Monopole pairing correlations with random interactions

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In this paper we study the monopole pairing correlation in 0^+ states of semimagic nuclei under random interactions. We calculate overlaps between S-pair wave functions and exact shell-model wave functions. No dominance of the S-pair components is shown: In a single-j shell 0^+ states have random overlaps with the S-pair wave function; in a multi-j space the 0_1^+ state has a small overlap with the S-pair wave function, and this is particularly pronounced if the single-particle splitting is small and/or the spins j of single-particle orbits are large. For the 0_1^+ state in a multi-j space, the generalized seniority scheme and the seniority truncation of the shell model provide similar results. We study the odd-even staggering of one-neutron separation energies under random interactions. The odd-even staggering phenomenon survives in random samples where all the even members have spin-zero ground states but disappears in random samples where all the even members have spin-nonzero ground states.

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I. INTRODUCTION

The pairing correlation plays an important role in low-lying states of atomic nuclei [1,2]. The ground states of even-even nuclei always have spin and parity $I^\pi = 0^+$ without any exception. One-proton separation energies of even-Z nuclei are systematically larger than those of neighboring odd-Z nuclei, and one-neutron separation energies of even-N nuclei are systematically larger than those of neighboring odd-N nuclei. Proton-proton or neutron-neutron pairs with spin $J = 0$ (namely S pairs) are found to be dominant ingredients in low-lying states of semimagic nuclei [3]. These phenomena were believed to be consequences of the strong and attractive monopole pairing interaction between like nucleons.

In 1997, Johnson, Bertsch, and Dean found the dominance of 0^+ ground states (0 g.s.) in even-even nuclei under two-body random ensembles (TBREs) [4]. Since then, many efforts have been devoted to understanding this puzzle under random interactions, as well as to studying collective motions in atomic nuclei. For example, in Ref. [5] Bijker and Frank studied vibrational and rotational motions in the interacting boson model with random interactions; in Refs. [6,7] Zhao *et al.* studied collective motions in nucleon-pair approximations of the shell model; in Ref. [8] Johnson and Nam studied correlation of yrast states using the Mallmann plot [9], and the correlation was further investigated in Refs. [10–13]. See Refs. [14–17] for a comprehensive review.

The monopole pairing correlation in semimagic nuclei was studied in the framework of the shell model with random interactions by Johnson *et al.* [18]. They found that one-neutron separation energies of even-N calcium isotopes are systematically larger than those of the neighboring odd-N calcium isotopes in the random samplings where all the even members have 0 g.s. and that the pair-transfer fractional collectivity between the ground states of n particles and $n-2$ particles is close to 1. In Ref. [19] Lei *et al.* calculated overlaps between the lowest-seniority wave function and the shell-model wave function for the 0_1^+ and 2_1^+ states of six neutrons in the sd shell and in the pf shell with random interactions. They found that the S pairs are very important as single-particle splittings are large. These results seem to indicate that the pairing correlation remains important in random quantum systems. However, in Ref. [20] Zhao *et al.* calculated the seniority number for 0_1^+ states of four and six neutrons in a single-j shell with random interactions and found no dominance of low seniority.

The purpose of this paper is to understand the inconsistency of the previous conclusions. We systematically study the monopole pairing correlation in 0^+ states of semimagic nuclei under random interactions. We focus attention on the structure of wave functions, viz., comparing S-pair wave functions and shell-model wave functions. The main results of our analysis are (1) in a single-j shell, 0^+ states have essentially random overlaps with the S-pair wave function; (2) random interactions favor one or more particular states as shell-model solutions, but the S-pair wave function is not so favored; (3) in a multi-j space, the 0_1^+ state has a small overlap with the S-pair wave function; (4) in a multi-j space, the generalized seniority scheme and the seniority truncation of the shell model provides similar results for the 0_1^+ state; (5) in a multi-j space, the overlap between the S-pair wave function and the shell-model wave function is relatively larger for the lowest and the highest 0^+ states; (6) the famous odd-even staggering of one-nucleon separation energies is seen in random interactions when the even nuclei have 0 g.s., but disappears when the ground states have non-0 spins.

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This paper is organized as follows. In Sec. II we introduce random interactions, S-pair wave functions, and random matrices. In Sec. III we present the overlap between S-pair wave functions and shell-model wave functions for 0^+ states of semimagic nuclei in a single-j shell and in a multi-j shell; we study the odd-even staggering of one-neutron separation energies under both the requirement of 0 g.s. and that of non-0 g.s. In Sec. IV we summarize our results.

II. FRAMEWORK

A. The random interaction

In random two-body interactions, we treat two-body matrix elements $V^{(JT)}_{j_1j_2j_3j_4} = \langle j_1j_2JT|V|j_3j_4JT\rangle$ as random parameters which follow the Gaussian distribution

$$\rho\left(V^{(JT)}_{j_1j_2j_3j_4}\right) = \frac{1}{\sqrt{2\pi x}}\exp\left[-\frac{\left(V^{(JT)}_{j_1j_2j_3j_4}\right)^2}{2x}\right], \quad (1)$$

where

$$x = \begin{cases} 1 & \text{if } |\langle j_1j_2JT|j_3j_4JT\rangle| = 1, \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad (2)$$

In this paper we use two random ensembles of Hamiltonian as follows.

(a) We use TBRE for a single-j shell or the pf shell: The single-particle energies are zero, and the two-body matrix elements are defined by Eq. (1).

(b) We use two-body random ensemble with random single-particle energies (TBRE-RSPE) for a two-j shell (j_1 and j_2): The two-body matrix elements are defined by Eq. (1), and the single-particle splitting, $\varepsilon \equiv \varepsilon_{j_2} - \varepsilon_{j_1}$, follows the uniform distribution between -10 and 10.

B. The S-pair wave function

In this work we obtain the S-pair wave function by using two pair-truncation schemes: the seniority truncation of the shell model and the generalized seniority scheme.

In the seniority truncation of the shell model [21] (it is also called the noncollective S-pair approximation), the building blocks are S pairs of single-j orbits, i.e.,

$$S^\dagger_j = (a^\dagger_j \times a^\dagger_j)^{(J=0)}, \quad (3)$$

where ${a_j}^\dagger$ denotes a nucleon on orbit j. The basis states of $2N$ neutrons with spin 0 are constructed by such pairs, i.e.,

$$\frac{1}{\mathcal{N}}\left[S^\dagger_{j_1} \times S^\dagger_{j_2} \times \cdots \times S^\dagger_{j_N}\right]^{(I=0)}|0\rangle, \quad (4)$$

where $\mathcal{N}$ is a normalization factor. By taking all possible combinations of the j_i, one gets a complete basis set for the seniority-zero space, and we diagonalize the Hamiltonian matrix in this space. The dimension is denoted by $d_{\rm ST}$, and the calculated wave functions are denoted by $|\Psi_{\rm ST}(0^+_i)\rangle$, with $i = 1,2,\ldots,d_{\rm ST}$, where we use "ST" to represent "seniority truncation."

In the generalized seniority scheme [3] (it is also called the collective S-pair approximation), the building blocks are collective (alternatively, correlated) S pairs, i.e.,

$$S^\dagger = \sum_j y_j S^\dagger_j, \quad (5)$$

where y_j are a set of structure coefficients to be determined properly. The wave function of $2N$ neutrons with spin 0 is constructed by the collective S pairs, i.e.,

$$|\Psi_{\rm GS}\rangle \equiv \frac{1}{\mathcal{N}}(S^\dagger \times S^\dagger \times \cdots \times S^\dagger)^{(I=0)}|0\rangle, \quad (6)$$

where we use the subscript "GS" to represent "generalized seniority." The structure coefficients of the collective S pairs, y_j, are determined by minimizing the expectation value of the Hamiltonian, $\langle\Psi_{\rm GS}|\hat{H}|\Psi_{\rm GS}\rangle$, through an iterative procedure. In this work we compute y_j using a calculating technique in Ref. [22].

In a multi-j shell the generalized seniority scheme can be regarded as an approximate solution of the seniority truncation of the shell model. In a single-j shell both the generalized seniority scheme and the seniority truncation are reduced to the "standard" seniority scheme [23]; i.e., we have $d_{\rm ST} = 1$ and $|\Psi_{\rm GS}\rangle = |\Psi_{\rm ST}(0^+_1)\rangle$.

We calculate 0^+ states of semimagic nuclei in the full shell-model space. The dimension of the 0^+ states is denoted by $d_{\rm SM}$ (in the j-j coupled scheme), and the calculated wave functions are denoted by $|\Psi_{\rm SM}(0^+_i)\rangle$, with $i = 1,2,\ldots,d_{\rm SM}$, where we use "SM" to represent "shell model."

C. The random matrix

Random matrices were introduced into physics by Wigner in the 1950s, which was used to understand the level spacings in resonances of slow-neutron scattering on heavy nuclei [24]. It is a useful tool to study spectral fluctuations, symmetries, and quantum chaos. See Refs. [16,17] for a comprehensive review.

We denote the random Hamiltonian matrix by H, the matrix elements by H_{ij}, and the dimension of H by d_H. In the Gaussian orthogonal ensemble (GOE), the matrix elements follow three properties:

(a) $H_{ij} = H_{ji} = {H_{ij}}^*$;

(b) H_{ij} ($i \leqslant j$) are independent;

(c) because the original basis states are arbitrarily chosen, the distribution of H is invariant under an arbitrary unitary transformation of the basis states [25].

To satisfy property (c), the distribution of the matrix elements is given by

$$\rho(H_{ij}) = \frac{1}{\sqrt{2\pi x}}\exp\left[-\frac{(H_{ij})^2}{2x}\right], \quad (7)$$

where

$$x = \begin{cases} 1 & \text{if } i = j, \\ \chi^2/2 & \text{if } i < j, \end{cases} \quad \text{and } \chi = 1. \quad (8)$$

If one takes a smaller value of χ ($0 < \chi < 1$), the magnitude of the off-diagonal matrix elements is smaller, which leads to weaker configuration mixings between the original basis states, and the distribution of H is not unitary invariant.

The submatrix of H is simply defined as

$$H' = (H_{ij}), \quad i,j = 1,2,\ldots,d_{H'}, \tag{9}$$

where $d_{H'}$ is the dimension of the submatrix ($0 < d_{H'} \leqslant d_H$).

III. RESULTS

In this paper we call the quantity $\langle\Psi_a|\Psi_b\rangle^2$ overlap between two wave functions $|\Psi_a\rangle$ and $|\Psi_b\rangle$. We investigate 0^+ states of four and six neutrons ($n = 4$ and 6) in a single-j shell and in a multi-j shell. We calculate overlaps between S-pair wave functions and shell-model wave functions with the TBRE or the TBRE-RSPE.

A. The single-j shell

Let us begin with random ensembles in a single-j shell. It is worth noting that there is only one 0^+ state for a semimagic even-even nucleus in the single-$j \leqslant 7/2$ shell, and this state has seniority quantum number $v = 0$ trivially. Thus, we take $j \geqslant 9/2$. We investigate the cases of $j = 9/2$, $11/2$, $13/2$, $15/2$, $17/2$, and $21/2$. In each case we diagonalize 1000 sets of the TBRE matrices and calculate overlaps between the S-pair wave function and the shell-model wave functions, $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$ with $i = 1,2,\ldots,d_{SM}$.

In Fig. 1 one sees that distributions of $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$ with different i are very similar to each other. This means S pairs do not favor the lowest 0^+ state in a single-j shell under random interactions. For the cases with $d_{SM} = 2$, i.e., four neutrons in the $j = 9/2$, $j = 11/2$, and $j = 13/2$ shells, we find two statistical peaks at $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 = 0$ and $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 = 1$, respectively. For the cases with $d_{SM} \geqslant 3$, i.e., four neutrons in the $j = 15/2$ shell and in the $j = 21/2$ shells and six neutrons in the $j \geqslant 11/2$ shells, we find one sharp peak at $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 = 0$, and the statistical probability decreases rapidly as $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$ increases. For $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 = 1$, the probability is close to 0.

The above results can be modeled by behavior of random unit vectors in a d_{SM}-dimensional space. We regard the S-pair wave function as a fixed unit vector and the shell-model wave functions obtained under the TBRE as random unit vectors in a d_{SM}-dimensional space. The exact distribution of overlaps between a fixed unit vector and random unit vectors is presented in Fig. 1 (see the Appendix for the analytic formula); we find that it reasonably explains the distribution of $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$. This means the validity of the S-pair approximation for 0^+ states of semimagic nuclei in a single-j shell is essentially random. We also see small deviations from the random distribution; for example, for the cases with $d_{SM} = 2$ [see Figs. 1(a)–1(c)], the probability of $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 \approx 0.5$ is higher than the purely random probability; for six neutrons in the $j = 11/2$ shell and in the $j = 13/2$ shell [see Figs. 1(f) and 1(g)], the probability of $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2 \approx 1$ is higher than the random probability.

One might ask whether the shell-model wave functions obtained under the TBRE are random unit vectors in the d_{SM}-dimensional Hilbert space. We investigate three cases: four neutrons in the $j = 13/2$ shell ($d_{SM} = 2$), four neutrons in the $j = 15/2$ shell ($d_{SM} = 3$), and six neutrons in the $j = 21/2$ shell ($d_{SM} = 13$).

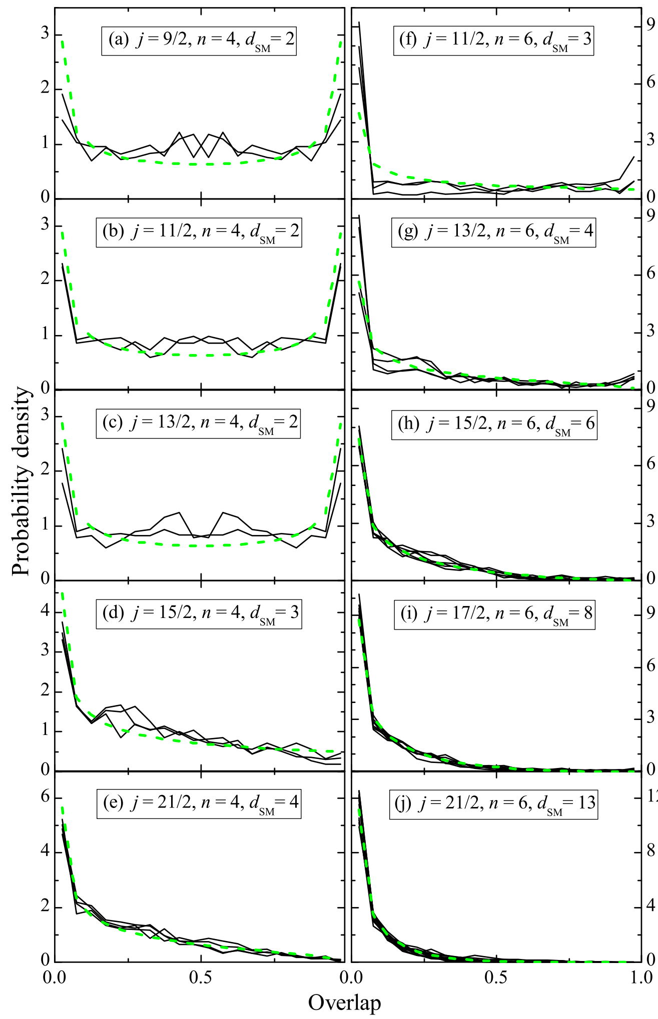

FIG. 1. Distributions of overlaps between the S-pair wave function and the shell-model wave functions, $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$, with $i = 1,2,\ldots,d_{SM}$, for four and six neutrons in a single-j shell with the TBRE. The average value of the probability density is set to 1. n is the neutron number, and d_{SM} is the dimension of the shell-model space for 0^+ states. The curves in black are distributions of $\langle\Psi_{GS}|\Psi_{SM}(0_i^+)\rangle^2$ for the random ensemble. The dashed curve in green is the exact distribution of overlaps between a fixed unit vector and random unit vectors in a d_{SM}-dimensional space.

For four neutrons in the $j = 13/2$ shell, we diagonalize 10 000 sets of TBRE matrices. The 0_1^+-state shell-model wave functions, as well as the S-pair wave function, are represented by the polar coordinate φ. The distribution is presented in Fig. 2(a), and the probability density is approximately given by the following equation:

$$\rho(\varphi) \approx 1 + 0.440\sin(4\varphi - 0.829). \tag{10}$$

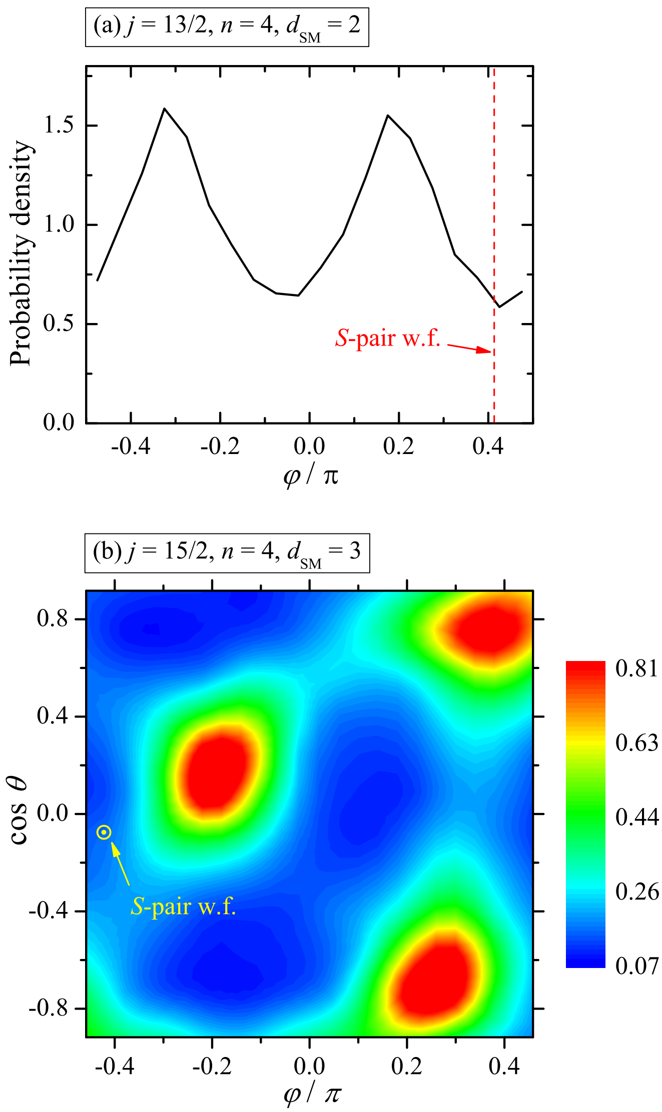

FIG. 2. Distribution of 0_1^+-state shell-model wave function obtained under the TBRE in the d_{SM}-dimensional Hilbert space. The average value of the probability density is set to 1 in panel (a) and $Y_{0,0}$ ($= 1/\sqrt{4\pi} = 0.282$) in panel (b). The coordinates of the S-pair wave function are also presented.

There is a correlation between the shell-model wave functions, and it is broadened by randomness. For a two-dimensional system, if a particular state is preferred, as either the lowest state or the highest state, the orthogonal state must be preferred. This is because as the Hamiltonian is random it can be multiplied by an overall sign. In Fig. 2(a) one sees two statistical peaks around $\varphi = -0.31\pi$ and 0.19π. The S-pair wave function is found at $\varphi = 0.41\pi$, and it is not favored by the TBRE. The above feature leads to the small deviations from the purely random distribution in Figs. 1(a)–1(c).

For four neutrons in the $j = 15/2$ shell ($d_{SM} = 3$), we diagonalize 7.5×10^4 sets of TBRE matrices. The 0_1^+-state wave functions, as well as the S-pair wave function, are represented by the spherical coordinates θ and φ. The distribution is presented in Fig. 2(b), and the probability density is approximately given by the multipole expansion in the real spherical-harmonic basis, i.e.,

$$\begin{aligned}\rho(\theta,\varphi) \approx\ & Y_{0,0} + 0.018Y_{2,-2} + 0.024Y_{2,-1} \\ & + 0.088Y_{2,0} - 0.002Y_{2,1} - 0.014Y_{2,2} \\ & - 0.181Y_{4,-4} + 0.243Y_{4,-3} + 0.462Y_{4,-2} \\ & + 0.172Y_{4,-1} - 0.051Y_{4,0} - 0.139Y_{4,1} \\ & - 0.154Y_{4,2} + 0.041Y_{4,3} - 0.206Y_{4,4}. \end{aligned} \quad (11)$$

A strong hexadecapole oscillation shows up. In Fig. 2(b) one sees that there are three particular states that are favored in this three-dimensional system, and they are orthogonal to each other. This leads to the small deviations from randomness in Fig. 1(d).

For six neutrons in the $j = 21/2$ shell ($d_{SM} = 13$), we diagonalize 7.5×10^5 sets of TBRE matrices. We calculate the average overlap of the 0_1^+-state shell-model wave functions with a given state $|\Psi'\rangle$. In this work we randomly choose ten of the 0_1^+-state shell-model wave functions as our $|\Psi'\rangle$, and the resulting average overlaps are equal to 0.108, 0.159, 0.112, 0.112, 0.118, 0.103, 0.105, 0.115, 0.127, 0.112, respectively. These values are larger than one would expect in the case of purely random vectors which would give the average overlap $1/d_{SM}$ (=0.077 in our case) [26,27]. If we choose the S-pair wave function as $|\Psi'\rangle$, the average overlap is equal to 0.112. In Ref. [28] Horoi *et al.* chose the 0_1^+-state wave function obtained by realistic interactions as $|\Psi'\rangle$ and found that the average overlap is larger than $1/d_{SM}$. It indicates that the "realistic" 0_1^+ state is favored by random interactions.

B. The multi-j shell

Now we come to random ensembles in multi-j shells. Here we investigate 0^+ states of four and six neutrons in the schematic $g_{9/2}i_{11/2}$ shell and in the pf shell. For the case of the $g_{9/2}i_{11/2}$ shell, we diagonalize 5000 sets of the TBRE-RSPE matrices. For the case of the pf shell, we diagonalize 1000 sets of the TBRE matrices.

First let us focus on the 0_1^+ state. For the case of the $g_{9/2}i_{11/2}$ shell, we present the distribution of the overlap between the seniority-truncated wave function and the shell-model wave function, $\langle\Psi_{ST}(0_1^+)|\Psi_{SM}(0_1^+)\rangle^2$, versus the single-particle splitting, $\varepsilon = \varepsilon_{i_{11/2}} - \varepsilon_{g_{9/2}}$, in Fig. 3. We find a statistical valley at $[\varepsilon, \langle\Psi_{ST}(0_1^+)|\Psi_{SM}(0_1^+)\rangle^2] = (0,1)$ and a statistical peak around (0,0). This means that the seniority truncation is not favored in the 0_1^+ state with small single-particle splittings. As the single-particle splitting increases, $\langle\Psi_{ST}(0_1^+)|\Psi_{SM}(0_1^+)\rangle^2$ increases statistically. If the splitting is very large (e.g., $|\varepsilon| \sim 10$), the distribution of $\langle\Psi_{ST}(0_1^+)|\Psi_{SM}(0_1^+)\rangle^2$ in the $g_{9/2}i_{11/2}$ shell is reduced to that in the single-$g_{9/2}$ or single-$i_{11/2}$ shell, which exhibits random behavior.

In Ref. [19], Lei *et al.* present the overlap between the generalized-seniority wave function and the shell-model wave function, $\langle\Psi_{GS}|\Psi_{SM}(0_1^+)\rangle^2$, for six neutrons in the sd shell and in the pf shell, under random interactions. They found that the 0_1^+ ground state is very well described by the generalized seniority scheme when single-particle splittings are very large. This result can be understood as follows. When single-particle

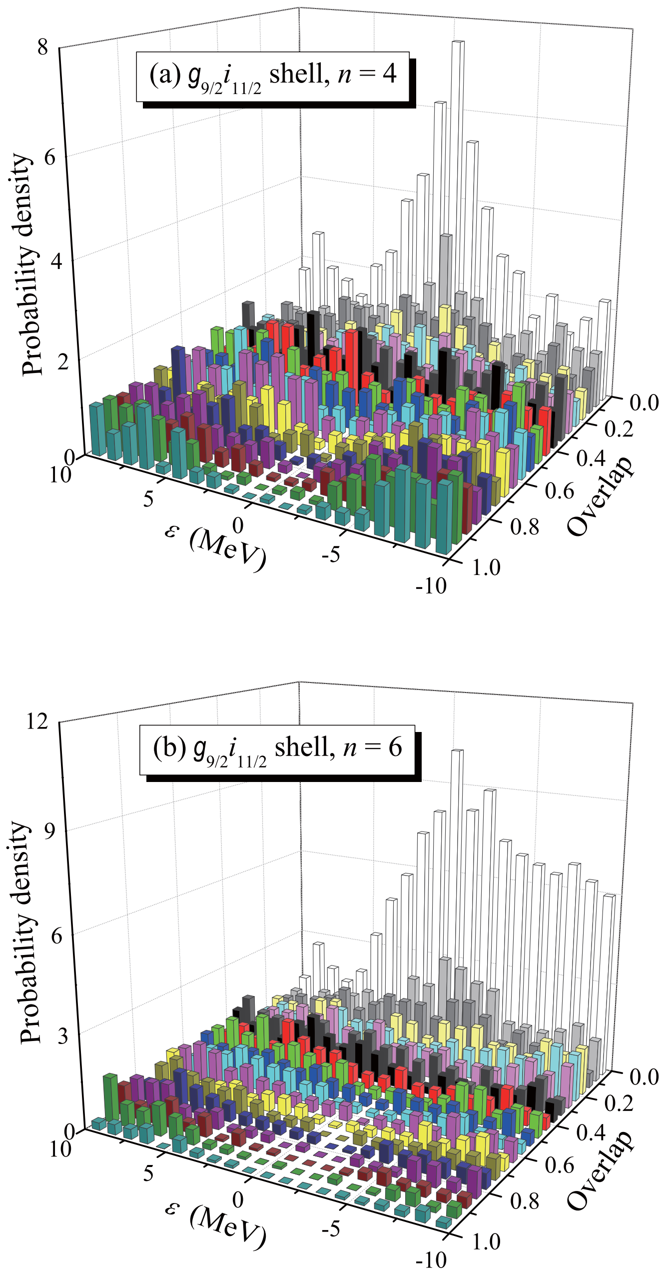

FIG. 3. Distributions of the overlap between the seniority-truncated wave function and the shell-model wave function, $\langle\Psi_{\mathrm{ST}}(0_1^+)|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$, for semimagic nuclei in the schematic $g_{9/2}i_{11/2}$ shell under the TBRE-RSPE. Here $\varepsilon = \varepsilon_{i_{11/2}} - \varepsilon_{g_{9/2}}$. The average value of the probability density is set to 1.

splittings are very large, low-lying states calculated in a multi-j shell can be approximately represented by those in a single-j shell. In the sd and pf shells spins j of the single-particle orbits are not larger than 7/2. Thus, there is only one 0^+ state in the single-j shell, which has seniority quantum number $\nu = 0$.

For six neutrons in the $g_{9/2}i_{11/2}$ shell, the 0_1^+ state is not well described by the seniority truncation even if the single-particle splitting is very large [see Fig. 3(b)]. The 0_1^+ state has a smaller overlap with the S-pair wave function when the

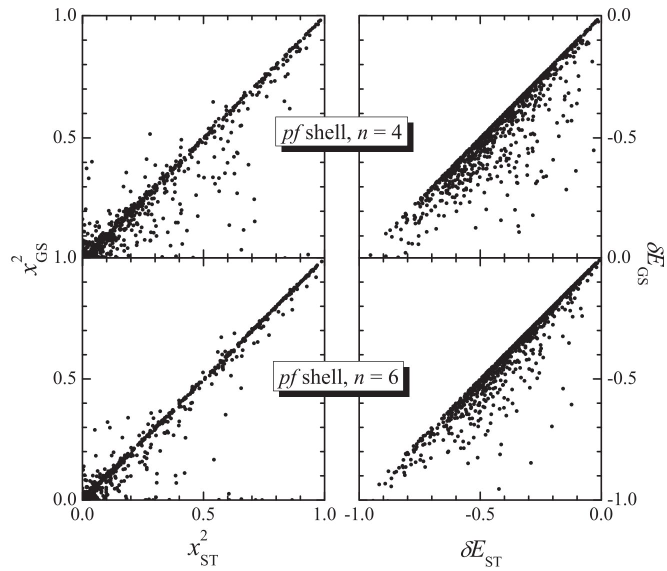

FIG. 4. Comparison of the generalized seniority scheme and the seniority truncation of the shell model for the 0_1^+ state of semimagic nuclei in the pf shell under the TBRE: the left two panels show $\langle\Psi_{\mathrm{GS}}|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$ versus $\langle\Psi_{\mathrm{ST}}(0_1^+)|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$, and the right two panels show δE_{GS} versus δE_{ST}. The generalized seniority scheme and the seniority truncation of the shell model provide very similar results.

spins j of single-particle orbits are larger. This conclusion is also consistent with the following numerical experiment. We calculate the overlap $\langle\Psi_{\mathrm{GS}}|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$ for six neutrons in the $g_{9/2}i_{11/2}$ shell ($d_{\mathrm{SM}} = 55$) and in the pf shell ($d_{\mathrm{SM}} = 137$), respectively, with the TBRE. The probability of large overlap (let us say the overlap >0.9) is equal to 6.7% for the case in the pf shell, and is lower than 0.1% for the case in the $g_{9/2}i_{11/2}$ shell.

We compare the generalized seniority scheme and the seniority truncation of the shell model for the 0_1^+ state. For the case of the pf shell under the TBRE, we calculate overlaps, ${x_{\mathrm{GS}}}^2 = \langle\Psi_{\mathrm{GS}}|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$ and ${x_{\mathrm{ST}}}^2 = \langle\Psi_{\mathrm{ST}}(0_1^+)|\Psi_{\mathrm{SM}}(0_1^+)\rangle^2$, and relative energy errors which are defined by

$$\delta E_{\mathrm{GS}} = \frac{E_{\mathrm{GS}} - E_{\mathrm{SM}}(0_1^+)}{E_{\mathrm{SM}}(0_1^+)}, \tag{12}$$

$$\delta E_{\mathrm{ST}} = \frac{E_{\mathrm{ST}}(0_1^+) - E_{\mathrm{SM}}(0_1^+)}{E_{\mathrm{SM}}(0_1^+)}. \tag{13}$$

The results are presented in Fig. 4. We find that most of the random samples follow the relations ${x_{\mathrm{GS}}}^2 \approx {x_{\mathrm{ST}}}^2$ and $\delta E_{\mathrm{GS}} \approx \delta E_{\mathrm{ST}}$, which means the generalized seniority scheme and the seniority truncation of the shell model provide very similar results.

Now let us consider not only the 0_1^+ state but also excited 0^+ states. For the cases of the $g_{9/2}i_{11/2}$ and pf shells, we calculate overlaps between seniority-truncated wave functions and shell-model wave functions, $\langle\Psi_{\mathrm{ST}}(0_i^+)|\Psi_{\mathrm{SM}}(0_j^+)\rangle^2$, with $1 \leqslant i \leqslant d_{\mathrm{ST}}$ and $1 \leqslant j \leqslant d_{\mathrm{SM}}$, under random interactions. For each combination of i and j, we obtain a distribution of the overlap; here we focus our attention on the probability of $\langle\Psi_{\mathrm{ST}}(0_i^+)|\Psi_{\mathrm{SM}}(0_j^+)\rangle^2 > 0.9$. In Table I one sees the probability

TABLE I. Probability of $\langle\Psi_{\rm ST}(0_i^+)|\Psi_{\rm SM}(0_j^+)\rangle^2 > 0.9$ ($i = 1,2,\ldots,d_{\rm ST}$ and $j = 1,2,\ldots,d_{\rm SM}$) under random interactions. Here $d_{\rm SM}$ is the dimension of the shell-model space for 0^+ states, and $d_{\rm ST}$ is the dimension of the seniority-truncated space. Only a few combinations of i and j are shown in this table, because for the rest of the combinations the probability of $\langle\Psi_{\rm ST}(0_i^+)|\Psi_{\rm SM}(0_j^+)\rangle^2 > 0.9$ is very close to zero.

i	j	Probability (%)
	$n = 4$, $g_{9/2}i_{11/2}$ shell ($d_{\rm SM} = 12$, $d_{\rm ST} = 3$), TBRE-RSPE	
1	1	5.8
1	2	5.0
3	11	5.0
3	12	6.4
	$n = 6$, $g_{9/2}i_{11/2}$ shell ($d_{\rm SM} = 55$, $d_{\rm ST} = 4$), TBRE-RSPE	
1	1	2.5
1	2	3.0
4	54	3.1
4	55	2.9
	$n = 4$, pf shell ($d_{\rm SM} = 28$, $d_{\rm ST} = 9$), TBRE	
1	1	1.5
9	28	2.1
	$n = 6$, pf shell ($d_{\rm SM} = 137$, $d_{\rm ST} = 15$), TBRE	
1	1	2.9
15	137	3.4

is lower than 7%, and this again demonstrates that S pairs are not dominant ingredients in wave functions of random many-body systems.

Interestingly, we find the probability of $\langle\Psi_{\rm ST}(0_i^+)|\Psi_{\rm SM}(0_j^+)\rangle^2 > 0.9$ is relatively higher if both i and j are very close to 1 or to the maximum. For example, for four neutrons in the pf shell, the probability of $\langle\Psi_{\rm ST}(0_1^+)|\Psi_{\rm SM}(0_1^+)\rangle^2 > 0.9$ is equal to 1.5%, and the probability of $\langle\Psi_{\rm ST}(0_9^+)|\Psi_{\rm SM}(0_{15}^+)\rangle^2 > 0.9$ is equal to 2.1%; for other combinations of i and j, the probability is very close to 0. Similarly, for six neutrons in the pf shell, the probability of $\langle\Psi_{\rm ST}(0_1^+)|\Psi_{\rm SM}(0_1^+)\rangle^2 > 0.9$ is equal to 2.9%, and the probability of $\langle\Psi_{\rm ST}(0_{15}^+)|\Psi_{\rm SM}(0_{137}^+)\rangle^2 > 0.9$ is equal to 3.4%; for other combinations of i and j, the probability is very close to 0. This phenomenon is very different from what we find in a single-j shell: The overlap between the S-pair wave function and the shell-model wave function has the same distribution for all 0^+ states. An interesting question is whether this phenomenon attributes to random behavior. Below we study this using random matrices.

The definition of the random Hamiltonian matrix H and its submatrix H' are given in Eqs. (7) and (9) with a parameter χ. For comparing with the case of four neutrons in the pf shell, we take $d_H = d_{\rm SM} = 28$, $d_{H'} = d_{\rm ST} = 9$, and $\chi = 0.27$; for comparing with the case of six neutrons in the pf shell, we take $d_H = 137$, $d_{H'} = 15$, and $\chi = 0.087$. We diagonalize H' and H and sort the eigenvectors by the eigenenergies from the smallest to the largest, respectively; the sorted eigenvectors are denoted by $|v_i'\rangle$ with $i = 1,2,\ldots,d_{H'}$ and $|v_j\rangle$ with $j = 1,2,\ldots,d_H$. We calculate overlaps $|\langle v_i'|v_j\rangle|^2$. In this work we consider 1000 sets of random matrices. For each combination of i and j we obtain a distribution of $|\langle v_i'|v_j\rangle|^2$. The probability of $|\langle v_i'|v_j\rangle|^2 > 0.9$ is presented in Table II. One sees a similar phenomenon: The probability of $|\langle v_i'|v_j\rangle|^2 > 0.9$ is relatively higher if both i and j are equal to 1 or to the maximum; for other combinations of i and j, the probability is very close to 0.

TABLE II. Probability that the overlap between the eigenvectors of a random matrix H and those of its submatrix H', $|\langle v_i'|v_j\rangle|^2$, is larger than 0.9 ($i = 1,2,\ldots,d_{H'}$ and $j = 1,2,\ldots,d_H$). H and H' are defined by Eqs. (7) and (9); d_H is the dimension of H; $d_{H'}$ is the dimension of H'; χ is a parameter which represents the magnitude of the off-diagonal matrix elements. Only a few combinations of i and j are shown in this table, because for the rest of the combinations the probability of $|\langle v_i'|v_j\rangle|^2 > 0.9$ is very close to zero.

i	j	Probability (%)
	$d_H = 28$, $d_{H'} = 9$, $\chi = 0.27$	
1	1	2.1
9	28	1.7
	$d_H = 137$, $d_{H'} = 15$, $\chi = 0.087$	
1	1	3.0
15	137	2.7

We also investigate the GOE matrices (namely, we set $\chi = 1$). The only difference between the GOE matrices and random matrices with $\chi < 1$ is that configuration mixings between the original basis states in the latter are weaker than configuration mixings in the former. Our numerical experiment shows that for all combinations of i and j the probability of $|\langle v_i'|v_j\rangle|^2 > 0.9$ is very close to 0 in the GOE.

C. One-neutron separation energy

In the final part of this section, we arrive at the one-neutron separation energy. It is well known that one-neutron (-proton) separation energies of even-N (-Z) nuclei are systematically larger than those of neighboring odd-N (-Z) nuclei, which is attributed to the monopole pairing correlation. In Ref. [18] Johnson *et al.* found that the odd-even staggering phenomenon survives with random interactions, under the requirement of 0 g.s.

In this work we study the odd-even staggering phenomenon under both the requirement of 0 g.s. and that of non-0 g.s. The odd-even staggering is well represented by the so-called three-point formula [29]

$$\begin{aligned}\Delta_n^{(3)}(Z,N) &\equiv \frac{(-)^{N+1}}{2}[S_n(Z,N+1) - S_n(Z,N)]\\ &= \frac{(-)^{N+1}}{2}[B(Z,N+1) + B(Z,N-1)\\ &\quad - 2B(Z,N)], \end{aligned} \tag{14}$$

where B is the binding energy, and S_n is the one-neutron separation energy, i.e., $S_n(Z,N) = B(Z,N) - B(Z,N-1)$. We investigate systems with neutron number $n = 3$–6 in the single-$j = 11/2$ shell. We calculate ground state energies and $\Delta_n^{(3)}$ under 1000 sets of the TBRE. The probability that both the

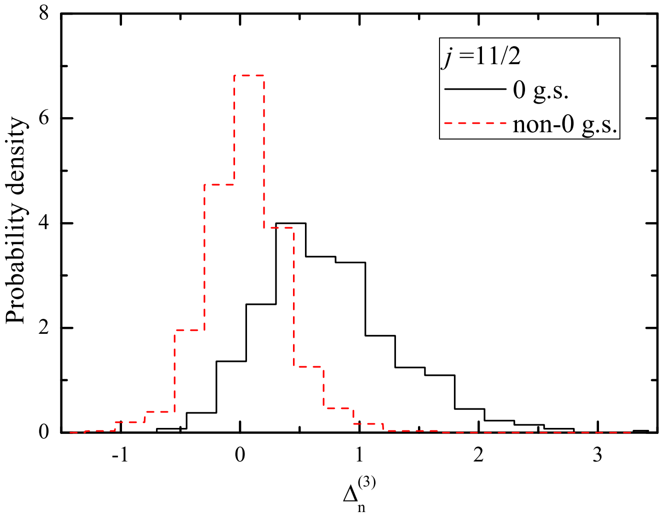

FIG. 5. Distributions of $\Delta_n^{(3)}$ for three to six neutrons in the single-$j = 11/2$ shell under TBRE. The curve in black represents the random ensembles that both the $n = 4$ and 6 systems have 0 g.s., and the curve in red represents those that both the $n = 4$ and 6 systems have non-0 g.s. The average value of the probability density is set to 1.

systems with $n = 4$ and 6 have 0 g.s. is ∼26%; the probability that both the even systems have non-0 g.s. is ∼30%.

In Fig. 5 the distribution of $\Delta_n^{(3)}$ of the random samples that both the even systems have 0 g.s. is presented by the curve in black. $\Delta_n^{(3)}$ has a very high probability to be positive, which indicates that the odd-even staggering of one-nucleon separation energy survives. This result agrees with that of Ref. [18]. The distribution of $\Delta_n^{(3)}$ of the random samples that both the even systems have non-0 g.s. is presented by the curve in red. Surprisingly, one sees a sharp peak at $\Delta_n^{(3)} = 0$; the odd-even staggering phenomenon disappears. Thus, the 0 g.s. dominance and the odd-even staggering of one-nucleon separation energy under random interactions may share the same origin.

IV. SUMMARY

In this paper, we study the monopole pairing correlation in 0^+ states of semimagic nuclei under random interactions. We investigate four and six neutrons in a single-j shell and in a multi-j shell. We present overlaps between the S-pair wave functions and the exact shell-model wave functions for the random ensembles. Our conclusions are as follows.

(1) In a single-j shell, the 0^+-state shell-model wave functions have random overlaps with the S-pair wave function. It is interpreted as the overlaps between random unit vectors and a fixed unit vector in a d_{SM}-dimensional space (see in Fig. 1). Small deviations from the purely random distribution are seen.

(2) The shell-model wave functions obtained by the TBRE are not purely random unit vectors in Hilbert space. There are one or more particular states that are favored by the random interactions, but the S-pair wave function is not so favored. This leads to the small deviations mentioned in (1).

(3) In a multi-j space, the 0_1^+-state shell-model wave function has a small overlap with the S-pair wave function. This is particularly pronounced if the single-particle splitting is small and/or the spins j of single-particle orbits are large. When the single-particle splitting is much larger than the magnitude of two-body interactions, the 0_1^+-state shell-model wave function is reduced to that in a single-j shell. For $j \leqslant 7/2$ there is only one 0^+ state in a single-j shell. It leads to the result of Ref. [19] that the 0_1^+ state of six neutrons in the sd and pf shells has seniority zero with a very large single-particle splitting.

(4) For the 0_1^+ state in a multi-j space, the level energy and the wave function obtained by the generalized seniority scheme are close to those obtained by the seniority truncation of the shell model.

(5) For not only the 0_1^+ state but also excited 0^+ states in a multi-j space, we have investigated overlaps between the seniority-truncated wave functions and the shell-model wave functions, $\langle \Psi_{\mathrm{ST}}(0_i^+)|\Psi_{\mathrm{SM}}(0_j^+)\rangle^2$. The overlap is relatively larger (although still small) if both i and j are very close to 1 or to the maximum. This feature is also found in the random matrix, where weaker off-diagonal elements increase the probability of overlap of extremal vectors of the full matrix and its submatrix.

(6) The odd-even staggering of one-nucleon separation energies survives in random interactions when the even nuclei have 0 g.s., but disappears when the ground states have non-0 spins.

The 0 g.s. and the odd-even staggering of one-nucleon separation energies are regarded as two textbook phenomena of the strong monopole pairing correlation in atomic nuclei. Under random interactions, scientists find the dominance of 0 g.s. [4], and the odd-even staggering survives in 0 g.s. systems [18]. The results in this work show that there is no dominance of monopole pairing in random many-body systems, contrasting with the prior results of Refs. [18,19].

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APPENDIX: THE DISTRIBUTION OF OVERLAPS BETWEEN A FIXED UNIT VECTOR AND RANDOM UNIT VECTORS IN A d-DIMENSIONAL SPACE

Random unit vectors are uniformly distributed over a d-dimensional sphere. The d-dimensional sphere volume is given by

$$V_d(R) = a_d R^d, \quad \text{with} \quad a_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}, \tag{A1}$$

where R is the radius, and $\Gamma(k)$ is the Γ function. If k is a positive integer or a positive half integer, the Γ function is reduced to a simple formula,

$$\Gamma(k) = (k-1)\Gamma(k-1) = \cdots = \begin{cases} (k-1)(k-2)\cdots 1 \times \Gamma(1) & \text{if } k = 2,3,4,\ldots, \\ (k-1)(k-2)\cdots \frac{1}{2} \times \Gamma(\frac{1}{2}) & \text{if } k = \frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots, \end{cases}$$
$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{A2}$$

The d-dimensional Cartesian coordinates are denoted by $x_1,x_2,\ldots,x_d$. Assuming the center of the d-dimensional sphere is located at $x_1 = x_2 = \cdots = x_d = 0$, we cut the sphere into three pieces along planes perpendicular to the x_1 axis: $x_1 \leqslant -\sqrt{y}$, $x_1 \geqslant \sqrt{y}$, and $-\sqrt{y} < x_1 < \sqrt{y}$, where $0 \leqslant y \leqslant R$. Here we focus on the third piece, whose volume is calculated as follows:

$$\begin{aligned} V_d\left(R; x_1^2 < y\right) &= \int_{-\sqrt{y}}^{\sqrt{y}} V_{d-1}\left(\sqrt{R^2 - x_1^2}\right) dx_1 = a_{d-1} \int_{-\sqrt{y}}^{\sqrt{y}} \left(R^2 - x_1^2\right)^{\frac{d-1}{2}} dx_1 \\ &= a_d \times \left[\sum_{r=\frac{1}{2} \text{ or } 1}^{d/2} \frac{\sqrt{y}\,\Gamma(r)}{\sqrt{\pi}\,\Gamma\left(r+\frac{1}{2}\right)} (R^2 - y)^{r-\frac{1}{2}} R^{d-2r} + \frac{1+(-)^d}{\pi} \arcsin\left(\frac{\sqrt{y}}{R}\right) R^d \right]. \end{aligned} \tag{A3}$$

In deriving the above equation, we use the result of Eqs. (A1) and (A2). The surface area is as follows:

$$\begin{aligned} S_d\left(R; x_1^2 < y\right) &= \frac{dV_d\left(R; x_1^2 < y\right)}{dR} \\ &= a_d \times \left\{ \sum_{r=\frac{1}{2} \text{ or } 1}^{d/2} \frac{\sqrt{y}\,\Gamma(r)}{\sqrt{\pi}\,\Gamma\left(r+\frac{1}{2}\right)} \left[(2r-1)(R^2 - y)^{r-\frac{3}{2}} R^{d-2r+1} + (d-2r)(R^2-y)^{r-\frac{1}{2}} R^{d-2r-1}\right] \right. \\ &\quad \left. + \frac{1+(-)^d}{\pi} \left[d \arcsin\left(\frac{\sqrt{y}}{R}\right) R^{d-1} - \sqrt{y}(R^2-y)^{-\frac{1}{2}} R^{d-1} \right] \right\}. \end{aligned} \tag{A4}$$

The overlap between a fixed unit vector, $\vec{v}$, and random unit vectors, $\vec{r}$, in a d-dimensional space is denoted by $o = |\vec{v} \cdot \vec{r}|^2$. The probability of the overlap $o < y$ (where $0 \leqslant y \leqslant 1$) is

$$\begin{aligned} P_d(o < y) &= \left.\frac{S_d(R; {x_1}^2 < y)}{S_d(R)}\right|_{R=1} = \frac{S_d\left(R; x_1^2 < y\right)\big|_{R=1}}{a_d d} \\ &= d^{-1} \times \left\{ \sum_{r=\frac{1}{2} \text{ or } 1}^{d/2} \frac{\sqrt{y}\,\Gamma(r)}{\sqrt{\pi}\,\Gamma\left(r+\frac{1}{2}\right)} \left[(2r-1)(1-y)^{r-\frac{3}{2}} + (d-2r)(1-y)^{r-\frac{1}{2}}\right] \right. \\ &\quad \left. + \frac{1+(-)^d}{\pi} \left[d \arcsin(\sqrt{y}) - \sqrt{y}(1-y)^{-\frac{1}{2}} \right] \right\}. \end{aligned} \tag{A5}$$

[1] D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. **75**, 607 (2003).
[2] Y. M. Zhao and A. Arima, Phys. Rep. **545**, 1 (2014).
[3] I. Talmi, Rev. Mod. Phys. **34**, 704 (1962).
[4] C. W. Johnson, G. F. Bertsch, and D. J. Dean, Phys. Rev. Lett. **80**, 2749 (1998).
[5] R. Bijker and A. Frank, Phys. Rev. Lett. **84**, 420 (2000); Phys. Rev. C **62**, 014303 (2000).

[6] Y. M. Zhao, S. Pittel, R. Bijker, A. Frank, and A. Arima, Phys. Rev. C **66**, 041301(R) (2002).
[7] Y. M. Zhao, J. L. Ping, and A. Arima, Phys. Rev. C **76**, 054318 (2007).
[8] C. W. Johnson and H. A. Nam, Phys. Rev. C **75**, 047305 (2007).
[9] C. A. Mallmann, Phys. Rev. Lett. **2**, 507 (1959).
[10] Y. Lei, Y. M. Zhao, N. Yoshida, and A. Arima, Phys. Rev. C **83**, 044302 (2011).
[11] G. J. Fu, Y. M. Zhao, J. L. Ping, and A. Arima, Phys. Rev. C **88**, 037302 (2013).
[12] Y. Lu, Y. M. Zhao, N. Yoshida, and A. Arima, Phys. Rev. C **90**, 064313 (2014).
[13] G. J. Fu, Y. M. Zhao, and A. Arima, Phys. Rev. C **90**, 064320 (2014).
[14] V. Zelevinsky and A. Volya, Phys. Rep. **391**, 311 (2004).
[15] Y. M. Zhao, A. Arima, and N. Yoshinaga, Phys. Rep. **400**, 1 (2004).
[16] T. Papenbrock and H. A. Weidenmueller, Rev. Mod. Phys. **79**, 997 (2007).
[17] H. A. Weidenmueller and G. E. Mitchell, Rev. Mod. Phys. **81**, 539 (2009).
[18] C. W. Johnson, G. F. Bertsch, D. J. Dean, and I. Talmi, Phys. Rev. C **61**, 014311 (1999).
[19] Y. Lei, Z. Y. Xu, Y. M. Zhao, S. Pittel, and A. Arima, Phys. Rev. C **83**, 024302 (2011).
[20] Y. M. Zhao, A. Arima, and N. Yoshinaga, Phys. Rev. C **66**, 064322 (2002).
[21] E. Caurier, G. Martínez-Pinedo, F. Nowacki, A. Poves, and A. P. Zuker, Rev. Mod. Phys. **77**, 427 (2005).
[22] L. Y. Jia, J. Phys. G: Nucl. Part. Phys. **42**, 115105 (2015).
[23] G. Racah, Phys. Rev. **62**, 438 (1942); **63**, 367 (1943).
[24] E. P. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957); **67**, 325 (1958).
[25] A. Bohr and B. R. Mottelson, *Nuclear Structure*, Vol. 1 (World Scientific, Singapore, New Jersey, London, Hong Kong, 1998).
[26] V. V. Flambaum and O. P. Sushkov, Nucl. Phys. A **412**, 13 (1984).
[27] V. Zelevinsky, Annu. Rev. Nucl. Part. Sci. **46**, 237 (1996).
[28] M. Horoi, B. A. Brown, and V. Zelevinsky, Phys. Rev. Lett. **87**, 062501 (2001).
[29] W. Satula, J. Dobaczewski, and W. Nazarewicz, Phys. Rev. Lett. **81**, 3599 (1998).