

APPENDIX: EXACT EVOLUTION FOR THE SYMMETRIC DEGENERATE CASE

In this section, we show that the combinatorial approach used in the present work can be motivated in some way by the exact case in the perturbative regime. We consider here that the two systems A and B are fully degenerated and governed by a pure pairing Hamiltonian. We follow Ref. [38] and use the compact notation $H_0 = H_A + H_B$. We assume that the system is described initially by the state $|n_A, n_B\rangle = |n_A\rangle \otimes |n_B\rangle$, where $|n_A\rangle$ and $|n_B\rangle$ are, respectively, the seniority zero ground states of A and B. The symbol $n_{A/B} = N_{A/B}/2$ denotes here the number of pairs. Since, the Hamiltonian conserves the seniority and the total number of particles, in the simple symmetric degenerate case (with initial half filling), one can introduce the set of states

$$|k\rangle \equiv |n_A + k, n_B - k\rangle, \quad -n_A \leqslant k \leqslant +n_B,$$

associated to the unperturbed energy E_n and decompose the time-dependent state $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = \sum_k c_k(t)|k\rangle, \qquad (A1)$$

with $|\Psi(t_0)\rangle = |0\rangle$. Introducing the set of parameters $b_k(t) = c_k(t)e^{+iE_kt/\hbar}$, the coupled equations on the b_k components can be written as

$$i\hbar\frac{d}{dt}b_l(t) = v(t)\sum_k b_k(t)e^{i\omega_{kl}t}\langle l|(C^\dagger + C)|k\rangle,$$

with $\omega_{kl} = (E_l - E_k)/\hbar$ and where we have introduced the notation $C^\dagger = \sum_{\alpha\beta} a_\alpha^\dagger a_{\bar\alpha}^\dagger b_{\bar\beta}b_\beta$. One can then introduce two matrices D^+ and D^- with components

$$D_{lk}^+ = \langle l|C^\dagger|k\rangle = \delta_{l,k+1}D_k^+,$$
$$D_{lk}^- = \langle l|C|k\rangle = \delta_{l,k-1}D_k^-. \qquad (A2)$$

Note that D^+ (respectively, D^-) only connects the state $|k\rangle$ with the state $|k+1\rangle$ (respectively, $|k-1\rangle$). We first assume that the transition frequencies are constant, with

$$\omega_{kl} = \pm\omega. \qquad (A3)$$

On top of that, we assume that the perturbation is time symmetric and take for convenience $t_\infty = -t_0$. From this, we obtain a compact expression for the b vector,

$$b(t_\infty) = \exp(2iz[D^+ + D^-])\,b(t_0), \qquad (A4)$$

with

$$z = -\frac{1}{\hbar}\int_{t_0}^{t_\infty} ds\, v(s)\cos(\omega s). \qquad (A5)$$

The final probability to have k pairs transferred, denoted again by P_{2kn} from the initial state $|0\rangle$ is then given by

$$P_{2kn} = |b(t_\infty)|^2$$
$$= \left|\sum_{J=0}^{+\infty}\frac{(2iz)^J}{J!}[(D^+ + D^-)^J]b(t_0)\right|^2$$
$$= \left|\sum_{J=0}^{+\infty}\frac{(2iz)^J}{J!}\left[\sum_{I=0}^{J}C_J^I(D^+)^I(D^-)^{(J-I)}\right]b(t_0)\right|^2. \quad (A6)$$

In general, the transfer probabilities result from a rather complicated interference effect between different paths depicted in Fig. 12. In this double sum, all terms with $I - (J - I) = 2I - J = k$ will contribute to the probability P_{2kn}. However, noting that $z \propto v_0$, the first term feeding the state $|k\rangle$ in Fig. 12 will dominates the probability in the perturbative regime. This term corresponds always to the lowest or highest branch in this figure. Neglecting all other terms in the expansion yields the simplified expressions

$$P_{2kn} \simeq \frac{|2z|^{2k}}{(k!)^2}|D_0^+|^2\cdots|D_{k-1}^+|^2,$$

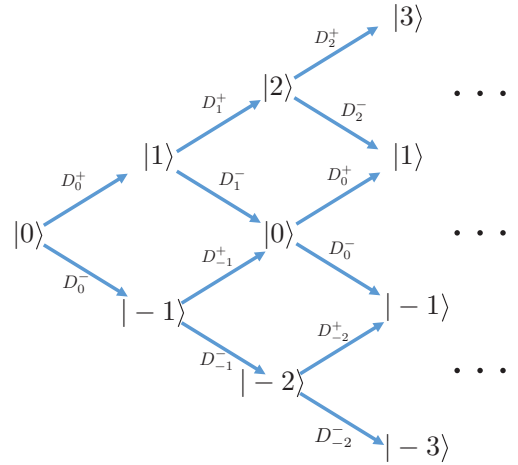$$P_{-2kn} \simeq \frac{|2z|^{2k}}{(k!)^2}|D_0^-|^2\cdots|D_{-(k-1)}^-|^2.$$

FIG. 12. Schematic view of the different contributions to the population of a state $|k\rangle$ in Eq. (A6).

These formulas induce the recurrence relations

$$P_{2(k+1)n} = |2z|^2\frac{|D_k^+|^2}{(k+1)^2}P_{2kn},$$

$$P_{-2(k+1)n} = |2z|^2\frac{|D_k^-|^2}{(k+1)^2}P_{-2kn}.$$

The matrix elements of D^+, D^- read [33][3]

$$|D_k^+|^2 = (\Omega_A - n_A - k)(n_A + k + 1)$$
$$\times (n_B - k)(\Omega_B - n_B + k + 1), \qquad (A7)$$

$$|D_{-k}^-|^2 = (\Omega_B - n_B - k)(n_B + k + 1)$$
$$\times (n_A - k)(\Omega_A - n_A + k + 1), \qquad (A8)$$

n_A and n_B being the initial number of pairs in A and B, respectively. We finally deduce the compact expressions

$$P_{2kn} = |2z|^{2k}C_{\Omega_A - n_A}^k C_{n_B}^k \frac{(n_A + k)!}{n_A!}\frac{(\Omega_B - n_B + k)!}{(\Omega_B - n_B)!},$$

$$P_{-2kn} = |2z|^{2k}C_{\Omega_B - n_B}^k C_{n_A}^k \frac{(n_B + k)!}{n_B!}\frac{(\Omega_A - n_A + k)!}{(\Omega_A - n_A)!}.$$

In the above expressions, we recognize the combinatorial factor that has been introduced in the main text. Choosing $p \simeq |2z|^2$,

---

[3]As a side remark, we note that if the exact state with particles N_A, N_B is systematically replaced by a BCS/HFB quasiparticle state with average particle number N_A, N_B, we obtain

$$|D_0^+|^2 = \left(\frac{\Delta_A(N_A)\Delta_B(N_B)}{g_A g_B}\right)^2$$
$$= n_A n_B(\Omega_A - n_A)(\Omega_B - n_B),$$

where $\Delta_A^{N_A}$ and $\Delta_B^{N_B}$. Similarly, we have

$$|D_k^+|^2 = (n_A + k)(\Omega_A - n_A - k)(n_B - k)(\Omega_B - n_B + k),$$
$$|D_{-k}^+|^2 = (n_A - k)(\Omega_A - n_A + k)(n_B + k)(\Omega_B - n_B - k),$$
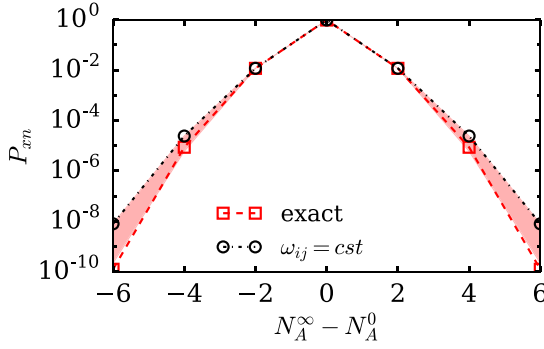
which is rather close to the exact formula.

FIG. 13. Asymptotic probabilities to transfer one or several pairs as a function of the quantity $N_A^\infty - N_A^0$ obtained from a symmetric degenerate situation with $\Omega_A = \Omega_B = N_A^0 = N_B^0 = 6$. The colored bands span the area obtained with Eq. (41) assuming Eq. (A9) (upper limit) or the same expression divided by $(k!)^2$ (lower limit). The distributions are obtained by using the exact second moment μ_2^{sc} to determine $p = q$. Approximate probabilities are compared with the exact values (red dashed line) and the leading order of the perturbative approach computed with the assumption $\omega_{ij} = cst$ (black dotted line). All calculations are performed for the coupling strength $v_0/g = 2.10^{-2}$.

we obtain indeed

$$W_k = C_{\Omega_A - n_A}^k C_{n_B}^k \frac{(n_A + k)!}{n_A!} \frac{(\Omega_B - n_B + k)!}{(\Omega_B - n_B)!}, \quad (A9)$$

$$W_{-k} = C_{\Omega_B - n_B}^k C_{n_A}^k \frac{(n_B + k)!}{n_B!} \frac{(\Omega_A - n_A + k)!}{(\Omega_A - n_A)!}, \quad (A10)$$

for the factors $W_{k/-k}$ involved in Eq. (41).

In Fig. 13, the method proposed in Sec. III C to obtain the transferred probabilities with two prescriptions for the W_k is tested against the exact results. In this figure, we also display the results obtained from the coupled equation enforcing that all transition frequencies are constant, as it was supposed in the derivation here. By construction, the results from the combinatorial method using Eq. (42) for the factors W_k matches the results from the coupled equation resolution with constant transition frequencies. On the other hand, this method tends to overestimate the transfer probabilities compared to the exact case especially when the number of pairs transferred increases. We conclude from this that (i) the approximation Eq. (A3) is too crude for the present situation; (ii) there is a rather close connection between the combinatorial factor and the set of values of transition frequencies at play during multiple pair transfer. Starting from a situation of a single degenerate shell with initial half-filling and neglecting pairing correlations, the addition or removal of two particles to the shell will indeed lead to $\omega_{kl} = \pm\omega = \pm 2\varepsilon$, where ε is the single-particle energy in the shell. However, when pairing plays a role, the energies are distorted by correlations. In particular, starting from the exact energies of the degenerate system with initial half-filling, it can be shown that the energies verifies $\omega_{k,k+1} \simeq \pm k\omega$ and therefore increases when the number of pairs increases. The combinatorial factor should take into account this aspect in some way.

To obtain a more realistic expression of $W_{k,-k}$ for superfluid system, we start back from the time-dependent perturbation theory. Without any assumption on the transition frequencies, we can show that the leading order contribution to b_k with $k > 0$ can be written as

$$b_k(t_\infty) = \hbar^{-k} e^{-i(\omega_k t_\infty - \omega_0 t_0)} \int_{\tau_{k-1}}^{t_\infty} d\tau_k \int_{\tau_{k-2}}^{\tau_k} d\tau_{k-1}$$

$$\cdots \int_{t_0}^{\tau_2} d\tau_1 \, v(\tau_k) \cdots v(\tau_1) e^{-i\omega_{k-1,k}\tau_k}$$

$$\cdots e^{-i\omega_{1,0}\tau_1} D_{k-1}^+ \cdots D_0^+.$$

To further progress, we make the simplifying assumption that

$$\int_{\tau_{k-1}}^{t_\infty} d\tau_k \int_{\tau_{k-2}}^{\tau_k} d\tau_{k-1} \cdots \to \frac{1}{k!} \int_{t_0}^{t_\infty} d\tau_k \int_{t_0}^{t_\infty} d\tau_{k-1} \cdots,$$

where the factor $1/k!$ accounts properly for the change of volume of integration in the time hyper-space $(\tau_1, \ldots, \tau_k)$. We further simplify the interaction and assume $v(t) = v_0 \Theta(\tau_c/2 - |t|)$, where τ_c is the interaction time and Θ is the Heavyside function. With this, we can explicitly perform the time integration and obtain

$$|b_k(t_\infty)| = \frac{1}{k!} \left(\frac{2v_0}{\hbar}\right)^k \prod_{l=1}^{k} \left[\frac{\sin(\omega_{l,l-1}\tau_c/2)}{\omega_{l,l-1}}\right] D_{k-1}^+ \cdots D_0^+.$$

We then see that the probability to transfer k pairs can be rewritten as a product,

$$P_{2kn} = \mathcal{P}_{+1} \cdots \mathcal{P}_{+k}, \quad (A11)$$

with

$$\mathcal{P}_{+k} = \left(\frac{2v_0}{k\hbar} \frac{\sin(\omega_{k,k-1}\tau_c/2)}{\omega_{k,k-1}}\right)^2 |D_{k-1}^+|^2. \quad (A12)$$

This probability is, in general, rather complicated since it contains the information on the initial and final system after transfer as well as the information on the time-dependent interaction.

In the case where all the transition frequencies are equal to ω, we recover the previous expression provided that

$$|2z| = \frac{|2v_0|}{\hbar} \left|\frac{\sin(\omega\tau_c/2)}{\omega}\right|.$$

In a different situation where the transition frequencies behave as $\omega_{l,l-1} = l\omega$, we obtain a new expression with

$$b_k(t_\infty) = \frac{1}{k!} \left(\frac{v_0 \tau_c}{\hbar}\right)^k D_{k-1}^+ \cdots D_0^+ \prod_{l=1}^{k} j_0\left(l\omega\frac{\tau_c}{2}\right),$$

where j_0 is the first spherical Bessel function, $j_0(x) = \sin(x)/x$. If $|\omega\tau_c/2| \ll 1$, we will always have $|j_0(l\omega\tau)| < |j_0(\omega\tau)|$ that explains empirically why the approximations Eq. (A10) leads systematically to an overestimation of the exact

probabilities. In particular, we obtain an upper bound for the absolute value of $|b_k|$ that is given by

$$\frac{|b_k|^2}{|D_0^+|^2 \cdots |D_k^+|^2} \leqslant \frac{1}{(k!)^4}\left(\frac{2v_0}{\hbar\omega}\right)^{2k},$$

This expression contains an extra $1/(k!)^2$ compared to the case where we assumed simply that all frequencies are the same.

From this last remarks, we empirically assume that the proper combinatorial factors are those given by Eqs. (A10) divided by $(k!)^2$, leading to

$$W_k = C_{\Omega_A - n_A}^k C_{n_B}^k C_{n_A + k}^k C_{\Omega_B - n_B + k}^k, \quad (A13)$$

$$W_{-k} = C_{\Omega_B - n_B}^k C_{n_A}^k C_{n_B + k}^k C_{\Omega_A - n_A + k}^k. \quad (A14)$$

This combinatorial factor improves significantly the description of the transfer probabilities and is the one retained in this work.

[1] R. A. Broglia and V. Zelevinsky (eds.) *Fifty Years of Nuclear BCS: Pairing in Finite Systems* (World Scientific, Singapore, 2013).

[2] D. M. Brink and R. A. Broglia, *Nuclear Superfluidity: Pairing in Finite Systems* (Cambridge University Press, Cambridge, UK, 2005).

[3] W. von Oertzen and A. Vitturi, Rep. Prog. Phys. **64**, 1247 (2001).

[4] M. Grasso, D. Lacroix, and A. Vitturi, Phys. Rev. C **85**, 034317 (2012).

[5] R. A. Broglia, O. Hansen, and C. Riedel, in *Two-Neutron Transfer Reactions and the Pairing Model*, edited by M. Baranger and E. Vogt, Advances in Nuclear Physics (Springer, Boston, MA, 1973), pp. 287–457.

[6] D. R. Bès and R. A. Broglia, Nucl. Phys. **80**, 289 (1966).

[7] D. Gambacurta and D. Lacroix, Phys. Rev. C **86**, 064320 (2012).

[8] G. Potel, A. Idini, F. Barranco, E. Vigezzi, and R. A. Broglia, Phys. Rev. C **96**, 034606 (2017).

[9] L. Corradi *et al.*, Phys. Rev. C **84**, 034603 (2011).

[10] D. Montanari *et al.*, Phys. Rev. Lett. **113**, 052501 (2014).

[11] D. Montanari *et al.*, Phys. Rev. C **93**, 054623 (2016).

[12] D. C. Rafferty *et al.*, Phys. Rev. C **94**, 024607 (2016).

[13] G. Scamps, V. V. Sargsyan, G. G. Adamian, N. V. Antonenko, and D. Lacroix, Phys. Rev. C **94**, 064606 (2016).

[14] B. Avez, C. Simenel, and P. Chomaz, Phys. Rev. C **78**, 044318 (2008).

[15] S. Ebata, T. Nakatsukasa, T. Inakura, K. Yoshida, Y. Hashimoto, and K. Yabana, Phys. Rev. C **82**, 034306 (2010).

[16] I. Stetcu, A. Bulgac, P. Magierski, and K. J. Roche, Phys. Rev. C **84**, 051309 (2011).

[17] Y. Hashimoto, Eur. Phys. J. A **48**, 55 (2012).

[18] Y. Hashimoto, Phys. Rev. C **88**, 034307 (2013).

[19] G. Scamps and D. Lacroix, Phys. Rev. C **87**, 014605 (2013).

[20] Y. Hashimoto and G. Scamps, Phys. Rev. C **94**, 014610 (2016).

[21] K. Sekizawa, P. Magierski, and G. Wlazlowski, in *Proceedings of the International Nuclear Physics Conference (INPC'16)*, Vol. 281 (Proceedings of science (PoS), Trieste, Italy, 2017).

[22] P. Magierski, K. Sekizawa, and G. Wlazlowski, Phys. Rev. Lett. **119**, 042501 (2017).

[23] A. Bulgac and S. Jin, Phys. Rev. Lett. **119**, 052501 (2017).

[24] M. Dasgupta, D. J. Hinde, N. Rowley, and A. M. Stefanini, Ann. Rev. of Nucl. Part. Science **48**, 401 (1998).

[25] N. Rowley, G. R. Satchler, and P. H. Stelson, Phys. Lett. B **254**, 25 (1991).

[26] R. A. Broglia and A. Winther, *Heavy Ion Reactions: The Elementary Processes Part I and II, Frontiers in Physics* (Westview Press, Boulder, CO, 1991).

[27] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, Berlin, 2000).

[28] M. Bender, P.-H. Heenen, and P-G. Reinhard, Rev. Mod. Phys. **75**, 121 (2003).

[29] J. Luis Egido, Phys. Scr. **91**, 073003 (2016).

[30] C. Simenel, Phys. Rev. Lett. **105**, 192701 (2010).

[31] G. Scamps and Y. Hashimoto, in *Proceedings of the European Physical Journal Web of Conferences (Fusion'17)*, Vol. 163 (EDP Sciences, Les Ulis Cedex A, France, 2017), p. 00049.

[32] G. Scamps and Y. Hashimoto, Phys. Rev. C **96**, 031602(R) (2017).

[33] K. Dietrich, Phys. Lett. B **32**, 428 (1970).

[34] A. Volya, B. A. Brown, and Z. Zelevinsky, Phys. Lett. B **509**, 37 (2001); T. Sumaryada and A. Volya, Phys. Rev. C **76**, 024319 (2007).

[35] J. von Delft and D. C. Ralf, Phys. Rep. **345**, 61 (2001).

[36] D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. **75**, 607 (2003).

[37] J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. **76**, 643 (2004).

[38] K. Dietrich, K. Hara, and F. Weller, Phys. Lett. B **35**, 201 (1971).

[39] K. Hara, Phys. Lett. B **35**, 198 (1971).

[40] J. Blocki and H. Flocard, Nucl. Phys. A **273**, 45 (1976).

[41] A. Bulgac, Phys. Rev. C **41**, 2333 (1990).

[42] H. Esbensen, A. Winther, R. A. Broglia, and C. H. Dasso, Phys. Rev. Lett. **41**, 296 (1978).

[43] S. Ayik, B. Yilmaz, and D. Lacroix, Phys. Rev. C **81**, 034605 (2010).

[44] S. Ayik, Phys. Lett. B **658**, 174 (2008).

[45] D. Lacroix and S. Ayik, Eur. Phys. J. A **50**, 95 (2014).

[46] D. Lacroix, D. Gambacurta, and S. Ayik, Phys. Rev. C **87**, 061302 (2013).

[47] Y. Castin and J. Dalibard, Phys. Rev. A **55**, 4330 (1997).

[48] H. Kurkjian, Y. Castin, and A. Sinatra, Phys. Rev. A **88**, 063623 (2013).

[49] J. Randrup and W. J. Swiatecki, Ann. Phys. (NY) **125**, 193 (1980); Nucl. Phys. A **429**, 105 (1984).

[50] H. Feldmeier, Rep. Prog. Phys. **50**, 915 (1987).

[51] K. Washiyama, S. Ayik, and D. Lacroix, Phys. Rev. C **80**, 031602(R) (2009).