Thermodynamic properties of a neutral vector boson gas in a constant magnetic field

G. Quintero Angulo*
Facultad de Física, Universidad de la Habana, San Lázaro y L, Vedado, La Habana 10400, Cuba

A. Pérez Martínez† and H. Pérez Rojas‡
Instituto de Cibernética, Matemática y Física (ICIMAF), Calle E esq a 15, Vedado, La Habana 10400, Cuba
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The thermodynamical properties of a neutral vector boson gas in a constant magnetic field are studied starting from the spectrum given by Proca formalism. Bose-Einstein condensation (BEC) and magnetization are obtained in the limit of low temperature. In this limit, the condensation is reached not only by decreasing the temperature or augmenting the density but also by increasing the magnetic field. The magnetization turns out to be a positive quantity that increases with the field; under certain conditions self-magnetization is possible. The anisotropy in the pressures due to the axial symmetry imposed to the system by the magnetic field is also discussed. Astrophysical implications are commented.

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I. INTRODUCTION

There is a diversity of structures associated with a wide range of magnetic fields (10^{-9} to 10^{15} G) coexisting in our universe. Salient examples are large objects such as galaxies (radius 1.5×10^{18} km) and compact objects such as neutron stars (radius 20 km). The internal composition of neutron stars, still poorly understood, is described by all sorts of exotic dense matter in form of hyperons, a superfluid of paired neutrons and/or protons, a Bose-Einstein condensate (BEC) of mesons, or deconfined quark matter in the presence of strong magnetic fields [1]. Size and shape of a compact object depend on its composition but also on the magnetic field [2]. There are also some phenomena at an astrophysical scale that do not have explanations, as jets of pulsars, where magnetic fields might be relevant [3,4].

Even though some theories have been proposed to explain the origin of such magnetic fields, this issue is far from exhausted and it is still under great debate. In this regard, spin-1 bosons seem to be good candidates for magnetic field sources since, at low temperature, they are known to show spontaneous magnetization. As a consequence, under certain conditions, a gas of bosons can generate and sustain its own magnetic field [5,6].

The study of BEC and magnetization for a charged scalar or vector boson gas in the presence of a constant magnetic field was tackled in Refs. [7–12]. For low temperatures, the charged vector boson gas is paramagnetic, can be self-magnetized, and undergoes a diffuse phase transition to a Bose-Einstein condensate. For a diffuse phase transition, there is not a critical temperature but an interval of temperatures along which the transition occurs gradually [13]. In particular, a diffuse BEC phase is characterized by the presence of a finite fraction of the total particle density in the ground state and in states on its neighborhood at some temperature $T > 0$ [7,8,11].

Although both charged and neutral vector bosons could be relevant participants in astronomical phenomena, the thermodynamics of the neutral vector boson gas has been less studied. An effect analogous to self-magnetization, named BE ferromagnetism, was found in Ref. [5] for a gas of nonrelativistic neutral bosons with spin 1. In Refs. [14] and [15], magnetic-field-induced superconductivity and superfluidity are obtained for a gas of charged and neutral vector mesons, but this ignores the weak coupling between the neutral mesons and the magnetic field. More recently, BEC for a gas of interacting vector bosons at zero magnetic field was studied in Ref. [16].

Hence, the aim of this paper is to study the thermodynamical properties of a neutral vector boson gas (NVBG) in a constant magnetic field. We will deal with its phenomenology in the framework of Proca theory, independently of the realistic conditions in which it may appear. Neutral vector bosons can be mesons, atoms, and other paired fermions with zero net charge and total integer spin. For numerical calculations, we use positronium gas parameters, characterized by a mass of approximately $2m_e$ (m_e is the electron mass) and twice the electron magnetic moment $\kappa = 2\mu_B$, with μ_B being the Bohr magneton. Since we are focused on possible astrophysical applications, we will consider systems of densities in the range of 10^{30} to 10^{34} cm^{-3}.

We found that the phase transition to the BEC is driven by temperature, particle density, and magnetic field. For sufficiently low temperatures, self-magnetization arises. An analysis of this phenomenon leads us to the conditions for the appearance of a self-sustained field. The axial symmetry of the magnetic field is reflected in the particle spectra and in the energy-momentum tensor of the system which becomes anisotropic. For that reason, we also study the splitting of the parallel and perpendicular pressures with respect to the direction of the magnetic field.

Our paper is organized as follows. In Sec. II, we present the equation of motion and spectrum of neutral vector boson with magnetic moment. Section III contains a derivation of the thermodynamical potential, particle density, BEC, internal energy, entropy, and specific heat for the NVBG. Magnetization,

*gquintero@fisica.uh.cu
†aurora@icimaf.cu
‡hugo@icimaf.cu

self-magnetization, and anisotropic pressures are also discussed. Section IV is devoted to conclusions. Appendixes A and B contain some details of the calculations.

II. EQUATION OF MOTION OF A NEUTRAL VECTOR BOSON BEARING A MAGNETIC MOMENT

Neutral spin-1 bosons with magnetic moment that move in a magnetic field can be described by an extension of the original Proca Lagrangian for spin-1 particles that includes particle-field interactions [17,18]:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\rho^{\mu\nu}\rho_{\mu\nu} + m^2\rho^\mu\rho_\mu + im\kappa(\rho^\mu\rho_\nu - \rho^\nu\rho_\mu)F_{\mu\nu}. \quad (1)$$

In Eq. (1), the indices μ and ν run from 1 to 4, $F^{\mu\nu}$ is the electromagnetic tensor, and $\rho^{\mu\nu}$ and ρ^μ are independent field variables that follow [17]

$$\partial_\mu\rho^{\mu\nu} - m^2\rho^\nu + 2i\kappa m\rho^\mu F_\mu{}^\nu = 0, \quad \rho^{\mu\nu} = \partial^\mu\rho^\nu - \partial^\nu\rho^\mu. \quad (2)$$

A variation of the Lagrangian with respect to the field ρ_μ gives us the equations of motion that in the momentum space read

$$[(p_\mu p^\mu + m^2)\delta_\mu^\nu - p^\nu p_\mu - 2i\kappa m F_\mu{}^\nu]\rho^\mu = 0. \quad (3)$$

Thus, the boson propagator is

$$D^{-1}_{\mu\nu} = (p_\mu p^\mu + m^2)\delta_\mu^\nu - p^\nu p_\mu - 2i\kappa m F_\mu{}^\nu. \quad (4)$$

Considering the magnetic field uniform, constant, and in p_3 direction $\mathbf{B} = B\mathbf{e}_3$, one can start from Eq. (2) and obtain the generalized Sakata-Taketani Hamiltonian for the six-component wave equation of the system [17,18] following the same procedure of Ref. [17]. The Hamiltonian reads

$$H = \sigma_3 m + (\sigma_3 + i\sigma_2)\frac{\mathbf{p}^2}{2m} - i\sigma_2\frac{(\mathbf{p}\cdot\mathbf{S})^2}{m} - (\sigma_3 - i\sigma_2)\kappa\mathbf{S}\cdot\mathbf{B}, \quad (5)$$

with $\mathbf{p} = (p_\perp, p_3)$ and $p_\perp = p_1^2 + p_2^2$. σ_i are the 2×2 Pauli matrices, S_i are the 3×3 spin-1 matrices in a representation in which S_3 is diagonal, and $\mathbf{S} = \{S_1, S_2, S_3\}$.[1]

The equations for $\mathbf{p}$ and $\mathbf{r}$ are obtained from Eq. (5) and read

$$\frac{\partial\mathbf{p}}{\partial t} = i[H,\mathbf{p}] = \mathbf{0},$$
$$m\frac{\partial\mathbf{r}}{\partial t} = i[H,\mathbf{p}] = (\sigma_3 - i\sigma_2)\mathbf{p} + i\sigma_2[\mathbf{S},\mathbf{p},\mathbf{S}]. \quad (6)$$

From Eq. (6), it follows that the neutral bosons move freely in the direction parallel to the field as well as in the perpendicular one. Therefore, all the momentum components are preserved as quantum observables. This is a main difference with respect to the charged vector boson gas, in which the charge–magnetic field interaction leads to a quantization in the transversal momentum component (Landau levels) [19].

The eigenvalues of (5) are

$$\varepsilon(p_3,p_\perp,B,s) = \sqrt{m^2 + p_3^2 + p_\perp^2 - 2\kappa s B\sqrt{p_\perp^2 + m^2}}, \quad (7)$$

where $s = 0, \pm 1$ are the spin eigenvalues. Note that although the transverse momentum component is not quantized, the magnetic field intensity B enters in the energy spectrum of the neutral bosons coupled with it (see the last term in the previous equation), as happens for the charged bosons [19]. In both cases, this coupling reflects the axial symmetry imposed on the system by the magnetic field.

The ground-state energy of the neutral spin-1 boson ($s = 1$ and $p_3 = p_\perp = 0$) is

$$\varepsilon(0,B) = \sqrt{m^2 - 2\kappa Bm} = m\sqrt{1-b}, \quad (8)$$

with $b = \frac{B}{B_c}$ and $B_c = \frac{m}{2\kappa}$.

The rest energy of the system Eq. (8) decreases with the magnetic field and becomes zero for $B = B_c$. For the values of m and κ, we are considering ($m = 2m_e$ and $\kappa = 2\mu_B$), $B_c = m_e^2/e = 4.41\times 10^{13}\,G$, which is the Schwinger critical field. Let us note that the ground state of the charged vector boson has a similar instability (see Ref. [7]).

Equation (8) allows us to define an effective magnetic moment as

$$d = -\frac{\partial\varepsilon(0,B)}{\partial B} = \frac{\kappa m}{\sqrt{m^2 - 2m\kappa B}} = \frac{\kappa}{\sqrt{1-b}}. \quad (9)$$

The system has a paramagnetic behavior because $d > 0$. It will be also important for the discussion below that d grows with the increase of the magnetic field and diverges for $b \to 1$ ($B \to B_c$).

III. THERMODYNAMICAL PROPERTIES

The general expression for the thermodynamical potential of the NVBG has the form

$$\Omega(B,\mu,T) = \sum_{s=-1,0,1}\frac{1}{\beta}\left[\sum_{p_4}\int_{-\infty}^{\infty}\frac{p_\perp dp_\perp dp_3}{(2\pi)^2}\ln\det D^{-1}(\overline{p}^*)\right]. \quad (10)$$

Here $D^{-1}(\overline{p}^*)$ is the neutral boson propagator given by (4), $\beta = 1/T$ denotes the inverse temperature, μ is the boson chemical potential, and $\overline{p}^* = (ip_4 - \mu, 0, p_\perp, p_3)$. After doing the Matsubara sum, Eq. (10) becomes

$$\Omega(B,\mu,T) = \Omega_{\rm st} + \Omega_{\rm vac}, \quad (11)$$

where $\Omega_{\rm st}$ is the statistical contribution of bosons and antibosons that depends on B, T, and μ:

$$\Omega_{\rm st}(B,\mu,T) = \sum_{s=-1,0,1}\frac{1}{\beta}\left\{\int_0^\infty\frac{p_\perp dp_\perp dp_3}{(2\pi)^2}\ln[(1 - e^{-[\varepsilon(p_3,p_\perp,B,s)-\mu]\beta})(1 - e^{-[\varepsilon(p_3,p_\perp,B,s)+\mu]\beta})]\right\}, \quad (12)$$

[1] $S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $S_2 = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$, $S_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$.

and $\Omega_{\rm vac}$ is the vacuum term which is only B dependent:[2]

$$\Omega_{\rm vac} = \sum_{s=-1,0,1} \int_0^\infty \frac{p_\perp dp_\perp dp_3}{(2\pi)^2} \varepsilon(p_3,p_\perp,B,s). \tag{13}$$

We can rewrite $\Omega_{\rm st}$ as

$$\Omega_{\rm st}(B,\mu,T) = \sum_{s=-1,0,1} \Omega_{\rm st}(s), \tag{14}$$

with

$$\Omega_{\rm st}(s) = \frac{1}{\beta}\left\{ \int_0^\infty \frac{p_\perp dp_\perp dp_3}{(2\pi)^2} \ln[(1-e^{-[\varepsilon(p_3,p_\perp,B,s)-\mu]\beta})(1-e^{-[\varepsilon(p_3,p_\perp,B,s)+\mu]\beta})]\right\} \tag{15}$$

being the contribution of each spin state to the statistical part of the potential.

Using the Taylor expansion of the logarithm, Eq. (15) can be written as

$$\Omega_{\rm st}(s) = -\frac{1}{4\pi^2\beta}\sum_{n=1}^\infty \frac{e^{n\mu\beta}+e^{-n\mu\beta}}{n}\int_0^\infty p_\perp dp_\perp \times \int_{-\infty}^\infty dp_3 e^{-n\beta\varepsilon(p_3,p_\perp,B,s)}. \tag{16}$$

The term $e^{n\mu\beta}$ stands for the particles and the term $e^{-n\mu\beta}$ is for the antiparticles.

In Eq. (16), the integration in p_3 can be completely carried out, while the integration in $p_\perp$ can be only partially done, and we obtain for the thermodynamical potential the expression

$$\Omega_{\rm st}(s) = -\frac{z_0^2}{2\pi^2\beta^2}\sum_{n=1}^\infty \frac{e^{n\mu\beta}+e^{-n\mu\beta}}{n^2}K_2(yz_0) - \frac{\alpha}{2\pi^2\beta}\sum_{n=1}^\infty \frac{e^{n\mu\beta}+e^{-n\mu\beta}}{n} \times \int_{z_0}^\infty dz \frac{z^2}{\sqrt{z^2+\alpha^2}}K_1(yz), \tag{17}$$

where $K_n(x)$ is the McDonald function of order n, $y=n\beta$, $z_0 = m\sqrt{1-sb}$, and $\alpha = smb/2$. To get Eq. (17), the change of variables $z^2 = (m^2+p_\perp^2+\alpha^2)^2-\alpha^2$ was done.

To compute the integral in the second term of Eq. (17),

$$I = \int_{z_0}^\infty dz \frac{z^2}{\sqrt{z^2+\alpha^2}}K_1(yz), \tag{18}$$

we follow the procedure described in Appendix A. Finally, $\Omega_{\rm st}(s)$ reads

$$\Omega_{\rm st}(s) = -\frac{z_0^2}{2\pi^2\beta^2}\left(1+\frac{\alpha}{\sqrt{z_0^2+\alpha^2}}\right)\sum_{n=1}^\infty \frac{e^{n\mu\beta}+e^{-n\mu\beta}}{n^2} \times K_2(yz_0) - \frac{\alpha z_0^2}{\pi^2\beta^2\sqrt{z_0^2+\alpha^2}} \times \sum_{n=1}^\infty \frac{e^{n\mu\beta}+e^{-n\mu\beta}}{n^2}\sum_{w=1}^\infty \frac{(-1)^w(2w-1)!!}{(z_0^2+\alpha^2)^w} \times \left(\frac{z_0}{y}\right)^w K_{-(w+2)}(yz_0). \tag{19}$$

In the low-temperature limit, $T \ll m$ (which for $m = 2m_e \cong 1$ MeV means $T \ll 10^{10} K$), $\Omega_{\rm st}(-1)$ and $\Omega_{\rm st}(0)$ vanish. Therefore, in this limit $\Omega_{\rm st} \simeq \Omega_{\rm st}(1)$. This means that all the particles are in the spin ground state $s = 1$. The leading term of $\Omega_{\rm st}(1)$ is the first one in Eq. (19). Since it admits a further simplification, for the assumed low temperatures the statistical part of the thermodynamical potential is

$$\Omega_{\rm st}(B,\mu,T) = -\frac{\varepsilon(0,B)^{3/2}}{2^{1/2}\pi^{5/2}\beta^{5/2}(2-b)}Li_{5/2}(e^{\beta\mu'}) = -\frac{\left(m\sqrt{1-b}\right)^{3/2}}{2^{1/2}\pi^{5/2}\beta^{5/2}(2-b)}Li_{5/2}(e^{\beta\mu'}), \tag{20}$$

where $Li_n(x)$ is the polylogarithmic function of order n and $\mu' = \mu - \varepsilon(0,B)$. The quantity μ' is a function of the temperature and the magnetic field that leads to the critical condition for the BEC: The existence of a nonzero temperature $T_{\rm cond}$ for which $\mu' = 0$.

The vacuum contribution to the thermodynamical potential after being regularized is (see Appendix B):

$$\Omega_{\rm vac} = -\frac{m^4}{288\pi}[b^2(66-5b^2)-3(6-2b-b^2)(1-b)^2 \times \log(1-b) - 3(6+2b-b^2)(1+b)^2\log(1+b)]. \tag{21}$$

By adding Eqs. (22) and (21), we get the total thermodynamical potential for the NVBG in the limit of low temperatures, Eq. (22):

$$\Omega(B,\mu,T) = -\frac{\left(m\sqrt{1-b}\right)^{3/2}}{2^{1/2}\pi^{5/2}\beta^{5/2}(2-b)}Li_{5/2}(e^{\beta\mu'}) - \frac{m^4}{288\pi} \times [b^2(66-5b^2)-3(6-2b-b^2)(1-b)^2 \times \log(1-b) - 3(6+2b-b^2)(1+b)^2 \times \log(1+b)]. \tag{22}$$

[2]The vacuum term is important, for instance, in the positronium case. It would represent a correction to the usual Euler-Heisenberg term in which the electron-positron pairs bosonize by coupling, for instance, through Coulomb force.

A. Particle density and Bose-Einstein condensation

To obtain the particle density, we compute the derivative of Eq. (22) with respect to the chemical potential μ

$$N = -\frac{\partial\Omega(B,\mu,T)}{\partial\mu} = \frac{\left(m\sqrt{1-b}\right)^{3/2}}{2^{1/2}\pi^{5/2}\beta^{3/2}(2-b)} Li_{3/2}(e^{\mu'\beta}). \tag{23}$$

By setting $\mu' = 0$ in the previous equation, we get the BEC critical temperature $T_{\rm cond}$:

$$T_{\rm cond} = \frac{1}{m\sqrt{1-b}}\left[\frac{2^{1/2}\pi^{5/2}(2-b)N}{\zeta(3/2)}\right]^{2/3}, \tag{24}$$

where $\zeta(x)$ is the Riemann ζ function.

Although the behavior of Eq. (24) resembles the one obtained for bosons at zero magnetic field—the functional relation between $T_{\rm cond}$ and N is the same—when the field is present the critical temperature depends on it (through b), and $T_{\rm cond}$ diverges when $b \to 1$ ($B \to B_c$). The dependence of the critical temperature on b evidences that the condensation can be reached not only by decreasing the temperature or augmenting the density but also by increasing the magnetic field. This can be easily seen if we compute the density of particles out of the condensate $N_{\rm oc}$ (for $T < T_{\rm cond}$):

$$N_{\rm oc} = \frac{\left(m\sqrt{1-b}\right)^{3/2} T^{3/2}}{2^{1/2}\pi^{5/2}(2-b)} Li_{3/2}(e^{\mu'\beta}) = N\left(\frac{T}{T_{\rm cond}}\right)^{3/2}, \tag{25}$$

because from Eq. (25) follows that $N_{\rm oc} \to 0$ when $T \to 0$ but also when $b \to 1$.

Given that there is a critical temperature for each value of the field (as well as a critical field for each temperature), it is possible to draw a T vs b phase diagram. We did so in Fig. 1 for two fixed values of the densities: $N = 10^{30}$ cm^{-3} and $N = 10^{32}$ cm^{-3}, respectively. We can see from the graphics how $T_{\rm cond}$ grows with the augment of the density and diverges when $b \to 1$ ($B \to B_c$). The values of the critical temperatures (the dotted lines that separate the region where the BEC appears from that where there is no BEC) are in the range of $T = 10^7$ to 10^9, K which are typical of compact objects.

We can also examine the transition to the condensate through the behavior of the specific heat at constant volume:

$$C_v = \frac{\partial E}{\partial T}, \tag{26}$$

where $E = TS + \Omega + \mu N$ is the internal energy of the system.

The entropy of the vector bosons gas is

$$S = -\frac{\partial\Omega}{\partial T} = -\beta\left(\mu' N + \frac{5}{2}\Omega_{\rm st} + \beta\frac{\partial\mu'}{\partial\beta}N\right), \tag{27}$$

with

$$\mu' \cong -\frac{\zeta(3/2)T}{4\pi}\left[1-\left(\frac{T_{\rm cond}}{T}\right)^{3/2}\right]\Theta(T-T_{\rm cond}) \tag{28}$$

in the low-temperature limit. Here $\Theta(x)$ is the Heaviside θ function.

With the use of Eq. (27), the internal energy can be written as

$$E = m\sqrt{1-b}N + \Omega_{\rm vac} - \frac{3}{2}\Omega_{\rm st} - \beta\frac{\partial\mu'}{\partial\beta}N. \tag{29}$$

Equation (29) allows us to obtain for the specific heat the following expression:

$$C_v = -\beta\left(\frac{15}{4}\Omega_{\rm st} + \frac{3}{2}\mu' N + \frac{1}{2}\beta\frac{\partial\mu'}{\partial\beta}N - \beta^2\frac{\partial^2\mu'}{\partial\beta^2}N\right). \tag{30}$$

The specific heat has been plotted in Fig. 2 as a function of the temperature for a fixed value of the density $N = 10^{32}$ cm^{-3} and three values of the magnetic field. As it is apparent, it has a maximum that is a fingerprint of the BEC phase transition. The maximum decreases, smoothes with the increment of the magnetic field, and is expected to disappear for $B \to B_c$ ($b \to 1$), because when $B = B_c$ the gas is condensed at any temperature and density.

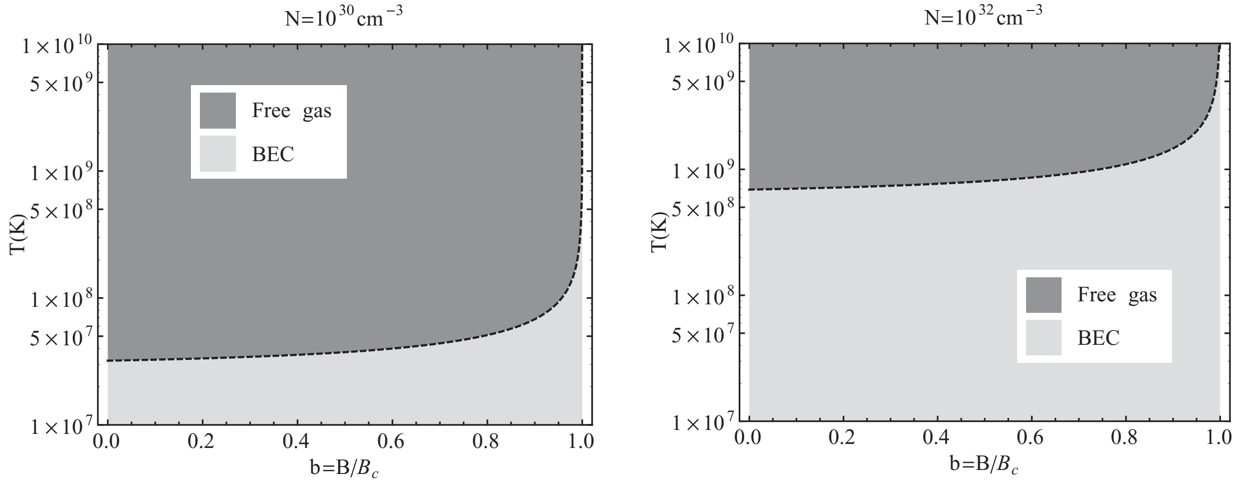

FIG. 1. The phase diagram in the T-b plane for different values of particle density. The black dashed lines are the critical curves that separates the region of T and b when there is condensate (light gray region) from the region in which there is not (dark gray region). Note that the critical line also depends on the particle density.

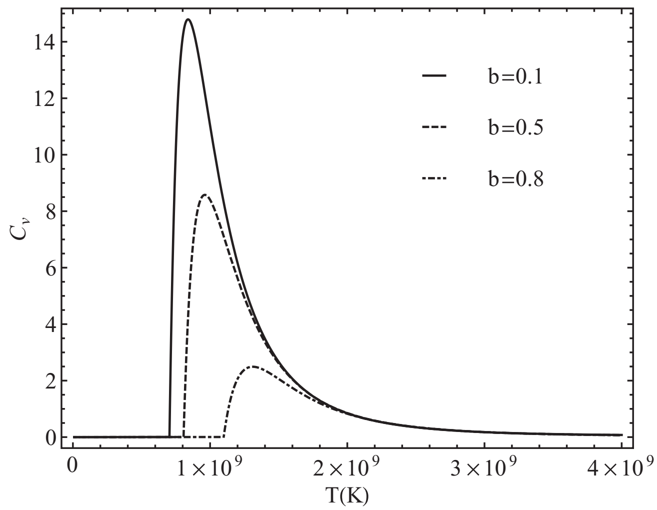

FIG. 2. The specific heat as a function of the temperature for $N = 10^{32}$ cm^{-3} and several values of the magnetic field.

B. Magnetization

We can obtain the magnetization of the system if we derive Eq. (22) with respect to the magnetic field:

$$M = dN_0 - \frac{\partial \Omega_{st}}{\partial B} - \frac{\partial \Omega_{vac}}{\partial B}. \tag{31}$$

In Eq. (31), N_0 is the number of particles in the condensate and d is the effective magnetic moment, Eq. (9). This term has to be added because in the low-temperature limit all the bosons are aligned to the field and contribute to the magnetization, but for temperatures under T_{cond}, Ω_{st} only accounts for the particles that are out of the condensate.

For the statistical contribution to the magnetization, we have the expression

$$M_{st} = dN_0 - \frac{\partial \Omega_{st}}{\partial B} = \frac{\kappa}{m\sqrt{1-b}}N - \frac{2\kappa m T^{5/2}}{(4\pi)^{5/2}(2-b)^2\left(m\sqrt{1-b}\right)^{1/2}} Li_{5/2}(e^{\beta\mu'}), \tag{32}$$

while the vacuum contribution is

$$M_{vac} = -\frac{\kappa m^3}{72\pi}[7b(b^2-6) + 3(2b^2+2b-7)(1-b) \times \log(1-b) - 3(2b^2-2b-7)(1+b)\log(1+b)]. \tag{33}$$

It is possible to show that for $T \ll m$ the second term in Eq. (32) is negligible, and one can prove that the vacuum magnetization in Eq. (33) is only relevant for low particle densities at very high fields, so it can be also neglected. Finally, the total magnetization of the NVBG is

$$M = \frac{\kappa}{\sqrt{1-b}}N = dN. \tag{34}$$

The previous expression is expected because at $T \ll m$ all the particles are in the $s = 1$ state. It is nothing else but the product of the effective magnetic moment by the particle density. However, an increase in the field still augments the magnetization because the effective magnetic moment d grows with B and diverges when $B \to B_c$ $(b \to 1)$. Since d is strictly positive for all values of B, the magnetization is always positive and different from zero even if $B = 0$ $[M(B=0) = \kappa N]$. This is an evidence of ferromagnetic response of the NVBG at low temperature. This behavior described for M is shown in the left panel of Fig. 3.

In the search for one of our main objectives, astrophysical magnetic field sources, we are interested in exploring if the system reaches the self-magnetization condition; i.e., whether or not the solid line in left panel of Fig. 3 intersects the curves of the magnetization. To do that, we consider $H = B - 4\pi M$ with no external magnetic field $H = 0$, and solve the self-consistent equation $B = 4\pi M$. This is a cubic equation due to the nonlinear dependency of the magnetization on the field. In the right panel of Fig. 3, its three solutions have been plotted but only one of them is physically meaningful. For one of the roots, the magnetic field is negative (see the dotted line), while for another, it decreases with the increasing density, reaching B_c when N goes to zero (dot-dashed line). These solutions imply that the magnetization also decreases with N; hence, they are contrary to Eq. (34) and must be discarded. Therefore, the only admissible solution of the self-magnetization equation is the one given by the solid line. The points of this line are the values of the self-sustained magnetic field. Nevertheless, this solution becomes complex for densities higher than $N_c = 7.14 \times 10^{34}$ cm^{-3}. N_c bounds the values of the particle density for which self-magnetization is possible. The maximum field that could be self-sustained by the gas corresponds to the critical density and has a magnitude of $2/3 \times B_c$. The values of B and N for which a self-magnetization may occur are in the order of those typical of compact objects. The maximum field that can be self-maintained by the NVBG is the same as that obtained for a gas of charged vector bosons with the same mass and magnetic moment, but in this case the critical particle density is of the order of 10^{32} cm^{-3} [6].

C. Anisotropic pressures

We will consider the energy momentum tensor and the anisotropic pressures of the system. The total energy momentum tensor of matter plus vacuum will be obtained as a diagonal tensor whose spatial part contains the pressures and the time component is the internal energy density E. One gets from the thermodynamical potential

$$T^i_j = \frac{\partial \Omega}{\partial a_{i,\lambda}} a_{j,\lambda} - \Omega\delta^i_j, \qquad T^4_4 = -E, \tag{35}$$

where a_i denotes the boson or fermion fields [20]. For a thermodynamical potential that depends on an external field, Eq. (35) leads to pressure terms of form

$$T^i_j = -\Omega - F^i_k\left(\frac{\partial \Omega}{\partial F^j_k}\right), \quad i = j. \tag{36}$$

Computing the pressures along each direction makes the anisotropy explicit:

$$\begin{aligned} P_3 &= T_3 = -\Omega = -\Omega_{st} - \Omega_{vac}, \\ P_\perp &= T^1_1 = T^2_2 = T_\perp = -\Omega - BM = P_3 - BM. \end{aligned} \tag{37}$$

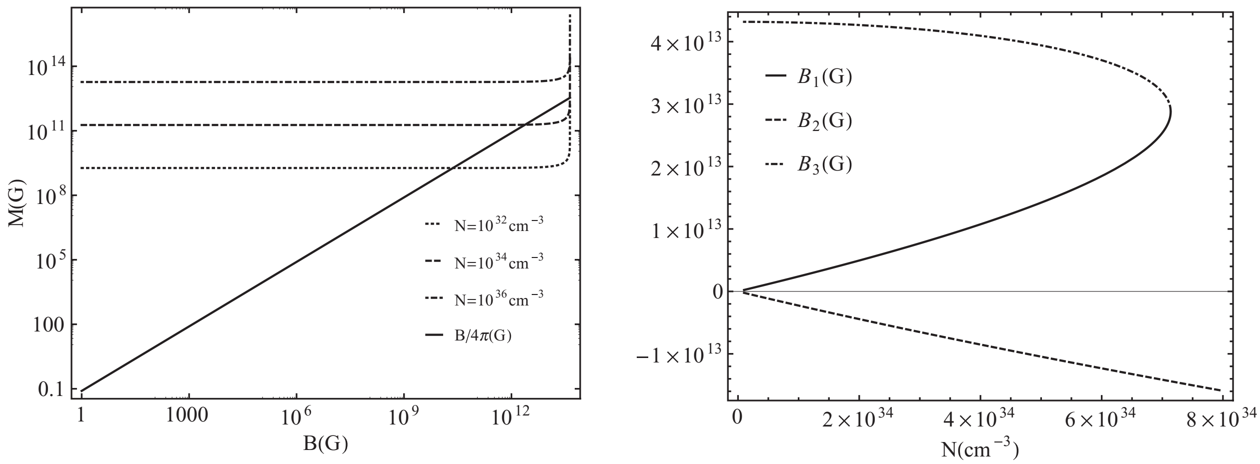

FIG. 3. Magnetization as a function of magnetic field for several values of the particle density (left panel). We have also plotted the $B/4\pi$. The solutions of self-magnetization equation as a function of particle density (right panel).

In the left panel of Fig. 4, the perpendicular and parallel pressures are depicted as function of the field [Eqs. (37)] for $T = 10^9$ K and $N = 10^{33}$ cm^{-3}. We also show the statistical and the vacuum parts of the parallel pressure in dashed and dot-dashed lines respectively. The values of the parallel pressure and its statistical part ($-\Omega_{st}$) coincides for $B = 0$, but their behavior is different when the field grows. Both are always positive but the total parallel pressure increases with the field and tends to the vacuum contribution $-\Omega_{vac}$, while its statistical part decreases and goes to zero for $B = B_c$: When all the particles are in the condensate they exert no pressure. We would like to remark that the parallel pressure remains different from zero due to the vacuum contribution.

On the contrary, the perpendicular pressure decreases (dotted line in left panel of Fig. 4) with the magnetic field and eventually reaches negative values. This is because the main contribution to $P_\perp$ comes from the term $-MB$, which is always negative and diverges in the critical field. A similar result is obtained for fermion gases in a magnetic field [20–22]. In this frame, negative pressure can be interpreted as the system becoming unstable. Because the effect of the negative perpendicular pressure is to push the particles inward to the magnetic field axis, we could be in the presence of a transversal magnetic collapse [21].

Whether the transversal pressure is negative or not depends on the field but also on the temperature and the particle density. This can be seen if we examine this pressure in more detail for the self-magnetized NVBG. To do that, we substitute in $P_\perp$ the solution of the self-magnetization condition $B = 4\pi M$ and plot the perpendicular pressure as a function of the particle density for several values of T (right panel of Fig. 4). If we start from the lower values of N, adding particles to the system increments the parallel pressure, but it also increases the self-produced magnetic field and T_{cond}. Once T_{cond} becomes higher than the gas temperature, the BEC phase appears and the pressure diminishes because a fraction of the particles fall in the condensate. Besides, as the self-generated field becomes higher, the contribution to $P_\perp$ of the negative term $-MB$

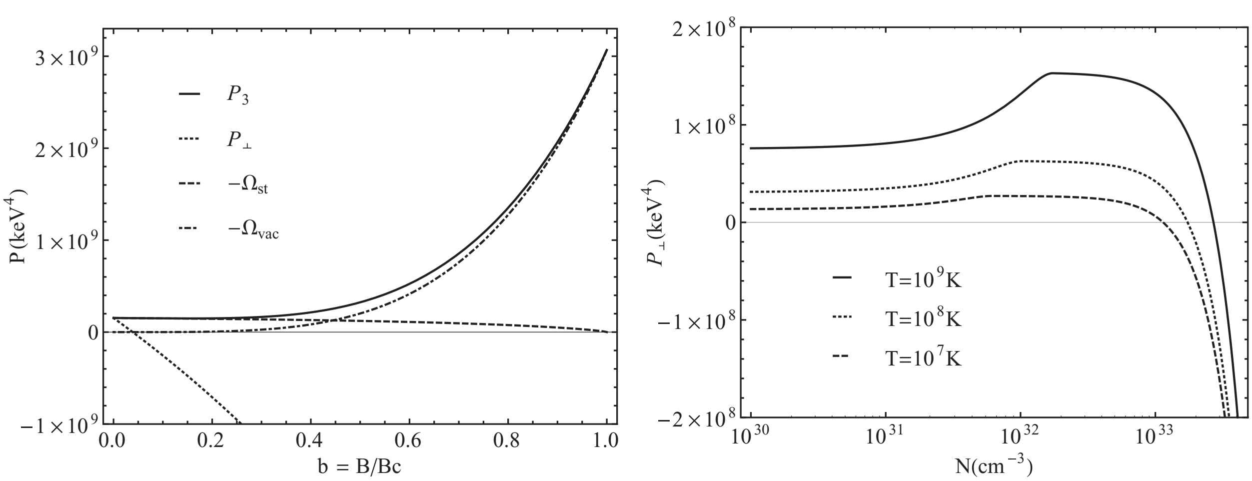

FIG. 4. The pressure as a function of the magnetic field for several values of temperature (left panel); the statistical and the vacuum contributions to the pressure are also plotted in dashed and dot-dashed lines. The perpendicular pressure of the self-magnetized gas as a function of the particle density for several values of temperature (right panel).

becomes more relevant until eventually adding more particles makes the system unstable. A decrease in the temperature lowers the value of particle density where the instability starts.

When the gas is not self-magnetized but subject to an external magnetic field, a continuous increment in the density also leads the system to the instability. In this case, the increase of N does not augment the field, but it still increments the magnetization and the condensation temperature. Therefore, the NVBG will be unstable depending on the values of the temperature, the density and the magnetic field, regardless of whether the field is self-produced or not.

The arising of an instability in the magnetized NVBG might be relevant in the description of some phenomena, as jets, that are related to the exertion of mass and radiation out of astronomical objets [6].

IV. CONCLUSIONS

Starting from the Proca formalism [17,18], we computed the spectrum of a gas of neutral vector bosons in a constant magnetic field. The effective rest energy Eq. (8) turns out to be a decreasing function of the magnetic field that becomes zero when it reaches certain critical value $B_c = \frac{m}{2k}$. As a consequence, the phase transition of the neutral vector boson gas to the BEC depends not only on the temperature or the particle density but also on the magnetic field, in a way that increasing the field drives the system to the condensate. When the magnetic field reaches its critical value, the gas is condensed for any temperature and density.

The magnetization of the gas is a positive quantity that increases with the field and diverges when $B = B_c$. For particle densities under a critical value $N_c \cong 7.14 \times 10^{34}$ cm^{-3}, the self-magnetization condition is fulfilled and the gas can maintain a self-generated magnetic field. The maximum field that can be reached by self-magnetization turns out to be $2/3 \times B_c \sim 10^{13} G$.

The change from spherical to axial symmetry induced by the magnetic field is explicitly manifested in the spectrum of the NVBG (through the asymmetry in the momentum components) and also in the splitting of the pressures in the parallel and perpendicular directions to the field. For low values of the field, the pressure exerted by the particles has the main role in both components. However, when the magnetic field grows, the increasing parallel pressure is dominated by the positive vacuum pressure term, while the decreasing perpendicular pressure is determined by the negative magnetic pressure term $-MB$. For magnetic fields and particle densities high enough, or sufficiently low temperatures, the perpendicular pressure becomes negative and an instability emerges in the system that turns out to be susceptible to suffer a transversal magnetic collapse.

All these phenomena that NVBG undergo (BEC, self-sustained magnetic field, and the collapsing of the transverse pressure) appear for typical values of densities and magnetic fields in compact objects. Therefore, they could be relevant in modeling jets as well as the mechanism that sustain the strong magnetic field in compact objects. These models deserve a separated treatment which is in progress.

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APPENDIX A: CALCULATION OF I

In order to compute the integral of the second term of Eq. (17),

$$I = \int_{z_0}^{\infty} dz \frac{z^2}{\sqrt{z^2+\alpha^2}} K_1(yz), \tag{A1}$$

let us introduce the following form for $K_1(yz)$:

$$K_1(yz) = \frac{1}{yz} \int_0^{\infty} dt e^{-t-\frac{y^2 z^2}{4t}}. \tag{A2}$$

If we substitute (A2) in (A1), the integration over z can be carried out:

$$I = \frac{\sqrt{\pi}}{y^2} \int_0^{\infty} dt \sqrt{t} e^{-t+\frac{y^2\alpha^2}{4t}} erfc\left(\frac{y\sqrt{z^2+\alpha^2}}{2\sqrt{t}}\right). \tag{A3}$$

To integrate over t in (A3), we replace the complementary error function erf$c(x)$ by its series expansion:

$$\text{erf}c(x) \backsimeq \frac{e^{-x^2}}{\sqrt{\pi}x}\left[1 - \sum_{w=1}^{\infty} \frac{(-1)^w (2w-1)!!}{(2x^2)^w}\right]. \tag{A4}$$

After the replacement and integration, I_3 can be written as

$$I = \frac{z_0^2}{y\sqrt{z_0^2+\alpha^2}} K_2(yz_0) - \frac{z_0^2}{y\sqrt{z_0^2+\alpha^2}} \sum_{w=1}^{\infty} \frac{(-1)^w (2w-1)!!}{(z_0^2+\alpha^2)^w} \times \left(\frac{z_0}{y}\right)^w K_{-(w+2)}(yz_0). \tag{A5}$$

APPENDIX B: VACUUM THERMODYNAMICAL POTENTIAL

To obtain Eq. (21) for the vacuum contribution to the thermodynamical potential, we start from its definition

$$\Omega_{\text{vac}} = \sum_{s=-1,0,1} \int_0^{\infty} \frac{p_\perp dp_\perp dp_3}{(2\pi)^2} \varepsilon,$$

where $\varepsilon(p_\perp, p_3, B, s) = \sqrt{p_3^2 + p_\perp^2 + m^2 - 2\kappa s B\sqrt{p_\perp^2+m^2}}$.

To integrate over p_3 and $p_\perp$, we use the equivalence

$$\sqrt{a} = -\frac{1}{2\sqrt{\pi}} \int_0^{\infty} dy y^{-3/2}(e^{-ya} - 1) \tag{B1}$$

and introduce the small quantity δ as lower limit of the integral to regularize the divergence of the a-dependent term and eliminate the term that does not depends on a:

$$\sqrt{a(\delta)} = -\frac{1}{2\sqrt{\pi}} \int_{\delta}^{\infty} dy y^{-3/2} e^{-ya}. \tag{B2}$$

Now, let us make $a(\delta) = \varepsilon^2 = p_3^2 + p_\perp^2 + m^2 - 2\kappa s B\sqrt{p_\perp^2 + m^2}$. Consequently,

$$\varepsilon = -\frac{1}{2\sqrt{\pi}}\int_\delta^\infty dy y^{-3/2} e^{-y(p_3^2+p_\perp^2+m^2-2\kappa s B\sqrt{p_\perp^2+m^2})}. \tag{B3}$$

By substituting Eq. (B3) in Eq. (B1), we obtain for the vacuum thermodynamical potential:

$$\Omega_{\rm vac} = -\frac{1}{8\pi^{5/2}}\sum_{s=-1,0,1}\int_\delta^\infty dy y^{-3/2}\int_0^\infty dp_\perp p_\perp \int_{-\infty}^\infty dp_3 e^{-y(p_3^2+p_\perp^2+m^2-2\kappa s B\sqrt{p_\perp^2+m^2})} \tag{B4}$$

After doing the Gaussian integral over p_3, Eq. (B4) reads

$$\Omega_{\rm vac} = -\frac{1}{8\pi^2}\sum_{s=-1,0,1}\int_\delta^\infty dy y^{-2}\int_0^\infty dp_\perp p_\perp e^{-y(p_\perp^2+m^2-2\kappa s B\sqrt{p_\perp^2+m^2})}. \tag{B5}$$

If we introduce the new variable $z = \sqrt{m^2 + p_\perp^2} - s\kappa B$, Eq. (B5) becomes

$$\Omega_{\rm vac} = -\frac{1}{8\pi^2}\sum_{s=-1,0,1}\left\{\int_\delta^\infty dy y^{-3} e^{-y(m^2-2ms\kappa B)} + s\kappa B\int_\delta^\infty dy y^{-2}\int_{z_1}^\infty dz e^{-y(z^2-s^2\kappa^2B^2)}\right\}, \tag{B6}$$

where $z_1 = m - s\kappa B$.

Equation (B6) admits a further simplification if we perform a second change of variables $w = z - z_1$ in its last term, sum over the spin, and remember that $b = B/B_c$ with $B_c = m/2\kappa$:

$$\Omega_{\rm vac} = -\frac{1}{8\pi^2}\left\{\int_\delta^\infty dy y^{-3} e^{-ym^2}(1 + 2\cosh[m^2by]) + mb\int_\delta^\infty dy y^{-2}\int_0^\infty dw e^{-y(m-w)^2}\sinh[mb(m-w)y]\right\}. \tag{B7}$$

To take the limit $\delta \to 0$, we subtract from $1 + 2\cosh[m^2by]$ and $\sinh[mb(m-w)y]$ the first terms in their series expansion and obtain for the vacuum thermodynamical potential the expression

$$\Omega_{\rm vac} = -\frac{1}{8\pi^2}\int_0^\infty dy y^{-3} e^{-ym^2}\{2\cosh[m^2by] - 2 - m^4b^2y^2\} - -\frac{mb}{8\pi^2}\int_0^\infty dy y^{-2}\int_0^\infty dw e^{-y(m-w)^2}$$
$$\times\left\{\sinh[mb(m-w)y] - mb(m-w)y - \frac{[mb(m-w)y]^3}{6}\right\}$$

that leads to Eq. (21) after integration.

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