Statistical ensembles and fragmentation of finite nuclei

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(Received 16 February 2017; revised manuscript received 30 June 2017; published 15 September 2017)

Statistical models based on different ensembles are very commonly used to describe the nuclear multifragmentation reaction in heavy ion collisions at intermediate energies. Canonical model results are more appropriate for finite nuclei calculations while those obtained from the grand canonical ones are more easily calculable. A transformation relation has been worked out for converting results of finite nuclei from grand canonical to canonical and vice versa. The formula shows that, irrespective of the particle number fluctuation in the grand canonical ensemble, exact canonical results can be recovered for observables varying linearly or quadratically with the number of particles. This result is of great significance since the baryon and charge conservation constraints can make the exact canonical calculations extremely difficult in general. This concept developed in this work can be extended in future for transformation to ensembles where analytical solutions do not exist. The applicability of certain equations (isoscaling, etc.) in the regime of finite nuclei can also be tested using this transformation relation.

DOI: 10.1103/PhysRevC.96.034609

I. INTRODUCTION

The statistical models based on canonical and grand canonical ensembles have been used successfully to describe the nuclear multifragmentation reaction in heavy ion collisions at intermediate energies [1–4]. The basic assumption behind this is attainment of statistical equilibrium at the freeze-out stage. In such models of nuclear disassembly, the population of different channels is solely decided by their statistical weights in available phase space. The microcanonical ensemble is applicable in the case of fixed particle number and fixed energy, but any practical calculation based on this is extremely difficult because of these two constraints [3,5]. The canonical model is applicable when the number of particles is finite (as it would be in experiments) but the energy fluctuates though the average can be constrained to a given value [2]. The grand canonical model on the other hand is applicable for both varying particle number as well as energy. The grand canonical version of the model for nuclear multifragmentation has been known for long time and is the most commonly used [6,7] since it is easy to implement. But it is more important to know how to treat an exact number of particles rather than an ensemble of particle numbers since the given dissociating system (finite nucleus) has a fixed number of particles. The answer is the canonical ensemble which deals with a fixed number of neutrons and protons. It is equally important to have knowledge about dealing with a particular energy rather than a spectrum of energies, and the answer is the microcanonical ensemble. These constraints of mass, charge, and energy conservation put severe restrictions on the calculation of the partition sum, which led to the more frequent use of the grand canonical ensemble for describing the fragmentation of finite nuclei. The main motivation of this work is to formulate a transformation relation for the finite nuclei calculation so that results from one ensemble can be converted to the other easily with the help of such a relation. Since both canonical and grand canonical versions [2] of the thermodynamical model are analytically solvable, they constitute an ideal framework to test the quality of the approximate transformations developed in this work. Very good agreement has been obtained using the transformation relation for different observables, and this is of great significance since it allows one to compute the results of any observable in the canonical enemble if the corresponding result in the grand canonical ensemble is known. The natural continuation of this work will be to exploit such transformations to account for situations where no analytical solutions exists. With this one can avoid in general the calculation of the computer intensive partition sum in the canonical ensemble, yet directly arrive at those results from the grand canonical ensemble ones and thus account for the conservation law (lepton and baryon) relevant for finite nuclei. In particular, applying the constraint of energy (microcanonical ensemble) requires numerically heavy Monte Carlo techniques with all the associated convergence problems. The transformation equations can handle the situation in a much easier way by an approximate implementation of these conservation laws through Lagrange parameters. This idea of using the transformation relation to obtain results in ensembles (for which solutions do not exist) can be extended to other domains of nuclear physics as well as other branches of physics, since most models in statistical mechanics cannot be solved analytically in a particular ensemble.

Analytical transformation relations connecting the grand canonical and the canonical ensembles have already been developed in our group [8], but that was confined to one kind of particle; that is, without distinguishing between the neutron and proton. This work is actually an extension of the earlier work and here isospin is successfully included and thus can be applied to results from fragmentation of finite nuclei. This is extremely important for several reasons. As stated earlier, calculations in the canonical ensemble are in general difficult due to the constraints. The most important application of this transformation equation is of course to obtain these canonical results directly through the grand canonical ensemble and the transformation equation without calculating the canonical partition sum. In the case of our thermodynamical model [2], the existence of the recursion relation enables solving the canonical model analytically, but such recursion relations are in general not available in all cases, and there the appropriate transformation relation will be of significant use. It is well known that results from canonical and grand canonical ensembles agree in the thermodynamic limit; that is, when the number of particles become infinite [9]. But in the case of finite nuclei too, they converge under certain conditions. These conditions of equivalence or convergence can be easily tested if one has a direct transformation relation connecting the two without directly calculating the observables using both the ensembles [10]. Another important domain where this transformation relation can be immensely useful is when dealing with symmetry energy [11] from isoscaling [12,13] and isobaric yield ratio parameters [13,14], and also temperature measurement by the double isotope ratio method [15]. These isoscaling and isobaric yield ratio equations as well as the equation for measuring the temperature have been derived using the yields of the fragments in the framework of the grand canonical ensemble. The actual experimental yield on the other hand is much closer to the canonical values. Hence the applicability of those equations in extracting different parameters in case of finite nuclei is limited and is not valid for all energies, source sizes, as well as asymmetry ratios. Different parameters deduced using these equations need to be corrected for finite nuclei, and the transformation relations can play a significant role in determining these correction factors.

The transformation relations connecting the two ensembles is not completely valid in the temperature or density regime where the liquid-gas phase transition occurs. Fluctuations become high in this domain, which limits the applicability of the formula. But since the Coulomb interaction quenches the liquid-gas phase transition, the formula works remarkably well in most of the thermodynamic region associated with the multifragmentation phenomenon. Another region where this formula is of limited applicability is when the observable value is very small and thehigher order correction terms become more significant. As stated earlier, since analytical solutions exist for both canonical and grand canonical ensembles in our thermodynamical model for fragmentation of nuclei, it is possible to test the efficiency of the developed transformation formula by applying it to observables significant for nuclear fragmentation as well as for the liquid-gas phase transition. This work can be a stepping stone for extension of this concept to other applications in future.

The paper is structured as follows. The next section gives a brief introduction to the models. Derivation of the ensemble transformation relation for realistic nuclei is described in Sec. III. In the Sec. IV results are presented, and finally the last section gives the summary.

II. CANONICAL AND GRANDCANONICAL MODEL

In a canonical model [2], the partitioning is done such that all partitions have the correct A_0, Z_0 (equivalently N_0, Z_0). The canonical partition function is given by

$$Q_{N_0,Z_0} = \sum \prod \frac{\omega_{N,Z}^{n_{N,Z}}}{n_{N,Z}!}, \tag{1}$$

where the sum is over all possible channels of breakup (the number of such channels is enormous) satisfying $N_0 = \sum N \times n_{N,Z}$ and $Z_0 = \sum Z \times n_{N,Z}$; $\omega_{N,Z}$ is the partition function of the composite with N neutrons and Z protons and n_{NZ} is its multiplicity. The partition function Q_{N_0,Z_0} is calculated using a recursion relation [2]. From Eq. (1), the average number of composites with N neutrons and Z protons is given by

$$\langle n_{N,Z}\rangle_c = \omega_{N,Z}\frac{Q_{N_0-N,Z_0-Z}}{Q_{N_0,Z_0}}. \tag{2}$$

In the grand canonical ensemble, if the neutron and proton fugacities are f_n and f_z ($f_n = \beta\mu_n$ and $f_z = \beta\mu_z$; μ_n and μ_z are the neutron and proton chemical potentials and $\beta = 1/T$) corresponding to average number of particle $\langle N_0\rangle$ and $\langle Z_0\rangle$, then they can be expressed in terms of grand canonical partition sum Q_{f_n,f_z} by the relations

$$\langle N_0\rangle_{f_n,f_z} = \frac{\partial ln Q_{f_n,f_z}}{\partial f_n}, \qquad \langle Z_0\rangle_{f_n,f_z} = \frac{\partial ln Q_{f_n,f_z}}{\partial f_z}. \tag{3}$$

The average number of composites with N neutrons and Z protons is given by [7]

$$\langle n_{N,Z}\rangle_{gc} = e^{f_n N + f_z Z}\omega_{N,Z}. \tag{4}$$

In both the models, the partition function of a composite having N neutrons and Z protons is a product of two parts. One is due to the the translational motion and the other is the intrinsic partition function of the composite:

$$\omega_{N,Z} = \frac{V}{h^3}(2\pi mT)^{3/2}A^{3/2} \times z_{N,Z}(\text{int}), \tag{5}$$

where V is the volume available for translational motion. Note that V will be less than V_f, the volume to which the system has expanded at breakup (freeze-out volume). We use $V = V_f - V_0$, where V_0 is the normal volume of a nucleus with Z_0 protons and N_0 neutrons. In this work the freeze-out volume is kept constant at $3V_0$. For nuclei in isolation, the internal partition function is given by $z_{N,Z}(\text{int}) = \exp[-\beta F(N,Z)]$, where $F = E - TS$. For mass number $A \geqslant 5$, we use the liquid-drop formula for calculating the binding energy, and the contribution for excited states is taken from the Fermi-gas model. The properties of the composites used in this work are listed in detail in [2].

III. THEORETICAL FORMALISM OF ENSEMBLE TRANSFORMATION

The partition functions in canonical and grand canonical ensembles are related to each other other through the following equation, and this connection forms the basis of the transformation formula for converting results of observables from one ensemble to another:

$$Q_{f_n,f_z} = \sum_{N_0,Z_0=0}^{\infty} Q_{N_0,Z_0}\exp\{f_n N_0 + f_z Z_0\},$$

$$P_{f_n,f_z}(N_0,Z_0) = \frac{Q_{N_0,Z_0}\exp\{f_n N_0 + f_z Z_0\}}{Q_{f_n,f_z}}, \tag{6}$$

where Q_{N_0,Z_0} is the canonical partition function defined in Eq. (1) and Q_{f_n,f_z} is the grand canonical partition function, which can be deduced by the condition of normalization

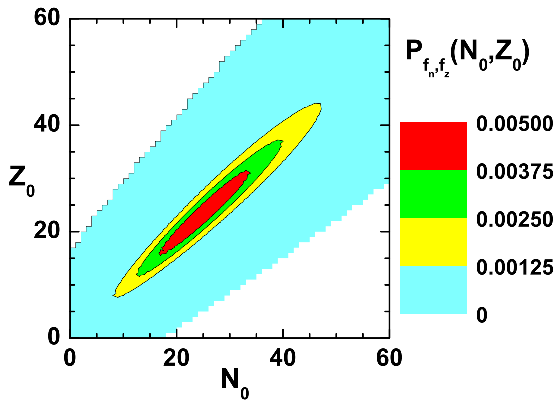

FIG. 1. Grand canonical proton and neutron number distributions for fragmenting source $\langle Z_0\rangle = 28$, $\langle N_0\rangle = 30$ at temperature $T = 8$ MeV.

of probabilities. From the definition of the grand canonical partition function Q_{f_n,f_z} in Eq. (6), it is evident that it is a combination of different canonical sources Q_{N_0,Z_0} of varying particle number with different probabilities P_{f_n,f_z}. This is shown in Fig. 1. Hence in realistic modeling of nuclear systems which are not coupled to any particle bath, the ideal ensemble cannot be grand canonical. Using similar arguments for energy (as for particle number) one can say that since nuclear systems are not connected to any heat bath, the ideal ensemble should be microcanonical, which corresponds to a fixed energy instead of a combination of all possible energies with varying weights.

$P_{f_n,f_z}(N,Z)$ in Eq. (6) is the particle number distribution in the grand canonical ensemble, and using this one can express the average number of protons and neutrons as

$$\begin{aligned}\langle N_0\rangle_{f_n,f_z} &= \sum_{N_0,Z_0=0}^{\infty} N_0 P_{f_n,f_z}(N_0,Z_0),\\ \langle Z_0\rangle_{f_n,f_z} &= \sum_{N_0,Z_0=0}^{\infty} Z_0 P_{f_n,f_z}(N_0,Z_0).\end{aligned} \tag{7}$$

Using $P_{f_n,f_z}(N,Z)$, the definition of particle number fluctuation in the grand canonical ensemble can be introduced as follows:

$$\sigma_n^2 = \frac{\partial^2 \ln Q_{f_n,f_z}}{\partial f_n^2} = \sum_{N_0,Z_0=0}^{\infty} (N_0 - \langle N_0\rangle_{f_n,f_z})^2 P_{f_n,f_z}(N_0,Z_0), \tag{8}$$

$$\sigma_z^2 = \frac{\partial^2 \ln Q_{f_n,f_z}}{\partial f_z^2} = \sum_{N_0,Z_0=0}^{\infty} (Z_0 - \langle Z_0\rangle_{f_n,f_z})^2 P_{f_n,f_z}(N_0,Z_0), \tag{9}$$

$$\begin{aligned}\sigma_{nz} &= \frac{\partial^2 \ln Q_{f_n,f_z}}{\partial f_n \partial f_z}\\ &= \sum_{N_0,Z_0=0}^{\infty} (N_0 - \langle N_0\rangle_{f_n,f_z})(Z_0 - \langle Z_0\rangle_{f_n,f_z}) P_{f_n,f_z}(N_0,Z_0).\end{aligned} \tag{10}$$

The analytical connection between the canonical and grand canonical ensembles [as in Eq. (6)] suggests that one should be able to extract grand canonical results from canonical ones and vice versa, provided the probability distribution is completely described by a limited number of moments. Consider any observable which can be studied in both canonical and grand canonical ensembles at a given temperature T and freeze-out volume V_f, denoted by $R_c(N_0,Z_0)$ and $R_{gc}(f_n,f_z)$ respectively. By using the exact relation connecting canonical and grand canonical ensembles,

$$R_{gc}(f_n,f_z) = \sum_{N_0,Z_0=0}^{\infty} P_{f_n,f_z}(N_0,Z_0) R_c(N_0,Z_0). \tag{11}$$

By doing Taylor series expansion of R_c around $(\langle N_0\rangle,\langle Z_0\rangle)$ and truncating at the second order,

$$\begin{aligned}R_{gc}(f_n,f_z) \approx{}& R_c(\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z})\\ &+ \frac{1}{2}\sigma_n^2 \frac{\partial^2 R_c}{\partial N_0^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}\\ &+ \frac{1}{2}\sigma_z^2 \frac{\partial^2 R_c}{\partial Z_0^2}\bigg|_{\langle N_0\rangle_{f_n,f_n},\langle Z_0\rangle_{f_n,f_z}}\\ &+ \sigma_{nz} \frac{\partial^2 R_c}{\partial N_0 \partial Z_0}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}.\end{aligned} \tag{12}$$

Now, similar to R_c, another observable is defined $T_c^{n^2}(N_0,Z_0) = \frac{\partial^2 R_c(N_0,Z_0)}{\partial N_0^2}$. By making the Taylor expansion of $T_c^{n^2}$ and substituting for the same, one gets

$$\begin{aligned}&\frac{1}{2}\sigma_n^2 \frac{\partial^2 R_c}{\partial N_0^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}\\ &\approx \frac{1}{2}\sigma_n^2 \frac{\partial^2 R_{gc}}{\partial \langle N_0\rangle^2}\bigg|_{f_n,f_z} - \frac{1}{4}(\sigma_n^2)^2 \frac{\partial^4 R_c}{\partial N_0^4}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}\\ &\quad - \frac{1}{4}\sigma_n^2\sigma_z^2 \frac{\partial^4 R_c}{\partial N_0^2 \partial Z_0^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}\\ &\quad - \frac{1}{2}\sigma_n^2\sigma_{nz} \frac{\partial^4 R_c}{\partial N_0^3 \partial Z_0}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}.\end{aligned} \tag{13}$$

Considering the limit of small particle number fluctuation $\sigma_n^2/\langle N_0\rangle_{f_n,f_z}^2 < 1$, $\sigma_z^2/\langle Z_0\rangle_{f_n,f_z}^2 < 1$, and $\sigma_{nz}/\langle N_0\rangle_{f_n,f_z}\langle Z_0\rangle_{f_n,f_z} < 1$, one can neglect the terms containing $(\sigma_n^2)^2$, $\sigma_n^2\sigma_z^2$, and $\sigma_n^2\sigma_{nz}$; therefore

$$\frac{1}{2}\sigma_n^2 \frac{\partial^2 R_c}{\partial N_0^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}} \approx \frac{1}{2}\sigma_n^2 \frac{\partial^2 R_{gc}}{\partial \langle N_0\rangle^2}\bigg|_{f_n,f_z}. \tag{14}$$

Similarly, by considering, $T_c^{z^2}(N_0,Z_0) = \frac{\partial^2 R_c(N_0,Z_0)}{\partial Z_0^2}$ and $T_c^{nz}(N_0,Z_0) = \frac{\partial^2 R_c(N_0,Z_0)}{\partial N_0 \partial Z_0}$ and repeating the same algebra for these variables, one finally gets the transformation equation

$$R_c(\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}) \approx R_{gc}(f_n,f_z) - \frac{1}{2}\sigma_n^2 \frac{\partial^2 R_{gc}}{\partial\langle N_0\rangle^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}} - \frac{1}{2}\sigma_z^2 \frac{\partial^2 R_{gc}}{\partial\langle Z_0\rangle^2}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}} - \sigma_{nz}\frac{\partial^2 R_{gc}}{\partial\langle N_0\rangle\partial\langle Z_0\rangle}\bigg|_{\langle N_0\rangle_{f_n,f_z},\langle Z_0\rangle_{f_n,f_z}}. \quad (15)$$

The right-hand side of the expression can be entirely calculated from the grand canonical ensemble, and then by applying this transformation relation one can calculate the canonical ensemble result from the grand canonical ensemble. The quantity R in Eq. (15) can represent different observables, and in the next section we examine the ones most relevant for nuclear miltifragmentation.

IV. RESULTS AND DISCUSSIONS

In this section the results will be displayed for different fragmentation observables in order to test the predictability of the formula [Eq. (15)] developed for converting results from grand canonical to canonical ones. Since both grand canonical and canonical ensembles have analytical solutions in this particular case, they serve as a suitable platform for testing the accuracy of the formula. The mass distribution of the fragments formed as a result of disassembly of the nucleus is one of the most important observable of nuclear multifragmentation. It is accessible to almost all experiments and also can serve as an indicator of the liquid-gas phase transition [16,17]. In Fig. 2, the results for mass distribution are plotted for two different temperatures, 6 and 8 MeV, from fragmentation of

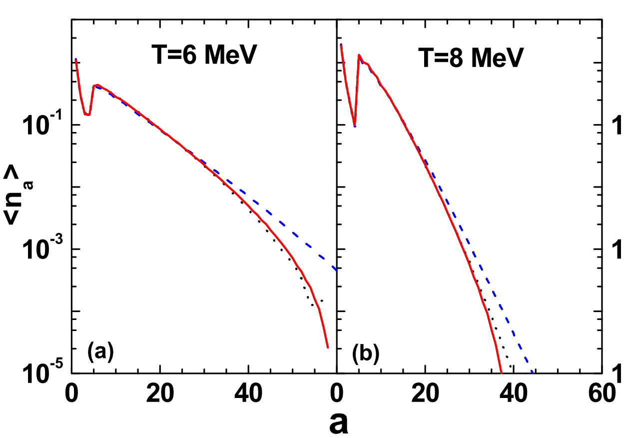

FIG. 2. Mass distribution of fragments produced from disassembly of a particular source of mass number 58 and proton number 28, calculated from canonical (black dotted line) and grand canonical (blue dashed line) models for two different temperatures, $T = 6$ MeV (left panel) and 8 MeV (right panel). The red solid lines represent the canonical result obtained from the grand canonical model by using Eq. (15).

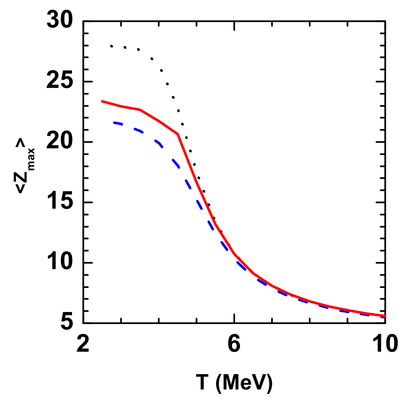

FIG. 3. Variation of average size of the largest cluster ($\langle Z_{\max}\rangle$) with temperature (T) for the fragmenting system of charge 28 and mass 58 calculated from canonical (black dotted line) and grand canonical (blue dashed line) models. The red solid lines represent the canonical result obtained from the grand canonical model by using Eq. (15).

a particular source of mass number 58 and proton number 28 at constant freeze-out volume $3V_0$. For each temperature, three lines are shown: one is from canonical model calculation, another is from grand canonical model calculation, and the third is the canonical result obtained using the expression for grand canonical to canonical transformation. It is seen that results from the transformation formula agree with those of the canonical model to a large extent at both the temperatures. Slight deviation is seen when the cross section (multiplicity) becomes very small, and in such cases the formula is not strictly valid since the higher order correction terms can no longer be neglected. The formula works remarkably well over an appreciable mass range, and one can thus reliably use this for converting results to canonical if one has access to the grand canonical results.

The average size of the largest cluster formed as a result of fragmentation of nuclei serves as a good order parameter for the nuclear liquid-gas phase transition both theoretically as well as experimentally, and hence calculation of this observable is of great significance. Figure 3 displays the variation of the size of the largest cluster with temperature for both canonical and grand canonical model calculations as well as that using the transformation formula.

The results from the formula agrees very well with the canonical model results, once again confirming its validity above the phase transition region. The reason for the disagreement of the formula with the canonical model results around the phase transition temperature is explained in the next figure.

If a system undergoes liquid-gas phase transition, then the particle number fluctuation suddenly increases near the phase transition temperature and then it again decreases. One can have some idea of the phase transition temperature from this variation. In Fig. 4 we show the results of σ_n^2, which actually indicate the fluctuation of the particle number with temperature. The results agree with our understanding that fluctuation reaches a maximum in the phase transition region and is less on either side of the phase transition temperature. The transformation formulas are not applicable when the

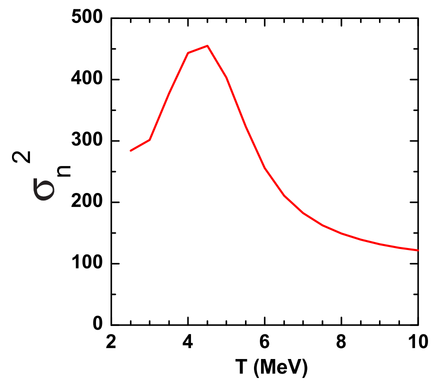

FIG. 4. Variation of σ_n^2 (red solid line) with temperature (T) for the fragmenting system of charge 28 and mass 58.

fluctuation is very high, since higher order terms in the formula can no longer be neglected. Although Coulomb interaction quenches the phase transition to some extent in fragmentation reactions, still one should apply the formula beyond the transition temperature in order to get better agreement. This also explains the disagreement at lower temperature in Fig. 3.

Isotopic distribution is another very important observable studied from nuclear multifragmentation, and is also measured in most experiments. The yields of fragments with different proton and neutron numbers form the basis of calculation and extraction of important parameters such as isoscaling, isobaric yield ratio, temperature, and others. The symmetry energy coefficient is often calculated using the isoscaling and isobaric yield ratio parameters [12–14]. Most of these depend on the ratios of yields of the fragments at varying conditions, and hence accurate calculation of the yields is extremely important. Figure 5 displays the isotopic distribution for $Z = 7$ and 12 at two fixed temperatures and for two different sources with the same proton number but different neutron number. Here also the transformation formula does a remarkably good in reproducing the values of the observables in the canonical ensemble starting from the grand canonical ones.

The last figure (Fig. 6) displays the isoscaling results as a function of neutron number (left panel) and proton number (right panel) for different Z and N values respectively. It is observed both theoretically and experimentally [12,13,18–21] that the ratio of multiplicities $R_{21} = n_2(N,Z)/n_1(N,Z)$ from two reactions 1 and 2 having different isospin asymmetry (2 is more neutron rich than 1) exhibit an exponential relationship as a function of N and Z, i.e.,

$$R_{21} = n_2(N,Z)/n_1(N,Z) = C\exp(\alpha N + \beta Z), \qquad (16)$$

where α and β are isoscaling parameters and C is a normalization constant. The isoscaling parameters are nothing but the slopes of the lines as displayed in the figure; α for the left panel and β for the right one. The results from the grand canonical ensembles are straight lines parallel to each other as expected [20,21]. The results from the canonical ensembles deviate from the straight lines (fixed slope) and their variation is very nicely reproduced by the values obtained from the transformation formula, once again establishing its validity. The derivation of the isoscaling

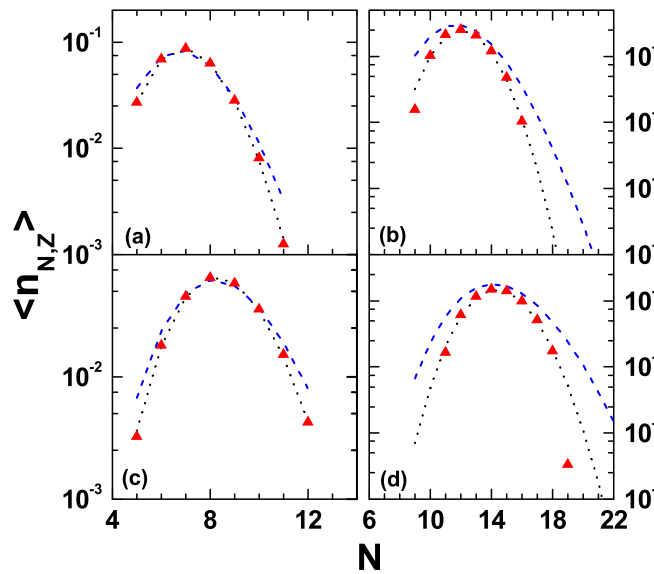

FIG. 5. Multiplicities of $Z = 7$ (left panels) and $Z = 12$ (right panels) isotopes produced from two fragmenting systems of the same atomic number 28, but different mass numbers 58 (upper panels) and 64 (lower panels), calculated from canonical (black dotted line) and grand canonical (blue dashed line) models. The freeze-out temperature for both the system is $T = 8$ MeV. The red triangles represent the canonical result obtained from the grand canonical model by using Eq. (15).

equation is based on the use of the grand canonical yields while experimental results are expected to be closer to those of the canonical ensemble [21]. The correction factors in the isoscaling parameters (slopes) due to this deviation can be incorporated with the help of the transformation relation. A similar correction is also mandatory for other parameters, such as temperature, calculations of which are based on grand

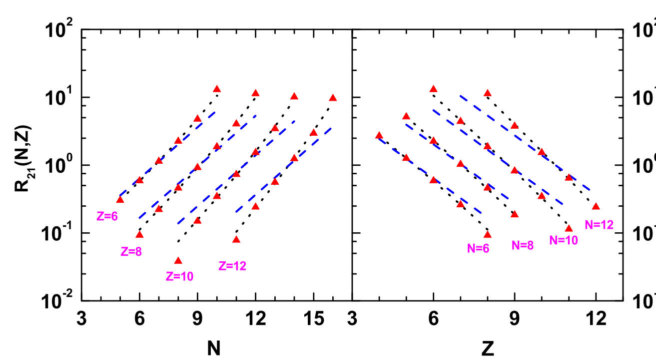

FIG. 6. Ratios (R_{21}) of multiplicities of fragments (N,Z) where mass and charge of the fragmenting system for reaction 1 are 58 and 28 respectively and those for reaction 2 are 64 and 28. The freeze-out temperature for both the fragmenting systems is $T = 8$ MeV. The left panel shows the ratios as a function of neutron number N for fixed Z values, while the right panel displays the ratios as a function of proton number Z for fixed neutron numbers (N) calculated from canonical (black dotted line) and grand canonical (blue dashed line) models. The red triangles represent the canonical result obtained from the grand canonical model by using Eq. (15).

TABLE I. The grand canonical result, as well as the approximation, Eq. (15), of the canonical result from the grand canonical ensemble are compared to the exact canonical calculation for different observables obtained from fragmentation of the source of mass number 58 and proton number 28 at freeze-out volume $V_f = 3V_0$ and two different temperatures $T = 6$ and 8 MeV.

Observables	Temperature (MeV)	Grand canonical model result	Canonical model result	Transformation relation [Eq. (15)]
$\langle n\rangle_{\mathrm{tot}}$	6	5.994	6.155	6.116
	8	9.131	9.184	9.171
$\langle Z_{\mathrm{max}}\rangle$	6	10.293	10.752	10.724
	8	6.653	6.796	6.798
α	6	0.668	0.958	0.942
	8	0.578	0.786	0.801
β	6	−0.780	−1.035	−1.048
	8	−0.670	−0.856	−0.867

canonical yields [15]. Table I displays the results of different observables at two different temperatures using canonical and grand canonical ensembles and also from the transformation formula for conversion from canonical to grand canonical. The observables which have been examined here are total multiplicity $\langle n\rangle_{\mathrm{tot}}$, charge of the largest cluster $\langle Z_{\mathrm{max}}\rangle$, and the isoscaling parameters α and β. The agreement is very good at both the temperatures irrespective of the observable used, which ensures the accurateness of the ensemble transformation relation for finite nuclei formed in the fragmentation reactions at intermediate energies.

V. SUMMARY

A transformation relation for converting results in finite nuclei from grand canonical to canonical ensemble and vice versa has been devised and tested for different observables relevant to nuclear multifragmentation. The results thus obtained from the formula are found to be in excellent agreement with those of the exact canonical results obtained through calculation of the partition sum. This is of great significance since it allows one to have access to the canonical results from the grand canonical ones. In the intermediate energy domain, the nuclear systems formed in the laboratory are coupled to neither the particle reservoir nor the heat bath. Hence the ideal ensemble to work with would be microcanonical, but the strict conservation laws render the calculation extremely difficult. Since the transformation relation has been proved to be very efficient in this work, the extension of this transformation relation for conversion of results from canonical to microcanonical (where no analytical solution exist) would be the motivation for a future work.

ACKNOWLEDGMENT

The authors gratefully acknowledge important discussions with Prof. Francesca Gulminelli, University of Caen, France.

[1] J. P. Bondorf, A. S. Botvina, A. S. Iljinov, I. N. Mishustin, and K. Sneppen, Phys. Rep. **257**, 133 (1995).
[2] C. B. Das, S. Das Gupta, W. G. Lynch, A. Z. Mekjian, and M. B. Tsang, Phys. Rep. **406**, 1 (2005).
[3] D. H. E. Gross, Phys. Rep. **279**, 119 (1997).
[4] A. J. Cole, *Statistical Models for Nuclear Decay: From Evaporation to Vaporization* (Institute of Physics, London, 2000).
[5] S. Pratt and S. Das Gupta, Phys. Rev. C **62**, 044603 (2000).
[6] S. Das Gupta and A. Z. Mekjian, Phys. Rep. **72**, 131 (1981).
[7] G. Chaudhuri and S. Das Gupta, Phys. Rev. C **80**, 044609 (2009).
[8] G. Chaudhuri, F. Gulminelli, and S. Mallik, Phys. Lett. B **724**, 115 (2013).
[9] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw Hill, New York, 1965).
[10] S. Mallik and G. Chaudhuri, Phys. Lett. B **718**, 189 (2012).
[11] B.-A. Li, L.-W. Chen, and C. M. Ko, Phys. Rep. **464**, 113 (2008).
[12] M. B. Tsang, W. A. Friedman, C. K. Gelbke, W. G. Lynch, G. Verde, and H. S. Xu, Phys. Rev. Lett. **86**, 5023 (2001).
[13] S. Mallik and G. Chaudhuri, Phys. Rev. C **87**, 011602(R) (2013).
[14] M. Huang, Z. Chen, S. Kowalski, Y. G. Ma, R. Wada, T. Keutgen, K. Hagel, M. Barbui, A. Bonasera, C. Bottosso, T. Materna, J. B. Natowitz, L. Qin, M. R. D. Rodrigues, P. K. Sahu, and J. Wang, Phys. Rev. C **81**, 044620 (2010).
[15] S. Albergo, S. Costa, E. Costanzo, and A. Rubbino, Nuovo Cimento A **89**, 1 (1985).
[16] S. Das Gupta, A. Z. Mekjian, and M. B. Tsang, in *Advances in Nuclear Physics*, Vol. 26, edited by J. W. Negele and E. Vogt (Plenum, New York, 2001), p. 89.
[17] S. Mallik, S. Das Gupta, and G. Chaudhuri, Phys. Rev. C **91**, 034616 (2015).
[18] M. B. Tsang, T. X. Liu, L. Shi, P. Danielewicz, C. K. Gelbke, X. D. Liu, W. G. Lynch, W. P. Tan, G. Verde, A. Wagner, H. S. Xu, W. A. Friedman, L. Beaulieu, B. Davin, R. T. deSouza, Y. Larochelle, T. Lefort, R. Yanez, V. E. Viola, R. J. Charity, and L. G. Sobotka, Phys. Rev. Lett. **92**, 062701 (2004).
[19] A. S. Botvina, O. V. Lozhkin, and W. Trautmann, Phys. Rev. C **65**, 044610 (2002).
[20] S. Mallik and G. Chaudhuri, Phys. Lett. B **727**, 282 (2013).
[21] G. Chaudhuri, S. Das Gupta, and M. Mocko, Nucl. Phys. A **813**, 293 (2008).