Extrapolation of scattering data to the negative-energy region. II. Applicability of effective range functions within an exactly solvable model

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A problem of analytical continuation of scattering data to the negative-energy region to obtain information about bound states is discussed within an exactly solvable potential model. This work is continuation of the previous one by the same authors [L. D. Blokhintsev *et al.*, Phys. Rev. C **95**, 044618 (2017)]. The goal of this paper is to determine the most effective way of analytic continuation for different systems. The $d + \alpha$ and $\alpha + {}^{12}\mathrm{C}$ systems are considered and, for comparison, an effective-range function approach and a recently suggested Δ method [O. L. Ramírez Suárez and J.-M. Sparenberg, Phys. Rev. C **96**, 034601 (2017).] are applied. We conclude that the Δ method is more effective for heavier systems with large values of the Coulomb parameter, whereas for light systems with small values of the Coulomb parameter the effective-range function method might be preferable.

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I. INTRODUCTION

Asymptotic normalization coefficients (ANCs) are fundamental nuclear characteristics important both in nuclear reaction and nuclear structure physics. They determine amplitudes of the asymptotic forms of bound-state nuclear wave functions in binary channels. The ANC for a virtual $a \leftrightarrow b + c$ process is related directly to the residue of the elastic $b + c$ scattering amplitude at the pole in the energy plane corresponding to the bound state of nucleus a (see, e.g., Ref. [1]).

The ANCs naturally appear in the expressions for the cross sections of nuclear reactions between charged particles at low energies when, due to the Coulomb barrier, the reactions occur at large distances between colliding nuclei [2]. Astrophysical nuclear reactions represent the most important type of such reactions. The role of the ANCs in nuclear astrophysics was first discussed in Refs. [3–5], where it was emphasized that the ANC determines the overall normalization of peripheral radiative capture reactions (see also Refs. [6,7]). The ANC method provides a powerful indirect technique in nuclear astrophysics.

There are different ways to determine the ANCs from experimental data. From the peripheral reactions the ANCs can be extracted directly by normalizing the calculated cross sections to the experimental data. However, it is impossible to directly determine the ANCs from elastic scattering data, which are measured at positive energies while the ANCs are related to the residues of the poles of the bound states at negative energies. Nevertheless, there is an indirect way to determine the ANC from experiment: The ANC $C_{a\to bc}$ can be determined from experimental data by extrapolating, in the plane of the center-of-mass (c.m.) energy E, the partial-wave amplitude of the elastic $b + c$ scattering, obtained by the phase-shift analysis, to the pole corresponding to the bound state a and lying at $E < 0$. The conventional procedure for such an extrapolation is the analytic approximation of the experimental values of the effective-range function (ERF) $K_l(E)$ with the subsequent continuation to the pole (here l is the orbital angular momentum). The ERF method has been successfully employed to determine the ANCs for bound (as well as resonant) nuclear states in a number of works (see, e.g., Refs. [8–10] and references therein).

The ERF is expressed in terms of scattering phase shifts. In the case of charged particles, the ERF for the short-range interaction should be modified. Such modification generates additional terms in the ERF. These terms depend only on the Coulomb interaction and may far exceed, in the absolute value, the informative part of the ERF containing the phase shifts. This fact hampers the practical procedure of the analytic continuation and affects its accuracy. In Ref. [11] it was suggested to use for the analytic continuation the quantity $\Delta_l(E)$ [which is defined below in Sec. II] rather than the ERF $K_l(E)$. Quantity $\Delta_l(E)$, which we will call a Δ function, does not contain the pure Coulomb terms. However, the validity of employing $\Delta_l(E)$ was not obvious, and this resulted in some discussions. It was demonstrated in Ref. [12] that the $\Delta_l(E)$ function suggested in Ref. [11] can be smoothly continued from the positive- to the negative-energy region along the real E axis (see also Ref. [13]). In what follows, using the $\Delta_l(E)$ function for extrapolation to the negative-energy region to find the ANC is referred to as a Δ method.

The present work can be considered as a natural development and extension of Ref. [12] by the same authors. Here we calculate the scattering phase shifts and the functions $K_l(E)$ and $\Delta_l(E)$ using an analytic solution of the

Schrödinger equation at $E > 0$ with an adopted potential in the form of the square well plus the Coulomb interaction. To the authors' knowledge, the square-well potential is the only local potential which, with the added Coulomb interaction, permits the analytic solution of the Schrödinger equation at any value of the orbital angular momentum. In this approach, our results are vigorous and obtained without any approximation. The calculated functions $K_l(E)$ and $\Delta_l(E)$ are approximated by Taylor polynomials in E and extrapolated to the negative-energy region including the bound-state poles of the system under consideration. This procedure imitates the approach to determining ANCs by the analytic approximation of experimental scattering data. The approximated values of $K_l(E)$, $\Delta_l(E)$, and the resulting ANCs are compared to the exact values following from the exact solution of the Schrödinger equation. This comparison allows one to evaluate the quality of the approximation and to compare the effectiveness of the ERF and Δ methods.

Note that the simplicity of our potential model is justified by the fact that at very low energies, which we are interested in, the wave length (the reciprocal of the relative momentum of the interacting nuclei) becomes much larger than the radius of the nuclear interaction potential, making the results insensitive to the specific shape of the used potential, whether it is Woods-Saxon, square well, δ function, or anything else.

In the present paper, the procedure described above is applied to two different nuclear systems: The $d + \alpha$ system and the $\alpha + {}^{12}$C system. These systems differ in the value of the Coulomb (Sommerfeld) parameter, which is much larger for the latter. One more qualitative distinction between these systems is that the $d + \alpha$ system has only one bound state corresponding to the ground state of ^{6}Li whereas the $\alpha + {}^{12}$C system possesses two bound states in the 0^+ channel. One of the main results of the present paper is the conclusion that the Δ method is more effective for heavier systems with large values of the Coulomb parameter whereas for light systems with small values of the Coulomb parameter the ERF method might be preferable.

The paper is organized as follows. Section II provides a brief outline of the general formalism of the elastic scattering for the superposition of a short-range and the Coulomb interactions, which is necessary for the subsequent discussion. Sections III and IV deal with the $d + \alpha$ and $\alpha + {}^{12}$C systems, respectively. The problem of the convergence of the approximate expressions for the Δ function is discussed in Sec. V and in the appendix.

Throughout the paper, we use the system of units in which $\hbar = c = 1$.

II. BASIC FORMALISM

In this section, we recapitulate basic formulas which are necessary for the subsequent discussion. The formalism has been published in more detail in Ref. [12].

The Coulomb-nuclear amplitude of elastic scattering of particles b and c is of the form

$$f_{NC}(\mathbf{k}) = \sum_{l=0}^{\infty}(2l+1)\exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}P_l(\cos\theta). \quad (1)$$

Here $\mathbf{k}$ is the relative momentum of b and c, θ is the c.m. scattering angle, $\sigma_l = \arg\Gamma(l+1+i\eta)$ and δ_l are the pure Coulomb and Coulomb-nuclear phase shifts, respectively, and $\Gamma(z)$ is the Γ function.

$$\eta = Z_b Z_c e^2 \mu/k \quad (2)$$

is the Coulomb parameter for the $b + c$ scattering state with the relative momentum k related to the energy by $k = \sqrt{2\mu E}$, $\mu = m_b m_c/(m_b + m_c)$, and m_i and $Z_i e$ are the mass and the electric charge of particle i.

The behavior of the Coulomb-nuclear partial-wave amplitude $f_l = [\exp(2i\delta_l) - 1]/2ik$ is irregular near $E = 0$. Therefore, one has to introduce the renormalized Coulomb-nuclear partial-wave amplitude $\tilde{f}_l$ [14–16]

$$\tilde{f}_l = \exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}\left[\frac{l!}{\Gamma(l+1+i\eta)}\right]^2 e^{\pi\eta}. \quad (3)$$

Equation (3) can be rewritten as

$$\tilde{f}_l = \frac{\exp(2i\delta_l)-1}{2ik}C_l^{-2}(\eta), \quad (4)$$

where $C_l(\eta)$ is the Coulomb penetration factor (or Gamow factor) determined by

$$C_l(\eta) = \left[\frac{2\pi\eta}{\exp(2\pi\eta)-1}v_l(\eta)\right]^{1/2}, \quad (5)$$

$$v_l(\eta) = \prod_{n=1}^{l}(1+\eta^2/n^2)\ (l>0), \quad v_0(\eta) = 1. \quad (6)$$

It was shown in Ref. [14] that the analytic properties of $\tilde{f}_l$ on the physical sheet of E are analogous to the ones of the partial-wave scattering amplitude for the short-range potential and it can be analytically continued into the negative-energy region.

The amplitude $\tilde{f}_l$ can be expressed in terms of the Coulomb-modified ERF $K_l(E)$ [14,16] by

$$\tilde{f}_l = \frac{k^{2l}}{K_l(E) - 2\eta k^{2l+1}h(\eta)v_l(\eta)} \quad (7)$$

$$= \frac{1}{kC_l^2(\eta)(\cot\delta_l - i)} \quad (8)$$

$$= \frac{1}{v_l^2\Delta_l(E) - ikC_l^2(\eta)}, \quad (9)$$

where

$$K_l(E) = k^{2l+1}\left[C_l^2(\eta)(\cot\delta_l - i) + 2\eta h(k)v_l(\eta)\right], \quad (10)$$

$$h(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta), \quad (11)$$

$$\Delta_l(E) = kC_0^2(\eta)\cot\delta_l, \quad (12)$$

and $\psi(x)$ is the digamma function. $\Delta_l(E)$ is the Δ function introduced in Ref. [11].

It was shown in Ref. [14] that function $K_l(E)$ defined by (10) is analytic near $E = 0$ and can be expanded into Taylor series in E. In the absence of the Coulomb interaction ($\eta = 0$), $K_l(E) = k^{2l+1}\cot\delta_l(k)$.

If the $b + c$ system has in the partial wave l the bound state a with the binding energy $\varepsilon = \varkappa^2/2\mu > 0$, then the amplitude $\tilde{f}_l$ has a pole at $E = -\varepsilon$. The residue of $\tilde{f}_l$ at this point is expressed in terms of the ANC $C^{(l)}_{a\to bc}$ [15] as

$$\mathrm{res}\,\tilde{f}_l(E)|_{E=-\varepsilon} = \lim_{E\to -\varepsilon}[(E+\varepsilon)\tilde{f}_l(E)] \tag{13}$$

$$= -\frac{1}{2\mu}\left[\frac{l!}{\Gamma(l+1+\eta_b)}\right]^2\left[C^{(l)}_{a\to bc}\right]^2, \tag{14}$$

where $\eta_b = Z_bZ_ce^2\mu/\varkappa$ is the Coulomb parameter for the $b + c$ bound state a.

In what follows, the short-range nuclear interaction between particles b and c is described by the square-well potential

$$V(r) = \begin{cases} -V_0 & \text{if} \quad 0 \leqslant r \leqslant R \\ 0 & \text{if} \quad r > R \end{cases}, \tag{15}$$

where R is the radius of the square well and $V_0 > 0$ is its depth.

The solution of the Schrödinger equation for the potential (15) plus the Coulomb interaction results in the following expression for the phase shift δ_l [12]:

$$\cot\delta_l = \frac{\frac{d\hat{G}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{G}_{l,\eta}(k,R)}{\frac{d\hat{F}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{F}_{l,\eta}(k,R)}. \tag{16}$$

Here $K = \sqrt{2\mu(E+V_0)}$, $\hat{F}_{l,\eta}(q,r) = F_l(\eta,qr)/qr$, $\hat{G}_{l,\eta}(q,r) = -G_l(\eta,qr)/qr$, and $F_l(\eta,\rho)$ and $G_l(\eta,\rho)$ are the regular and irregular Coulomb functions, respectively [17].

Equation (16) allows one to calculate the functions $K_l(E)$ and $\Delta_l(E)$ using Eqs. (10) and (12). Detailed derivation and explicit analytic expressions for $K_l(E)$ and $\Delta_l(E)$ are given in Ref. [12].

III. $d + \alpha$ SYSTEM

Consider the $d + \alpha$ system having one bound state corresponding to the ground-state of ^{6}Li with $l = 0$. For this system $m_b = m_d = 1877.79$ MeV, $m_c = m_\alpha = 3727.379$ MeV, $m_a = m_{^6\mathrm{Li}} = 5601.518$ MeV, $Z_bZ_c = 2$, binding energy $\varepsilon = m_d + m_\alpha - m_{^6\mathrm{Li}} = 1.474$ MeV.

Parameters of the square well $V_0 = 7.400955728$ MeV and $R = 3.963659401$ fm were found by fitting the binding energy and the ANC $C^{(0)}_{^6\mathrm{Li}\to\alpha d} = 2.29\,\mathrm{fm}^{-1/2}$ obtained in Ref. [8]. For brevity ANC $C^{(0)}_{^6\mathrm{Li}\to\alpha d}$ will be denoted as C.

A. Approximation of the ERF for the $d + \alpha$ system by the Taylor series

Consider first the approximation of the ERF $K_0(E)$ by the Taylor series in E at $E = 0$. Expansion into the Taylor series is performed using analytic expressions (10) and (16). In fact, we limit ourselves by the first several terms of the expansion. A polynomial obtained this way is then continued analytically to the negative-energy region to the bound-state pole.

Two versions of the approximation are considered:

TABLE I. Approximation of $K_0(E)$ for the $d + \alpha$ system.

	Version 1		Version 2	
N	ε (MeV)	C (fm$^{-1/2}$)	ε (MeV)	C (fm$^{-1/2}$)
2	1.4546	2.256	1.474	2.894
3	1.4729	2.2858	1.474	2.2902
4	1.4744	2.2917	1.474	2.28997
Exact value	1.474	2.29	1.474	2.29

Version 1. Both the binding energy and the ANC are found from the approximated form of $K_0(E)$.

Version 2. The binding energy is preset ($\varepsilon = 1.474$ MeV) and only the ANC is sought.

Actually, in the second version we approximate the function $F(E) = [K_0(E) - K_b]/(E + \varepsilon)$, where $K_b = 2\eta kh(\eta)|_{E=-\varepsilon}$ is the value of $K_0(E)$ at $E = -\varepsilon$. Function $F(E)$ is finite at $E = -\varepsilon$ and its approximation by the Taylor series guarantees the correct value of $K_0(E)$ at $E = -\varepsilon$, which is the correct position of the pole of the scattering amplitude corresponding to the bound state.

The results of the calculation of the binding energy (in the first version) and the ANC are presented in Table I. In this table, as well in all the following tables, N denotes the power of the approximating Taylor polynomial. The exact values of the corresponding quantities obtained by the exact calculations within the model used are shown in the last line of Table I. One can see that the convergence in N is quite good, especially within the second version.

The exact function $K_0(E)$ for the $d + \alpha$ system and its approximations by the Taylor polynomial of the third power in E ($N = 3$) are shown in Fig. 1 for two versions of the approximation. It is seen that $N = 3$ ensures a quite good description of the exact ERF $K_0(E)$ over the wide energy interval.

B. Approximation of the Δ function for the $d + \alpha$ system by the Taylor series

In this subsection, we will consider the function $\mathrm{Re}[D_0(E)] = K_0(E) - \mathrm{Re}[2\eta kh(\eta)]$, which is the real part of the denominator $D_0(E)$ of the partial-wave amplitude $\tilde{f}_0(E)$ for the $d + \alpha$ system. At $E < 0$ $\mathrm{Re}[D_0(E)] = \tilde{f}_0^{-1}(E)$ and the condition $\mathrm{Re}[D_0(E)] = 0$ is the condition of a pole of $\tilde{f}_0(E)$ corresponding to the bound state. At $l = 0$ $\mathrm{Re}[D_0(E)]$ coincides with the function $\Delta_l(E)$ [see Eq. (12)] introduced in Ref. [11]. Therefore, in what follows we will use the notation $\Delta_0(E)$ instead of $\mathrm{Re}[D_0(E)]$.

As in the case of $K_0(E)$ (see Sec. III A), we will approximate $\Delta_0(E)$ by the Taylor series in E at $E = 0$ with the subsequent continuation to the negative-energy region. We consider the same two versions of the approximation as in Sec. III A; however, in the first version we now use $\Delta_0(E)$ rather than $K_0(E)$. In addition, in the second version we actually approximate the function $\Delta_0(E)/(E + \varepsilon)$.

The results of the approximation of $\Delta_0(E)$ by the first several terms of the Taylor series are presented in Table II. An

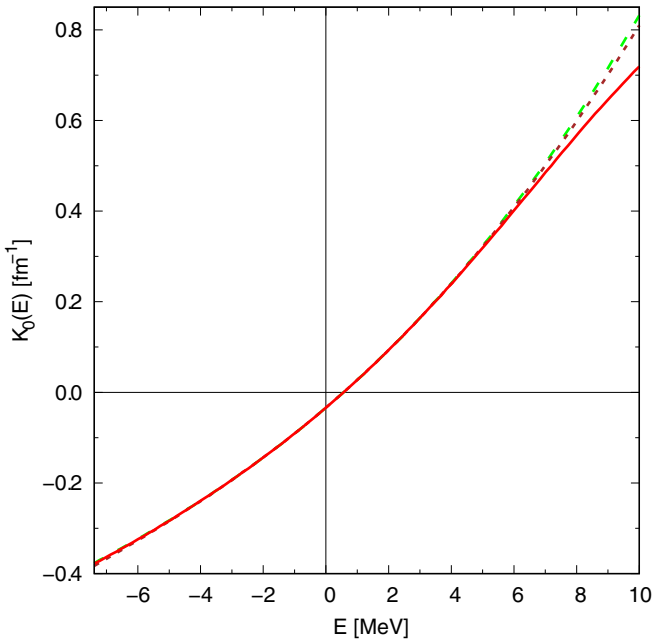

FIG. 1. The $K_0(E)$ function for the $d+\alpha$ system. The solid red line is the exact $K_0(E)$ function; the green dashed line is the approximation of $K_0(E)$ by the Taylor polynomial when the binding energy and the ANC of the bound state $(d\,\alpha)$ are not fixed (version 1); the brown dotted line is approximation of K_0 by the Taylor polynomial of the third order when only the binding energy of $(d\,\alpha)$ is fixed (version 2).

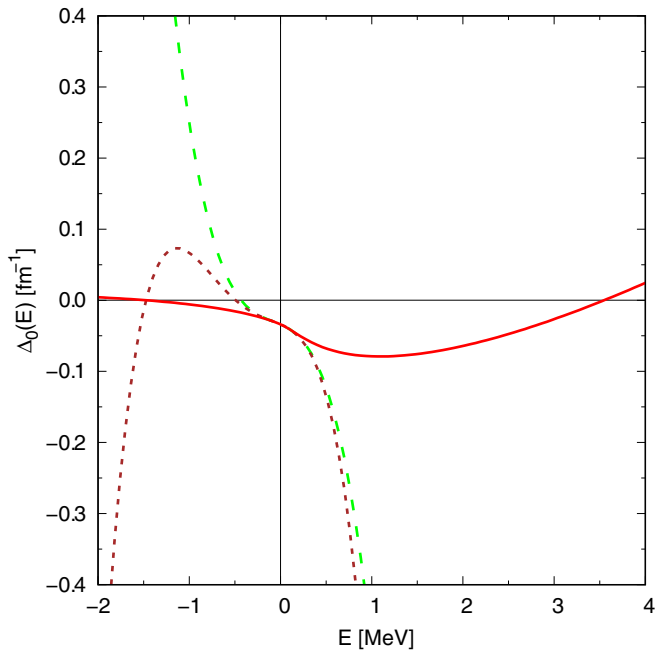

FIG. 2. The $\Delta_0(E)$ function for the $d+\alpha$ system. The notations are the same as in Fig. 1 but for $\Delta_0(E)$.

empty cell means that the given approximation does not lead to the bound state. The result marked by an asterisk is related to the fact that in the $N=3$ approximation the function $\Delta_0(E)$ turns into zero to the right of the point $E=-1.474$ MeV.

The exact function $\Delta_0(E)$ for the $d+\alpha$ system and its approximations by the Taylor polynomial of the third power in E $(N=3)$ are shown in Fig. 2 for two versions of the approximation. It is seen from Table II and Fig. 2 that the employed approximation of $\Delta_0(E)$ is absolutely unsatisfactory.

IV. $\alpha + {}^{12}$C SYSTEM WITH TWO BOUND 0^+ STATES

The goal of this paper is to find out which of the two extrapolation methods, the Coulomb-modified ERF $K_0(E)$ or the Ramírez Suárez-Sparenberg function $\Delta_0(E)$ [11], works better for the $\alpha + {}^{12}$C system in the $l=0$ partial wave with the ground and excited 0^+ bound states. To determine it, we use the same simple model as for the $d+\alpha$ system, namely

TABLE II. Approximation of $\Delta_0(E)$ for the $d+\alpha$ system.

	Version 1		Version 2	
N	ε (MeV)	C (fm$^{-1/2}$)	ε (MeV)	C (fm$^{-1/2}$)
2			1.474	0.799
3	0.432	0.565	0.493*	0.669
4			1.474	0.087
Exact values	1.474	2.29	1.474	2.29

a square-well nuclear potential plus the Coulomb interaction acting between two pointlike particles α and ^{12}C. In the realistic potential approach, the wave function of the relative motion of $\alpha + {}^{12}$C has two nodes at $r>0$ for the ground bound state. In our simplified model, we use one potential supporting two 0^+ bound states, the ground and the first excited ones. In this simplified approach, the ground bound-state wave function of the $\alpha + {}^{12}$C system is nodeless at $r>0$ while the wave function of the first excited state has one node at $r>0$.

For the $\alpha + {}^{12}$C system, we have $m_b = m_\alpha = 3727.379$ MeV, $m_c = m_{^{12}\mathrm{C}} = 11174.862$ MeV, $m_a = m_{^{16}\mathrm{O}} = 14895.079$ MeV, and $Z_b Z_c = 12$.

We adopt the square well potential with parameters $V_0 = 13.70363036$ MeV and $R = 6.009708703$ fm. The sum of this nuclear potential and the Coulomb interaction leads to two bound 0^+ states with the binding energies $\varepsilon_1 = 1.113$ MeV and $\varepsilon_2 = 7.162$ MeV. These binding energies coincide with the experimental ones. The ANC values for such a potential are $C_1 = 3218.458518$ fm$^{-1/2}$ and $C_2 = 3475.353169$ fm$^{-1/2}$ for the excited and ground states, respectively. Because we use a simplified potential model these ANCs should not be considered as realistic ones but they will help us to identify which extrapolation method works better for the $\alpha + {}^{12}$C system.

Note that in principle one may use an alternative way to find the parameters V_0 and R, namely, by fitting them to the value of ε_1 and to the value of C_1 obtained from the analysis of experimental data, e.g., from Ref. [18]. The qualitative results stated below do not depend on the way how the square-well parameters are chosen.

Since the considered $\alpha + {}^{12}$C system has two bound states, the ERF $K_0(E)$, as well as function $\Delta_0(E)$, has two poles: one at negative energy (E_{i2}) and another at positive energy (E_{i1}).

TABLE III. Approximation of $K_0(E)$ for the $\alpha + {}^{12}$C system.

	Version 1		Version 2	
N	ε_1 (MeV)	C_1 (fm$^{-1/2}$)	ε_1 (MeV)	C_1 (fm$^{-1/2}$)
2	0.457	14361	1.113	10928
3			1.113	3090.07
4	1.042	3060.34	1.113	3230.43
5	1.122	3265.97	1.113	3217.94
6	1.1126	3215.71	1.113	3216.71
Exact values	1.113	3218.46	1.113	3218.46

The pole at negative energy leads to the change of the sign of the partial-wave amplitude $\tilde{f}_0(E)$ in the interval between the points corresponding to the two bound states. This guarantees the correct signs of the residues of $\tilde{f}_0$ at both poles $E = -\varepsilon_1$ and $E = -\varepsilon_2$ (see Refs. [19,20]). As is seen from Eq. (14), the sign of both residues should be negative in order to guarantee that the ANC is real. The pole at $E > 0$ is due to the Levinson theorem. The above mentioned values of V_0 and R result in $E_{i2} = -4.48135$ MeV and $E_{i1} = 25.315$ MeV.

A. Approximation of the ERF for the $\alpha + {}^{12}$C system: Search for the parameters of the excited 0^+ state

The approximation versions 1 and 2 are similar to those for the $d + \alpha$ system. Within the version 2 the binding energy ε_1 of the excited state is fixed. The presence of the ground state and of the pole at E_{i2} is not taken into account explicitly.

The results of the calculations are presented in Table III and Figs. 3 and 4. The exact function $K_0(E)$ for the $\alpha + {}^{12}$C system in the 0^+ channel is shown in Fig. 3 in a wide energy interval.

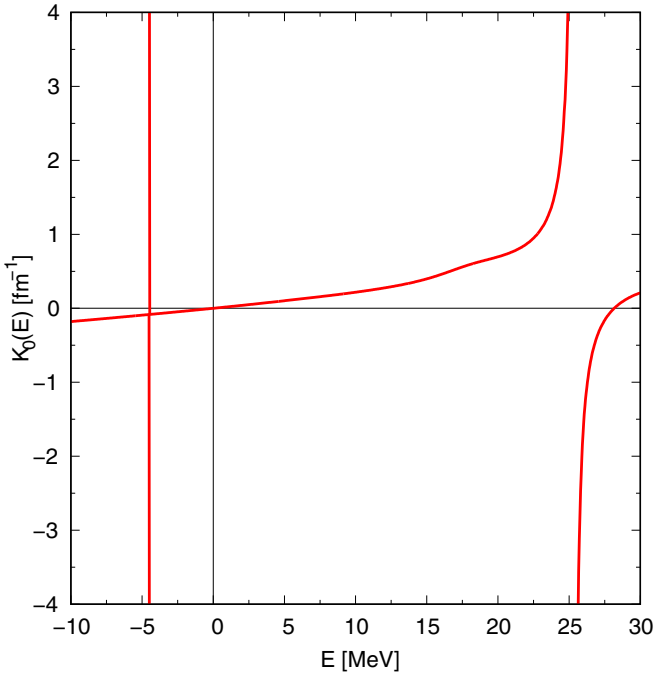

FIG. 3. The exact $K_0(E)$ function for the system $\alpha + {}^{12}$C with two 0^+ bound states. The pole at the negative energy is very narrow.

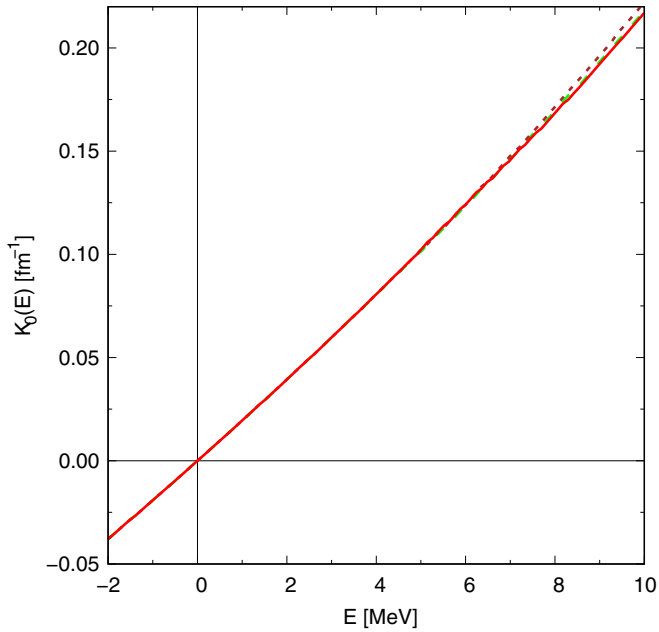

FIG. 4. The same as in Fig. 1 but for the system $\alpha + {}^{12}$C.

Note that $K_0(E)$ is not equal to zero at $E = 0$; however, it is rather small: $K_0 = -1.609\,10^{-6}$ fm^{-1}. This fact leads to a large value of the scattering length. In Fig. 4, we present the exact function $K_0(E)$ and its approximation by the Taylor polynomial of the third power in E for two versions of the approximation. The energy interval is much more narrow than in Fig. 3 and does not include the poles of $K_0(E)$ which cannot be described by the Taylor polynomial approximation. It is seen from Table III that, although the results are quite satisfactory, the convergence to the exact values is slower than in the case of the $d + \alpha$ system.

B. Approximation of the Δ function for the $\alpha + {}^{12}$C system by the Taylor series: Search for the parameters of the excited 0^+ state

Consider three versions of the approximation:

Version 1. Both the binding energy and the ANC are found from the approximated form of $\Delta_0(E)$.

Version 2. The binding energy is preset ($\varepsilon_1 = 1.113$ MeV) and only the ANC is sought. Function $\Delta_0(E)/(E + \varepsilon_1)$ is approximated.

Version 3. The binding energy is preset ($\varepsilon_1 = 1.113$ MeV) and only the ANC is sought. Function $\ln(\Delta_0(E)/(E + \varepsilon_1))$ is approximated by the Taylor expansion.

Using version 3 is related to the fact that the Δ function changes drastically near $E = 0$.

The results of the calculations are presented in Table IV. The exact $\Delta_0(E)$ function for the $\alpha + {}^{12}$C system is shown in Figs. 5 and 6 for different energy intervals. Figure 7 presents the exact $\Delta_0(E)$ function for the $\alpha + {}^{12}$C system and its approximation by the Taylor polynomial of the third power in E corresponding to the aforementioned three versions of the approximation. As

TABLE IV. Approximation of $\Delta_0(E)$ for the $\alpha + {}^{12}$C system.

	Version 1		Version 2		Version 3	
N	ε_1 (MeV)	C_1 (fm$^{-1/2}$)	ε_1 (MeV)	C_1 (fm$^{-1/2}$)	ε_1 (MeV)	C_1 (fm$^{-1/2}$)
2			1.113	2813.41	1.113	3211.95
3	0.915	3296.90	1.113	3421.48	1.113	3224.88
4			1.113	3153.03	1.113	3216.54
5	1.064	3048.47	1.113	3245.73	1.113	3219.89
6	1.147	3476.67	1.113	3205.76	1.113	3217.52
7	1.100	3131.55	1.113	3226.06	1.113	3219.26
Exact values	1.113	3218.46	1.113	3218.46	1.113	3218.46

one can see from Table IV and Fig. 7, the Taylor polynomial approximation of the $\Delta_0(E)$ function for the $\alpha + {}^{12}$C system, in contrast to the lighter $d + \alpha$ system, turns out to be a quite good approximation.

C. Approximation of the Δ function for the $\alpha + {}^{12}$C system by the Taylor series: Search for the parameters of the ground 0^+ state

If one intends to determine the ANC C_2 for the ground 0^+ state, it is necessary to explicitly include in the approximation form of $\Delta_0(E)$ the presence of the pole E_{i2} at $E < 0$.

Consider two versions of the approximation:

Version 1. Approximation of the function $\Delta_0(E)(E - E_{i2})/(E + \varepsilon_1)(E + \varepsilon_2)$

Version 2. Approximation of the function $\ln[-\Delta_0(E)(E - E_{i2})/(E + \varepsilon_1)(E + \varepsilon_2)]$.

Within both versions the positions of two bound states and of the pole E_{i2} are preset. The pole E_{i1} lies far from the negative-energy region and its influence can be ignored.

As before, the approximation is based on the Taylor expansion at $E = 0$. The results of the calculations with the two versions of the approximation are presented in Table V and in Fig. 8. An empty cell in the table means that the given version of the approximation gives a wrong sign for the derivative of $\Delta_0(E)$ at $E = -\varepsilon_2$ and, therefore, does not lead to a genuine bound state. It is clear from Table V that the approximation used here does not allow one to obtain any reasonable result for the ANC C_2 corresponding to the ground state of ^{16}O even if one presets explicitly the position of the pole of $\Delta_0(E)$ at $E < 0$. This is not surprising since the ground state is located far from the point $E = 0$ at which the expansion in E is performed. The situation gets much worse if one tries to determine C_2 by extrapolating the experimental data since the position of the pole E_{i2} is not known from the experiments.

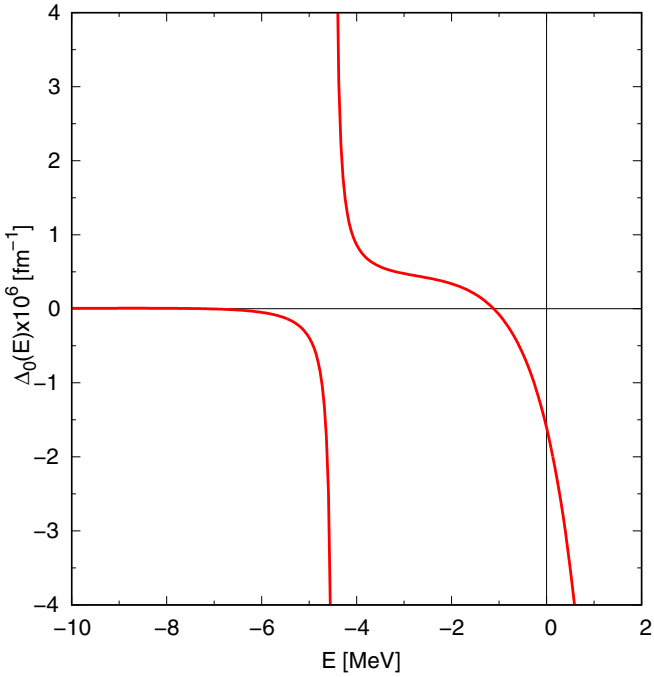

FIG. 5. The exact $\Delta_0(E)$ function for the system $\alpha + {}^{12}$C with two bound states. The pole at negative energy is located at $E_{i2} = -4.48135$ MeV.

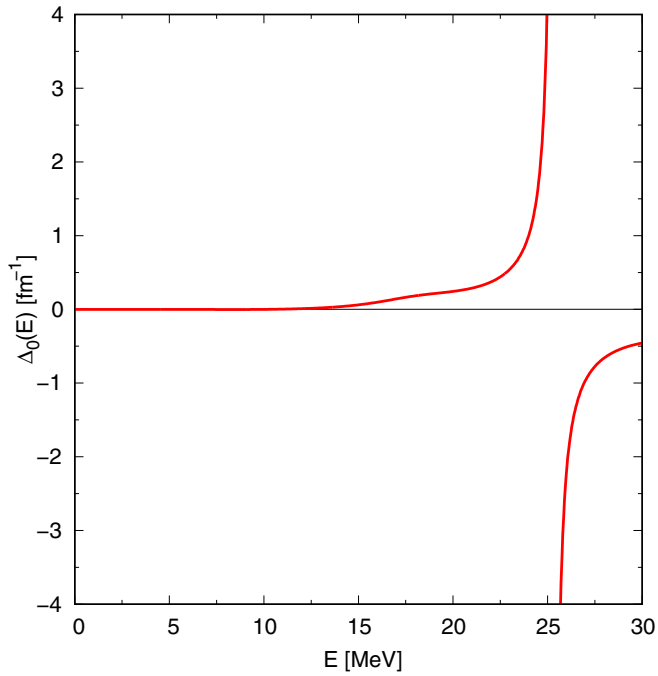

FIG. 6. The exact $\Delta_0(E)$ function for the system $\alpha + {}^{12}$C with two bound states. The pole at positive energy is located at $E_{i1} = 25.315$ MeV.

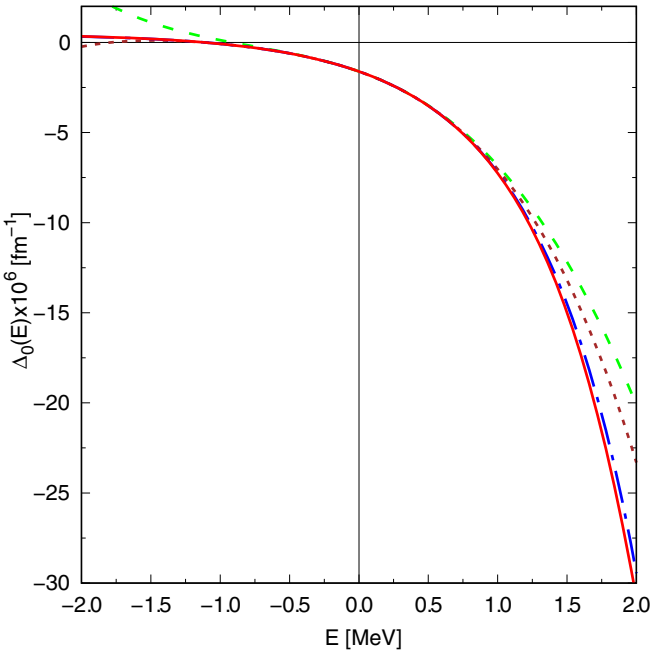

FIG. 7. The $\Delta_0(E)$ function for the $\alpha + {}^{12}$C system. The solid red line is the exact $\Delta_0(E)$ function; the green dashed line is the approximation of the $\Delta_0(E)$ function by the Taylor polynomial of the third order when the binding energy and the ANC of the bound state ($\alpha + {}^{12}$C) are not fixed; the brown dotted line is approximation of $\Delta_0(E)$ by the Taylor polynomial of the third order when the binding energy of ($\alpha + {}^{12}$C) is fixed while the ANC is a fitting parameter; and the blue dash-dotted line is obtained using the approximation of $\ln(\Delta_0(E)/(E+\varepsilon_1))$ by the third-order Taylor polynomial when the binding energy and the ANC of the bound state ($\alpha + {}^{12}$C) are not fixed.

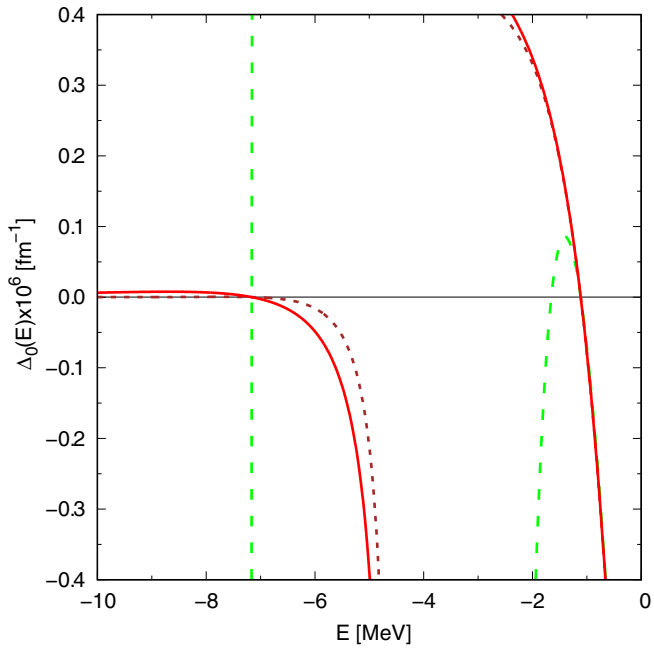

FIG. 8. The $\Delta_0(E)$ function for the system $\alpha + {}^{12}$C with two 0^+ bound states. The solid red line is the exact $\Delta_0(E)$; the brown dotted line is obtained using approximation of the function $\Delta_0(E)(E-E_{i2})/(E+\varepsilon_1)(E+\varepsilon_2)$ by the Taylor polynomial of the third order; and the green dashed line corresponds to the approximation of the function $\ln[-\Delta_0(E)(E-E_{i2})/(E+\varepsilon_1)(E+\varepsilon_2)]$ by the Taylor polynomial of the third order.

Note that the attempts to determine C_2 by extrapolating the function $K_0(E)$ or $\Delta_0(E)$ from the positive to the negative energy region were made in Refs. [13,21]. In these papers, the parameters of the analytic approximation of $K_0(E)$ and $\Delta_0(E)$ were fitted to the results of the phase-shift analysis of the elastic $\alpha + {}^{12}$C scattering at low energies. However, the C_2 values presented in these papers could hardly be taken seriously for the following reasons. In Ref. [21], while continuing the $K_0(E)$ function to the point corresponding to the ground state, the authors ignored the presence of the excited 0^+ state which affects significantly the behavior of $K_0(E)$ at $E < 0$. In Ref. [13], the excited state was taken into account; however, the approximated analytic form of $\Delta_0(E)$ used by the authors ignored the existence of the pole of $\Delta_0(E)$ at $E < 0$. This fact is the reason for the wrong sign of the residue of the partial-wave scattering amplitude $\tilde{f}_0$ at the pole corresponding to the ground state. It leads to an unphysical imaginary value of the ANC C_2. Furthermore, the real value of C_2 presented in Ref. [13] is also erroneous. This is due to the improper manipulation with the absolute value sign for the residue of $\tilde{f}_0$.

It is worth mentioning that the exact partial-wave $\alpha + {}^{12}$C scattering amplitudes, in contrast to our theoretical model, possess a number of singularities (branching points) situated at $E < 0$ between the ground and excited 0^+ state poles. These singularities are due to the following Feynman diagrams contributing to the elastic $\alpha + {}^{12}$C scattering amplitude:

(1) The loop diagram describing two-pion exchange between α and ^{12}C.
(2) The pole diagram describing the ^{8}Be transfer process (or the loop diagram describing two-α transfer).
(3) The triangle diagrams describing scattering of an α particle on virtual nucleons containing in ^{12}C.

It is obvious that the approximation of the $K_0(E)$ or $\Delta_0(E)$ function by Taylor polynomials or rational functions cannot take into account the presence of these singularities. Moreover, we think that, even in the absence of the excited 0^+ state, the extrapolation distance ($\approx$7 MeV) would be too large to obtain sensible results for the ANC. In the realistic case, when extrapolating the scattering phase shift to the ground bound

TABLE V. Approximation of $\Delta_0(E)$ for the $\alpha + {}^{12}$C system taking into account the ground 0^+ state.

	Version 1		Version 2	
N	C_1 (fm$^{-1/2}$)	C_2 (fm$^{-1/2}$)	C_1 (fm$^{-1/2}$)	C_2 (fm$^{-1/2}$)
2	2714.48	72.32	3204.12	1953.26
3	3496.00		3223.23	9520.82
4	3132.61	29.62	3216.19	223.77
5	3254.24		3219.81	5.6×10^7
Exact values	3218.46	3475.35	3218.46	3475.35

state in the $\alpha + {}^{12}$C system, one has to take into account the singularities of the aforementioned diagrams.

V. CONVERGENCE OF THE APPROXIMATION FOR THE Δ FUNCTION

The renormalized Coulomb-nuclear partial-wave scattering amplitude $\tilde{f}_0(E)$ can be written as follows ($l = 0$):

$$\tilde{f}_0(E) = 1/D_0(E), \tag{17}$$

where

$$D_0(E) = K_0(E) - R(E), \tag{18}$$

$$R(E) = 2\alpha_1 h(\eta), \tag{19}$$

$$h(\eta) = 2\alpha_1[\psi(i\eta) - \ln(i\eta) + 1/(2i\eta)], \tag{20}$$

and $\alpha_1 = z_b z_c e^2 \mu > 0$, $\eta = \alpha_1/\sqrt{2\mu E}$. We note that the Δ function for $l = 0$, $\Delta_0(E)$, which we are interested in, is directly related to $D_0(E)$: $\Delta_0(E) = \mathrm{Re}[D_0(E)]$.

It is known that the ERF $K_0(E)$ can be expanded in powers of E. In order to decide on the problem of similar power expansion and the Taylor polynomial approximation for the whole denominator $D_0(E)$ [and hence for $\Delta_0(E)$], we consider the properties of the function $h(\eta)$.

Since at $E \to 0$ $\eta \to \infty$, one may use the asymptotic expansion for $\psi(i\eta)$ [22] which results in the following expansion of $h(\eta)$:

$$h(\eta) = -\sum_{\nu=1}^{\infty} \frac{B_{2\nu}}{2\nu(i\eta)^{2\nu}} \tag{21}$$

$$= -\sum_{\nu=1}^{\infty} \frac{B_{2\nu}}{2\nu}\left(\frac{-2\mu E}{\alpha_1^2}\right)^{\nu} \tag{22}$$

$$= -\sum_{\nu=1}^{n-1} \frac{B_{2\nu}}{2\nu}\left(\frac{-2\mu E}{\alpha_1^2}\right)^{\nu} - U_n(E) \tag{23}$$

$$\equiv h_n(\eta) - U_n(E), \tag{24}$$

where $B_{2\nu}$ are the Bernoulli numbers. At $n = 1$, the sum in (23) is equal to zero. The form and the features of the residual term $U_n(E)$ are considered in the appendix. In the present section, we consider the separate terms of the expansion (23).

The series (23) can be considered as the expansion in E. However, due to the features of the Bernoulli numbers, the series (23) is asymptotic, that is, divergent. Nevertheless, it is worthwhile to investigate the first few terms in (23) which contribute to the Taylor polynomial approximation of $\Delta_0(E)$. The rate of convergence of the series (23) at given E is determined by the quantity $\alpha_2 = 2\mu/\alpha_1^2$. As α_2 gets smaller, the convergence becomes faster.

For the $d + \alpha$ system, the value of α_2 is rather large: $\alpha_2 = 7.53$ MeV^{-1}. As a result, the approximation of $h(\eta)$ by the first terms of the series (23) is poor. This is seen in Fig. 9 which displays the real part $\mathrm{Re}[R(E)]$ for the $d + \alpha$ system. However, for the heavier $\alpha + {}^{12}$C system α_2 is two orders of magnitude smaller than for the $d + \alpha$ system: $\alpha_2 = 0.0933$ MeV^{-1}. Therefore, for this system $h(\eta)$ can be successfully approximated by first few terms of the expansion (23) in a wide energy interval. This result is illustrated in Fig. 10 which displays the real part $\mathrm{Re}[R(E)]$ for the $\alpha + {}^{12}$C system.

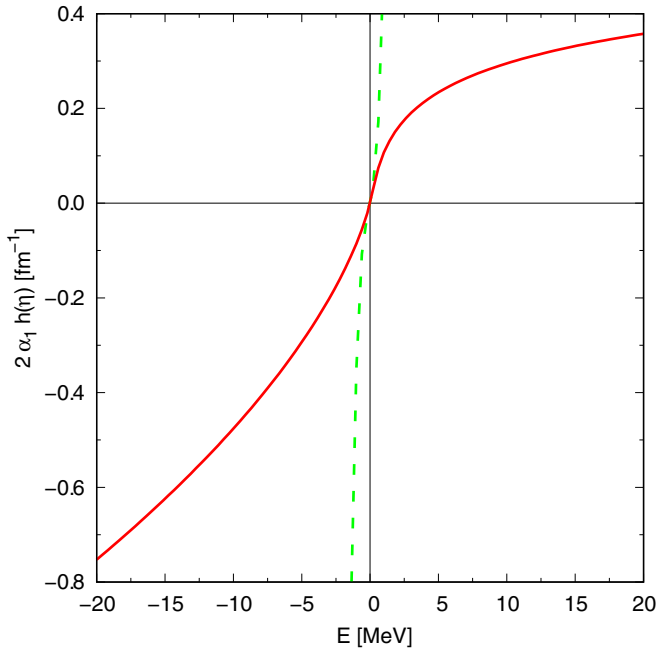

FIG. 9. The real part $\mathrm{Re}[R(E)]$ for the $d + \alpha$ system. The solid red line is the exact result; the dashed green line is the asymptotic expansion of $\mathrm{Re}[R(E)]$ up to E^3.

Thus, one can conclude that the Taylor polynomial approximation of the function $R(E) = 2\alpha_1 h(\eta)$ and hence of the functions $D_0(E)$ and $\Delta_0(E)$ is more effective for systems

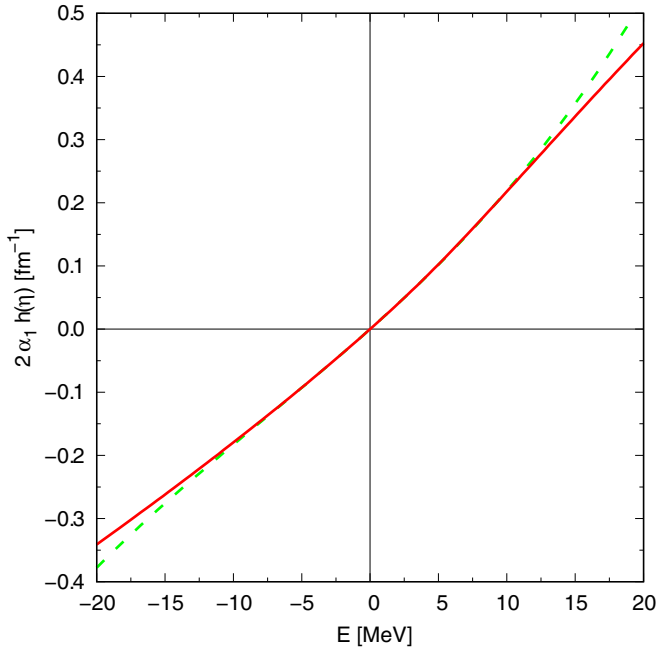

FIG. 10. The real part $\mathrm{Re}[R(E)]$ for the $\alpha + {}^{12}$C system. The solid red line is the exact result; the dashed green line is the asymptotic expansion of $\mathrm{Re}[R(E)]$ up to E^3.

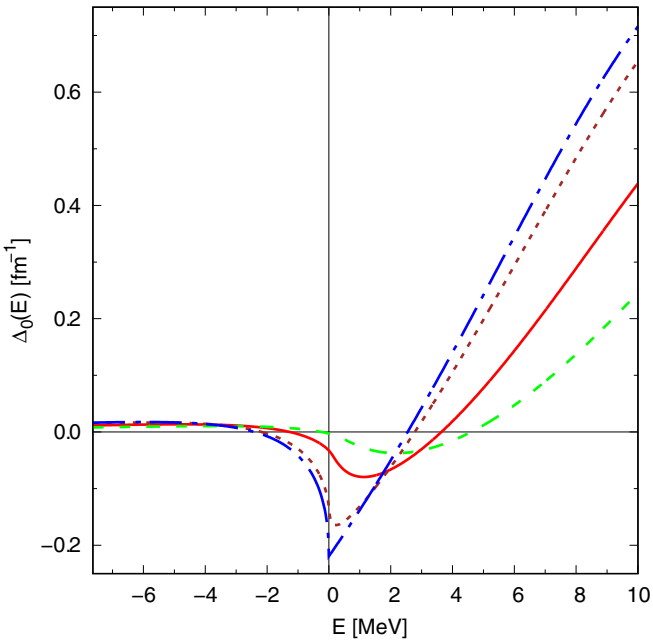

FIG. 11. The dependence of the $\Delta_0(E)$ function on the charge-scaling factor β (see the text) for $d+\alpha$. Solid red line, $\beta = 1$; dashed green line, $\beta = 2$; dotted brown line, $\beta = 0.2$; dash-dotted blue line, $\beta = 0$.

with larger values of the product of charges $Z_b Z_c e^2$ and the reduced mass μ. This inference is clearly demonstrated in Fig. 11, which displays the calculations of $\Delta_0(E)$ for the $d+\alpha$ system obtained by substituting the quantity $Z_b Z_c$ by $\beta Z_b Z_c$ where the correction factor β assumes the values 0, 0.2, 1, and 2. It is seen that the smaller β is, the less smooth is the joining of two parts of the curves of $\Delta_0(E)$ corresponding to $E > 0$ and $E < 0$ at $E = 0$. Naturally, the effectiveness of the Taylor polynomial approximation of the function $\Delta_0(E)$ also drops with decreasing β. At $\beta = 0$ (the Coulomb interaction is switched off), the $\Delta_0(E)$ function turns into the ERF $k \cot \delta$ and ceases to coincide with the denominator of the amplitude $\tilde{f}_0(E)$ at $E < 0$.

The results obtained in this section corroborate and elucidate the conclusion drawn from the results of Secs. III and IV, namely, that the Taylor polynomial approximation of the Δ function is more effective for heavier nuclear systems with larger values of the Coulomb parameter η.

VI. CONCLUSIONS

In the present paper, within an exactly solvable model, we have investigated the applicability of the effective range function (ERF) and the Δ function suggested in Ref. [11] for continuation of scattering data to the negative-energy region in order to determine ANCs. The $d+\alpha$ and $\alpha + {}^{12}$C systems have been considered. It is demonstrated that if the system under consideration features two bound states with the same quantum numbers, then the ERF and Δ functions have two poles: One in the positive-energy region and the other in the negative-energy region, between the energies corresponding to the two bound states. It is also shown that if the system has more than one bound state with the same quantum numbers, then the method of the continuation in energy of the ERF or Δ functions practically allows one to determine the binding energy and the ANC for the highest state only. To determine the features of other (lower lying) bound states, one should apply alternative methods, e.g., the method of analytic continuation of differential cross sections of transfer reactions to the pole in the scattering angle or find peripheral transfer reactions populating the bound states of interest.

It is demonstrated that the approximation of the Δ function by the first several terms of its Taylor expansion can be successfully used to determine binding energies and ANCs for the nuclear systems with sufficiently large Z. The procedure is less effective for the systems with small Z. The criterion for the applicability of such an approximation is derived.

The renormalized Coulomb-nuclear amplitude $\tilde{f}_l(E)$ was introduced in Ref. [14]. It was shown that the analytic properties of $\tilde{f}_l(E)$ on the physical sheet are similar to those of the scattering amplitude generated by the short-range potential. On the other hand, it was also stated [14] that $\tilde{f}_l(E)$ possesses the essential singularity at $E = 0$. These two assertions contradict each other since the scattering amplitude for the short-range potential does not possess an essential singularity at $E = 0$. It is known that an arbitrary function $\varphi(z)$ has no definite limit at $z \to z_0$ if z_0 is a point of an essential singularity. In the vicinity of the essential singularity, the function may take any value. The calculations performed within the model used in the present paper have shown that the amplitude $\tilde{f}_0(E)$ has a definite limit at $E \to 0$ that does not depend on the direction from which E approaches zero. It means that the point $E = 0$ is not an essential singularity point of $\tilde{f}_0(E)$. The amplitude $\tilde{f}_0(E)$ possesses the unitary cut $0 \leqslant E < \infty$ on which $\mathrm{Im}[\tilde{f}_0(E)]$ has a discontinuity.

In the present paper, the approximate versions of the ERF and Δ functions have been constructed on the basis of Taylor expansions at zero energy. Of course, there are alternative ways to construct the approximate forms of these functions, e.g., by rational functions in the form of Padé approximants. We expect that using Padé approximants should not change the qualitative conclusions made above. The test calculations using Padé approximants did not improve appreciably the unsatisfactory results obtained in Sec. III B for the Taylor polynomial approximation of the Δ function. Furthermore, though all calculations were performed for $l = 0$, we believe the inferences made in the present paper should be valid for arbitrary l.

ACKNOWLEDGMENTS

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APPENDIX

Consider in more detail the function $R(E)$ [see Eq. (19)] discussed in Sec. V. Using the asymptotic expansion of $\psi(z)$ at $|z| \to \infty$ [22] and inserting $\eta = \alpha_1/\sqrt{2\mu E}$, one can write $h(\eta)$ in the form of Eq. (23), where the residual term $U_n(z)$ is subject to

$$|U_n(z)| \leqslant \frac{|B_{2n}|}{2n\cos^{2n+1}(\arg(z)/2)|z|^{2n}}, \ |\arg(z)| < \pi. \quad \text{(A1)}$$

For positive energies ($E > 0$), $z = i\eta = i\alpha_1/\sqrt{2\mu E}$. Therefore, $\arg(z) = \pi/2$. Then, taking into account $\cos(\pi/4) = 1/\sqrt{2}$, we can write

$$|U_n(E)| \leqslant \frac{\sqrt{2}|B_{2n}|2^n}{2n}\left(\frac{2\mu E}{\alpha_1^2}\right)^n. \quad \text{(A2)}$$

For negative energies ($E < 0$), $z = i\eta = \alpha_1/\sqrt{2\mu|E|}$. Therefore, $\arg(z) = 0$. Then, using $\cos(0) = 1$, we have

$$|U_n(E)| \leqslant \frac{|B_{2n}|}{2n}\left(\frac{2\mu|E|}{\alpha_1^2}\right)^n. \quad \text{(A3)}$$

If the series (23) were convergent, then at $n \to \infty$ $U_n(z) \to 0$. However, the series (23) is asymptotic and the residual term behaves differently. With increasing n, $|U_n(E)|$ decreases but beginning with some n it starts to grow unrestrictedly. The corresponding value of n depends on E. It is useless to increase this value of n since at this value the partial sum of the series (23) is the best approximation of the exact value of $h(\eta)$. It is natural to set this value equal to the maximal value of n at which the following condition holds:

$$\left|\frac{U_{n+1}(E)}{U_n(E)}\right| < 1. \quad \text{(A4)}$$

Evaluation of the residual term allows one to evaluate n by setting $U_n(E)$ equal to its maximal value. Such evaluation is very strict; nevertheless it makes finding the upper boundary for n possible.

For positive energies the condition (A4) takes the form

$$\frac{2n|B_{2n+2}|}{(n+1)|B_{2n}|}\left(\frac{2\mu E}{\alpha_1^2}\right) < 1. \quad \text{(A5)}$$

For negative energies Eq. (A4) becomes

$$\frac{n|B_{2n+2}|}{(n+1)|B_{2n}|}\left(\frac{2\mu|E|}{\alpha_1^2}\right) < 1. \quad \text{(A6)}$$

Condition (A5) is more strict than (A6). If Eq. (A5) holds for some values of n and $E > 0$, then condition (A6) also holds for the same n but for $E' = -E < 0$. Therefore, in what follows, we will use the more strict condition (A5) to analyze specific systems.

The maximal value of n at given E and, vice versa, the maximal value of E at given n depend on the quantity $\alpha_2 = 2\mu/\alpha_1^2$. As α_2 gets smaller, n becomes larger for given E or E becomes larger for given n. This means that the smaller α_2 is, the better the exact function $h(\eta)$ is approximated by the function $h_n(\eta)$ [(see Eq. (24), which is the partial sum of the series (23)].

For the $d + \alpha$ system $\alpha_2 = 7.53$ MeV^{-1}. Let us approximate the function $R(E)$ by the Taylor polynomial of the second power in E, that is, by the first three terms of the sum (23). In that case $n = 4$. The maximal value of energy $E_n > 0$, at which the condition (A5) holds, is determined by the equation

$$E_n = \frac{1}{\alpha_2}\frac{(n+1)|B_{2n}|}{2n|B_{2n+2}|}. \quad \text{(A7)}$$

At $n = 4$ Eq. (A7) results in $E_4 = 0.036532$ MeV. Thus the energy interval, in which the employed approximation can satisfactorily describe the exact function $R(E)$, is extremely narrow and is not seen in Fig. 9. At the same time, for the $\alpha + {}^{12}$C system $\alpha_2 = 0.0933$ MeV^{-1} and $E_4 = 2.9460$ MeV. Therefore, the favorable energy interval is by two orders broader than for the $d + \alpha$ system, which results in the successful Taylor polynomial approximation of $R(E)$ (see Fig. 10).

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