Reexamination of a novel determination of density, temperature, and symmetry energy based on a modified Fisher model

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In this article, a mistake in the formulation of the modified Fisher model (MFM) derived in the pioneering works of the Purdue group is addressed and corrected by reversing the sign of the mixing entropy term in the original formulation. The errors in the results of the previous MFM-related studies, such as isotopic yield distribution, isobaric yield ratios, isoscaling, m scaling, self-consistent determination of density, symmetry energy, and temperature, and density and temperature determination related to the intermediate mass fragment (IMF) freezeout, are quantitatively analyzed. It is found that the errors originating from the mistake in sign of the mixing entropy term are generally small and even have no effect in some cases.

I. INTRODUCTION

In 1967, M. E. Fisher proposed a droplet model of a second-order phase transition to describe the power law behavior of the "fragment" mass distribution around the critical point for a liquid-gas phase transition [1]. In the early 1980s, based on the Fisher model (FM), the Purdue group generated a novel classical droplet model, which was the so-called modified Fisher model (MFM), and introduced it into nuclear physics [2–4]. Taking into account basic nuclear properties, such as the Coulomb force, pairing effect, and proton-neutron two-component mixture, the MFM is capable of describing the general features of mass and isotopic yields [2–4]. Recently, a series of experimental and theoretical investigations based on the MFM have been carried out to explore the symmetry energy of the nuclear equation of state and the critical behavior of hot fragmenting matter [5–32].

However, the MFM formulation from the pioneering works of the Purdue group in Refs. [2–4] contained a mistake which originated from the wrong sign in front of the mixing entropy term and caused some errors. In this work, we address this mistake in their formulation, and quantitative analyses are given for the errors originating from the mistake in typical MFM-related studies. This article is organized as follows. In Sec. II we briefly describe the formulism of FM and MFM, and provide a corrected formulation of the MFM. In Sec. III, the resultant changes in isotopic yield distribution, isobaric yield ratios, isoscaling, m scaling, self-consistent determination of density, symmetry energy, and temperature, and density and temperature determination related to the intermediate mass

fragment (IMF) freezeout are quantitatively discussed. Finally, a summary is given in Sec. IV.

II. FORMULISM OF FM AND MFM

A. FM and MFM

In FM [1], for a single constituent system, a parent system with $A + B$ particles undergoes a phase transition into a gas phase containing B particles and a droplet containing A particles. The free energy of the system in the initial and final phases can be written as

$$F_{\text{initial}} = \mu_g(A + B) - TS, \tag{1}$$

$$F_{\text{final}} = \mu_l A + \mu_g B + 4\pi R^2 \sigma - T(S - \tau \ln A). \tag{2}$$

Here the Helmholtz free energy is used in this context. In Eqs. (1) and (2), μ_l and μ_g are, respectively, the chemical potentials of the liquid and gas phases, and S is the total entropy of the initial state. The third term in Eq. (2) is the surface contribution for a spherical droplet with radius R ($R = r_0 A^{1/3}$) and surface tension parameter σ. Near the critical point, σ can be expressed as a function of temperature as

$$\sigma(T, T_c) = \begin{cases} \sigma_0 \left(1 + \frac{3T}{2T_c}\right)\left(1 - \frac{T}{T_c}\right)^{3/2} & (T \geqslant T_c), \\ 0 & (T \geqslant T_c), \end{cases} \tag{3}$$

where T_c is the temperature at the critical point. The last term in Eq. (2) originates from the entropy change of the system when the droplet is formed. The term $\tau \ln A$ is the entropy change caused by liquefaction, introduced by Fisher [1], where τ is the critical exponent. $\tau \ln A$ with $\tau > 0$ is subtracted from the total entropy because when liquefaction occurs the entropy of the system decreases.

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The free energy of the droplet can be obtained as the difference between Eqs. (1) and (2):

$$F_{\text{droplet}} = F_{\text{final}} - F_{\text{initial}}$$
$$= (\mu_l - \mu_g)A + 4\pi r_0^2 \sigma A^{2/3} + T\tau \ln A. \quad (4)$$

In a canonical ensemble, free energy can be deduced as

$$-F_{\text{droplet}} = T \ln(\mathbf{Z}), \quad (5)$$

where $\mathbf{Z}$ is the partition function and it is proportional to the yield $Y(A)$ of a given type of droplets with A particles:

$$Y(A) \propto \mathbf{Z} = \exp\left(-\frac{F_{\text{droplet}}}{T}\right). \quad (6)$$

Equation (6) is the mathematical expression of Fisher model.

In order to apply the Fisher model to a nuclear multifragmentation process, two constituents (neutrons and protons) and the characteristics of nuclear force have to be taken into account in the model. In the framework of MFM, from analogy to Eq. (4), the free energy of an isotopic fragment with mass number A and $I = N - Z$ (N neutrons and Z protons) is expressed as

$$F(I,A) = [-W(I,A) - \mu_n N - \mu_p Z] - T(-\tau \ln A + S_{\text{mix}}). \quad (7)$$

Inserting Eq. (7) into Eq. (6), the yield of an isotope with A and $I = N - Z$ produced in a multi-fragmentation reaction, can be written as

$$Y(I,A) = Y_0 A^{-\tau} \exp\left[\frac{W(I,A) + \mu_n N + \mu_p Z}{T} + S_{\text{mix}}\right]. \quad (8)$$

In Eq. (8), the critical exponent τ is often taken as $\tau = 2.3$ in our previous works [5–14], according to Ref. [15]. $W(I,A)$ is given, utilizing the generalized Weizsäcker-Bethe semiclassical mass formula [33,34], and can be approximated as

$$W(I,A) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}}$$
$$- a_{\text{sym}} \frac{I^2}{A} - a_p \frac{\delta_p}{A^{1/2}},$$
$$\delta_p = -\frac{(-1)^Z + (-1)^N}{2}, \quad (9)$$

where a_v, a_s, a_c, a_{sym}, and a_p are coefficients for quantifying the contributions of volume, surface, Coulomb, symmetry energy, and pairing effects. μ_n (μ_p) is the neutron (proton) chemical potential. S_{mix} is the mixing entropy.

The MFM is formulated at a critical temperature where one can expect the nucleons are in a gas phase in the initial stage and transition into a cluster-gas mixed phase. The mixing entropy term provides a simple expression to describe the entropy change originating from the component transition from a single-component system to a proton-neutron two-component system for such a phase transition. One should note that in Eq. (8) the mixing entropy term has a positive contribution to the total entropy, compared to $\tau \ln A$, since the entropy always increases during the component transition. However, in the original work of the Purdue group [2–4], this mixing entropy

term has a negative contribution. In the following section, the derivation of S_{mix} is introduced and the mistake that the Purdue group made by putting a wrong sign in front of S_{mix} in the MFM formulation is addressed.

B. Mixing entropy

Following the FM and MFM scenario, the mixing entropy can be derived within a classical approach. For a classical system, the total number of microstates, Ω, is expressed as

$$\Omega = \frac{N_0!}{\prod a_l!} \prod \omega_l^{a_l}, \quad (10)$$

where N_0 is the particle number and a_l is the particle number at the l state, such that $N_0 = \Sigma a_l$. ω_l is the degeneracy of the l state. Going to a nuclear system and ignoring the spin, nucleons only have two states, proton and neutron, defined as n state and p state here. The degeneracies for both states are 1. Therefore for a fragment with Z protons ($a_p = Z$) and N neutrons ($a_n = N$), the total number of microstates becomes

$$\Omega(N,Z) = \frac{A!}{N!Z!}. \quad (11)$$

Thus the mixing entropy is simply calculated following Boltzmann's entropic equation as $S_{\text{mix}}(N,Z) = \ln[\Omega(N,Z)]$. After applying Stirling's approximation for the factorial of a large nucleon number, $S_{\text{mix}}(N,Z)$ is further reduced as

$$S_{\text{mix}}(N,Z) = \ln(A!) - \ln(N! \cdot Z!)$$
$$= \left[A(\ln A - 1) + \frac{1}{2}\ln(2\pi A)\right]$$
$$- \left[N(\ln N - 1) + \frac{1}{2}\ln(2\pi N)\right.$$
$$\left. + Z(\ln Z - 1) + \frac{1}{2}\ln(2\pi Z)\right]$$
$$\approx A(\ln A - 1) - [N(\ln N - 1) + Z(\ln Z - 1)]$$
$$= -\left[N \ln\left(\frac{N}{A}\right) + Z \ln\left(\frac{Z}{A}\right)\right]. \quad (12)$$

It is well known that upon adding one component (neutron or proton) to the other (proton or neutron), the fractions of the two components are both less than 1. Therefore, $S_{\text{mix}}(N,Z)$ is always positively defined in this expression, suggesting an entropy increase due to the neutron-proton mixing mathematically. This positive $S_{\text{mix}}(N,Z)$ expression has been widely applied to both ideal solutions and ideal gases.

For comparison, the mixing entropy is also derived within a quantum approach. In the quantum framework, for an ideal Fermi gas, the average number of fermions in a single-particle state i is given by the Fermi-Dirac distribution as

$$f_i = \frac{1}{e^{(\epsilon_i - \mu)/T} + 1}, \quad (13)$$

where T is the temperature, ϵ_i is the energy of the single-particle state i, and μ is the chemical potential. The number of states between ϵ and $\epsilon + d\epsilon$ is

$$D(\epsilon)d\epsilon = g\frac{2\pi V}{h^3}(2m_0)^{3/2}\epsilon^{1/2}d\epsilon, \quad (14)$$

where g is the degeneracy factor, V is the system volume, and m_0 is the mass of a fermion. The density ρ, total particle number N_0, and total energy U of the free Fermi gas are given by

$$\rho = g \frac{2\pi}{h^3} (2m_0 T)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^{x-\mu/T} + 1}, \tag{15}$$

$$N_0 = g \frac{2\pi V}{h^3} (2m_0 T)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^{x-\mu/T} + 1}, \tag{16}$$

$$U = g \frac{2\pi V}{h^3} (2m_0 T)^{3/2} T \int_0^\infty \frac{x^{3/2} dx}{e^{x-\mu/T} + 1}. \tag{17}$$

Then the entropy per particle of the free Fermi gas is given as

$$\frac{S}{N_0} = \frac{U - F}{N_0 T} = \frac{U + PV - \mu N_0}{N_0 T}$$

$$= \frac{\frac{5}{3} U - \mu N_0}{N_0 T} = \frac{5}{3} \frac{\int_0^\infty \frac{x^{3/2} dx}{e^{x-\mu/T}+1}}{\int_0^\infty \frac{x^{1/2} dx}{e^{x-\mu/T}+1}} - \frac{\mu}{T}, \tag{18}$$

where $F = \mu N_0 - PV$ is the free energy. $P = \frac{2}{3} \frac{\partial U}{\partial V}$ is the pressure of the fermion system. S/N_0 is a function of temperature and density through the chemical potential μ.

Assuming the emitted fragment to be an ideal Fermi gas system with a fixed density of neutrons and protons ($\rho = \rho_n + \rho_p$), we have $m = \frac{\rho_n - \rho_p}{\rho}$, where ρ_n, ρ_p, and ρ are the neutron, proton, and total nucleon densities, respectively. Therefore, the mixing entropy per nucleon of the fermion system is

$$\frac{S_{\text{mix}}}{N_0}(\rho_n, \rho_p) = \left[\frac{\rho_n}{\rho} \frac{S}{N}(\rho_n) + \frac{\rho_p}{\rho} \frac{S}{Z}(\rho_p) \right] - \frac{S}{N_0}(\rho)$$

$$= \left[\frac{1+m}{2} \frac{S}{N}(\rho_n) + \frac{1-m}{2} \frac{S}{Z}(\rho_p) \right] - \frac{S}{N_0}(\rho). \tag{19}$$

In Eq. (19), the chemical potentials for neutrons and protons are, respectively, defined as μ_n and μ_p, and determined by solving Eq. (15) for a given temperature and density of the system. Inserting Eq. (18) into Eq. (19), one can obtain the numerical values of the mixing entropy of the nuclear system for a given density and temperature.

Figure 1 shows the calculated values of S_{mix} per nucleon within both the classical [Eq. (12)] and quantum [Eq. (19)] approaches as a function of m. The quantum $S_{\text{mix}}(\rho_n, \rho_p)$ per nucleon is calculated for $T = 1, 3, 5, 7$, and 30 MeV at a given density $\rho = \rho_0/6$ and the results are shown by different symbols in the figure. The solid line corresponds to $S_{\text{mix}}(N, Z)$ per nucleon calculated using the classical approach, where m is calculated as $m = \frac{N-Z}{A} = \frac{\rho_n - \rho_p}{\rho}$. As T increases, the quantum result gradually approaches the classical one, and both results become consistent with each other at the classical limit of high temperature and low density. Along with FM, the MFM was originally formulated at the classical limit near the critical point [2–4] and used to describe the critical properties of the hot nuclear matter created in nuclear collisions in a wide energy region under a coarse approximation; i.e., first based on the MFM, the Purdue group studied the power law behavior of the experimentally measured inclusive fragment mass distributions, which is a natural result of the MFM model
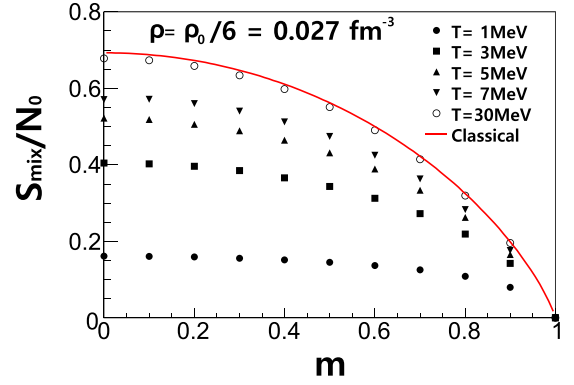

FIG. 1. Mixing entropy per nucleon as a function of m. The quantum results from Eq. (19) are shown by different symbols for $T = 1$ MeV (dots), 3 MeV (solid squares), 5 MeV (solid triangles), 7 MeV (solid inverted triangles), and 30 MeV (open circles) at density $\rho = \rho_0/6$. The solid line corresponds to the mixing entropy per nucleon calculated using the classical formula, Eq. (12).

at the critical point, from the reactions of proton on Xe and Kr at 80 to 350 GeV/c [2–4]. The power law behavior has also been demonstrated from the heavy ion reactions around the Fermi energy in some of our previous works in more detail [6,15]. In our present MFM formulation, the classical mixing entropy is therefore adopted for consistency.

Therefore, following this classical scenario and inserting Eq. (12) into Eq. (8), our present formulation of MFM is written as

$$Y(I, A) = Y_0 A^{-\tau} \exp\left[\frac{W(I, A) + \mu_n N + \mu_p Z}{T} \right.$$

$$\left. - N \ln\left(\frac{N}{A}\right) - Z \ln\left(\frac{Z}{A}\right) \right]. \tag{20}$$

Comparing Eq. (20) and the formulation of the Purdue group, one should note that our present formulation of the MFM is identical to that of the Purdue group in Refs. [2–4], except for the opposite sign of the mixing entropy term. Clearly, the negative expression of $S_{\text{mix}}(N, Z)$ in their formulation is wrong, since it goes against the nature of the mixing entropy increase as a single-component system transforms into a two-component (proton-neutron) system. The errors of the results related to the MFM studies originating from this mistake in the sign of $S_{\text{mix}}(N, Z)$ are quantitatively discussed in the following section. They include isotopic yield distribution, isobaric yield ratios, isoscaling, m scaling, self-consistent determination of density, symmetry energy, and temperature, and density and temperature determination related to the IMF freezeout.

III. RESULTS AND DISCUSSIONS

A. Isotopic yield distribution

After the formulation of the MFM, the Purdue group first applied it to reproduce the fragment isotopic yields [2–4]. Figure 2 shows the isotopic yield of all available fragments in the $p + $ Xe system at incident momenta between 80 and 350 GeV/c (dots) and their corresponding fitting results (triangles). In their fitting, a_v is fixed as 14.1 MeV without
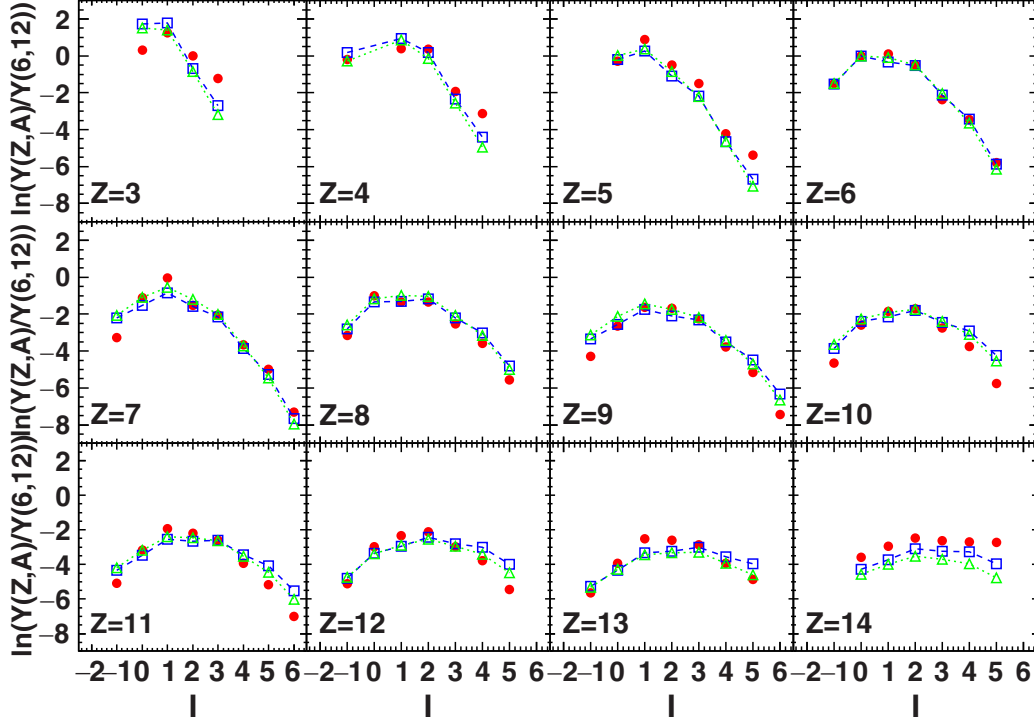
FIG. 2. Isotopic yields as a function of I. Dots: isotopic yield of all available measured fragments in $p + Xe$ system at the incident momentum between 80 and 350 GeV/c. Triangles: fitting results from the original MFM formulation of the Purdue group. Squares: fitting results from the corrected MFM formulation, Eq. (20).

assuming a volume heat contribution of 8 MeV to the volume free energy and τ is fixed as 2.64 [2–4]. Other parameters, a_s, a_c, a_{sym}, a_p, μ_p, μ_n, and $T(1/\beta)$, are set free. The fitting is preformed among the 60 isotopic yields with $12 \leqslant A \leqslant 31$. Also note that the Coulomb term and paring term of Eq. (9) are generated as $a_c(Z^2/A^{1/3})$ and $a_p(\delta_p/A^{0.75})$, respectively, in their formulation. Following the same approach, we refit the data using the present MFM formulation with a positive mixing entropy term, according to Eq. (20). Our results are shown by squares in Fig. 2. As shown in the figure, the experimental isotopic yield distributions are well reproduced with both formulations by adjusting the parameters. The resultant parameters are summarized in Table I. The present parameter values optimized from the corrected formulation

TABLE I. Values for the parameters in the fits to the isotopic yields using the original and present MFM formulas and values of the normal nuclei taken from Refs. [2–4].

	Purdue	Present	Normal nuclei
τ	2.64	2.64	
a_v (MeV)	14.1	14.1	14.1
a_s (MeV)	5.35	12.12	13.0
a_c (MeV)	0.489	0.510	0.595
a_{sym} (MeV)	22.60	19.13	19.00
a_p (MeV)	5.92	10.45	33.5
μ_p (MeV)	−11.32	−14.34	
μ_n (MeV)	−7.59	−10.46	
T (MeV)	3.28	3.56	

(third column) are more or less comparable to those presented in Refs. [2–4] (second column), except for the surface and pairing coefficients, a_s and a_p; that is, the present a_s and a_p values are closer to those of cold nuclear nuclei. Rather large values of a_s and a_p are inconsistent with the fact that at a critical temperature the surface energy and the pairing correlation should become negligible. This indicates that the experimentally measured isotopic yields reflect the nuclear natures at the final stage, rather than at the critical point, due to the secondary decay process [5,10,11,14].

B. Isobaric yield ratios

In Ref. [5], isobaric yield ratios were utilized to study the a_{sym}, a_c, $\Delta\mu$ ($\Delta\mu = \mu_n - \mu_p$), and a_p values, relative to temperature, in the MFM framework for the first time. Based on the MFM formulation, $R(I + 2, I, A)$, the isobaric yield ratio between isobars differing by 2 units of I, i.e., $I + 2$ and I, is defined as

$$R(I + 2, I, A)$$
$$= Y(I + 2, A)/Y(I, A)$$
$$= \exp\{[\Delta\mu + W(I + 2, A) - W(I, A)]/T + \Delta\}. \quad (21)$$

Taking the logarithm of $R(I + 2, I, A)$, one can obtain

$$\ln[R(I + 2, I, A)]$$
$$= [\Delta\mu + W(I + 2, A) - W(I, A)]/T + \Delta, \quad (22)$$

where Δ is the mixing entropy difference between isobars with $I + 2$ and I. Following the corrected formulation of the MFM
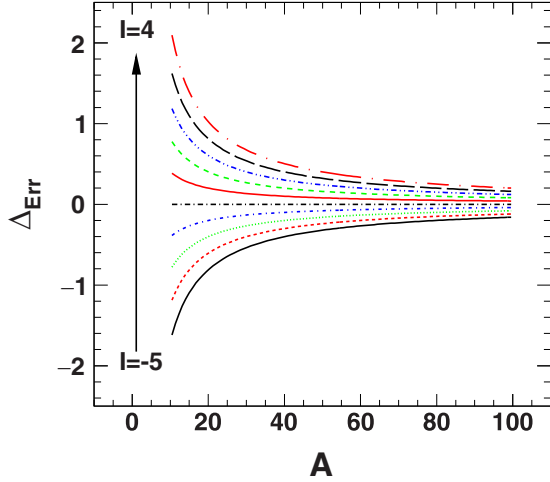
FIG. 3. Numerical values of Δ_{Err} calculated using Eq. (24) as a function of A with $I = -5$ to 4 from the bottom to the top.

FIG. 4. Numerical values of $\Delta(a_{\rm sym}/T)$ calculated using Eq. (26) as a function of A with $I = 1$–9.

[Eq. (20)], Δ in the corrected present formula, $\Delta_{\rm Pres}$, is written as

$$
\Delta_{\rm Pres} = -\left[\frac{A+I+2}{2} \ln\left(\frac{A+I+2}{2A}\right)\right.
$$
$$
+ \frac{A-I-2}{2} \ln\left(\frac{A-I-2}{2A}\right)\Big]
$$
$$
+ \left[\frac{A+I}{2} \ln\left(\frac{A+I}{2A}\right)\right.
$$
$$
\left.+ \frac{A-I}{2} \ln\left(\frac{A-I}{2A}\right)\right]. \tag{23}
$$

For the original MFM formulation, $\Delta_{\rm Orig}$ is expressed as $\Delta_{\rm Orig} = -\Delta_{\rm Pres}$. Thus the error occurring from the mistake in the sign of the mixing entropy term, Δ_{Err}, is given as the difference between $\Delta_{\rm Orig}$ and $\Delta_{\rm Pres}$,

$$
\Delta_{Err} = \Delta_{\rm Orig} - \Delta_{\rm Pres} = -2\Delta_{\rm Pres}. \tag{24}
$$

The numerical values of Δ_{Err} are calculated as a function of A with $I = -5$ to 4 using Eq. (24) and the results are shown in Fig. 3. As seen in the figure, the error decreases rapidly as A increases, and the errors for the isobars with N closer to Z are smaller. For the mirror isobars (black dotted-dashed line), the errors fully cancel out.

In our recent work [7], the ratio of symmetry energy relative to temperature, $a_{\rm sym}/T$, was extracted from the cold isotopic yields of 140 MeV/nucleon ^{40,48}Ca + ^{9}Be and ^{58,64}Ni + ^{9}Be from the experiments performed by Mocko *et al.* at the National Superconducting Cyclotron Laboratory (NSCL) at Michigan State University (MSU) [35], using the equation

$$
\frac{a_{\rm sym}}{T} = \frac{A}{4(I+1)}\{[\Delta\mu + 2a_c(Z-1)/A^{1/3}]/T
$$
$$
- \ln[R(I+2,I,A)] + \Delta\}. \tag{25}
$$

According to Eq. (25), the percentage error contribution to the final result, $\Delta(a_{\rm sym}/T)$, can be defined as

$$
\Delta(a_{\rm sym}/T) = \frac{A}{4(I+1)} \frac{|\Delta_{Err}|}{a_{\rm sym}/T} \times 100\%. \tag{26}
$$

Taking the ^{64}Ni + ^{9}Be system used in Ref. [7] as an example, $\Delta(a_{\rm sym}/T)$ is calculated for all available isotopic yields. The resultant $\Delta(a_{\rm sym}/T)$ values are plotted as a function of A in Fig. 4. In the figure, $\Delta(a_{\rm sym}/T)$ exhibits a behavior similar to that of Δ_{Err} shown in Fig. 3, and rather small errors, ~3%–15%, are found in this analysis. Similarly, the analyses of the symmetry energy and pairing energy relative to temperature, the temperature of the cold fragment, neutron skin effects, and critical behavior of the multifragmentation have also been performed using the isobaric yield ratio observables [5,6,21–32]. In Refs. [5,6,21–32], intermediate and heavy mass isobars were generally utilized. Similar magnitudes of the errors are also found for these works. Therefore, we conclude that the conclusions of these works remain valid with small changes in the extracted quantitative values.

C. Isoscaling and *m* scaling

In Ref. [9], $a_{\rm sym}/T$ was experimentally extracted as a function of the fragment atomic number using isoscaling parameters. From the MFM formulation, R_{12}, the yield ratio for the same isotope from two similar reaction systems with different N/Z ratios, is written as

$$
R_{21}(N,Z) = C \exp(\alpha N + \beta Z). \tag{27}
$$

This relation is well known as the isoscaling relation [36]. The isoscaling parameters, $\alpha = (\mu_n^2 - \mu_n^1)/T$ and $\beta = (\mu_p^2 - \mu_p^1)/T$, are the differences of the neutron or proton chemical potentials between the systems 2 and 1, divided by the temperature. C is a constant. According to Eq. (27), the error in the MFM formulation does not contribute in the isoscaling parameters, since the mixing entropy cancels out in R_{12}. In the same work, the variance of the isotope distributions was also

TABLE II. Extracted a/T_0 and $\Delta\mu/T_0$ for the first round ($k = 0$) and the final round ($k = 0.0022$) using the self-consistent approach based on the corrected formulation of the MFM.

	$\widetilde{a}_v/T_0$	a_s/T_0	a_c/T_0	a_p/T_0	$\Delta\mu/T_0$
		$k = 0$			
g0	-0.239	0.000	0.182	0.582	0.604
g0AS	-0.282	0.000	0.164	0.691	0.480
g0ASS	-0.319	0.000	0.145	1.121	0.432
Expt.	-0.305	0.001	0.144	0.111	0.626
		$k = 0.0022$			
g0	-0.236	0.000	0.167	0.526	0.626
g0AS	-0.277	0.000	0.150	0.630	0.505
g0ASS	-0.308	0.000	0.133	1.046	0.451
Expt.	-0.304	0.004	0.126	0.093	0.676

utilized to extract a_{sym}/T. However, the method used is based on an approximation of the MFM formulation proposed by Ono *et al.* [37], in which the mixing entropy does not appear. Thus, the results and conclusions in the work are not affected.

In Refs. [8,15–20], A. Bonasera *et al.* proposed a Landau free-energy approach for describing the free energy in the exponent of Eq. (8) within the Landau $\mathcal{O}(m^6)$ theory, where $m = (N_f - Z_f)/A_f$ is the order parameter, a consequence of (one of) the symmetries of the nuclear Hamiltonian. Within this framework, isoscaling depends mainly on this order parameter through the "external (conjugate) field" H. The external field is just given by the difference in chemical potentials of the neutrons and protons of the two sources. To distinguish from previously employed isoscaling analyses, this approach is dubbed m scaling. For m scaling, the mixing entropy is absent from the free energy of Eq. (8), so that the mistake in the original MFM formulation does not affect the analyses in these papers. Therefore, the results and conclusions related to m scaling are fully valid.

D. Density, temperature, and symmetry energy determination

In Ref. [13], based on the original formulation of the MFM [2–4], a self-consistent approach was developed to extract the density, temperature, and symmetry energy for nuclear fragmentation, and was applied to the reconstructed hot isotope yields from $^{64}\mathrm{Zn} + {}^{112}\mathrm{Sn}$ at 40 MeV/nucleon, utilizing the simulations of antisymmetrized molecular dynamics (AMD) of Ono *et al.* [38,39]. The basic steps of the self-consistent procedure are briefly summarized as follows:

(1) Optimize $\Delta\mu/T_0$ and a_c/T_0 values from mirror isobars.
(2) Optimize $\tilde{a}_v/T_0$, a_s/T_0, and a_p/T_0 values from $N = Z$ isotopes, where $\tilde{a}_v = a_v + \frac{1}{2}(\mu_n - \mu_p)$.
(3) Using parameters determined in steps 1 and 2, a_{sym}/T_0 values are extracted from all available isotopes. Comparing the extracted a_{sym}/T_0 values to those of the AMD simulations with different Gogny interactions with three density dependent symmetry energy terms, i.e., the standard Gogny interaction which has an asymptotic soft symmetry energy (g0), the Gogny interaction with an asymptotic stiff symmetry energy

TABLE III. Extracted ρ/ρ_0 and symmetry energy from the first round ($k = 0$) and the final round ($k = 0.0022$) using the self-consistent approach based on the corrected formulation of the MFM.

	R_{sym}	ρ/ρ_0	a_{sym} (MeV)
	$k = 0$		
g0			24.2 ± 0.4
g0/g0AS	1.234 ± 0.025	0.529 ± 0.048	
g0AS			19.0 ± 1.3
g0/g0ASS	1.545 ± 0.034	0.568 ± 0.021	
g0ASS			15.8 ± 0.7
g0/Expt.	1.162 ± 0.025	0.562 ± 0.015	
Expt.			20.8 ± 0.6
	$k = 0.0022$		
g0			24.1 ± 0.4
g0/g0AS	1.237 ± 0.026	0.524 ± 0.049	
g0AS			18.9 ± 1.3
g0/g0ASS	1.553 ± 0.035	0.564 ± 0.020	
g0ASS			15.6 ± 0.7
g0/Expt.	1.161 ± 0.025	0.558 ± 0.015	
Expt.			20.8 ± 0.6

(g0AS), and the Gogny interaction with an asymptotic super-stiff symmetry energy (g0ASS) [37–39], the density of the fragmenting source is extracted. Using this density, the experimental value of the symmetry energy coefficient, a_{sym}, is determined. The temperature is then extracted following the relation $T_0 = a_{\mathrm{sym}}/(a_{\mathrm{sym}}/T_0)$.

Iteration of steps 1–3 is performed to take into account the difference of the apparent temperature T and the physical temperature T_0 in these steps [12]. Typically the iteration is repeated two to three times to get a reasonably flat distribution of T_0 as a function of the fragment mass. More detailed descriptions of the self-consistent approach can be found in Refs. [12,13]. Note that in Ref. [10], Eq. (25) was applied to extract the a_{sym}/T values instead of Steps. (2)-(3). As mentioned in Sec. III A, the a_{sym}/T errors caused by the mistake in the original Purdue formulation partially cancel each other and become small. A detailed error evaluation and the newly extracted values of the density, temperature, and symmetry energy in Ref. [10] are given in an erratum [40].

Here, the density, temperature, and symmetry energy values are recalculated based on the corrected formulation of the MFM using steps 1–3. In Tables II and III, the newly extracted values of the parameters $\tilde{a}_v/T_0$, a_s/T_0, a_c/T_0, a_p/T_0, and $\Delta\mu/T_0$, and symmetry energy and ρ/ρ_0, from both experimental data and AMD simulated events are summarized,

TABLE IV. Comparison of density, temperature, and symmetry energy from the corrected and original formulations of the MFM.

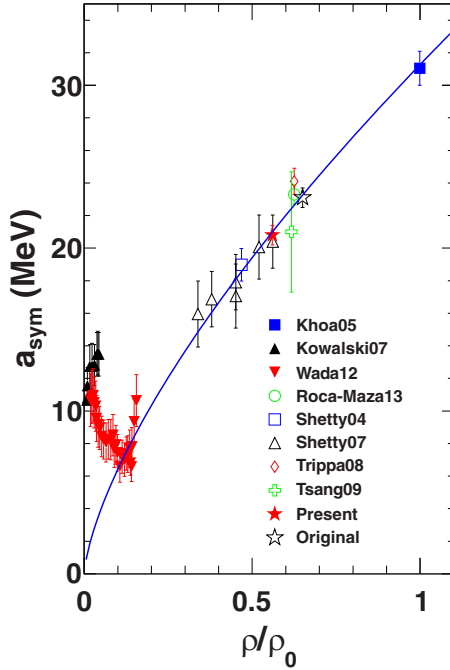|	Present	Original	Diffrence
ρ/ρ_0	0.56 ± 0.02	0.65 ± 0.02	16.1%
T_0	5.2 ± 0.6 MeV	5.0 ± 0.4 MeV	3.8%
a_{sym}	20.8 ± 0.6 MeV	23.1 ± 0.6 MeV	11.1%

FIG. 5. Comparisons between the present and original results together with available published results. The line is from the fitting of the available data using Eq. (28). Data are taken from Khoa *et al.* (2005) [41], Kowalski *et al.* (2007) [42], Wada *et al.* (2012) [43], Roca-Maza *et al.* (2013) [44], Shetty *et al.* (2004) [45], Shetty *et al.* (2007) [46], Trippa *et al.* (2008) [47], and Tsang *et al.* (2009) [48].

respectively, for the first iteration round ($k = 0$) and the final iteration round ($k = 0.0022$). In this analysis, the effect caused by the mistake is absorbed mostly in step 2, when $\tilde{a}_v/T_0$, a_s/T_0, and a_p/T_0 values are optimized as free parameters using the MFM formulation. In Table II, the newly obtained a_s/T_0 and a_p/T_0 values are close to 0. This is reasonable, since the reconstructed isotopic yields can reflect the characteristic nature around the critical point. The density, temperature, and symmetry energy values from the present and original formulations of the MFM are compared in Table IV. As presented in the table, the errors originating from the sign of the mixing entropy are rather small, i.e., 16.1% for the density and even smaller for the other deduced quantities.

In order to further investigate the effect of the mistake, the a_{sym} values from the corrected and original MFM formulations are compared together with other available published data

in Fig. 5. At $0.1 \lesssim \rho/\rho_0 \leqslant 1.0$, the existing data points are consistent with each other within the errors and distribute along a line systematically, which is optimized within the mean-field theory:

$$a_{sym}(\rho/\rho_0) = 31.5(\rho/\rho_0)^{0.69}. \qquad (28)$$

The present and original values are both along the same curve. This observation indicates that the errors caused by the mistake are of the order of 10%, but they do not change the basic conclusions extracted.

E. Density and temperature determination related to IMF Freezeout

In Ref. [14], for the central collision events of $^{40}Ca + ^{40}Ca$, generated by the AMD model in the intermediate energies of 35 to 300 MeV/nucleon, the density and temperature of a fragmenting source were extracted using the self-consistent method with the original MFM. The extracted density and temperature values are, respectively, $\rho/\rho_0 \sim 0.65$ to 0.7 and $T_0 \sim 5.9$ to 6.5 MeV. Here the density and temperature values are recalculated using the corrected formulation of the MFM, and the results are summarized in Table V, together with those obtained in Ref. [14]. Errors, $\sim$12–17% for density and $\sim$4–13% for temperature, are evaluated in the table. In spite of the errors originating from the sign of the mixing entropy, density and temperature show rather constant values as a function of the incident energy, indicating that there is a "freezeout" volume for the IMF production in the intermediate heavy-ion reactions, which is commonly used in the statistical multifragmentation model as the basic assumption. Thus, the conclusion drawn in Ref. [14] is still valid.

IV. SUMMARY

In this article, the formulation of the modified Fisher model is examined. A mistake in the formulation of the MFM derived in the pioneering works of the Purdue group in Refs. [2–4] is addressed. A corrected formulation of the MFM is presented by reversing the sign of the mixing entropy term in the original formulation. The errors from the mistake in the results of the previous MFM-related studies, such as isotopic yield distribution, isobaric yield ratios, isoscaling, *m* scaling, self-consistent determination of density, symmetry energy, and temperature, and density and temperature determination related to the IMF freezeout, are quantitatively evaluated. It is found that the errors originating from the mistake in sign of the mixing

TABLE V. Density and temperature extracted using the self-consistent method for the central collision events of $^{40}Ca + ^{40}Ca$ in the incident energies from 35 to 300 MeV/nucleon.

Energy (MeV/nucleon)	ρ_{Pres}/ρ_0	ρ_{Orig}/ρ_0	Diff.	T_{Pres} (MeV)	T_{Orig} (MeV)	Diff.
35	0.59 ± 0.02	0.67 ± 0.02	11.9%	6.3 ± 0.3	5.6 ± 0.2	12.5%
50	0.56 ± 0.02	0.64 ± 0.02	12.5%	6.6 ± 0.3	5.9 ± 0.3	11.9%
80	0.54 ± 0.02	0.65 ± 0.02	16.9%	6.7 ± 0.3	6.2 ± 0.2	8.1%
100	0.59 ± 0.02	0.69 ± 0.02	14.5%	6.8 ± 0.3	6.5 ± 0.3	4.6%
140	0.57 ± 0.02	0.67 ± 0.02	14.9%	6.9 ± 0.3	6.4 ± 0.3	7.8%
300	0.58 ± 0.02	0.68 ± 0.02	14.7%	7.1 ± 0.3	6.5 ± 0.3	9.2%

entropy term are generally small and even have no effect in some cases. The results and conclusions in the original papers are generally valid.

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