Viscosity of a net-baryon fluid near the QCD critical point

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In the dynamics of the QCD critical point, the net-baryon fluid, linked to the slow component of the order parameter, relaxes to a three-dimensional Ising system in equilibrium. An analytical study of shear and bulk viscosity, with constraints imposed by the dynamics of the critical net-baryon fluid, the universality property, and the requirements of a class of strong-coupling theories, is performed in the neighborhood of the critical point. It is found that the shear viscosity of the net-baryon fluid is restricted in the domain $1.6 \leqslant 4\pi \frac{\eta}{s} \leqslant 3.7$ for $T_c < T \leqslant 2T_c$, whereas the bulk viscosity is small, $4\pi \frac{\zeta}{s} < 0.05$ (for $T > 1.27T_c$), but rising towards the singularity at $T = T_c$.

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I. INTRODUCTION

Near the QCD critical point, the transport coefficients of a strongly interacting fluid, created in high-energy nuclear collisions, are expected to exhibit anomalous temperature dependence. This behavior is implied by the time evolution of the order parameter fluctuations after reaching equilibrium. At time scales of the order of the relaxation time, the order parameter fluctuations are governed by the fluctuations of the baryon number density n_b. The reason is that the latter is the slow component of the order parameter due to baryon number conservation [1,2]. In contrast, all other degrees of freedom, occurring in the energy-momentum tensor of the strongly interacting fluid, do not contribute to the singular behavior of the transport coefficients in the stage of relaxation to equilibrium. In particular the chiral condensate, the other component of the order parameter, is linked to a fast mode since the σ field becomes massive and therefore does not participate in the dynamics of the QCD critical point at long time scales [1]. As a result, one may claim that, although in the study of static properties of the QCD critical point (critical fluctuations, and divergence of baryon-number susceptibility) both components of the order parameter (σ, n_b) are relevant [3], in any attempt to investigate the dynamic properties of the QCD critical point (divergence of transport coefficients) only the baryon number density n_b is a relevant order parameter [1]. Experimentally, one may state that, in the search for the location of the critical point in the phase diagram, the investigation of critical fluctuations of $\pi^+\pi^-$ pairs may simulate σ-field critical fluctuations [4]. However, dynamic properties of the QCD critical point cannot be revealed within a study of transport coefficients (viscosity) of a meson gas produced in high-energy nuclear collisions [5].

In what follows, we consider the behavior of viscosity (shear and bulk) in the baryon-number fluid, near the QCD critical point in a process of relaxation towards a state of equilibrium described by a three-dimensional (3D) Ising universality class. Conventional systems (He, N_2, H_2O) of liquid-gas transition [6] also belong in the same universality class, and the basic ingredients in this process out of equilibrium are thermal diffusion and sound waves [7]. The Ising model description in equilibrium is characterized by the critical exponents of the divergent thermodynamic quantities and of the correlation length, near the critical point [8].

Within this context, we may conjecture that, approaching the critical point, the following thermodynamic quantities prevail in the description of shear (η) and bulk (ζ) viscosity: $\eta(T, v_s, \xi, \frac{c_P}{c_V}, \ldots)$ and $\zeta(\rho, v_s, \xi, \frac{c_P}{c_V}, \ldots)$. In fact, thermal diffusion produces an inverse relaxation time (τ^{-1}) for a disturbance, proportional to the specific heat coefficient, $\tau^{-1} \sim c_P$ [7], whereas the correlation length ξ represents the length scale near the critical point. The velocity of sound, v_s, is present since we have assumed that sound waves are among the nonequilibrium modes in this process. Finally, from the thermodynamic quantities in the equation of state, shear viscosity captures the dependence on the temperature (T), whereas bulk viscosity must depend on the mass density (ρ) of the medium (net-baryon fluid) in the bulk. In this approximation, no other quantities are considered in the description of shear and bulk viscosity during the relaxation stage. On the basis, now, of dimensional considerations, viscosity = energy density × time, one may obtain the following expressions in terms of singular quantities in the limit $T \to T_c$, $\mu_b = \mu_c$:

$$\frac{\eta}{s} = \frac{k_B T v_s^{-1}}{\xi^2 s} F^{(s)}\left(\frac{c_P}{c_V}\right), \quad \frac{\zeta}{s} = \frac{\rho v_s \xi}{s} F^{(b)}\left(\frac{c_P}{c_V}\right), \tag{1}$$

where we have introduced the entropy density (s) forming the dimensionless ratios (1) in the system of units $k_B = c = \hbar = 1$. The basic thermodynamics of the fluid is formulated in terms of the relations

$$c_P - c_V = T k_T \left(\frac{\partial P}{\partial T}\right)_V^2, \quad \frac{c_P}{c_V} = \frac{k_T}{k_S}, \quad v_s^2 = (\rho k_S)^{-1},$$
$$s = \frac{\varepsilon + P}{T} - \frac{\mu_b n_b}{T}, \tag{2}$$

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where k_T and k_S are the isothermal and adiabatic (isentropic) compressibility, ε the energy density, P the pressure, and μ_b the baryochemical potential.

A minimal requirement of relativistic thermodynamics leads to the identification of the mass density ρ of the fluid with the enthalpy density: $h = \varepsilon + P$ ($\rho = \frac{h}{c^2}$) in Eqs. (1) and (2) as a result of the properties of the energy-momentum tensor [9]. Moreover, in order to fix the amplitudes (scales) of the singular quantities, consistent with relativity, one may consider the net-baryon fluid consisting, in the quark phase ($T \gg T_c$), of the quark excess with conserved number density, following the equation of state of an ideal, massless, classical system and leading to

$$\varepsilon = 3P, \quad P = n_b T, \quad h = 4n_b T, \quad c_V = 3n_b, \quad c_P = 4n_b,$$
$$k_T = (n_b T)^{-1}, \quad s = \left(4 - \frac{\mu_b}{T}\right)n_b, \quad v_s = \frac{1}{\sqrt{3}}. \tag{3}$$

In this description the remaining balanced quarks and antiquarks (zero baryon number) form an environment for the net-baryon fluid which affects the dependence of the chemical potential μ_b on the temperature T.

II. CRITICAL EXPONENTS OF VISCOSITY

Introducing now the appropriate critical exponents and the corresponding amplitudes, we obtain in the limit $T \to T_c$, $\mu_b = \mu_c$, the power laws

$$c_V = A_\pm |t|^{-\alpha}, \quad k_T = \Gamma_\pm |t|^{-\gamma},$$
$$\xi = \xi_\pm |t|^{-\nu} \quad \left(t \equiv \frac{T - T_c}{T_c}\right), \tag{4}$$

where the indices ($\pm$) in the amplitudes correspond to the limits $t \to 0^+$ and $t \to 0^-$, respectively [8].

In Eqs. (4) not only are the critical exponents universal but also the ratios of the amplitudes $\frac{A_+}{A_-}$, $\frac{\Gamma_+}{\Gamma_-}$, and $\frac{\xi_+}{\xi_-}$, corresponding to the phases $T > T_c$ (net-baryon, quark-matter fluid) and $T < T_c$ (net-baryon, baryonic fluid), are fixed within the universality class of the critical point [8]. Moreover, following our discussion above, the amplitudes (A_+, Γ_+) in the quark-matter phase, representing the scales of the thermodynamic quantities (c_V, k_T) near the critical temperature, can be fixed with the help of Eqs. (3) assuming a continuous transition to the ideal behavior. To ensure the continuity of the sound velocity [see Eqs. (7) below], reaching the constant value $\frac{1}{\sqrt{3}}$ in the ideal regime, the matching of c_V, k_T has to be taken at $t = 1$ ($T = 2T_c$), leading to the relations

$$A_+ = 3n_c, \quad \Gamma_+ = (2n_c T_c)^{-1}, \tag{5}$$

where $n_c \equiv n_b(T_c) = n_b(2T_c)$ denotes the critical baryon number density. Here we have made use of the fact that the order parameter $n_b(T) - n_c$ vanishes in the symmetric phase $T > T_c$ [10]. We are aware of the fact that the extrapolation of the power laws (4), beyond the critical region, is a crude approximation. Our conjecture, however, is that the combination of the singular thermodynamic quantities to form expressions (1) of shear and bulk viscosity may lead to a solution, valid also in a distance from T_c, developing a noncritical behavior there. Obviously, this conjecture can only be verified *a posteriori*, at the end of our treatment.

In this framework one may proceed to a semiquantitative treatment of shear and bulk viscosity, near the QCD critical point, on the basis of Eqs. (1), (2), and (4). For the functions $F^{(i)}(\frac{c_P}{c_V})$, $i : (s,b)$ in Eqs. (1), we adopt a simple model inspired by a perturbative treatment of conventional fluids, in the vicinity of the liquid-gas critical point, sharing the same universality class (3D Ising) with QCD [7]: $F^{(i)}(\frac{c_P}{c_V}) = f^{(i)}\frac{c_P}{c_V}$, where the dimensionless constants $f^{(i)}$ are not universal—they depend on the nature and the length scale of the medium at microscopic level. With this choice, Eqs. (1) and (2) give

$$v_s^2 = \frac{k_T^{-1}}{h}\left[1 + Tk_T\left(\frac{\partial P}{\partial T}\right)_V^2 c_V^{-1}\right],$$
$$\frac{\eta}{s} = f^{(s)}\frac{T^{3/2}h^{1/2}}{s}\left(\frac{\partial P}{\partial T}\right)_V k_T \xi^{-2} c_V^{-1/2}$$
$$\times\left[1 + T^{-1}\left(\frac{\partial P}{\partial T}\right)_V^{-2}\frac{c_V}{k_T}\right]^{1/2},$$
$$\frac{\zeta}{s} = f^{(b)}\frac{T^{3/2}h^{1/2}}{s}\left(\frac{\partial P}{\partial T}\right)_V^3 k_T \xi c_V^{-3/2}$$
$$\times\left[1 + T^{-1}\left(\frac{\partial P}{\partial T}\right)_V^{-2}\frac{c_V}{k_T}\right]^{3/2}. \tag{6}$$

Incorporating the power laws (4) for c_V, k_T, and ξ in Eqs. (6) we obtain the singular forms in the limit $T \to T_c$:

$$v_s^2 = \frac{|t|^\alpha}{h}\left[\Gamma_\pm^{-1}|t|^{\gamma-\alpha} + A_\pm^{-1}T\left(\frac{\partial P}{\partial T}\right)_V^2\right],$$
$$\left(\frac{\eta}{s}\right)_\pm = f^{(s)}\frac{T_c^{3/2}h_c^{1/2}\lambda_c}{s_c}\left(\Gamma_\pm \xi_\pm^{-2}A_\pm^{-1/2}\right)$$
$$\times\left(1 + T_c^{-1}\lambda_c^{-2}A_\pm\Gamma_\pm^{-1}|t|^{\gamma-\alpha}\right)^{1/2}|t|^{-\gamma+2\nu+\frac{\alpha}{2}},$$
$$\left(\frac{\zeta}{s}\right)_\pm = f^{(b)}\frac{T_c^{3/2}h_c^{1/2}\lambda_c^3}{s_c}\left(\Gamma_\pm \xi_\pm A_\pm^{-3/2}\right)$$
$$\times\left(1 + T_c^{-1}\lambda_c^{-2}A_\pm\Gamma_\pm^{-1}|t|^{\gamma-\alpha}\right)^{3/2}|t|^{-\gamma-\nu+\frac{3\alpha}{2}}, \tag{7}$$

where $\lambda_c \equiv (\frac{\partial P}{\partial T})_V$ at $T = T_c$. In fact Eqs. (7) contain also a noncritical contribution increasing for $T > T_c$. This leads to a solution interpolating smoothly between the critical and the asymptotic ideal gas behavior, taking also the constraint of baryon-number conservation into account. The critical exponents (α,γ,ν) are not independent since they are constrained by the Josephson scaling law $\nu d = 2 - \alpha$. Therefore, the indices of the power laws in Eqs. (7) are given in terms of two independent critical exponents. In particular, the leading power laws of shear and bulk viscosity are

$$\eta \sim |t|^{1-\gamma+\frac{\nu}{2}}, \quad \zeta \sim |t|^{3-\gamma-\frac{11}{2}\nu}, \tag{8}$$

where the exponents (γ,ν) are expected to be compatible with Ising-like universality class in three dimensions. In fact the behavior of Eqs. (8) is universal; it is valid near the liquid-gas critical point of conventional matter [7] and also near the quark-hadron critical point of QCD matter. However, the behavior of Eqs. (8) must be in accordance with the dynamical aspects of the QCD critical point which suggest, according to a compilation of predictions [1,11–15], the singular behavior $\eta \sim \xi^{0.05}$, $\zeta \sim \xi^{2.8}$. Comparing these power laws with the behavior of Eqs. (8) we find ($\xi \sim |t|^{-\nu}$) $\gamma \simeq 1.34$, $\nu \simeq 0.61$, a solution compatible with the Ising-like universality class.

Finally, in order to verify explicitly the compatibility of the scaling relations (8) with the dynamics of the critical point, we consider the treatment in Ref. [14] in which the renormalized transport coefficients, computed in the ϵ expansion, behave as follows: $\eta_R \sim \xi^{\epsilon/19}$ and $\zeta_R(0) \sim \xi^{z-\alpha/\nu}$ in the slow mode $\omega = 0$. The dynamic critical exponent z is given by the expansion $z = 4 - \frac{18}{19}\epsilon + \cdots$ and for a 3D fluid and the Ising exponents $\nu \simeq 0.61$, $\alpha = 2 - \nu d$, one finds, to first order in $\epsilon = 4 - d$, $\eta_R \sim \xi^{0.053}$, $\zeta_R(0) \sim \xi^{2.77}$, in a very good agreement with Eqs. (8).

III. SINGULAR SOLUTIONS

The characteristic properties of viscosity near the QCD critical point, described by the solution (7), depend on a number of nonuniversal amplitudes which are fixed by the following constraints: (a) the assumption that in the quark-matter phase ($T \gtrsim T_c$) the amplitudes Γ_+ and A_+ are given by Eqs. (5), which are compatible with the equation of state of a noninteracting, massless, classical system with constant baryon number density $n_b(T) = n_c$ for $T \gg T_c$ and (b) the universality constraint imposed on the ratios of the Ising amplitudes: $\frac{A_+}{A_-} = 0.5 - 0.6$, $\frac{\xi_+}{\xi_-} = 2$, and $\frac{\Gamma_+}{\Gamma_-} = 4.5 - 5.0$ [16].

In fact, the solution (7) can be written in a simplified form:

$$\left(\frac{\eta}{s}\right)_\pm = f^{(s)} M_\pm (1 + \Lambda_\pm |t|^{\gamma+3\nu-2})^{1/2} |t|^{1-\gamma+\frac{\nu}{2}},$$
$$\left(\frac{\zeta}{s}\right)_\pm = f^{(b)} N_\pm (1 + \Lambda_\pm |t|^{\gamma+3\nu-2})^{3/2} |t|^{3-\gamma-\frac{11}{2}\nu}, \tag{9}$$

where

$$M_\pm \equiv \frac{T_c^{3/2} h_c^{1/2} \lambda_c}{s_c} \Gamma_\pm \xi_\pm^{-2} A_\pm^{-1/2},$$
$$N_\pm \equiv \frac{T_c^{3/2} h_c^{1/2} \lambda_c^3}{s_c} \Gamma_\pm \xi_\pm A_\pm^{-3/2}, \text{ and}$$

$$\Lambda_\pm \equiv T_c^{-1} \lambda_c^{-2} A_\pm \Gamma_\pm^{-1}, \tag{10}$$

with $h_c = 4n_c T_c$, $\lambda_c = n_c$, and $s_c = (4 - \frac{\mu_c}{T_c})n_c$, assuming the (approximate) validity of Eqs. (3) in the temperature range $(T_c, 2T_c)$ for all thermodynamic quantities which do not possess divergent singularities for $T \to T_c$. Constraints (a) and (b) lead to the following solution for the dimensionless amplitudes (10):

$$M_+ = \frac{1}{\sqrt{3}} \frac{\xi_+^{-2} T_c}{s_c}, \quad N_+ = \frac{1}{3\sqrt{3}} \frac{n_c \xi_+ T_c}{s_c}, \quad \Lambda_+ = 6,$$
$$M_- = \frac{4}{7\sqrt{3}} \frac{\xi_+^{-2} T_c}{s_c}, \quad N_- = \frac{1}{84\sqrt{3}} \frac{n_c \xi_+ T_c}{s_c}, \quad \Lambda_- = 60. \tag{11}$$

Here $\xi_+ \simeq 1$ fm is a typical scale of the correlation length and a set of critical values (T_c, μ_c, and n_c) can be taken from Ref. [10], where a study of baryon-number susceptibility near the critical point is performed and also from NA49 measurements in a search for critical fluctuations [4,17]: $T_c \simeq 160$ MeV, $\mu_c \simeq 220$ MeV, and $n_c \simeq 0.13$ fm^{-3}. The location of the critical point (T_c,μ_c) in the QCD phase diagram is still an open problem and the above values are only indicative. However, a change of these quantities affects only the actual values of the constants $f^{(i)}$ but not the solution (9). In fact, if we choose the critical values suggested by lattice QCD, $T_c \simeq 160$ MeV and $\mu_c \simeq 400$ MeV [18,19], the constants $f^{(i)}$ increase by a factor of 1.7 but the overall prefactors $f^{(s)} M_\pm$ and $f^{(b)} N_\pm$ in Eqs. (9) remain unchanged, fixed by the constraint of the Kovtun-Son-Starinets (KSS) bound, described in the discussion below.

To complete our treatment and determine the remaining constants $f^{(s)}$ and $f^{(b)}$ in Eqs. (9), we employ, as a final guiding principle, the KSS bound [20], which is assumed to be reached by the minimum of the ratio $\frac{\eta}{s}$, located very close to the critical temperature, in the hadronic phase ($t \simeq -2.9 \times 10^{-3}$) according to Eqs. (9). This constraint has its origin in a class of strong-coupling field theories (anti-de Sitter–conformal field theory (AdS/CFT) limit) and it is widely accepted that the formation of quark matter in high-energy nuclear collisions creates an ideal environment in order to test its validity [6]. Also, in the same framework, a constraint on the bulk viscosity can be obtained if we use the parametrization $\frac{\zeta}{s} = \frac{1}{8\pi}(\frac{1}{3} - v_s^2)$ introduced in Ref. [21] and take, for our purpose, the average in the domain $0.5 \leqslant t \leqslant 1$. From Eqs. (7) we have for $T_c \leqslant T \leqslant 2T_c$

$$v_s^2 = \frac{t^{2-3\nu}}{4}\left(\frac{2t^{\gamma+3\nu-2}}{1+t} + \frac{1}{3}\right), \quad \langle v_s^2 \rangle \simeq 0.27. \tag{12}$$

Thus, we obtain the final constraints,

$$\left(\frac{\eta}{s}\right)_{\min} = \frac{1}{4\pi} \quad (t \simeq 0^-), \quad \left\langle \left(\frac{\zeta}{s}\right)_+ \right\rangle \simeq \frac{0.030}{4\pi}, \tag{13}$$

which lead to the estimate $f^{(s)} \simeq 8.2 \times 10^{-2}$ and $f^{(b)} \simeq 2.0 \times 10^{-3}$.

IV. DISCUSSION AND CONCLUSIONS

The solution for shear viscosity is unstable at $T = T_c$, under small changes of the critical exponents (ν,γ). In fact if we consider the values of 3D Ising exponents given by recent theoretical studies [22], $\nu \simeq 0.63$ and $\gamma \simeq 1.24$, the weakly divergent singularity at $T = T_c$ discussed above (solution I), becomes a cusp singularity (solution II). This solution vanishes at $T = T_c$ but, for the same value of $f^{(s)}$, it rapidly approaches solution I as we depart from the critical temperature ($|t| \gg 0.04$), leaving, as a single imprint, a cusp at $t = 0$. Solution II violates the KSS bound and therefore the dimensionless constant $f^{(s)}$ remains a free parameter. In Fig. 1 we show, for illustration, three different solutions of type II, sharing the same cusp singularity and corresponding to the values $f^{(s)} = 0.082, 0.130$, and 0.325. In fact, solutions of types I and II are presented in Fig. 1 and compared with other findings, not related to critical behavior. In the hadronic phase ($T < T_c$)

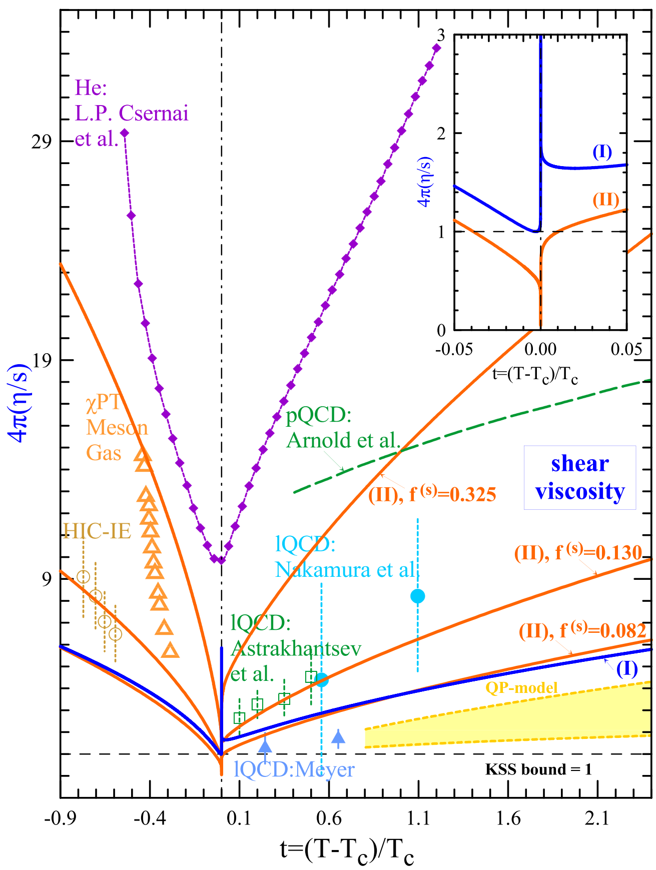

FIG. 1. Our solution of type I (continuous dark line) for the shear viscosity compared with the findings of Refs. [5] (open triangles), [23] (open circles), [24] (solid circles and solid triangles), [25] (open rectangles), [26] (dashed line), [27] (a quasiparticle model; band), and [6] (dotted line with solid rectangles). In the inset graph we focus on the shape of our solution in the vicinity of the critical temperature. Also solutions of type II (continuous light lines) are shown.

we found $1 \leqslant 4\pi\frac{\eta}{s} \leqslant 4.3$ for $\frac{T_c}{2} \leqslant T < T_c$ (in solution I), deviating from the behavior of chiral matter (meson gas in chiral perturbation theory) [5] and the behavior of $\frac{\eta}{s}$ extracted from heavy-ion collisions at intermediate energies (HIC-IE) [23]. For $T > T_c$ (quark matter) we found $1.6 \leqslant 4\pi\frac{\eta}{s} \leqslant 3.7$ for $T_c < T \leqslant 2T_c$ (in solution I), and a comparison with recent results of lattice QCD (lQCD) for the shear viscosity of gluonic matter [24] is illustrated. In the same figure, it is of particular interest to compare our solutions with the estimate of the ratio $\frac{\eta}{s}$ for QCD with dynamical $N_f = 3$ quarks given in Ref. [25]. Also in Fig. 1, predictions of perturbative QCD [26] and of a quasiparticle model [27] (band) are presented for comparison. Finally, it is of interest to note that the weakness of the singularity, at $T = T_c$, manifests itself as a two-minima structure, very close to the critical temperature (Fig. 1). The absolute minimum reaches the KSS bound in the hadronic phase and not in the quark-matter phase. This structure cannot be seen in the coarse data of conventional matter, as shown in Fig. 1 in the case of helium [6], and, certainly, is not expected to be observable in high-energy nuclear collisions either. We also observe that in the quark-matter phase ($T > T_c$) the critical

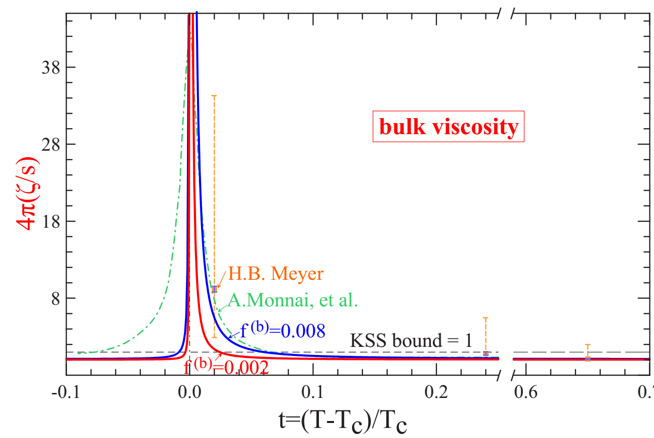

FIG. 2. Solutions for the bulk viscosity (continuous lines) compared with the findings of Refs. [11] (dot-dashed line) and [28] (solid rectangles) with systematic (large) and statistical (small) uncertainties.

behavior of the shear viscosity is confined in a very narrow region $\frac{\Delta T}{T_c} \simeq 10^{-2}$ corresponding to the position of the local minimum at $t_{\min} \simeq 2.0 \times 10^{-2}$. Departing from this region ($t \gg 10^{-2}$) the solution (9) may still be valid, dominated by the square-root term which leads to a smoothly increasing function with a noncritical behavior (Fig. 1). Moreover, in a distance from the critical point ($t \simeq 1$) the properties of shear viscosity are expected to deviate from the requirement of a strong-coupling regime and come close to the properties of a noninteracting system with conserved baryon number density (3). This observation justifies *a posteriori* the constraint (5) on the amplitudes A_+ and Γ_+. Similar remarks may apply to bulk viscosity, which remains practically constant beyond its critical region $\frac{\Delta T}{T_c} \simeq 10^{-2}$ (Fig. 2).

In Fig. 2 the bulk viscosity of net-baryon matter, in our solution, is presented (continuous light line). The behavior of bulk viscosity remains practically unchanged under small changes of the critical exponents (ν,γ). Despite the fact that it develops a strong singularity at the critical point ($\zeta \sim \xi^{2.8}$), it decreases rapidly and stays at an approximately constant value, smaller than the KSS bound, for $|t| \gg 0.025$. Our result is compared with the solution (dot-dashed line) in Ref. [11] where a dynamical treatment of enhanced bulk viscosity near the critical point is performed. In the same figure, the results of lattice QCD for gluonic matter are shown [28], whereas in a similar lQCD treatment [23] the results for the bulk viscosity of gluonic matter are compatible with zero, for $\frac{3T_c}{2} < T < 2T_c$, and are not shown in Fig. 2.

Finally, if we remove the constraint on bulk viscosity, inspired by gauge-gravity duality, given in Eq. (13), the constant $f^{(b)}$ remains a free parameter. In Fig. 2 a solution with $f^{(b)} = 0.008$ is also shown (continuous dark line), approaching, in the quark phase, the solution described in Ref. [11].

In summary, an analytical study of shear and bulk viscosity of net-baryon matter near the QCD critical point is performed. It is based on the assumption that the net-baryon fluid, associated with the slow order parameter (baryon number density n_b) of the critical phenomenon, relaxes, in a process out of equilibrium, to the Ising universality class in equilibrium. The universal indices (critical exponents, and ratios of the amplitudes of critical singularities) are basic ingredients in this approach, leading to a prediction of viscosity near the critical point. This becomes possible if we employ constraints inspired by gauge-gravity duality, in the strong-coupling regime, near the critical point (KSS bound for shear viscosity and a related parametrization for bulk viscosity). In fact, these constraints provide us with an estimate of the dimensionless constants $f^{(s)}$ and $f^{(b)}$ [Eq. (13)], something which at present is beyond our capability to calculate in QCD. If we remove these constraints, these constants remain free parameters. As a final conclusion, this study suggests that precision measurements of elliptic flow of net protons at the Super Proton Synchrotron (NA61 experiment) or at the Relativistic Heavy Ion Collider in the Beam Energy Scan program [29] are of particular importance since they are strongly linked to the dynamics of the QCD critical point.

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