Microscopic approach based on a multiscale algebraic version of the resonating group model for radiative capture reactions

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A microscopic approach to description of radiative capture reactions based on a multiscale algebraic version of the resonating group model is developed. The main idea of the approach is to expand wave functions of discrete spectrum and continuum for a nuclear system over different bases of the algebraic version of the resonating group model. These bases differ from each other by values of oscillator radius playing a role of scale parameter. This allows us in a unified way to calculate total and partial cross sections (astrophysical S factors) as well as branching ratio for the radiative capture reaction, to describe phase shifts for the colliding nuclei in the initial channel of the reaction, and at the same time to reproduce breakup thresholds of the final nucleus. The approach is applied to the theoretical study of the mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions, which are of great interest to nuclear astrophysics. The calculated results are compared with existing experimental data and with our previous calculations in the framework of the single-scale algebraic version of the resonating group model.

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I. INTRODUCTION

Radiative capture reactions are known as one of the major classes of fusion processes and are of great interest to nuclear astrophysics, particularly, in studies of kinetic processes inside stars, including the Sun, and stellar and primordial nucleosynthesises. As a rule, their cross sections are not available for reliable experimental measurements at low astrophysically relevant energies ("Gamow window") due to strong Coulomb repulsion. The mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions are important examples of the radiative capture [1–10]. The cross sections of these reactions in the low-energy range are needed to solve a number of problems connected with big-bang nucleosynthesis [1–6] (see Fig. 1). Moreover, the latter reaction plays an important role in solar core investigations [4–10] (see Fig. 2). Nevertheless, the cross sections for both the reactions at very low energies are still inaccessible for experiments. In the current situation, theoretical calculations based on microscopic approaches are supposed to be one of the most justified and promising lines of attack of the low-energy cross section problem.

Experimental investigations of the mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions have a long story that goes back to the end of the 1950s when the first experimental measurements of their cross sections were performed [11]. After this work, a number of experimental studies of the ^{3}H$(\alpha,\gamma)^7$Li [12–17] and ^{3}He$(\alpha,\gamma)^7$Be [18–36] reactions were carried out for the next decades. Nevertheless, "Gamow window" for the ^{3}He$(\alpha,\gamma)^7$Be reaction is still uncovered. Furthermore, values of the cross sections for both the reactions considered have significant scatter at energies available for the measurements.

As to theoretical studies for these reactions, there are a lot of works devoted to calculations of their cross sections or astrophysical S factors [37–76]. Among these calculations, there are ones based either on the direct capture model (DCM) [37–43] or the potential cluster model (PCM) [44–49], or the PCM modifications [50–53]. These are two-body calculations, which are not microscopic. Different semimicroscopic calculations are presented in Refs. [54–57]. The PCM is combined with the resonating group model (RGM) [77,78] in Ref. [54]. The orthogonality condition model (OCM) [79,80] and its algebraic version [81,82] are used in the works [55] and [56,57] respectively. Fully microscopic calculations based on the RGM are performed for example in Refs. [58–68]. In Ref. [69], the astrophysical S factors are extracted studying the electric dipole polarizability of the ^{7}Li nucleus. There are theoretical investigations combining the PCM for the two-body description of continuum for the colliding nuclei either with the variational Monte Carlo method (VMC) [70,71] or with the no-core shell model (NCSM) [72,73] for an *ab initio* description of discrete spectrum for the formed nuclei. In work [74], *ab initio* calculation was done in the framework of the fermionic molecular dynamics (FMD). Study [75,76] is based on solving the coupled Faddeev equations (FEs). It should be noted that the results from the theoretical works differ from each other. Moreover, there is no a microscopic calculation describing successfully both the normalization and the energy dependence of the modern cross section data on the ^{3}He$(\alpha,\gamma)^7$Be reaction [26–36] and of the most reliable cross section data on the ^{3}H$(\alpha,\gamma)^7$Li reaction [16] simultaneously. It means that questions concerning energy behavior of the cross sections for these reactions are still open.

In our previous works [83–87,56,57], a microscopic approach based on the single-scale algebraic version of the RGM (AVRGM) (the so-called conventional AVRGM approach) for describing the radiative capture reactions was proposed and implemented to calculate the energy dependences of the cross sections for the discussed reactions. In Ref. [88], the AVRGM was also utilized to treat these reactions but only at zero-point energy without a calculation of the cross section energy dependences.

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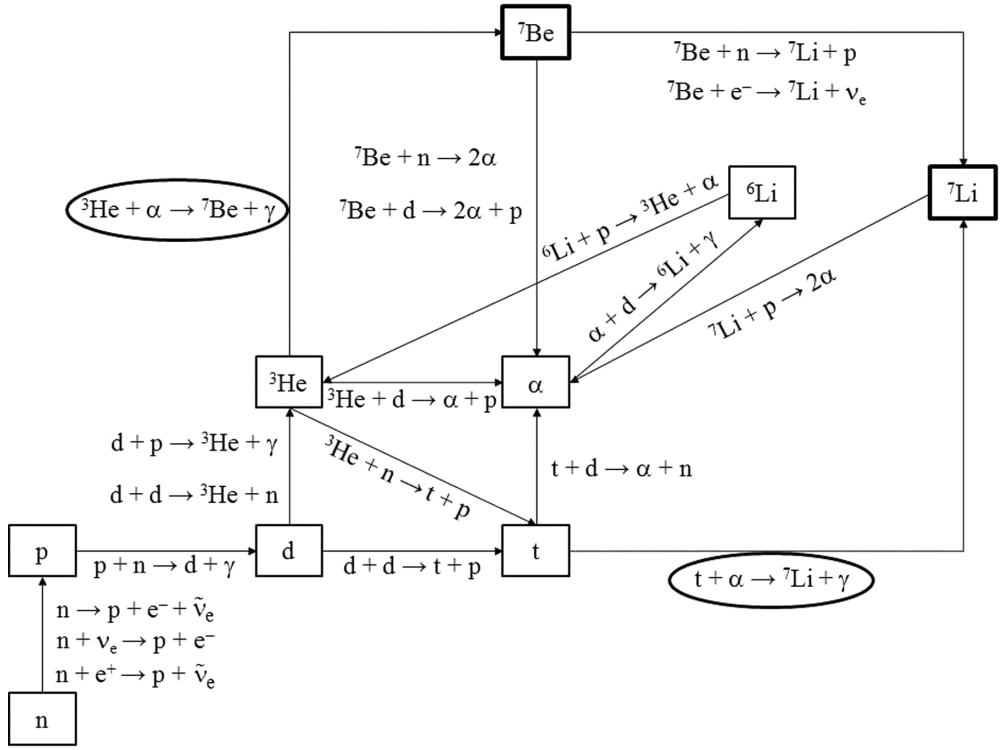

FIG. 1. The basic reactions of big-bang nucleosynthesis.

It should be mentioned that the AVRGM was first proposed by Filippov [89,90]. It is a special mathematical realization of the RGM suitable for calculations and having evident numerical advantages compared to the original RGM. Initially, the AVRGM was successfully applied to study nuclear processes in the binary collisions [91–97]. Later on, it allowed one to combine dynamics of cluster and collective degrees of freedom and to describe breakups of giant resonances in light nuclei [93,98]. This model was also used to consider three-cluster configurations [99–104] essential for many reactions, which

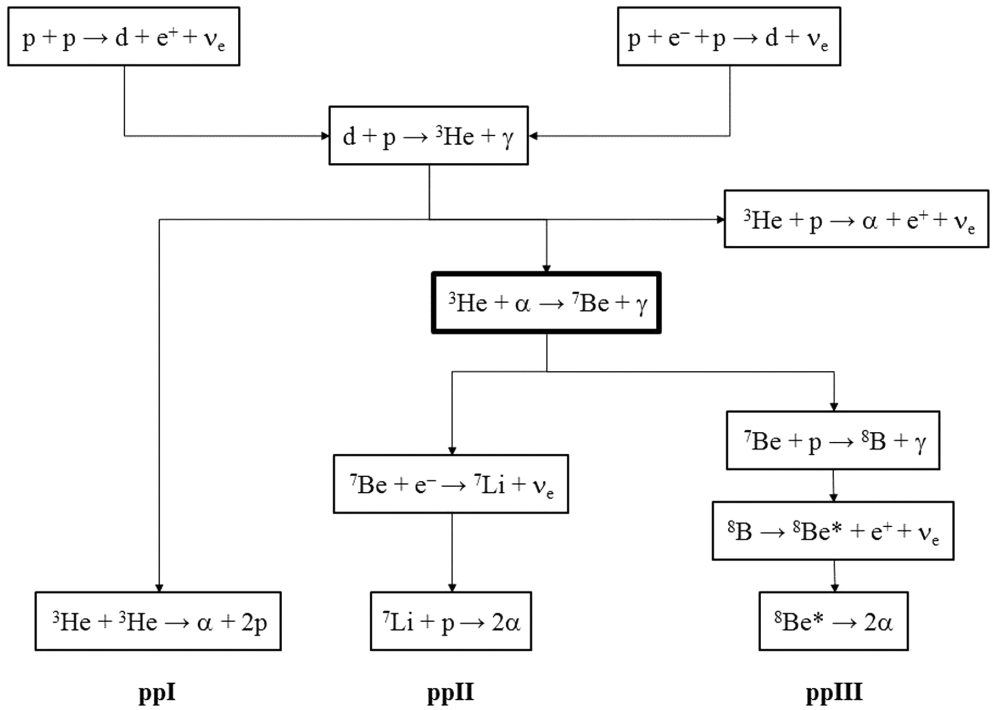

FIG. 2. The *pp* chain of the hydrogen burning in stars.

are of interest to nuclear astrophysics, and, recently, to study two-cluster systems dynamics in phase space [105].

In the present work, a microscopic approach based on the multiscale AVRGM (the so-called generalized AVRGM approach) to describe the radiative capture reactions is developed. An important feature of this microscopic approach is the capability of reproducing the final nucleus breakup thresholds into the initial colliding nuclei. The multiscale AVRGM approach is founded theoretically better than the single-scale one proposed previously. The former also makes it possible to describe a wider set of experimental data on the radiative capture reactions in a unified way. All these advantages enhance the predictive power of the novel approach and increase the reliability of the obtained theoretical results. The approach is applied to study the ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions at low energies.

It should be emphasized that one of the most important aims of modern nuclear theory is to build *ab initio* microscopic approaches. It is supposed that these approaches must describe a dynamics of all nucleons constituting a nuclear system, take into account the Pauli exclusion principle completely, treat the center-of-mass motion rigorously, and finally use realistic microscopic nuclear potentials. From the mathematical viewpoint, it means that the wave functions must depend on space and spin-isospin coordinates of all the nucleons of the system, be fully antisymmetrized for permutations of all pairs of the nucleons, and be translationally invariant. At the present moment, such approaches are continuously developed to describe nuclear structure and scattering (see Refs. [73,106] and references therein). Unfortunately, their computational complexities require supercomputing and generalization to reactions meets principal difficulties. For these reasons, the *ab initio* NCSM was combined with the RGM oriented to describing nuclear reactions, including ones with sufficiently heavy nuclei. This approach, named the *ab initio* NCSM/RGM, was proposed in works [107,108]. The different applications of the approach and its improvements can be found in Refs. [109–115]. That is why the multiscale AVRGM approach reviewed in the present work is of additional interest to modern nuclear theory and can be useful for calculations in the general case.

II. BASIC POINTS OF AVRGM

An effective tool for many nuclear problems is an expansion

$$\Psi(\mathbf{q}) = \sum_{\nu lm} C_{\nu lm} f_{\nu lm}(\mathbf{q}) \tag{1}$$

of a wave function over the basis of the eigenfunctions of a three-dimensional harmonic oscillator,

$$f_{\nu lm}(\mathbf{q}) = (-1)^{(\nu-l)/2} \sqrt{\frac{2\Gamma[(\nu-l+2)/2]}{r_0^3\Gamma[(\nu+l+3)/2]}}\,(q/r_0)^l \times L^{(l+1/2)}_{(\nu-l)/2}\left(q^2/r_0^2\right)\exp\left(-q^2/2\,r_0^2\right)Y_{lm}(\mathbf{n_q}), \tag{2}$$

where Γ, $L_n^{(\beta)}$, and Y_{lm} are the gamma function, the generalized Laguerre polynomial, and the spherical harmonic respectively; ν is the number of oscillator quanta; l and m are the orbital angular momentum and its projection respectively; r_0 is the oscillator radius; $\mathbf{q}$ is the spatial coordinates. The convergence of the expansion (1) for square-integrable functions is clear. The convergence properties of this expansion for functions, which do not belong to the space of the square-integrable ones, are not evident. Nevertheless, it was rather successfully applied to solve the Schrödinger equation in continuum. For the first time, such an approach, named the J-matrix method, was proposed in Ref. [116]. Later, a similar approach gave rise to the AVRGM [89], which was applied to consider multinucleon systems and to avoid a difficult numerical procedure for the solution of the integrodifferential RGM equations in coordinate representation. In the following works [117,118], the methods based on the expansions over the oscillator basis (2) were further developed. A comprehensive investigation of the convergence properties was undertaken in Ref. [119] where the asymptotic form of the expansion coefficients was established and the pointwise convergence of the expansions of the continuous spectrum wave functions was proved. This provided a true mathematical base for applications of the expansions in the series of the oscillator functions (2) in the continuous spectrum problem.

The basic idea of the AVRGM is to seek for the wave function of the relative motion of clusters (nuclei) in the form of the expansion (1). This leads to the expansion

$$\Psi = \sum_{J=J_0}^{\infty}\sum_{M=-J}^{J}\sum_{s=|s_1-s_2|}^{s_1+s_2}\sum_{l=|J-s|}^{J+s}\sum_{\nu=\nu_0}^{\infty} C_{J^\pi Mlsv}\,\Psi_{J^\pi Mlsv} \tag{3}$$

of the total many-nucleon wave function of the two-cluster system over the so-called AVRGM basis:

$$\Psi_{J^\pi Mlsv} = N_{J^\pi lsv}\,\hat{\mathcal{A}}\left\{\sum_{m+\sigma=M} C^{JM}_{lm\,s\sigma}\left[\phi^{(1)}_{s_1}\phi^{(2)}_{s_2}\right]_{s\sigma} f_{vlm}(\mathbf{q})\right\}. \tag{4}$$

Here $N_{J^\pi lsv}$ is the normalization; $\hat{\mathcal{A}}$ is the antisymmetrization operator; $\phi^{(1)}_{s_1}$ and $\phi^{(2)}_{s_2}$ are intrinsic wave functions of the clusters with the spins s_1 and s_2 coupled to the channel spin s with the projection σ; J and M are the total angular momentum and its projection respectively; π is the parity of the system; $C^{JM}_{lm\,s\sigma}$ is the Clebsch-Gordan coefficient; ν_0 is the minimum number of oscillator quanta allowed by the Pauli exclusion principle; $C_{J^\pi Mlsv}$ are unknown expansion coefficients; $\mathbf{q}$ is the Jacobi vector proportional to the relative distance between the cluster centers of mass:

$$\mathbf{q} = \sqrt{\frac{A_1A_2}{A_1+A_2}}\left(\frac{1}{A_1}\sum_{i=1}^{A_1}\mathbf{r}_i - \frac{1}{A_2}\sum_{i=A_1+1}^{A_1+A_2}\mathbf{r}_i\right), \tag{5}$$

$\mathbf{r}_i$ is the radius vector of ith nucleon, A_1 and A_2 are the mass numbers of the clusters. The intrinsic wave functions of the clusters are usually chosen in the form of the translationally invariant oscillator shell model wave functions for the lowest states compatible with the Pauli exclusion principle. Here we follow this choice.

The oscillator radius in the intrinsic wave functions and in the oscillator functions (2) is supposed to be the same. The corresponding choice of a value of this parameter partly

allows us to compensate for an inaccuracy of the approximate description of the internal cluster states. The main idea of the multiscale AVRGM is to use different values of the oscillator radius in the AVRGM basis for the expansions of the wave functions for bound states and continuum. Evidently, it is more natural for the wave functions of the different types. By this way, the mutual cluster influence resulting in monopole cluster deformations in the bound states can be taken to some extent into account. It should be emphasized that one of the principal advantages of the use of the different oscillator radii is an opportunity to adjust the breakup thresholds to their experimental values for the nuclei formed in the radiative capture reactions. This is an important feature, for example, because the corresponding breakup thresholds directly enter into the expression for the radiative capture cross section [see Eq. (24) in Sec. IV].

The expansion coefficients over the AVRGM basis (4) obey an infinite set of linear algebraic equations [89,91]

$$\begin{cases} \sum\limits_{s=|s_1-s_2|}^{s_1+s_2} \sum\limits_{l=|J-s|}^{J+s} \sum\limits_{\nu=\nu_0}^{\infty} \left(\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}|H|J^\pi Mls\nu\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right) C_{J^\pi Mls\nu} = 0, \\ \tilde{s} = |s_1-s_2|,\ldots,s_1+s_2, \quad \tilde{l} = |J-\tilde{s}|,\ldots,J+\tilde{s}, \quad \tilde{\nu} = \nu_0,\nu_0+2,\ldots, \end{cases} \tag{6}$$

where H and E are the Hamiltonian and the total energy of the system respectively; δ_{ij} is the Kronecker symbol. Equations (6) follow from the multiparticle Schrödinger equation after substituting the expansion (3) in it and projecting onto the corresponding basis functions (4).

For the bound states, the expansions over the AVRGM basis (4) can be truncated at a sufficiently large value $\nu = \nu_{\max}$, which depends on a desired accuracy. Thus, the problem reduces to a finite set of homogeneous algebraic equations

$$\begin{cases} \sum\limits_{s=|s_1-s_2|}^{s_1+s_2} \sum\limits_{l=|J-s|}^{J+s} \sum\limits_{\nu=\nu_0}^{\nu_{\max}} \left(\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}|H|J^\pi Mls\nu\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right) C^{(\mathrm{D})}_{J^\pi Mls\nu} = 0, \\ \tilde{s} = |s_1-s_2|,\ldots,s_1+s_2, \quad \tilde{l} = |J-\tilde{s}|,\ldots,J+\tilde{s}, \quad \tilde{\nu} = \nu_0,\nu_0+2,\ldots,\nu_{\max}, \end{cases} \tag{7}$$

and to a diagonalization of the Hamiltonian matrix in a truncated AVRGM basis. The negative sign of the difference between some eigenvalue and the sum of the intrinsic energies of the clusters calculated with the intrinsic cluster functions involved in Eq. (4) indicates a bound state.

The case of the continuous spectrum is somewhat more complicated. The expansion coefficients decrease rather slowly and the corresponding expansion terms cannot be neglected as in the case of the bound states. Moreover, such neglect is principally impossible due to the significant distortion of the nature of the continuous spectrum wave functions. An accurate approach in this case implies a replacement of the expansion coefficients by their asymptotic values [90,119]:

$$C^{(\mathrm{as})}_{J^\pi Mls\nu} = \delta_{ll_\mathrm{e}}\delta_{ss_\mathrm{e}} C^{(-)}_{l\nu} - S^{J^\pi}_{ls,\,l_\mathrm{e}s_\mathrm{e}} C^{(+)}_{l\nu}, \tag{8}$$

$$C^{(\pm)}_{l\nu} = \sqrt{\frac{2\pi m(2l_\mathrm{e}+1)}{\hbar k^3 q_0}}\, i^{l+1} r_0 \left[G_l(\eta,kq_0) \pm i F_l(\eta,kq_0)\right] \exp(\mp i\sigma_l), \tag{9}$$

starting from a sufficiently large value $\nu = \nu_{\mathrm{as}}$. Here l_e and s_e relate to the entrance channel; $S^{J^\pi}_{ls,\,l_\mathrm{e}s_\mathrm{e}}$ is the element of the scattering matrix; F_l and G_l are the regular and irregular Coulomb functions respectively; η is the Coulomb parameter; σ_l is the Coulomb phase shift; $q_0 = r_0\sqrt{2\nu+3}$ is the classical turning point for the oscillator; $k = \sqrt{2mE_{\mathrm{c.m.}}}/\hbar$ is the wave number; m is the nucleon mass; $E_{\mathrm{c.m.}}$ is the relative motion energy of the clusters in the center-of-mass system; $\hbar$ is the Planck constant. Expression (8) allows us to take into account the boundary condition in the oscillator representation for the continuum. It corresponds to the superposition of the incoming and outgoing Coulomb-distorted spherical waves in coordinate representation at large distances. It should be noted that a more precise expression for the asymptotic expansion coefficients can be found in Ref. [119]. So, in contrast to the previous case of the bound states, an inhomogeneity occurs in the equations:

$$\begin{cases} \sum\limits_{s=|s_1-s_2|}^{s_1+s_2} \sum\limits_{l=|J-s|}^{J+s} \sum\limits_{\nu=\nu_0}^{\nu_{\mathrm{as}}-2} \left(\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}|H|J^\pi Mls\nu\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right) C^{(\mathrm{C})}_{J^\pi Mls\nu} = F_{J^\pi M\tilde{l}\tilde{s}\tilde{\nu}}, \\ \tilde{s} = |s_1-s_2|,\ldots,s_1+s_2, \quad \tilde{l} = |J-\tilde{s}|,\ldots,J+\tilde{s}, \quad \tilde{\nu} = \nu_0,\nu_0+2,\ldots,\nu_{\mathrm{as}}, \end{cases} \tag{10}$$

$$F_{J^\pi M\tilde{l}\tilde{s}\tilde{\nu}} = -\sum_{s=|s_1-s_2|}^{s_1+s_2} \sum_{l=|J-s|}^{J+s} \sum_{\nu=\nu_{\mathrm{as}}}^{\nu'_{\max}} \langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}|H|J^\pi Mls\nu\rangle C^{(\mathrm{as})}_{J^\pi Mls\nu}. \tag{11}$$

Formally, the infinite sum is truncated in Eq. (11) at some upper limit $\nu = \nu'_{\max}$. This follows from the fact that the matrix elements entering into Eq. (11) decrease significantly for $\nu \to \infty$ [91] and the contribution of the corresponding terms is negligible. The elements of the scattering matrix and the expansion coefficients can be determined from Eqs. (8)–(11) by the usual way (see, for example, Refs. [89,90,97]).

Thus, the general scheme of the AVRGM has been fully described in this section. It is worthwhile to mention that the most complicated task for the AVRGM realization is to calculate the Hamiltonian matrix elements. The main complexity originates from the antisymmetrization of the total wave function with respect to every pair permutation of the nucleons. This leads to the great number of terms in bra and ket vectors of the matrix elements.

III. MATRIX ELEMENTS CALCULATION. THE GENERATING FUNCTIONS METHOD

A set of the Jacobi coordinates has a significantly more complicated transformation law under permutations of the nucleons than a set of the single-particle ones. For this reason, the latter is more preferable in the calculations of the matrix elements. To pass from the former set to the latter one, it is worthwhile to introduce into the functions (4) the additional factor representing the wave function of the total center-of-mass motion in the form of zero-point oscillations,

$$F_{000} = \exp\left[-\frac{A}{2}\left(\frac{\mathbf{r}_{\text{c.m.}}}{r_0}\right)^2\right], \tag{12}$$

where A is the mass number of the nuclear system and $\mathbf{r}_{\text{c.m.}}$ is the center-of-mass radius vector. So, it is the first simplification in the calculations of the matrix elements. Dependence on the center-of-mass motion arises in the matrix elements of a transnationally invariant operator in the form of the common multiplier $\langle F_{000} \mid F_{000}\rangle$, which can be extracted easily from the final results.

The further improvement of the calculations can be provided by the generating functions method [89,91–93]. The main idea of the method is to utilize the generating function

$$\exp\left(-q^2/2r_0^2 + \mathbf{qR}/r_0 - R^2/4\right) = \sum_{\nu lm} B_{\nu l} f_{\nu lm}(\mathbf{q}) \frac{R^\nu}{\nu!} Y^*_{lm}(\mathbf{n_R}), \tag{13}$$

$$B_{\nu l} = \frac{(\pi r_0)^{3/2}\nu!}{2^{\nu-1/2}\sqrt{\Gamma[(\nu-l+2)/2]\,\Gamma[(\nu+l+3)/2]}}, \tag{14}$$

for the oscillator functions (2), where $\mathbf{R}$ is the generating parameter. Replacing $f_{\nu lm}$ in Eq. (4) by the left-hand side of the relation (13), one can obtain the generating function for the AVRGM basis. Then, it is desirable to introduce instead of Eq. (12) to some extent a more complicated function:

$$F(\mathbf{r}_{\text{c.m.}}) = \exp\left[-\frac{A}{2}\left(\frac{\mathbf{r}_{\text{c.m.}}}{r_0}\right)^2 - \sqrt{\frac{A_2 A}{A_1}}\frac{\mathbf{R}\mathbf{r}_{\text{c.m.}}}{r_0} - \frac{A_2}{4A_1}R^2\right]. \tag{15}$$

The choice (15) has an advantage: it leads to the calculations with antisymmetrized many-particle functions, which can be expressed in the form of Slater determinants composed of single-particle states:

$$|i\rangle \equiv g_i(\mathbf{r}_i)|\xi_i\rangle, \tag{16}$$

where i is the number of a nucleon; $|\xi_i\rangle$ is the spin-isospin function ($|\uparrow p\rangle$, $|\downarrow p\rangle$, $|\uparrow n\rangle$, $|\downarrow n\rangle$) of ith nucleon; g_i is the coordinate function having either the form

$$\exp\left[-\frac{1}{2}\left(\frac{\mathbf{r}_i}{r_0}\right)^2\right] \tag{17}$$

for the nucleons of the first cluster or the form

$$\exp\left[-\frac{1}{2}\left(\frac{\mathbf{r}_i}{r_0}\right)^2 + \frac{\mathbf{r}_i\tilde{\mathbf{R}}}{r_0} - \frac{1}{4}\tilde{R}^2\right] \tag{18}$$

for the nucleons of the second one. Here the clusters are assumed to be s clusters and the scaled generating parameter

$$\tilde{\mathbf{R}} = -\sqrt{\frac{A}{A_1 A_2}}\,\mathbf{R} \tag{19}$$

is introduced (in the following, we omit the tilde sign for a simplification in the notation of the generating parameter). The coordinate dependences of g_i are rather convenient for analytical calculations.

The general properties of determinant allow us to orthogonalize the single-particle states (16) from the bra and ket vectors, which are the Slater determinants, and to reduce significantly difficulties in the calculations of the matrix elements of the translationally invariant operator

$$V = \sum_{i>j=1}^{A} V_{ij} \tag{20}$$

being the sum of the pair operators V_{ij}. The calculated matrix elements should be divided by the factor $\langle F \mid F\rangle$ to remove the admixture of the center-of-mass motion. So, the generating matrix elements $\langle \mathbf{Q}|V|\mathbf{R}\rangle$ for the operator V have been obtained. All its matrix elements between the AVRGM basis functions (4) can be derived from $\langle \mathbf{Q}|V|\mathbf{R}\rangle$ in accordance with Eq. (13):

$$\left\langle J_f^{\pi_f} M_f l_f s_f \nu_f \middle| V \middle| J_i^{\pi_i} M_i l_i s_i \nu_i \right\rangle = \frac{1}{\kappa_{\nu_f l_f s_f}\kappa_{\nu_i l_i s_i}\nu_f!\nu_i!}\frac{\partial^{\nu_f}}{\partial Q^{\nu_f}}\frac{\partial^{\nu_i}}{\partial R^{\nu_i}} I_{i\to f}(Q,R)\Big|_{Q=R=0}, \tag{21}$$

$$I_{i\to f}(Q,R) = \sum_{\substack{m_f+\sigma_f=M_f,\\ m_i+\sigma_i=M_i}} C^{J_f M_f}_{l_f m_f\, s_f\sigma_f} C^{J_i M_i}_{l_i m_i\, s_i\sigma_i} \iint Y^*_{l_f m_f}(\mathbf{n_Q})\langle \mathbf{Q}, s_f\sigma_f|V|\mathbf{R}, s_i\sigma_i\rangle Y_{l_i m_i}(\mathbf{n_R})\, d\mathbf{n_Q} d\mathbf{n_R}, \tag{22}$$

$$\kappa^2_{\nu l s} = \frac{2\pi}{(\nu!)^2}\frac{\partial^\nu}{\partial Q^\nu}\frac{\partial^\nu}{\partial R^\nu}\int_{-1}^{1}\langle \mathbf{Q}, s\sigma|\mathbf{R}, s\sigma\rangle P_l(t)\, dt\Big|_{Q=R=0}, \tag{23}$$

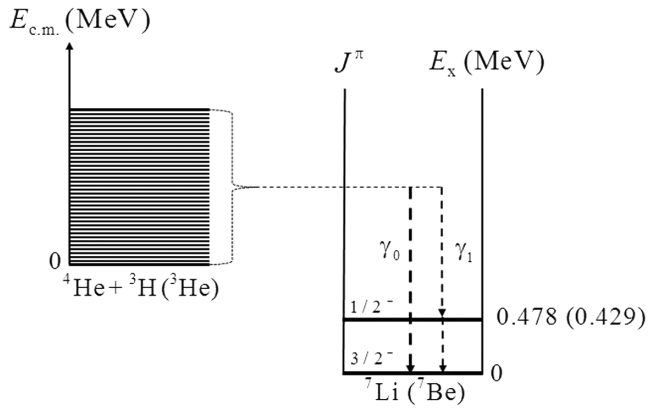

FIG. 3. Scheme of the mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions at low energies.

where $P_l(t)$ is the Legendre polynomial and $t = \cos\theta_{\mathbf{QR}}$, $\theta_{\mathbf{QR}}$ is the angle between the vectors $\mathbf{Q}$ and $\mathbf{R}$. The relation (23) is a consequence of the AVRGM basis orthonormality. An additional advantage of the generating functions method is that it allows us to apply the recurrence technique [91], which is very effective for the computations of the matrix elements in the AVRGM basis.

The generating matrix elements and expressions, which are needed to calculate the matrix elements of the Hamiltonians for the systems considered in this work, are presented in the Appendix.

IV. PARTIAL CROSS SECTIONS FOR THE MIRROR ^{3}H$(\alpha,\gamma)^7$Li AND ^{3}He$(\alpha,\gamma)^7$Be REACTIONS WITHIN THE MULTISCALE AVRGM APPROACH

To demonstrate capabilities of the developed approach, the mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions were adopted for consideration. As it is well known [63,70], electric dipole transitions ($E1$ transitions) of the ^{4}He + ^{3}H and ^{4}He + ^{3}He systems from their continuum to the ground and first excited states of the ^{7}Li and ^{7}Be nuclei respectively play the dominating role in those reactions at low energies (see Fig. 3). Their total cross sections can be represented as the sums of the partial ones characterizing the contributions of the corresponding partial waves in the expansion of the initial state. The colliding nuclei are supposed to be unpolarized and polarization of the emitted photons and the formed nuclei is not considered. Therefore, averaging over spin projections of the projectile and the target and summation over the spin projections of the final products should be performed. As a result, in the long-wavelength limit, the partial cross sections for the considered reactions within the AVRGM approach take the form [84–86]

$$\sigma_{i\to f}(E_{\text{c.m.}}) = \frac{8\pi}{9\hbar(2l_i+1)}\left(\frac{E_\gamma}{\hbar c}\right)^3 \left| \sum_{\nu_i,\nu_f} C^{(\text{D})}_{J_f^{\pi_f} l_f s \nu_f} \times \left\langle J_f^{\pi_f} l_f s \nu_f \middle\| M_1^{\text{E}} \middle\| J_i^{\pi_i} l_i s \nu_i \right\rangle C^{(\text{C})}_{J_i^{\pi_i} l_i s \nu_i} \right|^2, \quad (24)$$

where $C_f^{(\text{D})}$ and $C_i^{(\text{C})}$ are the expansion coefficients of the wave functions for the final and initial states of the nuclear system over the AVRGM bases with the different values of the oscillator radius [these coefficients are solutions of the AVRGM equations sets (7) and (10) respectively]; c is the light velocity; $E_\gamma = E_{\text{c.m.}} + \varepsilon_n$ is the energy of the emitted photon, ε_n is the breakup threshold for nth bound state of the ^{7}Li (^{7}Be) nucleus into the corresponding colliding nuclei ^{4}He + ^{3}H (^{4}He + ^{3}He); $M^{\text{E}}_{1\mu}$ is the electric dipole operator defined by

$$M^{\text{E}}_{1\mu} = e\sum_{i=1}^{A} g_l(i)|\mathbf{r}_i - \mathbf{r}_{\text{c.m.}}|Y_{1\mu}\left(\mathbf{n}_{\mathbf{r}_i-\mathbf{r}_{\text{c.m.}}}\right), \qquad g_l(i) = \frac{1}{2} - t_{3,i}, \quad (25)$$

in which e is the elementary charge ($e > 0$) and $t_{3,i}$ is the isospin projection operator for ith nucleon. The channel spin s is equal to 1/2 in the initial and final states (the clusters are considered to be unexcited). The partial wave functions of the initial state are assumed to be normalized to the unit flux density of the incident plane wave. In Eq. (24), the quantum numbers (J_i,l_i) are equal to (1/2, 0), (3/2, 2), and (5/2, 2) for the $E1$ captures proceeding to the ground state of the ^{7}Li (^{7}Be) nucleus with the quantum numbers $(J_f,l_f) = (3/2,1)$. In turn, (J_i,l_i) are equal to (1/2, 0) and (3/2, 2) for the $E1$ captures to the first excited state of the ^{7}Li (^{7}Be) nucleus with $(J_f,l_f) = (1/2,1)$.

The matrix elements for the operator $M^{\text{E}}_{1\mu}$ depend on the magnetic quantum numbers according to the Wigner-Eckart theorem. This allowed us to perform summations in Eq. (24) over these numbers analytically and to express the partial cross section via the reduced matrix elements of the operator (25). In the framework of the multiscale AVRGM approach, these reduced matrix elements are given by [120]

$$\left\langle J_f^{\pi_f} l_f s \nu_f \middle\| M_1^{\text{E}} \middle\| J_i^{\pi_i} l_i s \nu_i \right\rangle = \zeta(-1)^{J_i+l_f+s+1}\frac{e}{14}\left(\frac{2r_{01}r_{02}}{r_{01}^2+r_{02}^2}\right)^{10}\sqrt{\frac{3(2l_i+1)(2J_i+1)(2J_f+1)}{\pi}}\begin{Bmatrix} l_i & s & J_i \\ J_f & 1 & l_f \end{Bmatrix} \times \frac{C^{l_f 0}_{l_i 0\ 10}}{\kappa_{\nu_f l_f s}\kappa_{\nu_i l_i s}\nu_f!\nu_i!}\left(r_{01}\nu_f\frac{\partial^{\nu_f-1}}{\partial Q^{\nu_f-1}}\frac{\partial^{\nu_i}}{\partial R^{\nu_i}}U_{l_i}(Q,R) + r_{02}\nu_i\frac{\partial^{\nu_f}}{\partial Q^{\nu_f}}\frac{\partial^{\nu_i-1}}{\partial R^{\nu_i-1}}U_{l_f}(Q,R)\right)\Bigg|_{Q=R=0}. \quad (26)$$

Here

$$\zeta = \begin{cases} -1 & \text{for } {}^4\text{He} + {}^3\text{H} \text{ system}, \\ 1 & \text{for } {}^4\text{He} + {}^3\text{He} \text{ system}, \end{cases} \quad (27)$$

$$U_l(Q,R) = 2\pi \int_{-1}^{1} \exp\left(\frac{3\left(r_{01}^2 - r_{02}^2\right)\left(Q^2 - R^2\right) - 9 r_{01} r_{02} QRt}{7\left(r_{01}^2 + r_{02}^2\right)}\right)\left[\exp\left(\frac{r_{01} r_{02}}{r_{01}^2 + r_{02}^2} QRt\right) - 1\right]^3 P_l(t)\, dt, \tag{28}$$

$$\kappa_{\nu ls}^2 = \frac{2\pi}{\nu!}\left[\left(\frac{6}{7}\right)^{\nu} - 3\left(\frac{5}{14}\right)^{\nu} + 3\left(-\frac{1}{7}\right)^{\nu} - \left(-\frac{9}{14}\right)^{\nu}\right]\varepsilon_{\nu l}, \tag{29}$$

$$\varepsilon_{\nu l} = \begin{cases} \dfrac{2^{l+1}\nu!((\nu+l)/2)!}{(\nu+l+1)!((\nu-l)/2)!}, & l \leqslant \nu,\ l+\nu\ -\ \text{even}, \\ 0\,, & \text{in other cases}, \end{cases} \tag{30}$$

and $\left\{\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right\}$ is the $6j$ symbol. The oscillator radius used in the expansions of the wave functions over the AVRGM basis (4) has been denoted as r_{01} for the continuum of the $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems and as r_{02} for the bound states of the $^7\mathrm{Li}$ and $^7\mathrm{Be}$ nuclei. It should be noted that matrix elements for the electric quadrupole ($E2$) and magnetic dipole ($M1$) operators can be calculated using results of the work [120].

At low sub-barrier energies, a cross section of a reaction induced by charged particles rapidly decreases with the relative motion energy $E_{\mathrm{c.m.}}$. This strong dependence is predominantly caused by penetrability of the Coulomb barrier. The astrophysical S factor has smoother energy behavior than the cross section because the penetrability of the Coulomb barrier is explicitly extracted from it:

$$S(E_{\mathrm{c.m.}}) = E_{\mathrm{c.m.}} \exp\left(\sqrt{E_{\mathrm{G}}/E_{\mathrm{c.m.}}}\right)\sigma(E_{\mathrm{c.m.}}), \tag{31}$$

where

$$E_{\mathrm{G}} = 2mc^2\left(\frac{\pi e^2 Z_1 Z_2}{\hbar c}\right)^2 \frac{A_1 A_2}{A_1 + A_2} \tag{32}$$

is the Gamow energy for the colliding particles (nuclei) with the charge numbers Z_1 and Z_2. For example, the Gamow energy E_{G} is equal to 6.76 and 27.04 MeV for the considered $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems respectively.

V. RESULTS AND DISCUSSION

A. Binding energies and nuclear phase shifts for the $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems

The oscillator radius r_{01} involved in the expansions of the continuum wave functions of the $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems over the AVRGM basis was fixed at a value 1.386 fm to obtain α particle binding energy coinciding with its experimental value 28.296 MeV [121] and at the same time to reach triton and hellion binding energies, which are the closest to their experimental values 8.482 and 7.718 MeV respectively [121].

In order to describe nuclear interaction, the effective modified Hasegawa-Nagata potential [122] has been used with a single adjustable parameter—the intensity of the central Majorana force g_{c}. Energy dependences of s- and d-wave nuclear phase shifts for the $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems calculated by using that potential with the g_{c} values equal to 0.974, 0.977, and 0.980 are presented in Figs. 4 and 5 respectively. The curves for the s-wave phase shift $\delta_{1/2^+}$ calculated at the adopted g_{c} values differ only slightly from each other. The difference between the curves for the d-wave phase shift $\delta_{3/2^+}$ is more minor. The similar situation is in the case of the d-wave phase shift $\delta_{5/2^+}$. The phase shifts $\delta_{3/2^+}$ and $\delta_{5/2^+}$ calculated at the same value of the g_{c} parameter lie on top of each other in the figures and are almost indistinguishable from each other on the scale of the plots. Nevertheless, they are not identical because of the noncentral components of the nuclear potential [122]. Increasing the g_{c} values leads to decreasing the phase shift values. Only small variations of this parameter are usually assumed. That is why the g_{c} values remain close to unity [94,96].

As to data on the nuclear phase shifts $\delta_{1/2^+}$, $\delta_{3/2^+}$, and $\delta_{5/2^+}$ extracted from the $^4\mathrm{He} + {}^3\mathrm{H}$ [123,124] and $^4\mathrm{He} + {}^3\mathrm{He}$ [124–129] scattering experiments, they are characterized by a large scatter of values (see Figs. 4 and 5). Nevertheless, their energy dependences are described rather reasonably by the obtained curves for both the $^4\mathrm{He} + {}^3\mathrm{H}$ and $^4\mathrm{He} + {}^3\mathrm{He}$ systems simultaneously.

The nuclear phase shifts for the considered systems calculated with the parameter values $r_{01} = 1.22$ fm and $g_{\mathrm{c}} = 1.035$, which were used in our single-scale AVRGM calculations

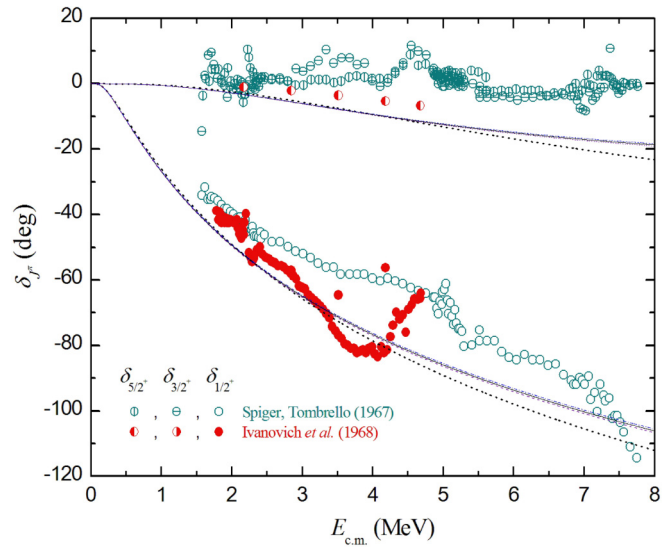

FIG. 4. Nuclear phase shifts for the $^4\mathrm{He} + {}^3\mathrm{H}$ system. Dash-dotted, solid, and dashed lines are the calculations with the g_{c} values equal to 0.974, 0.977, and 0.980 respectively ($r_{01} = 1.386$ fm). Dotted line is the calculation with $r_{01} = 1.22$ fm and $g_{\mathrm{c}} = 1.035$. The lower curves represent $\delta_{1/2^+}$. The upper ones assign to $\delta_{3/2^+}$. The $\delta_{5/2^+}$ curves lie on top of $\delta_{3/2^+}$ ones. Symbols are the data taken from works [123,124].

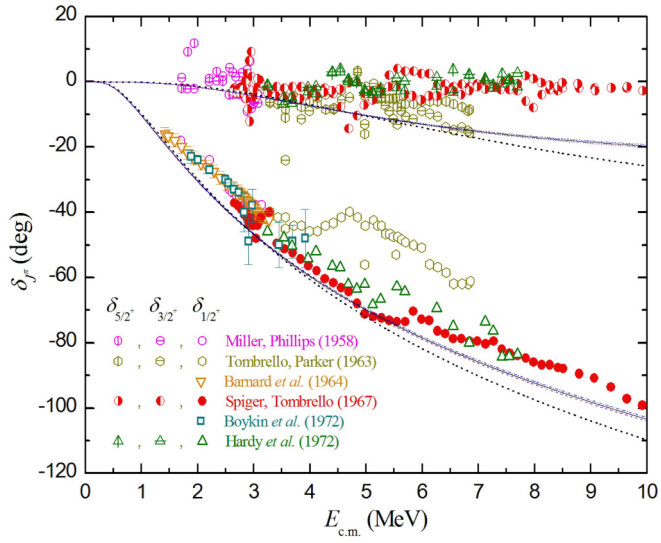

FIG. 5. Nuclear phase shifts for the ^{4}He + ^{3}He system. Lines: see Fig. 4 caption. Different symbols: the data [124–129].

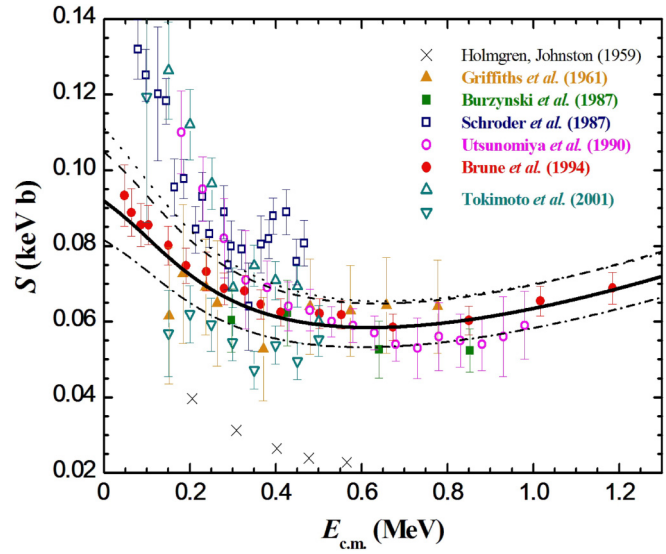

FIG. 6. Total astrophysical S factor for the ^{3}H$(\alpha,\gamma)^7$Li reaction. Dash-dotted, solid, and dashed lines are the multiscale AVRGM calculation with the g_c values equal to 0.974, 0.977, and 0.980 respectively ($r_{01} = 1.386$ fm). Dotted line is the single-scale AVRGM calculation with $r_0 = 1.22$ fm and $g_c = 1.035$. The experimental data are taken from Refs. [11–17].

performed in the previous studies [56,57], are presented in Figs. 4 and 5 by dotted lines. At relatively low energies, these curves lie very close to the others and to the data but they give a worse description of the data at higher energies than the other curves. Moreover, the given r_{01} value does not allow us to obtain cluster binding energies, including that for α particle, consistent with the experimental values. All the binding energies turn out to be underestimated [86].

B. The ^{3}H$(\alpha,\gamma)^7$Li reaction: Total and partial astrophysical S factors and branching ratio

Values of the oscillator radius r_{02} involved in the expansions of the wave functions of the ground and first excited states of the ^{7}Li nucleus over the AVRGM basis are tuned to reproduce the experimental values [130] for the breakup thresholds of the ground $\varepsilon_0^{(\alpha+t)}$ and first excited $\varepsilon_1^{(\alpha+t)}$ states of the ^{7}Li nucleus into the initial ^{4}He + ^{3}H fragments:

$$\varepsilon_0^{(\alpha+t)} = 2.467 \text{ MeV}, \quad \varepsilon_1^{(\alpha+t)} = 1.989 \text{ MeV}. \tag{33}$$

The total astrophysical S factor for the ^{3}H$(\alpha,\gamma)^7$Li reaction calculated in the framework of the multiscale AVRGM approach, using the r_{01} value 1.386 fm and the different g_c values 0.974, 0.977, and 0.980 adopted above in the calculations of the nuclear phase shifts, is shown in Fig. 6. As it can be seen in Fig. 6, the curve obtained for the g_c value equal to 0.977 is in a perfect agreement with some of the latest data [16] covering the widest energy range and having relatively small errors compared to the other data. The values of the oscillator radius r_{02}, which correspond to the adopted optimal r_{01} and g_c values, are presented in the second row of Table I.

Astrophysical S_0 and S_1 factors for the capture to the ground and first excited states of the ^{7}Li nucleus, respectively, as well as the partial astrophysical S factors calculated within the multiscale AVRGM approach at the parameter values $r_{01} = 1.386$ fm, $g_c = 0.977$, and r_{02} from Table I are shown in Fig. 7. Here it is a reasonable enough agreement between the obtained results for the astrophysical S_0 and S_1 factors and the data extracted from direct measurements [12–14,16]. At lower energies, the $E1$ captures from the s waves play the dominating role. At relatively higher energies, the $E1$ captures from the d waves also turn out to be significant.

The calculations of the total astrophysical S factor and of the astrophysical S_0 and S_1 factors for the ^{3}H$(\alpha,\gamma)^7$Li reaction on the basis of our single-scale AVRGM approach ($r_{01} \equiv r_{02} \equiv r_0 = 1.22$ fm, $g_c = 1.035$) [56,57] are respectively presented in Figs. 6 and 7 by dotted lines. The obtained curve for the total astrophysical S factor slightly overestimates the data [16] but, at the same time, the astrophysical S_0 and S_1 factors describe the data [16] quite accurately. There is a principal disadvantage of this single-scale AVRGM calculation: the obtained ^{7}Li breakup thresholds are underestimated ($\varepsilon_0^{(\alpha+t)} = 1.672$ MeV, $\varepsilon_1^{(\alpha+t)} = 1.546$ MeV) in comparison with their experimental values (33). It is also interesting to note that the multiscale AVRGM approach provides smoother low-energy behavior of the calculated astrophysical S_0 and S_1 factors in contrast to the single-scale one.

For the sake of completeness, the branching ratio between the radiative captures to the first excited and ground states

TABLE I. Values of the oscillator radius r_{02} (in fm) used in calculations with $r_{01} = 1.386$ fm and $g_c = 0.977$. The r_{02} values reproducing ε_0 and ε_1 experimental values [see Eqs. (33) and (34)] for the considered nuclei are denoted as $r_{02,0}$ and $r_{02,1}$ respectively.

Nucleus	$r_{02,0}$	$r_{02,1}$
^{7}Li	1.3030	1.2820
^{7}Be	1.3068	1.4205

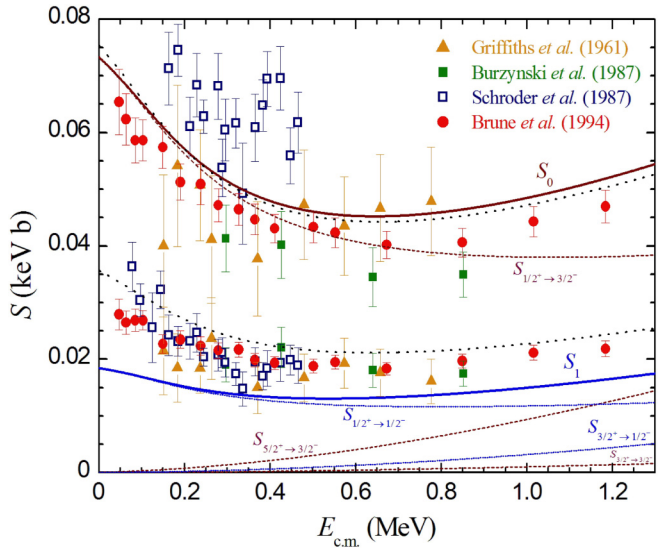

FIG. 7. Partial astrophysical S factors for the ^{3}H$(\alpha,\gamma)^7$Li reaction. Solid lines are the multiscale AVRGM calculation of the astrophysical S_0 and S_1 factors; dotted lines and symbols are the same quantities calculated within the single-scale AVRGM approach and extracted from experimental works [12–14,16] respectively. Short-dashed and short-dotted lines are the multiscale AVRGM calculation of the partial astrophysical S factors for capture to the ground and first excited states of the final nucleus respectively.

of ^{7}Li determined as $R = \sigma_1/\sigma_0$, where σ_0 and σ_1 are the radiative capture cross sections related to the corresponding astrophysical S_0 and S_1 factors by Eq. (31), is plotted in Fig. 8. The result of the single-scale AVRGM calculation (dotted line, $r_0 = 1.22$ fm, $g_c = 1.035$) [57] is seen to agree well with the experimental data [16] while the multiscale one (solid line, $r_{01} = 1.386$ fm, $g_c = 0.977$, r_{02}—see Table I) underestimates

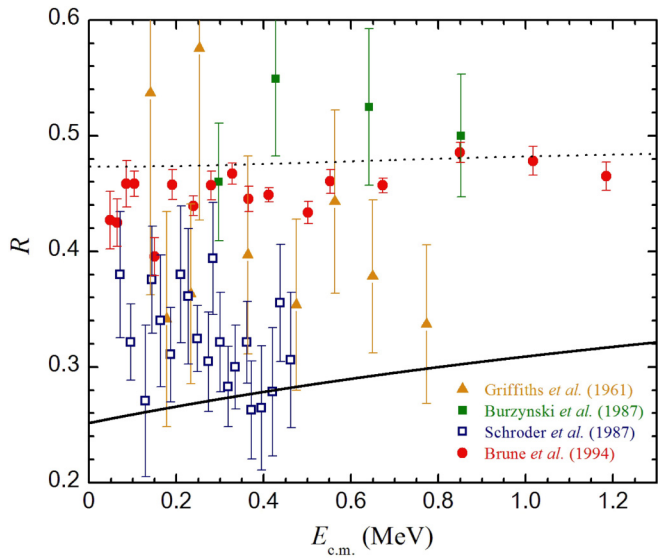

FIG. 8. Branching ratio for the ^{3}H$(\alpha,\gamma)^7$Li reaction. Solid and dotted lines are the multiscale and single-scale AVRGM calculations respectively. Symbols are the experimental data [12–14,16].

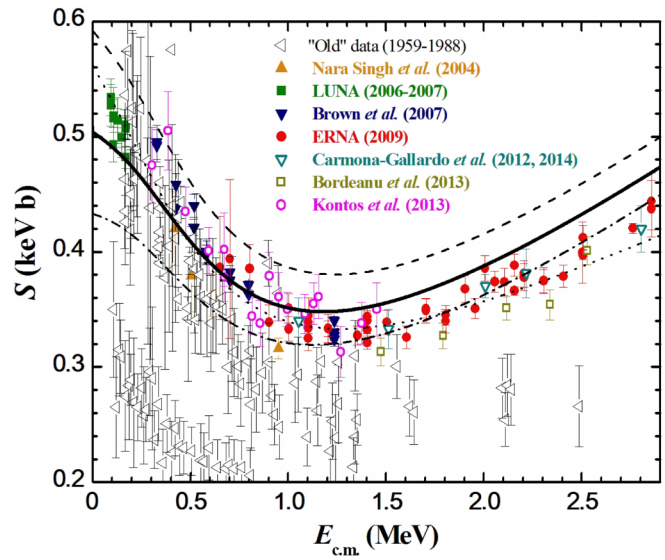

FIG. 9. Total astrophysical S factor for the ^{3}He$(\alpha,\gamma)^7$Be reaction. Lines: see Fig. 6 caption. Symbols: the data from "old" [11,18–21,23–25] and modern [26–36] experimental works.

these data. Moreover, the former is almost energy independent, but the latter demonstrates more evident energy dependence. In fact, these interesting findings are a result of the energy behavior of the astrophysical S_0 and S_1 factors.

C. The ^{3}He$(\alpha,\gamma)^7$Be reaction: Total and partial astrophysical S factors and branching ratio

Similarly to the mirror system considered above, values of the oscillator radius r_{02} involved in the expansions of the wave functions of the ground and first excited states of the ^{7}Be nucleus over the AVRGM basis are chosen to reproduce

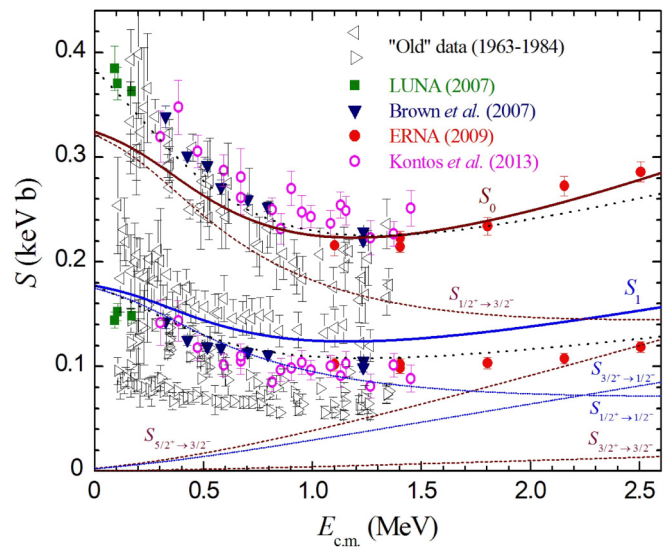

FIG. 10. Partial astrophysical S factors for the ^{3}He$(\alpha,\gamma)^7$Be reaction. Lines: see Fig. 7 caption. Symbols are the data extracted from "old" [18–20,23,24] and modern [29,30,32,34] direct measurements.

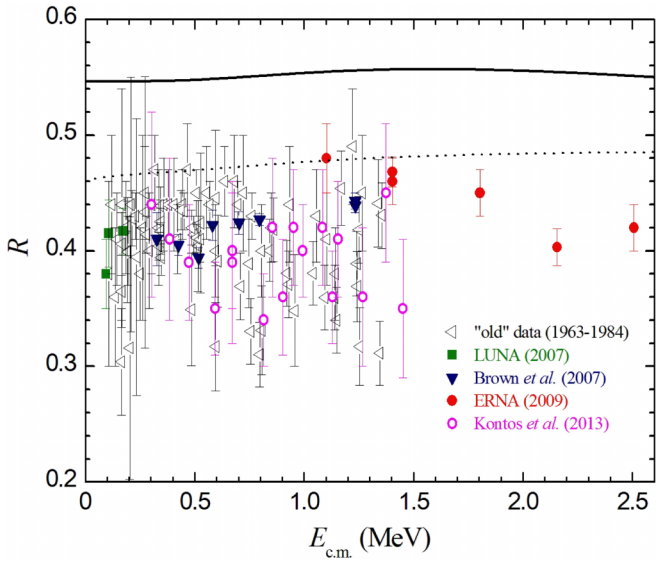

FIG. 11. Branching ratio for the ^{3}He$(\alpha,\gamma)^7$Be reaction. Lines: see Fig. 8 caption. Symbols: see Fig. 10 caption.

the experimental values [130] for the breakup thresholds of the ground $\varepsilon_0^{(\alpha+\mathrm{h})}$ and first excited $\varepsilon_1^{(\alpha+\mathrm{h})}$ states of the ^{7}Be nucleus into the initial ^{4}He + ^{3}He fragments:

$$\varepsilon_0^{(\alpha+\mathrm{h})} = 1.586 \text{ MeV}, \quad \varepsilon_1^{(\alpha+\mathrm{h})} = 1.157 \text{ MeV}. \tag{34}$$

Theoretical curves for the ^{3}He$(\alpha,\gamma)^7$Be total astrophysical S factor calculated in the framework of the multiscale AVRGM approach, using the fixed r_{01} and g_c values, are depicted in Fig. 9. The calculated total astrophysical S factor for the ^{3}He$(\alpha,\gamma)^7$Be reaction agrees with the existing modern experimental data [26–36] in the best way at the same values of the r_{01} and g_c parameters as in the case of the mirror reaction ($r_{01} = 1.386$ fm, $g_c = 0.977$). The values of the oscillator radius r_{02} for these optimal r_{01} and g_c values are given in the third row of Table I. The corresponding astrophysical S factors for the capture to the ground and first excited states of the ^{7}Be nucleus, including the partial contributions of the $E1$ captures from the s and d waves, are displayed in Fig. 10. The calculated astrophysical S_0 and S_1 factors show a rather reasonable agreement with the modern experimental data from direct measurements [29,30,32,34]. One can conclude analogously to the mirror ^{3}H$(\alpha,\gamma)^7$Li reaction (see Fig. 7) that the contributions of the $E1$ captures from the s waves to the total astrophysical S factor for the ^{3}He$(\alpha,\gamma)^7$Be reaction turn out to be crucial at very low energies because of the centrifugal barrier. However, the effect of the d-wave $E1$ captures increases at higher energies.

The results of the single-scale AVRGM calculation [56,57] of the total astrophysical S factor and of the astrophysical S_0 and S_1 factors for the ^{3}He$(\alpha,\gamma)^7$Be reaction, using the same parameter values as in the case of the mirror reaction, are also presented in Figs. 9 and 10 respectively. These astrophysical S factors give a good description of the modern data. However, the breakup thresholds obtained in this calculation turn out to be underestimated ($\varepsilon_0^{(\alpha+\mathrm{h})} = 0.828$ MeV, $\varepsilon_1^{(\alpha+\mathrm{h})} = 0.710$ MeV) compared with their experimental

TABLE II. The ^{3}H$(\alpha,\gamma)^7$Li zero-energy S-factor $S(0)$ and $S'(0)/S(0)$ values.

Authors, reference	Year	Method	$S(0)$ (keV b)	$S'(0)/S(0)$ (MeV^{-1})
		Experiment		
Holmgren, Johnston [11]	1959	direct measurement	0.051	−1.28
Griffiths *et al.* [12]	1961	direct measurement	0.064	
Schröder *et al.* [14]	1987	direct measurement	0.14 ± 0.02	
Brune *et al.* [16]	1994	direct measurement	0.1067 ± 0.0004	
		Theory		
Williams, Koonin [40]	1981	DCM		−2.034
Kajino [63]	1986	RGM	0.098 ± 0.006	−2.056 ± 0.123
Buck, Merchant [45]	1988	PCM	0.089 ± 0.030	−2.02
Kajino *et al.* [65]	1988	RGM	0.083 ÷ 0.15	
Altmeyer *et al.* [66]	1988	RGM	0.112; 0.124	
Kajino *et al.* [69]	1988	^{7}Li polarizability study	0.097 ± 0.038	
Chopovsky [88]	1989	AVRGM	0.154	
Kajino *et al.* [41]	1989	DCM	0.11 ± 0.03	
Mohr [42]	1993	DCM	0.100	−1.02
Csótó, Langanke [67]	2000	single-channel RGM	0.099 ÷ 0.131	−2.19 ÷ −1.82
		multichannel RGM	0.138 ÷ 0.184	−2.26 ÷ −2.07
Nollett [70]	2001	VMC+PCM	0.095	
Igamov, Yarmukhamedov [50]	2007	PCM modification	0.0974	−1.19
Mason *et al.* [53]	2009	PCM modification	0.13	
Sadeghi [76]	2013	FEs solution	0.107; 0.112	
Solovyev *et al.* [57]	2016	single-scale AVRGM	0.111	−1.188
Present paper	2017	multiscale AVRGM	0.092	−0.955

TABLE III. Theoretical predictions for the ^{3}He$(\alpha,\gamma)^7$Be zero-energy S-factor $S(0)$ and $S'(0)/S(0)$ values.

Authors, reference	Year	Method	$S(0)$ (keV b)	$S'(0)/S(0)$ (MeV^{-1})
Christy, Duck [37]	1961	DCM	1.24	
Kim *et al.* [55]	1981	OCM	0.48 ± 0.01	-0.60
Liu *et al.* [58]	1981	RGM	0.45; 0.61	
Williams, Koonin [40]	1981	DCM		-0.575
Walliser *et al.* [59]	1983	RGM	0.698	-0.607
Walliser *et al.* [60]	1984	RGM	0.598	-0.720
			0.621	-0.685
			0.698	-0.607
Buck *et al.* [44]	1985	PCM	0.47 ± 0.02	
Langanke [54]	1986	PCM+RGM	0.56	
Kajino [63]	1986	RGM	0.50 ± 0.03	-0.548 ± 0.033
Mertelmeier, Hofmann [64]	1986	single-channel RGM	0.53	
		multichannel RGM	0.58	
Buck, Merchant [45]	1988	PCM	0.51 ± 0.03	-0.53
Kajino *et al.* [65]	1988	RGM	$0.36 \div 0.63$	
Kajino *et al.* [69]	1988	^{7}Li polarizability study	0.49 ± 0.14	
Chopovsky [88]	1989	AVRGM	0.690	
Mohr [42]	1993	DCM	0.516	-0.711
Csótó, Langanke [67]	2000	single-channel RGM	$0.52 \div 0.70$	$-0.53 \div -0.50$
		multichannel RGM	$0.83 \div 1.16$	$-0.70 \div -0.57$
Nollett [70]	2001	VMC+PCM	0.40	
Arai *et al.* [68]	2002	single-channel RGM	0.52	-0.63
		multichannel RGM	0.63	-0.57
Mason *et al.* [53]	2009	PCM modification	0.42	
Mohr [43]	2009	DCM	$0.497 \div 0.611$	$-0.791 \div -0.536$
Neff [74]	2011	FMD	0.593	
Tursunmahatov, Yarmukhamedov [51]	2012	PCM modification	$0.613^{+0.026}_{-0.063}$	
Sadeghi [75]	2013	FEs solution	0.563; 0.581	
Solovyev *et al.* [57]	2016	single-scale AVRGM	0.561	-0.524
Present paper	2017	multiscale AVRGM	0.502	-0.225

values (34). Analogously to the considered mirror reaction, the astrophysical S_0 and S_1 factors within the multiscale AVRGM approach demonstrate smoother behavior at very low energies than those of the single-scale one.

The ^{3}He$(\alpha,\gamma)^7$Be branching ratio calculated within the single-scale [56,57] and multiscale AVRGM approaches is shown in Fig. 11. Both the calculated branching ratios provide relatively weak energy dependence, which agrees with the modern data [29,30,32,34], but the former is closer to these experimental results.

Thus, one can see that the well-founded multiscale AVRGM approach reproduces to a greater or lesser extent the whole set of the data on the considered dynamic quantities for the mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions and even very important kinematical quantities—the breakup thresholds of the final nuclei produced in both the reactions. Indeed, as it was shown, the experimental data on the total astrophysical S factors for the reactions and on the nuclear phase shifts in the initial channels are simultaneously described well enough. At the same time, the experimental values of the breakup thresholds for ^{7}Li and ^{7}Be are reproduced. Concerning the ^{3}H$(\alpha,\gamma)^7$Li calculated astrophysical S_0 and S_1 factors, the former slightly overestimates while the latter underestimates the data [16]. In case of the mirror reaction, the situation looks opposite: the calculated S_0 factor underestimates whereas the calculated S_1 factor slightly overestimates the modern data [29,30,32,34]. As a consequence, the ^{3}H$(\alpha,\gamma)^7$Li calculated branching ratio lies lower than the data [16] but the ^{3}He$(\alpha,\gamma)^7$Be branching ratio occurs higher than the modern data. Nevertheless, the qualitative description of the branching ratios for both the reactions looks rather reasonable. It should be emphasized that the problems of such type often arise in the microscopic studies based on the RGM (see, for example, [60,66–68]). These problems also accompany the mixed approaches, which combine either the VMC [70] or the NCSM [72] with the PCM, and even the fully *ab initio* FMD approach [74].

In turn, the single-scale AVRGM approach describes well both the absolute scale and the energy dependence of the data on the astrophysical S_0 and S_1 factors as well as on the branching ratios for both the considered reactions and this description on the whole looks slightly better than that of the multiscale one (see Figs. 7, 8, 10, and 11). However, the breakup thresholds calculated within the single-scale AVRGM approach underestimate the experimental values. Moreover,

TABLE IV. The ^{3}He$(\alpha,\gamma)^7$Be zero-energy S-factor $S(0)$ and $S'(0)/S(0)$ values obtained by extrapolation of the experimental data.

Authors, reference	Year	Experimental method	$S(0)$ (keV b)	$S'(0)/S(0)$ (MeV^{-1})
Holmgren, Johnston [11]	1959	direct measurement	1.20	−1.28
Parker, Kavanagh [18]	1963	direct measurement	0.47 ± 0.05	−0.596
Nagatani *et al.* [19]	1969	direct measurement	0.61 ± 0.07	−0.951
Kräwinkel *et al.* [20]	1982	direct measurement	0.30 ± 0.03	
Robertson *et al.* [21]	1983	activation measurement	0.63 ± 0.04	
Volk *et al.* [22]	1983	activation measurement	0.56 ± 0.03	
			0.38 ± 0.03	
Osborne *et al.* [23]	1982, 1984	direct and activation measurements	0.53 ± 0.03	
Alexander *et al.* [24]	1984	direct measurement	0.47 ± 0.04	
Hilgemeier *et al.* [25]	1988	direct measurement	0.53 ± 0.03	
Nara Singh *et al.* [26]	2004	activation measurement	0.53 ± 0.02	
Bemmerer *et al.* [27]	2006	activation measurement	0.547 ± 0.017	
Gyürky *et al.* [28]	2007	activation measurement	0.547 ± 0.017	
Confortola *et al.* [29]	2007	direct and activation measurements	0.560 ± 0.017	
Brown *et al.* [30]	2007	direct and activation measurements	0.595 ± 0.018	
Costantini *et al.* [31]	2008	direct and activation measurements	0.567 ± 0.018	
Di Leva *et al.* [32]	2009	recoil and activation measurements	0.590 ± 0.016	
Kontos *et al.* [34]	2013	direct measurement	0.554 ± 0.020	

this approach suffers some difficulties in the simultaneous description of the normalization and the energy behavior of the ^{3}He$(\alpha,\gamma)^7$Be total astrophysical S-factor modern data [26–36] and of the ^{3}H$(\alpha,\gamma)^7$Li total astrophysical S-factor most reliable data [16] (see Figs. 6 and 9). In order to avoid these problems, one has to use different sets of the parameter values in the calculations [84–86].

D. ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be zero-energy astrophysical S factors and their derivatives

In calculations of rates for reactions being of astrophysical interest, it is very important to know their zero-energy astrophysical S-factor values $S(0)$. A summary of existing results on the $S(0)$ value as well as on ratio of zero-energy derivative $S'(0)$ of the astrophysical S factor to the $S(0)$ value obtained by theoretical predictions and extrapolations of the experimental data is provided in Table II for the ^{3}H$(\alpha,\gamma)^7$Li reaction and in Tables III and IV for the ^{3}He$(\alpha,\gamma)^7$Be reaction. Moreover, recommended values for these quantities taken from the modern compilations [1,4,5,9,10] and evaluation [6] of the modern data are summarized in Table V. As it can be seen, Tables II–IV show a significant variation of the values while Table V demonstrates an overlap of all the recommended $S(0)$ values for the respective reactions but a discrepancy of the $S'(0)/S(0)$ values. A scatter of the theoretical predictions could be caused by various reasons: using different nuclear potentials, internal cluster wave functions, models with specific parametrizations, and so on.

Our result for the $S(0)$ and $S'(0)/S(0)$ values from the single-scale and multiscale AVRGM approaches are given in Tables II and III (the last two rows). As to the ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be $S(0)$ values from the multiscale AVRGM approach, our result is close to a large number of the presented values. In particular, the obtained $S(0)$ values for both the reactions are in good agreement with the RGM [63,65] and PCM [45] predictions as well as with the result of work [69]. Moreover, the $S(0)$ value for the ^{3}H$(\alpha,\gamma)^7$Li reaction agree well with the DCM prediction [41]. The $S(0)$ value obtained for the ^{3}He$(\alpha,\gamma)^7$Be reaction is also in good correspondence with the DCM prediction [43] and with the extrapolations of the direct measurements [18,23–25]. It should be emphasized

TABLE V. Evaluated $S(0)$ and $S'(0)/S(0)$ values for the ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions from the modern analyses of the data.

Reaction, authors, reference	Year	$S(0)$ (keV b)	$S'(0)/S(0)$ (MeV^{-1})
^{3}H$(\alpha,\gamma)^7$Li			
Angulo *et al.* [4]	1999	0.10 ± 0.02	−1.5
Descouvemont *et al.* [1]	2004	0.095 ± 0.005	
Xu *et al.* [5]	2013	$0.098^{+0.011}_{-0.008}$	
^{3}He$(\alpha,\gamma)^7$Be			
Adelberger *et al.* [9]	1998	0.53 ± 0.05	−0.566
Angulo *et al.* [4]	1999	0.54 ± 0.09	−0.963
Descouvemont *et al.* [1]	2004	0.51 ± 0.04	
Cyburt, Davids [6]	2008	0.58 ± 0.043	-0.92 ± 0.18
Adelberger *et al.* [10]	2011	0.56 ± 0.04	−0.643
Xu *et al.* [5]	2013	$0.056^{+0.05}_{-0.07}$	

that our $S(0)$ values coincide within the errors with all the recommended values for ^{3}H$(\alpha,\gamma)^7$Li and overlap with most values for ^{3}He$(\alpha,\gamma)^7$Be from Table V. The $S(0)$ and $S'(0)/S(0)$ values for both the reactions extracted from the single-scale AVRGM approach have larger magnitudes than those from the multiscale one. Nevertheless, the $S(0)$ values are also in agreement with many values and the $S'(0)/S(0)$ values are even closer to the variety of values presented in the tables. Reasons for the difference of our $S'(0)/S(0)$ values from the predictions could be the same that were mentioned above for a possible explanation of the discrepancy between these predictions. The difference between the results of our present and previous studies indicates that the breakup thresholds reproducing implemented in the present work is rather important for description of the radiative capture reactions. One should pay more attention to this feature, making a microscopic approach.

VI. CONCLUSION

In the present work, the microscopic approach to the description of the radiative capture reactions based on the multiscale AVRGM was proposed. The mirror ^{3}H$(\alpha,\gamma)^7$Li and ^{3}He$(\alpha,\gamma)^7$Be reactions were studied in the framework of this approach. The calculated energy dependences of the total astrophysical S factors for both the reactions describe the experimental data remarkably well. At the same time, the calculated breakup thresholds for the ground and first excited states of the ^{7}Li and ^{7}Be nuclei into the corresponding colliding nuclei reproduce their experimental values. In addition, the nuclear phase shifts and the astrophysical S factors for the captures to the ground and first excited states of the formed nuclei agree reasonably well with the data. The qualitative description of the data on the branching ratios is also reasonable enough but the quantitative one is to some extent not so perfect. Thus, the developed approach demonstrates its capability for the treatment of the radiative capture reactions induced by the light nuclei and makes it possible to achieve the unified description of a wide enough set of the experimental data. All these advantages produce good prospects and opportunities for its further applications.

There are two possible ways to improve the approach. The first one is to use more complicated intrinsic cluster wave functions taking into account excitations of the clusters and a greater number of the expansion terms. The second one is to apply the modern microscopic nuclear potentials. These improvements will allow us to cover a wider class of nuclear reactions and to increase the predictive power of the approach.

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APPENDIX

The generating matrix elements for the operators of the mirror ^{4}He + ^{3}H and ^{4}He + ^{3}He systems and expressions utilized in the calculations of the corresponding matrix elements between the AVRGM basis functions (4) are presented below.

The generating matrix elements for the unit operator read

$$\begin{aligned}\langle \mathbf{Q},s\sigma_f|\mathbf{R},s\sigma_i\rangle &= \langle \mathbf{Q}|\mathbf{R}\rangle\delta_{\sigma_f\sigma_i}\\ &= u^{-9/7}(u-1)^3\delta_{\sigma_f\sigma_i},\end{aligned} \tag{A1}$$

where

$$u = \exp(\mathbf{QR}/2). \tag{A2}$$

The generating matrix elements of the kinetic energy are expressed by

$$\langle \mathbf{Q},s\sigma_f|T-T_{\text{c.m.}}|\mathbf{R},s\sigma_i\rangle = \frac{\hbar^2}{2mr_0^2}\left(9-\frac{3}{7}\mathbf{Q}^2-\frac{3}{7}\mathbf{R}^2+t\frac{\partial}{\partial t}\right)\langle \mathbf{Q}|\mathbf{R}\rangle\delta_{\sigma_f\sigma_i}. \tag{A3}$$

The generating matrix elements of the Coulomb potential are given by

$$\langle \mathbf{Q},s\sigma_f|V_{\text{Coul}}|\mathbf{R},s\sigma_i\rangle = \sqrt{\frac{2}{\pi}}\frac{e^2}{r_0}\left[\langle \mathbf{Q}|\mathbf{R}\rangle + u^{-9/7}(u-1)^2\int_0^1\left(2u\,U^{(+)}-U^{(-)}-U^{(0)}\right)d\varsigma\right]\delta_{\sigma_f\sigma_i} \tag{A4}$$

for the ^{4}He + ^{3}H system and

$$\langle \mathbf{Q},s\sigma_f|V_{\text{Coul}}|\mathbf{R},s\sigma_i\rangle = 2\sqrt{\frac{2}{\pi}}\frac{e^2}{r_0}\left[\langle \mathbf{Q}|\mathbf{R}\rangle + u^{-9/7}\int_0^1\left(u(2u^2-3u+1)\,U^{(+)}-(u^2-3u+2)\,U^{(-)}-2u(u-1)\,U^{(0)}\right)d\varsigma\right]\delta_{\sigma_f\sigma_i} \tag{A5}$$

for the ^{4}He + ^{3}He system, where

$$U^{(\pm)} = \exp\left(-\frac{(\mathbf{R}\pm\mathbf{Q})^2}{8}\varsigma^2\right), \tag{A6}$$

$$U^{(0)} = \exp\left(-\frac{\mathbf{Q}^2}{8}\varsigma^2\right)+\exp\left(-\frac{\mathbf{R}^2}{8}\varsigma^2\right)-1. \tag{A7}$$

The generating matrix elements of the central nuclear interaction of the modified Hasegawa-Nagata potential [122] take the form

$$\langle \mathbf{Q},s\sigma_f|V_{\rm c}|\mathbf{R},s\sigma_i\rangle = 9\langle \mathbf{Q}|\mathbf{R}\rangle\delta_{\sigma_f\sigma_i}\sum_{n=1}^{3}\varsigma_{{\rm c},n}^{3/2}(\alpha_{{\rm c},n}-\delta_{{\rm c},n}) + 3\,u^{-9/7}\delta_{\sigma_f\sigma_i}\sum_{n=1}^{3}\varsigma_{{\rm c},n}^{3/2}\big\{[(4\alpha_{{\rm c},n}+2\beta_{{\rm c},n}+2\gamma_{{\rm c},n}+\delta_{{\rm c},n})\,u - 2\alpha_{{\rm c},n}-2\beta_{{\rm c},n}-2\gamma_{{\rm c},n}-3\delta_{{\rm c},n}]u(u-1)\,U_{{\rm c},n}^{(+)} + [3\alpha_{{\rm c},n}+2\beta_{{\rm c},n}+2\gamma_{{\rm c},n}+2\delta_{{\rm c},n} - (\alpha_{{\rm c},n}+2\beta_{{\rm c},n}+2\gamma_{{\rm c},n}+4\delta_{{\rm c},n})\,u](u-1)\,U_{{\rm c},n}^{(-)} - (\alpha_{{\rm c},n}-\delta_{{\rm c},n})\,(5u^2-6u+1)\,U_{{\rm c},n}^{(0)}\big\}. \tag{A8}$$

Here the following denotations are introduced:

$$\varsigma_{{\rm c},n} = \frac{a_{{\rm c},n}}{2r_0^2+a_{{\rm c},n}}, \quad a_{{\rm c},n} = \frac{1}{\mu_{{\rm c},n}}, \tag{A9}$$

$$\alpha_{{\rm c},n} = V_{{\rm c},n}[w_{{\rm c},n}+(1-g_{\rm c})\,m_{{\rm c},n}], \quad \beta_{{\rm c},n} = V_{{\rm c},n}\,b_{{\rm c},n}, \quad \gamma_{{\rm c},n} = -V_{{\rm c},n}\,h_{{\rm c},n}, \quad \delta_{{\rm c},n} = -g_{\rm c}V_{{\rm c},n}\,m_{{\rm c},n}, \tag{A10}$$

$$U_{{\rm c},n}^{(0,\pm)} = U^{(0,\pm)}\left(\varsigma = \sqrt{2r_0^2\varsigma_{{\rm c},n}/a_{{\rm c},n}}\right). \tag{A11}$$

See Ref. [122] for values of the parameters $V_{{\rm c},n}$, $w_{{\rm c},n}$, $m_{{\rm c},n}$, $b_{{\rm c},n}$, $h_{{\rm c},n}$, and $\mu_{{\rm c},n}$.

The general expression (21) for the matrix elements in the AVRGM basis can be simplified for the considered operators and written in the form

$$\left\langle J_f^{\pi_f}M_f l_f s_f \nu_f\middle|V\middle|J_i^{\pi_i}M_i l_i s_i \nu_i\right\rangle = 2\pi\frac{\delta_{J_fJ_i}\delta_{M_fM_i}\delta_{l_fl_i}\delta_{s_fs_i}}{\kappa_{\nu_f l_i s_i}\kappa_{\nu_i l_i s_i}\nu_f!\nu_i!}\frac{\partial^{\nu_f}}{\partial Q^{\nu_f}}\frac{\partial^{\nu_i}}{\partial R^{\nu_i}}\int_{-1}^{1}\langle \mathbf{Q},s_i\sigma_i|V|\mathbf{R},s_i\sigma_i\rangle P_{l_i}(t)dt\Big|_{Q=R=0}. \tag{A12}$$

Substituting the generating matrix elements presented above into Eq. (A12) and realizing the recurrence technique [91], one can calculate all the necessary matrix elements in the AVRGM basis for those operators.

The methods for calculating the matrix elements of the operators considered above can be generalized to the case of noncentral interaction. For this purpose, it is needed to transform the general expression (21) for this interaction to the form similar to Eq. (A12) [91,131]. For example, in the case of spin-orbit interaction of the modified Hasegawa-Nagata potential [122], Eq. (21) can be written

$$\left\langle J_f^{\pi_f}M_f l_f s_f \nu_f\middle|V_{ls}\middle|J_i^{\pi_i}M_i l_i s_i \nu_i\right\rangle = 2\pi\frac{\delta_{J_fJ_i}\delta_{M_fM_i}\delta_{l_fl_i}\delta_{s_fs_i}}{\kappa_{\nu_f l_i s_i}\kappa_{\nu_i l_i s_i}\nu_f!\nu_i!}\frac{\partial^{\nu_f}}{\partial Q^{\nu_f}}\frac{\partial^{\nu_i}}{\partial R^{\nu_i}}\int_{-1}^{1}\chi_{ls}(t)P_{l_i}(t)\,dt\Big|_{Q=R=0}, \tag{A13}$$

$$\chi_{ls}(t) = 7\left(J_i(J_i+1)-l_i(l_i+1)-\frac{3}{4}\right)u^{-9/7}\sum_{n=1}^{2}a_{ls,n}\varsigma_{ls,n}^{3/2} \times\left[\left(\frac{2\alpha_{ls,n}+\gamma_{ls,n}}{17r_0^2+12a_{ls,n}}u^2-\frac{2(\alpha_{ls,n}+2\gamma_{ls,n})}{3r_0^2+5a_{ls,n}}u-\frac{3\gamma_{ls,n}}{11r_0^2+2a_{ls,n}}\right)uU_{ls,n}^{(+)} + \left(\frac{\alpha_{ls,n}+2\gamma_{ls,n}}{17r_0^2+5a_{ls,n}}u^2-\frac{2(2\alpha_{ls,n}+\gamma_{ls,n})}{3r_0^2-2a_{ls,n}}u-\frac{3\alpha_{ls,n}}{11r_0^2+9a_{ls,n}}\right)U_{ls,n}^{(-)}\right], \tag{A14}$$

where

$$\varsigma_{ls,n} = \frac{a_{ls,n}}{2r_0^2+a_{ls,n}}, \quad a_{ls,n} = \frac{1}{\mu_{ls,n}}, \tag{A15}$$

$$\alpha_{ls,n} = V_{ls,n}\,w_{ls,n}, \quad \gamma_{ls,n} = -V_{ls,n}\,h_{ls,n}, \tag{A16}$$

$$U_{ls,n}^{(\pm)} = U^{(\pm)}\left(\varsigma = \sqrt{2r_0^2\varsigma_{ls,n}/a_{ls,n}}\right). \tag{A17}$$

Values of the parameters $V_{ls,n}$, $w_{ls,n}$, $h_{ls,n}$, and $\mu_{ls,n}$ can be found in Ref. [122]. As compared to Eq. (A12), the function χ_{ls} is involved in Eq. (A13) instead of the generating matrix elements. This function is determined by the corresponding generating matrix elements [131].

[1] P. Descouvemont, A. Adahchour, C. Angulo, A. Coc, and E. Vangioni-Flam, At. Data Nucl. Data Tables **88**, 203 (2004).

[2] F. Iocco, G. Mangano, G. Miele, O. Pisanti, and P. D. Serpico, Phys. Rep. **472**, 1 (2009).

[3] B. D. Fields, Annu. Rev. Nucl. Part. Sci. **61**, 47 (2011).

[4] C. Angulo, M. Arnould, M. Rayet *et al.*, Nucl. Phys. A **656**, 3 (1999).

[5] Y. Xu, K. Takahashi, S. Goriely, M. Arnould, M. Ohta, and H. Utsunomiya, Nucl. Phys. A **918**, 61 (2013).

[6] R. H. Cyburt and B. Davids, Phys. Rev. C **78**, 064614 (2008).

[7] J. N. Bahcall, A. M. Serenelli, and S. Basu, Astrophys. J. **621**, L85 (2005).
[8] J. N. Bahcall, A. M. Serenelli, and S. Basu, Astrophys. J. Suppl. **165**, 400 (2006).
[9] E. G. Adelberger, S. M. Austin, J. N. Bahcall *et al.*, Rev. Mod. Phys. **70**, 1265 (1998).
[10] E. G. Adelberger, A. García, R. G. H. Robertson *et al.*, Rev. Mod. Phys. **83**, 195 (2011).
[11] H. D. Holmgren and R. L. Johnston, Phys. Rev. **113**, 1556 (1959).
[12] G. M. Griffiths, R. A. Morrow, P. J. Riley, and J. B. Warren, Can. J. Phys. **39**, 1397 (1961).
[13] S. Burzyński, K. Czerski, A. Marcinkowski, and P. Zupranski, Nucl. Phys. A **473**, 179 (1987).
[14] U. Schröder, A. Redder, C. Rolfs, R. E. Azuma, L. Buchmann, C. Campbell, J. D. King, and T. R. Donoghue, Phys. Lett. B **192**, 55 (1987).
[15] H. Utsunomiya, Y.-W. Lui, D. R. Haenni *et al.*, Phys. Rev. Lett. **65**, 847 (1990); **69**, 863(E) (1990).
[16] C. R. Brune, R. W. Kavanagh, and C. Rolfs, Phys. Rev. C **50**, 2205 (1994).
[17] Y. Tokimoto, H. Utsunomiya, T. Yamagata, M. Ohta, Y.-W. Lui, R. P. Schmitt, S. Typel, Y. Aoki, K. Ieki, and K. Katori, Phys. Rev. C **63**, 035801 (2001).
[18] P. D. Parker and R. W. Kavanagh, Phys. Rev. **131**, 2578 (1963).
[19] K. Nagatani, M. R. Dwarakanath, and D. Ashery, Nucl. Phys. A **128**, 325 (1969).
[20] H. Kräwinkel, H. W. Becker, L. Buchmann *et al.*, Z. Phys. A **304**, 307 (1982).
[21] R. G. H. Robertson, P. Dyer, T. J. Bowles, R. E. Brown, N. Jarmie, C. J. Maggiore, and S. M. Austin, Phys. Rev. C **27**, 11 (1983).
[22] H. Volk, H. Kräwinkel, R. Santo, and L. Wallek, Z. Phys. A **310**, 91 (1983).
[23] J. L. Osborne, C. A. Barnes, R. W. Kavanagh, R. M. Kremer, G. J. Mathews, J. L. Zyskind, P. D. Parker, and A. J. Howard, Phys. Rev. Lett. **48**, 1664 (1982); Nucl. Phys. A **419**, 115 (1984).
[24] T. K. Alexander, G. C. Ball, W. N. Lennard, H. Geissel, and H.-B. Mak, Nucl. Phys. A **427**, 526 (1984).
[25] M. Hilgemeier, H. W. Becker, C. Rolfs, H. P. Trautvetter, and J. W. Hammer, Z. Phys. A **329**, 243 (1988).
[26] B. S. Nara Singh, M. Hass, Y. Nir-El, and G. Haquin, Phys. Rev. Lett. **93**, 262503 (2004).
[27] D. Bemmerer, F. Confortola, H. Costantini *et al.*, Phys. Rev. Lett. **97**, 122502 (2006).
[28] Gy. Gyürky, F. Confortola, H. Costantini *et al.*, Phys. Rev. C **75**, 035805 (2007).
[29] F. Confortola, D. Bemmerer, H. Costantini *et al.*, Phys. Rev. C **75**, 065803 (2007).
[30] T. A. D. Brown, C. Bordeanu, K. A. Snover, D. W. Storm, D. Melconian, A. L. Sallaska, S. K. L. Sjue, and S. Triambak, Phys. Rev. C **76**, 055801 (2007).
[31] H. Costantini, D. Bemmerer, F. Confortola *et al.*, Nucl. Phys. A **814**, 144 (2008).
[32] A. Di Leva, L. Gialanella, R. Kunz *et al.*, Phys. Rev. Lett. **102**, 232502 (2009); **103**, 159903(E) (2009).
[33] C. Bordeanu, Gy. Gyürky, Z. Halász, T. Szücs, G. G. Kiss, Z. Elekes, J. Farkas, Zs. Fülöp, and E. Somorjai, Nucl. Phys. A **908**, 1 (2013).
[34] A. Kontos, E. Uberseder, R. deBoer, J. Görres, C. Akers, A. Best, M. Couder, and M. Wiescher, Phys. Rev. C **87**, 065804 (2013).
[35] M. Carmona-Gallardo, B. S. Nara Singh, M. J. G. Borge *et al.*, Phys. Rev. C **86**, 032801(R) (2012).
[36] M. Carmona-Gallardo, A. Rojas, M. J. G. Borge, B. Davids, B. R. Fulton, M. Hass, B. S. Nara Singh, C. Ruiz, and O. Tengblad, Eur. Phys. J. Web Conf. **66**, 07003 (2014).
[37] R. F. Christy and I. Duck, Nucl. Phys. **24**, 89 (1961).
[38] T. A. Tombrello and G. C. Phillips, Phys. Rev. **122**, 224 (1961).
[39] T. A. Tombrello and P. D. Parker, Phys. Rev. **131**, 2582 (1963).
[40] R. D. Williams and S. E. Koonin, Phys. Rev. C **23**, 2773 (1981).
[41] T. Kajino, G. J. Mathews, and K. Ikeda, Phys. Rev. C **40**, 525 (1989).
[42] P. Mohr, H. Abele, R. Zwiebel, G. Staudt, H. Krauss, H. Oberhummer, A. Denker, J. W. Hammer, and G. Wolf, Phys. Rev. C **48**, 1420 (1993).
[43] P. Mohr, Phys. Rev. C **79**, 065804 (2009).
[44] B. Buck, R. A. Baldock, and J. A. Rubio, J. Phys. G. **11**, L11 (1985).
[45] B. Buck and A. C. Merchant, J. Phys. G **14**, L211 (1988).
[46] S. B. Dubovichenko and A. V. Dzhazairov-Kakhramanov, Phys. Part. Nucl. **28**, 615 (1997) [Fiz. Elem. Chastits At. Yadra **28**, 1529 (1997)].
[47] D. Baye and E. Brainis, Phys. Rev. C **61**, 025801 (2000).
[48] S. B. Dubovichenko, Phys. At. Nucl. **73**, 1526 (2010) [Yad. Fiz. **73**, 1573 (2010)].
[49] S. B. Dubovichenko and Yu. N. Uzikov, Phys. Part. Nucl. **42**, 251 (2011) [Fiz. Elem. Chastits At. Yadra **42**, 478 (2011)].
[50] S. B. Igamov and R. Yarmukhamedov, Nucl. Phys. A **781**, 247 (2007).
[51] Q. I. Tursunmahatov and R. Yarmukhamedov, Phys. Rev. C **85**, 045807 (2012).
[52] L. Canton and L. G. Levchuk, Nucl. Phys. A **808**, 192 (2008).
[53] A. Mason, R. Chatterjee, L. Fortunato, and A. Vitturi, Eur. Phys. J. A **39**, 107 (2009).
[54] K. Langanke, Nucl. Phys. A **457**, 351 (1986).
[55] B. T. Kim, T. Izumoto, and K. Nagatani, Phys. Rev. C **23**, 33 (1981).
[56] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Eur. Phys. J. Web Conf. **117**, 09017 (2016).
[57] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Bull. Russ. Acad. Sci. Phys. **80**, 290 (2016) [Izv. RAN, Ser. Fiz. **80**, 322 (2016)].
[58] Q. K. K. Liu, H. Kanada, and Y. C. Tang, Phys. Rev. C **23**, 645 (1981).
[59] H. Walliser, Q. K. K. Liu, H. Kanada, and Y. C. Tang, Phys. Rev. C **28**, 57 (1983).
[60] H. Walliser, H. Kanada, and Y. C. Tang, Nucl. Phys. A **419**, 133 (1984).
[61] H. Walliser, H. Kanada, and Y. C. Tang, Phys. Rev. Lett. **53**, 399 (1984).
[62] T. Kajino and A. Arima, Phys. Rev. Lett. **52**, 739 (1984).
[63] T. Kajino, Nucl. Phys. A **460**, 559 (1986).
[64] T. Mertelmeier and H. M. Hofmann, Nucl. Phys. A **459**, 387 (1986).
[65] T. Kajino, H. Toki, K.-I. Kubo, and I. Tanihata, Phys. Lett. B **202**, 475 (1988).
[66] T. Altmeyer, E. Kolbe, T. Warmann, K. Langanke, and H. J. Assenbaum, Z. Phys. A **330**, 277 (1988).
[67] A. Csótó and K. Langanke, Few-Body Syst. **29**, 121 (2000).

[68] K. Arai, D. Baye, and P. Descouvemont, Nucl. Phys. A **699**, 963 (2002).
[69] T. Kajino, G. F. Bertsch, and K.-I. Kubo, Phys. Rev. C **37**, 512 (1988).
[70] K. M. Nollett, Phys. Rev. C **63**, 054002 (2001).
[71] L. E. Marcucci, K. M. Nollett, R. Schiavilla, and R. B. Wiringa, Nucl. Phys. A **777**, 111 (2006).
[72] P. Navrátil, C. A. Bertulani, and E. Caurier, Nucl. Phys. A **787**, 539c (2007).
[73] P. Navrátil, S. Quaglioni, I. Stetcu, and B. R. Barrett, J. Phys. G **36**, 083101 (2009).
[74] T. Neff, Phys. Rev. Lett. **106**, 042502 (2011).
[75] H. Sadeghi, Astrophys. Space Sci. **347**, 261 (2013).
[76] H. Sadeghi, Chin. Phys. Lett. **30**, 102501 (2013).
[77] K. Wildermuth and Y. C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Braunschweig, 1977).
[78] Y. C. Tang, M. Lemere, and D. R. Thompson, Phys. Rep. **47**, 167 (1978).
[79] S. Saito, Prog. Theor. Phys. **40**, 893 (1968).
[80] S. Saito, Prog. Theor. Phys. **41**, 705 (1969).
[81] S. Yu. Igashov, Yu. F. Smirnov, and Yu. M. Tchuvil'sky, Bull. Russ. Acad. Sci. Phys. **73**, 756 (2009) [Izv. RAN, Ser. Fiz. **73**, 798 (2009)].
[82] S. Yu. Igashov and Yu. M. Tchuvil'sky, Phys. At. Nucl. **74**, 1588 (2011) [Yad. Fiz. **74**, 1615 (2011)].
[83] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Phys. At. Nucl. **77**, 1453 (2014) [Yad. Fiz. **77**, 1525 (2014)].
[84] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, J. Phys. Conf. Ser. **569**, 012020 (2014).
[85] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Bull. Russ. Acad. Sci. Phys. **78**, 433 (2014) [Izv. RAN, Ser. Fiz. **78**, 621 (2014)].
[86] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Bull. Russ. Acad. Sci. Phys. **79**, 499 (2015) [Izv. RAN, Ser. Fiz. **79**, 541 (2015)].
[87] A. S. Solovyev, S. Yu. Igashov, and Yu. M. Tchuvil'sky, Eur. Phys. J. Web Conf. **86**, 00054 (2015).
[88] L. L. Chopovsky, Phys. Lett. B **229**, 316 (1989).
[89] G. F. Filippov and I. P. Okhrimenko, Sov. J. Nucl. Phys. **32**, 480 (1980) [Yad. Fiz. **32**, 932 (1980)].
[90] G. F. Filippov, Sov. J. Nucl. Phys. **33**, 488 (1981) [Yad. Fiz. **33**, 928 (1981)].
[91] I. P. Okhrimenko, Nucl. Phys. A **424**, 121 (1984).
[92] G. F. Filippov, V. S. Vasilevsky, and L. L. Chopovsky, Sov. J. Part. Nucl. **15**, 600 (1984) [Fiz. Elem. Chastits At. Yadra **15**, 1338 (1984)].
[93] G. F. Filippov, V. S. Vasilevsky, and L. L. Chopovsky, Sov. J. Part. Nucl. **16**, 153 (1985) [Fiz. Elem. Chastits At. Yadra **16**, 349 (1985)].
[94] V. S. Vasilevsky, I. F. Gutich, and I. P. Okhrimenko, Sov. J. Nucl. Phys. **46**, 427 (1987) [Yad. Fiz. **46**, 757 (1987)].
[95] I. P. Okhrimenko, Few-Body Syst. **2**, 169 (1987).
[96] R. T. Tyapaev, Phys. At. Nucl. **56**, 516 (1993) [Yad. Fiz. **56**, 168 (1993)].
[97] S. Yu. Igashov, Bull. Russ. Acad. Sci. Phys. **65**, 817 (2001) [Izv. RAN, Ser. Fiz. **65**, 756 (2001)].
[98] A. Sytcheva, F. Arickx, J. Broeckhove, and V. S. Vasilevsky, Phys. Rev. C **71**, 044322 (2005).
[99] G. F. Filippov, A. V. Nesterov, I. Yu. Rybkin, and S. V. Korennov, Phys. Part. Nucl. **25**, 569 (1994) [Fiz. Elem. Chastits At. Yadra **25**, 1347 (1994)].
[100] V. Vasilevsky, A. V. Nesterov, F. Arickx, and J. Broeckhove, Phys. Rev. C **63**, 034606 (2001).
[101] V. Vasilevsky, A. V. Nesterov, F. Arickx, and J. Broeckhove, Phys. Rev. C **63**, 034607 (2001).
[102] V. Vasilevsky, A. V. Nesterov, F. Arickx, and J. Broeckhove, Phys. Rev. C **63**, 064604 (2001).
[103] Yu. A. Lashko and G. F. Filippov, Nucl. Phys. A **826**, 24 (2009).
[104] A. V. Nesterov, F. Arickx, J. Broeckhove, and V. S. Vasilevsky, Phys. Part. Nucl. **41**, 716 (2010) [Fiz. Elem. Chastits At. Yadra **41**, 1337 (2010)].
[105] Yu. A. Lashko, G. F. Filippov, and V. S. Vasilevsky, Nucl. Phys. A **941**, 121 (2015).
[106] T. Neff and H. Feldmeier, Eur. Phys. J. Spec. Top. **156**, 69 (2008).
[107] S. Quaglioni and P. Navrátil, Phys. Rev. Lett. **101**, 092501 (2008).
[108] S. Quaglioni and P. Navrátil, Phys. Rev. C **79**, 044606 (2009).
[109] P. Navrátil, R. Roth, and S. Quaglioni, Phys. Rev. C **82**, 034609 (2010).
[110] P. Navrátil and S. Quaglioni, Phys. Rev. C **83**, 044609 (2011).
[111] S. Baroni, P. Navrátil, and S. Quaglioni, Phys. Rev. C **87**, 034326 (2013).
[112] S. Quaglioni, C. Romero-Redondo, and P. Navrátil, Phys. Rev. C **88**, 034320 (2013).
[113] G. Hupin, J. Langhammer, P. Navrátil, S. Quaglioni, A. Calci, and R. Roth, Phys. Rev. C **88**, 054622 (2013).
[114] G. Hupin, S. Quaglioni, and P. Navrátil, Phys. Rev. C **90**, 061601(R) (2014).
[115] F. Raimondi, G. Hupin, P. Navrátil, and S. Quaglioni, Phys. Rev. C **93**, 054606 (2016).
[116] H. A. Yamani and L. Fishman, J. Math. Phys. **16**, 410 (1975).
[117] V. S. Vasilevsky and F. Arickx, Phys. Rev. A **55**, 265 (1997).
[118] W. Vanroose, J. Broeckhove, and F. Arickx, Phys. Rev. Lett. **88**, 010404 (2002).
[119] S. Yu. Igashov, in *The J-Matrix Method. Developments and Applications*, edited by A. D. Alhaidari, E. J. Heller, H. A. Yamani, and M. S. Abdelmonem (Springer, Berlin, 2008), p. 49.
[120] A. S. Solovyev and S. Yu. Igashov, Yad. Fiz. Inzhin. **4**, 989 (2013) (in Russian).
[121] G. Audi, A. H. Wapstra, and C. Thibault, Nucl. Phys. A **729**, 337 (2003).
[122] H. Kanada, T. Kaneko, S. Nagata, and M. Nomoto, Progr. Theor. Phys. **61**, 1327 (1979).
[123] M. Ivanovich, P. G. Young, and G. G. Ohlsen, Nucl. Phys. A **110**, 441 (1968).
[124] R. J. Spiger and T. A. Tombrello, Phys. Rev. **163**, 964 (1967).
[125] P. D. Miller and G. C. Phillips, Phys. Rev. **112**, 2048 (1958).
[126] T. A. Tombrello and P. D. Parker, Phys. Rev. **130**, 1112 (1963).
[127] D. M. Hardy, R. J. Spiger, S. D. Baker, Y. S. Chen, and T. A. Tombrello, Nucl. Phys. A **195**, 250 (1972).
[128] A. C. L. Barnard, C. M. Jones, and G. C. Phillips, Nucl. Phys. **50**, 629 (1964).
[129] W. R. Boykin, S. D. Baker, and D. M. Hardy, Nucl. Phys. A **195**, 241 (1972).
[130] D. R. Tilley, C. M. Cheves, J. L. Godwin, G. M. Hale, H. M. Hofmann, J. H. Kelley, C. G. Sheu, and H. R. Weller, Nucl. Phys. A **708**, 3 (2002).
[131] S. Yu. Igashov and R. T. Tyapaev, Phys. At. Nucl. **58**, 780 (1995) [Yad. Fiz. **58**, 843 (1995)].