New quasibound states of the compound nucleus in α-particle capture by the nucleus

Sergei P. Maydanyuk,[1,2,*] Peng-Ming Zhang,[1,†] and Li-Ping Zou[1,‡]
[1]*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, 730000, China*
[2]*Institute for Nuclear Research, National Academy of Sciences of Ukraine, Kiev, 03680, Ukraine*
(Received 20 September 2016; revised manuscript received 5 May 2017; published 6 July 2017)

We generalize the theory of nuclear decay and capture of Gamow that is based on tunneling through the barrier and internal oscillations inside the nucleus. In our formalism an additional factor is obtained, which describes distribution of the wave function of the the α particle inside the nuclear region. We discover new most stable states (called quasibound states) of the compound nucleus (CN) formed during the capture of α particle by the nucleus. With a simple example, we explain why these states cannot appear in traditional calculations of the α capture cross sections based on monotonic penetrabilities of a barrier, but they appear in a complete description of the evolution of the CN. Our result is obtained by a complete description of the CN evolution, which has the advantages of (1) a clear picture of the formation of the CN and its disintegration, (2) a detailed quantum description of the CN, (3) tests of the calculated amplitudes based on quantum mechanics (not realized in other approaches), and (4) high accuracy of calculations (not achieved in other approaches). These peculiarities are shown with the capture reaction of $\alpha + {}^{44}$Ca. We predict quasibound energy levels and determine fusion probabilities for this reaction. The difference between our approach and theory of quasistationary states with complex energies applied for the α capture is also discussed. We show (1) that theory does not provide calculations for the cross section of α capture (according to modern models of the α capture), in contrast with our formalism, and (2) these two approaches describe different states of the α capture (for the same α-nucleus potential).

DOI: 10.1103/PhysRevC.96.014602

I. INTRODUCTION

A traditional way in understanding of capture of α particles by nuclei is based on the idea of tunneling through a potential barrier [1] (see improved formalism in Ref. [2]). Evaluations of the α-particle capture rates indicate an important role of such reactions in stars [3–5]. There are intensive investigations [1,6–8] providing the most accurate potential of interactions between the α particles and nuclei basing on existed experimental information of α decay and α capture. Although approaches in determination of penetrabilities of the barrier are highly developed, there is no a generally accepted method to describe a fusion in this reaction. In heavy-ion collisions and scattering with fragments heavier than the α particle, an essential attention has been focused on understanding the mechanisms of the fusion (the current status in the experimental and theoretical investigations on this topic can be seen in the recent review [9], also in Refs. [10–24]). In the case of α capture, the model descriptions of the fusion of the α particle by the target nucleus inside a nuclear region are very simplified. The approach of sharp angular-momentum cutoff was proposed by Glas and Mosel [25,26]. Eberhard *et al.* proposed a relation that gives information about fusion in the α capture of the ^{40}Ca and ^{44}Ca nuclei. They compared calculated cross sections with experimental data at selected energies [27]. Recently, a more precise way to study the α-capture problem is proposed in Ref. [28]. In that paper we investigated a high-precision method (called the method of multiple internal reflections, MIR) to determine fusion in the capture of α particles by nuclei. With this method, we found new parametrization of the α-nucleus potential and fusion probabilities (see Fig. 6, Tables 2 and B.3 in Ref. [28]). Error in description of experimental data is decreased by 41.72 times for $\alpha + {}^{40}$Ca and 34.06 times for $\alpha + {}^{44}$Ca in comparison with previous results (see Fig. 5 and Table 1 in Ref. [28], for details). To date, this is the most accurate and successful approach in describing experimental data for α capture. Based on our fusion probability formula (see Eqs. (21)–(27) and Figs. 8 and 9 in Ref. [28]), we predicted cross sections for the α capture by the nucleus ^{46}Ca for future experimental tests.[1]

In frameworks of existing models of α capture, it is assumed that a complete fusion of the α particle and nucleus takes place after tunneling. Cross sections of the fusion are determined by the penetrabilities. The dependence of the penetrability on energy of the incident α particle is monotonic (without any minima and maxima) at each allowed orbital momentum (see Figs. 2 and 3 in Ref. [28] for the capture $\alpha + {}^{44}$Ca). This explains the absence of peaks in the calculated cross sections of the α capture and has been confirmed by existing experimental information, because we have the cross sections for capture of the α particles by the nuclei ^{40}Ca, ^{44}Ca [27], ^{59}Co [30], ^{208}Pb [31], and ^{209}Be [31]. This is why people assume there is no any

*maidan@kinr.kiev.ua
†zhpm@impcas.ac.cn
‡zoulp@impcas.ac.cn

[1]This nucleus ^{46}Ca is of research interest connected with discovery of new neutron magic numbers at $N = 16$ and $N = 26$ (see review [29] of this topic; here the standard theory gives us only seven experimentally known neutron numbers at 2, 8, 20, 28, 50, 82, 126). In this regard, it could be interesting for experimentalists to investigate the fusion process at the capture of the α particle by this nucleus based on our predictions.

state of possible formation of the compound-nuclear system when nucleons of the α particle and nucleus-target form the most stable bound nuclear system at some fixed energies.

Nowadays, two approximated approaches are very popular to determine the penetrability: (1) the Wentzel–Kramers–Brillouin (WKB) approximation and (2) replacing the original barrier by the inverse oscillator potential, which has solutions for the wave function (see the Hill–Wheeler approach [32] and Wong's formula [33] for calculating cross sections). Such approaches have been widely used in the basis of modern coupled-channel calculations for the study of fusion [9,20]. However, note that the region of applicability of both approaches is only near the barrier maximum. Both approaches completely ignore the shape of the internal nuclear region and the external tail of the potential. The WKB approximation cannot be applied for the main region of under-barrier energies, and the oscillator potential used in the second approach is completely different from the original barrier. In the framework of the WKB approach, a reflection from the barrier is not defined, so we cannot apply the test of quantum mechanics to check the results obtained. As a result, an essential part of under-barrier and above-barrier energy regions for the original barrier looks like black box and cannot be correctly studied by means of the two above approaches, and consequently the proposed result for the penetrability cannot be tested.

By such a motivation, we approximate the original potential barrier by a number of rectangular steps, for which there are exact analytical solutions for wave functions at any energy [28,34]. It turns out that approximation of this approach can be reduced up to zero by increasing the number of steps, all solutions for the wave function are convergent and fully satisfy all known tests of quantum mechanics (with an accuracy up to the first 15 digits). We study quantum processes both for deep under-barrier energies, and energies highly above the barrier maximum (that is a problem for both approaches mentioned above). This approach has been successfully applied for different tasks of quantum physics [35] with the barriers of very specific shape (note that the two approximate approaches mentioned above cannot be even applied for the proper determination of the penetrabilities of these barriers). It allows us to study the influence of the shapes of potential outside the tunneling region on the obtained penetrability. The analysis in Refs. [28,34] shows that such an influence is not small, and in some cases can change the penetrability more then 100%.[2] In the framework of our approach, we find that the penetrability depends on some new parameters. They could actually be more important than the nuclear deformations. However, these parameters are missed in the two approximate approaches above.

[2]In Fig. 1 of Ref. [28] (and in Fig. 1 of Ref. [34]) we shown variations of the penetrability of more than four times in dependence of the localization of the capture point (this is the internal boundary of the potential region with the barrier, for which we calculate the penetrability) at the same incident energy of 2 MeV of the α particle for the capture $\alpha + {}^{44}\mathrm{Ca}$ at $l = 0$.

According to quantum mechanics, consideration of evolution of the system up to the moment of propagation of the α particle through the barrier is not complete (that was analyzed in Refs. [28,34]). Conservation of a flux of wave function requires us to take a further evolution of this system into account. Our research in this paper starts from an analysis of this evolution of the compound system in that stage. It turns out that such a consideration leads to the appearance of oscillations of this system and its disintegration (and allows us to include mechanisms of fusion). In frameworks of unified formalism we join the tunneling processes and oscillations inside the internal nuclear region the first time. Note that the idea introduced by Gamow in 1928 applied to describe α decay [36], where these two processes were considered separately (and there is no approach combining these two processes) for determination of half-lives of decay. Until now, half-lives of the α decay of nuclei are determined on such a basis with inclusion of spectroscopic factors (see, for example, Refs. [37–50]).

Another implication of our method shown in this paper is the appearance of the maximally stable states of the compound system at some energies of the incident α particle (at monotonic penetrabilities of the barrier). The existence of such maximally stable states (we call them as quasibound) reflects the quantum nature of collisions of nuclei; however, it cannot be explained by traditional methods (see, for example, methods based on Ref. [27] for comparison). In this regard, new questions will appear. By how much do oscillations prevail, how fast does the fusion takes place, and in which space region does the fusion dominate? To clarify these questions, in this paper we improve the method proposed in Ref. [28] by including a new formalism of evolution of the compound system (after tunneling) with possible fusion.

II. METHOD

To clearly understand how the quasibound states of the compound-nuclear system appear with monotonic penetrabilities, let us consider the simplest picture of scattering of an α particle off a nucleus in a spherically symmetric scenario. It turns out that the simplest potential applicable for this aim and its corresponding general solution of the wave function (up to normalization) are

$$V(r) = \begin{cases} V_1 & \text{at } r_{\min} < r \leqslant r_1 \quad \text{(region 1)} \\ 0 & \text{at } r_1 \leqslant r \leqslant r_{\max} \quad \text{(region 2)}, \end{cases}$$

$$\psi(r,\theta,\varphi) = \frac{\chi(r)}{r} Y_{lm}(\theta,\varphi),$$

$$\chi(r) = \begin{cases} \alpha_1 e^{ik_1 r} + \beta_1 e^{-ik_1 r} & \text{(region 1)} \\ e^{-ik_2 r} + A_R e^{ik_2 r} & \text{(region 2)}, \end{cases} \tag{1}$$

where $V_1 < 0$, $r_{\min} \geqslant 0$, α_1, β_1, and A_R are unknown amplitudes, $Y_{lm}(\theta,\varphi)$ is the spherical function, and $k_j = \frac{1}{\hbar}\sqrt{2m(E - V_j)}$ are complex wave numbers ($j = 1,2$, $V_2 = 0$). We fix the normalization of the wave function so that the modulus of the amplitude of the incident wave $e^{-ik_2 r}$ equals unity.

According to the MIR method in Ref. [28] (see Refs. [51] also, for details), the scattering of the particle on the potential

is sequentially considered by steps of propagation of a wave packet relative to each boundary. In the first step we consider a wave $e^{-ik_2 r}$ in region 2, which is incident on the boundary at point r_1. This wave is transformed into a new wave $\beta_1^{(1)} e^{-ik_1 r}$ propagated to the center in region 1, and a new wave $\alpha_2^{(1)} e^{ik_2 r}$ reflected from the boundary and propagated into region 2. We have such a wave function for this process:

$$\chi^{(1)}(r) = \begin{cases} \beta_1^{(1)} e^{-ik_1 r} & \text{at } r_{\min} < r \leqslant r_1 \\ e^{-ik_2 r} + \alpha_2^{(1)} e^{ik_2 r} & \text{at } r_1 \leqslant r \leqslant r_{\max}. \end{cases} \tag{2}$$

The transmitted wave is formed in the internal nuclear region. Thus, it describes the formation of a compound nucleus and its further evolution. The reflected wave describes reflection of the particle by Coulomb forces of the nucleus. Therefore, in the framework of this extremely simple scheme, we have separated the scattering of particle off the nucleus into two physically different processes: (1) formation of the compound nucleus and its possible disintegration and (2) the potential scattering without compound-nucleus formation.

In the second step we consider the wave $\beta_1^{(1)} e^{-ik_1 r}$ in region 1 formed in the previous step. This wave propagates to center of the nucleus and is transformed into a new wave $\alpha_1^{(2)} e^{ik_1 r}$. In the third step, we consider the wave $\alpha_1^{(2)} e^{ik_1 r}$ which is incident on the boundary at r_1 and transformed into a new wave in region 2 (describing transmission through the boundary) which propagates outside, and another new wave in region 1 (describing reflection from the boundary) which propagates to center. One can describe these processes by wave functions:

$$\begin{aligned} &\chi^{(2)}(r) = \beta_1^{(1)} e^{-ik_1 r} + \alpha_1^{(2)} e^{ik_1 r} \text{ at } r_{\min} < r \leqslant r_1, \\ &\chi^{(3)}(r) = \begin{cases} \alpha_1^{(2)} e^{ik_1 r} + \beta_1^{(3)} e^{-ik_1 r} & \text{at } r_{\min} < r \leqslant r_1 \\ \alpha_2^{(3)} e^{ik_2 r} & \text{at } r_1 \leqslant r \leqslant r_{\max}. \end{cases} \end{aligned} \tag{3}$$

Here, $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are unknown amplitudes (we add upper index i denoting step number, and bottom index j denoting region number). We find the following recurrent relations from conditions of continuity of the full wave function and its derivative:

$$\begin{aligned} &\alpha_2^{(1)} = R_1^-, \quad \beta_1^{(1)} = T_1^-, \quad \alpha_1^{(2)} = R_0 \beta_1^{(1)}, \\ &\alpha_2^{(3)} = \alpha_1^{(2)} T_1^+, \quad \beta_1^{(3)} = \alpha_1^{(2)} R_1^+, \quad R_1^- = \frac{k - k_1}{k + k_1} e^{-2ikr_1}, \\ &T_1^- = \frac{2k}{k + k_1} e^{-i(k-k_1)r_1}, \quad R_0 = -e^{-2ik_1 r_{\min}}, \\ &T_1^+ = \frac{2k_1}{k + k_1} e^{i(k_1-k)r_1}, \quad R_1^+ = \frac{k_1 - k}{k + k_1} e^{2ik_1 r_1}. \end{aligned} \tag{4}$$

Here, we introduce new amplitudes T_1^- and R_1^+, describing transmission and reflection concerning the boundary (bottom index "1" or "0" indicates the number of the boundary, upper sign "−" or "+" indicates the negative or positive radial direction, respectively, of the incident wave in determination of the amplitude). Each next step in such a consideration for propagation of waves is similar to one of these three steps. With the above analysis we find recurrent relations for new unknown amplitudes and calculate the following summations of all waves:

$$\begin{aligned} &\sum_{i=1} \beta_1^{(i)} = A_{\text{osc}} T_1^-, \quad \sum_{i=1} \alpha_1^{(i)} = R_0 \sum_{i=1} \beta_1^{(i)}, \\ &\sum_{i=2} \alpha_2^{(i)} = A_{\text{osc}} T_1^- R_0 T_1^+, \\ &A_{\text{osc}} = \left(1 + \sum_{i=1} (R_0 R_1^+)^i\right) = \frac{1}{1 - R_0 R_1^+}. \end{aligned} \tag{5}$$

The factor A_{osc} describes oscillation of waves inside internal region 1 (so we call it the *amplitude of oscillations*). At $R_0 = -1$ we obtain

$$\begin{aligned} &\sum_{i=1} \beta_1^{(i)} = -\sum_{i=1} \alpha_1^{(i)} = A_{\text{osc}} \frac{2k e^{-i(k-k_1)r_1}}{k + k_1}, \\ &\sum_{i=2} \alpha_2^{(i)} = -A_{\text{osc}} \frac{4k k_1 e^{2i(k_1-k)r_1}}{(k + k_1)^2}, \\ &A_{\text{osc}} = \frac{k + k_1}{(k + k_1) + (k_1 - k) e^{i2k_1 r_1}}. \end{aligned} \tag{6}$$

Note that full flux of all outgoing waves equals the flux of incident waves (k and k_1 are real): $|\alpha_2^{(1)} + \sum_{i=2} \alpha_2^{(i)}|^2 = 1$.

Let us calculate integral of the square of the modulus of the wave function over the region 1 (at $R_0 = -1$):

$$\begin{aligned} &P_{\text{cn}} = \int_0^{r_1} |\varphi(r)|^2 \, dr = P_{\text{osc}} T_{\text{bar}} P_{\text{loc}}, \\ &P_{\text{osc}} = |A_{\text{osc}}|^2, \quad T_{\text{bar}} \equiv \frac{k_1}{k_2} |T_1^-|^2, \\ &P_{\text{loc}} = 2\frac{k_2}{k_1} \left(r_1 - \frac{\sin(2k_1 r_1)}{2k_1}\right). \end{aligned} \tag{7}$$

This integral is interpreted as the probability of existence of the compound nucleus formed (in space region up to r_1) during the scattering. One can see from Fig. 1 that this probability depends on the energy of the α particle and it has maxima and minima (for the same fixed normalization of the incident wave). This is because P_{cn} is the explicit multiplication of the penetrability T_{bar}, coefficient of oscillations P_{osc}, and one additional new factor P_{loc}. Thus, we have obtained a generalization of Gamow's idea in determination of half-life of nuclear decay, that based on the penetrability of barrier and internal oscillations inside the internal region. But here we obtain also a new factor P_{loc}, which can be interpreted as space distribution of the α particle inside the nuclear region (at one oscillation) described via the wave function. We call it the "coefficient of localization."

Moreover, there is an interference term between the wave reflected in the first step from the boundary r_1 (describing the potential scattering without compound-nucleus formation) and summation of all other waves outgoing to region 2 (which are formed in formation of the compound nucleus and its decay).

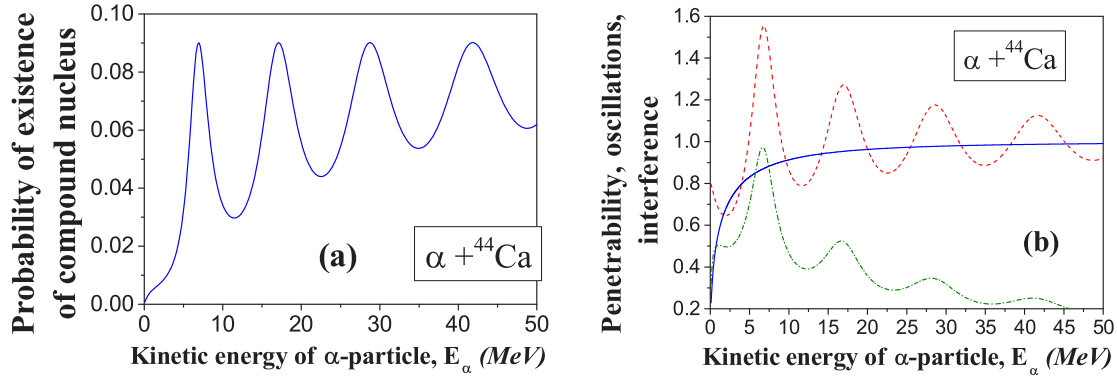

FIG. 1. (a) Probability of existence of the compound nucleus P_{cn} defined by Eq. (7), (b) penetrability of the boundary T_{bar} (blue solid line) defined by Eq. (7), modulus of the amplitude of oscillations A_{osc} (red dashed line) defined in Eq. (6), and interference term P_{interf} (green dash-dotted line) defined in Eq. (8) as a function of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca at $l = 0$ (parameters V_1 and r_1 are used concerning the depth of the well and the coordinate of the maximum of the realistic radial barrier at parametrization [6] for this reaction: we obtain $V_1 = -23.73$ MeV and $r_1 = 8.935$ fm; in all calculations presented the test is satisfied with the coincidence of the first 14 digits). One can see a clear presence of maxima of the probability of existence of the compound nucleus, the amplitude of oscillations, and the interference term (energies of maxima of these functions are very close, but not coincident; these functions are principally different near zero energy), whereas the penetrability is a smooth monotonic function. Without inclusion of the function describing the internal processes, the cross section of fusion defined only on the basis of the penetrabilities of the barrier (for example, as in the approach used in Ref. [27]) cannot give information about these maximally stable states of compound nuclei. Factor P_{cn} has the same maxima, its oscillatory behavior is explained mainly by the amplitude of oscillations A_{osc}.

We have ($R_0 = -1$)

$$P_{\text{interf}} \equiv 2\left|\alpha_2^{(1)*}\sum_{i=2}\alpha_2^{(i)}\right| = \frac{4\sqrt{2}kk_1|k-k_1|}{(k+k_1)^2} \times \frac{1}{\sqrt{k^2[1-\cos(2k_1r_1)]+k_1^2[1+\cos(2k_1r_1)]}}. \tag{8}$$

For instance, in Fig. 1 we present the coefficients for the reactions of $\alpha + {}^{44}$Ca at $l = 0$.

The complete fusion could be described via a requirement: *the flux of each wave propagating inside the nuclear region is suppressed up to zero*. Mathematically, this condition can be realized by

$$R_0 \to 0, \quad \sum_{i=1}\beta_1^{(i)} = T_1^-, \quad \sum_{i=1}\alpha_1^{(i)} = 0, \quad \sum_{i=2}\alpha_2^{(i)} = 0,$$
$$A_{\text{osc}} = 1, \quad P_{\text{cn}} = \frac{k_2r_1}{k_1}T_{\text{bar}}. \tag{9}$$

According to the amplitudes obtained, this fusion is fast and complete. It takes place from the moment after waves tunneling the barrier, and there are no further oscillations of waves inside nuclear region. If we construct the compound nucleus without fast complete fusion, we should partially suppress fluxes inside the internal region, i.e., it needs to make condition (9) less strict. Thus, we introduce a new coefficient p_1 in region 1 and redefine amplitude R_0 as

$$R_0^{(\text{new})} \equiv R_0^{(\text{old})}(1-p_1), \quad 0 \leqslant p_1 \leqslant 1. \tag{10}$$

At $p_1 = 1$ Eq. (10) is transformed to Eq. (9) and fast complete fusion is obtained, while at $p_1 = 0$ we have the compound nucleus without fusion.

Now we would like to generalize the idea presented above for a realistic potential of α capture with barrier of complicated shape, which has successfully been approximated by a sufficiently large number N of rectangular steps in Ref. [28] (see logic of this method, tests, details, and reference therein). In addition to our previous study [28], in this paper we consider that the capture of the α particle by the nucleus takes place in a region with number N_{cap} after its tunneling through the barrier from the right part of potential and next propagations of all waves along the potential are possible, which follows from conservation of full flux based on the full wave function. A general solution of the radial wave function (up to its normalization) for the above-barrier energies has the form of Eqs. (6) and (7) from Ref. [28]. We have fixed a normalization of the wave function so that the modulus of the amplitude of the incident wave e^{-ik_Nr} (in region with number N) equals unity. We shall search a solution of the unknown amplitudes of the wave function by the MIR approach. Each step in such a consideration of packet propagation is similar to one of the first independent $2N-1$ steps. Amplitudes $T_1^{\pm},\ldots,T_{N-1}^{\pm}$ and $R_1^{\pm},\ldots,R_{N-1}^{\pm}$ are expressed as

$$T_j^+ = \frac{2k_j}{k_j+k_{j+1}}e^{i(k_j-k_{j+1})r_j}, \quad T_j^- = \frac{2k_{j+1}}{k_j+k_{j+1}}e^{i(k_j-k_{j+1})r_j},$$
$$R_j^+ = \frac{k_j-k_{j+1}}{k_j+k_{j+1}}e^{2ik_jr_j}, \quad R_j^- = \frac{k_{j+1}-k_j}{k_j+k_{j+1}}e^{-2ik_{j+1}r_j}. \tag{11}$$

Now, let us find a wave propagating to the left in the region with number $j-1$, which is formed after transmission through the boundary r_{j-1} of all possible incident waves, produced as result of all possible reflections and transmissions of any waves in the right part of the potential from this boundary. The amplitude of this wave can be determined from a summation of the amplitudes of all the waves incident on the boundary

at point r_{j-1} multiplied by a factor T^-_{j-1}. Note that any wave incident on boundary at r_{j-1} can be further reflected from this boundary, then can be reflected from the boundary at r_j and is incident on the boundary at r_{j-1} once again. We write

$$\tilde{T}^-_{j-1} = \tilde{T}^-_j T^-_{j-1}\left[1+\sum_{m=1}^{+\infty}(R^-_{j-1}\tilde{R}^+_j)^m\right] = \frac{\tilde{T}^-_j T^-_{j-1}}{1-R^-_{j-1}\tilde{R}^+_j}. \tag{12}$$

Here, we use a summarized reflection amplitude $\tilde{R}^+_j$ (which takes into account transmission of waves through boundary at r_j, then after further reflections and transmissions they return back to the region with number j):

$$\begin{aligned}\tilde{R}^+_{j-1} &= R^+_{j-1} + T^+_{j-1}\tilde{R}^+_j T^-_{j-1}\left[1+\sum_{m=1}^{+\infty}(\tilde{R}^+_j R^-_{j-1})^m\right] \\ &= R^+_{j-1} + \frac{T^+_{j-1}\tilde{R}^+_j T^-_{j-1}}{1-\tilde{R}^+_j R^-_{j-1}}.\end{aligned} \tag{13}$$

We choose $\tilde{R}^+_{N-1} = R^+_{N-1}$ and $\tilde{T}^-_{N-1} = T^-_{N-1}$ and consequently calculate all amplitudes $\tilde{R}^+_{N-2},\ldots,\tilde{R}^+_{N_{\rm cap}}$, and $\tilde{T}^-_{N-2},\ldots,\tilde{T}^-_{N_{\rm cap}}$ by using the recurrent relations above. We define the summarized amplitude A_T of the transition through the barrier via all waves transmitted through the potential region with the barrier from $r_{\rm cap}$ to r_{N-1}: $A_{T,\rm bar} = \tilde{T}^-_{N_{\rm cap}}$.

To sum all waves reflected from the boundary at point r_{j+1} and propagating to the right, we calculate a summarized amplitude of reflection as

$$\begin{aligned}\tilde{R}^-_{j+1} &= R^-_{j+1} + T^-_{j+1}\tilde{R}^-_j T^+_{j+1}\left[1+\sum_{m=1}^{+\infty}(R^+_{j+1}\tilde{R}^-_j)^m\right] \\ &= R^-_{j+1} + \frac{T^-_{j+1}\tilde{R}^-_j T^+_{j+1}}{1-R^+_{j+1}\tilde{R}^-_j}.\end{aligned} \tag{14}$$

On such a basis, we define the amplitude of reflection from the potential region with the barrier from $r_{\rm cap}$ to r_{N-1} as $A_{R,\rm bar} = \tilde{R}^-_{N-1}$, where $\tilde{R}^-_{N_{\rm cap}} = R^-_{N_{\rm cap}}$. We find a summarized amplitude $A_{R,\rm ext}$ of all waves reflected from the external barrier region (from the external turning point $r_{\rm tp,ext}$ to r_{N-1}) and propagated outside as $A_{R,\rm ext} = \tilde{R}^-_{N-1}$ where $\tilde{R}^-_{N_{\rm tp,ext}} = R^-_{N_{\rm tp,ext}}$. Moreover, we find another summarizing amplitude $A_{R,\rm tun}$ of all waves which are reflected just inside the potential region from $r_{\rm cap}$ to the external turning point $r_{\rm tp,ext}$ (i.e., they propagate through the external barrier region without any reflection, tunnel under the barrier, and may propagate to the boundary $r_{\rm cap}$ and further be reflected back from this boundary) as $A_{R,\rm tun} = A_{R,\rm bar} - A_{R,\rm ps}$. We estimate the amplitude of oscillations $A_{\rm osc}$ in the region of capture with number $N_{\rm cap}$ as $A_{\rm osc}(N_{\rm cap}) = (1-\tilde{R}^-_{N_{\rm cap}-1}\tilde{R}^+_{N_{\rm cap}})^{-1}$.

In the framework of the MIR formalism, we define the penetrability $T_{\rm bar}$ and the full reflection $R_{\rm bar}$ concerning the barrier (i.e., the region from $r_{\rm cap}$ to r_{N-1}), the coefficient of reflection $R_{\rm ext}$ of the external part of the barrier (i.e., the region from $r_{\rm tp,ext}$ to r_{N-1}), and the coefficient of reflection $R_{\rm tun}$ of the barrier without the external part (i.e., the region from $r_{\rm cap}$ to $r_{\rm tp,ext}$) as

$$\begin{aligned}T_{\rm bar} &\equiv \frac{k_{\rm cap}}{k_N}|A_{T,\rm bar}|^2, \quad R_{\rm bar} \equiv |A_{R,\rm bar}|^2, \\ R_{\rm ext} &\equiv |A_{R,\rm ext}|^2, \quad R_{\rm tun} \equiv |A_{R,\rm tun}|^2.\end{aligned} \tag{15}$$

We check the property $T_{\rm bar} + R_{\rm bar} = 1$, which indicates whether the MIR method gives proper solutions for the wave function. We calculate the summations of amplitudes $\alpha^{(i)}_j$ and $\beta^{(i)}_j$ for arbitrary region with number j:

$$\begin{aligned}\beta_j &\equiv \sum_{i=1}\beta^{(i)}_j = \tilde{T}^-_j\left[1+\sum_{i=1}(\tilde{R}_{j-1}\tilde{R}^+_j)^i\right] = \frac{\tilde{T}^-_j}{1-\tilde{R}_{j-1}\tilde{R}^+_j}, \\ \alpha_j &\equiv \sum_{i=1}\alpha^{(i)}_j = \tilde{R}_{j-1}\sum_{i=1}\beta^{(i)}_j = \frac{\tilde{R}_{j-1}\tilde{T}^-_j}{1-\tilde{R}_{j-1}\tilde{R}^+_1}.\end{aligned} \tag{16}$$

We define *the probability of existence of a compound nucleus via integral over a space region between two internal turning points $r_{\rm int,1}$ and $r_{\rm int,2}$, where point $r_{\rm int,2}$ is the middle turning point concerning the barrier for under-barrier energy, or the coordinate of the maximum of the barrier for above-barrier energy*:

$$\begin{aligned}P_{\rm cn} \equiv \int_{r_{\rm int,1}}^{r_{\rm int,2}}|R(r)|^2\,r^2dr = \sum_{j=1}^{n_{\rm int}}\Bigg\{&(|\alpha_j|^2+|\beta_j|^2)\Delta r \\ &+\frac{\alpha_j\beta^*_j}{2ik_j}e^{2ik_jr}\Big|_{r_{j-1}}^{r_j} - \frac{\alpha^*_j\beta_j}{2ik_j}e^{-2ik_jr}\Big|_{r_{j-1}}^{r_j}\Bigg\}.\end{aligned} \tag{17}$$

There is a traditional definition for the cross section σ of fusion in α capture that is based on the penetrabilities $T_{\rm bar,1}$ and probabilities of fusion P_l. It is assumed that fusion takes place completely after the α particle tunnels through the barrier; for example, see Ref. [27]:

$$\sigma_{\rm fus}(E) = \sum_{l=0}^{+\infty}\sigma_l(E), \quad \sigma_l(E) = \frac{\pi\hbar^2}{2mE}(2l+1)T_{{\rm bar},l}(E)P_l, \tag{18}$$

where σ_l is the partial cross section at l, and E is energy of the relative motion of the α particle with respect to the nucleus. To study the compound nucleus, we introduce a definition for the partial cross section of fusion via the probability of existence of the compound nucleus (17):

$$\begin{aligned}\sigma_l &= \frac{\pi\hbar^2}{2mE}(2l+1)f_l(E)P_{\rm cn}(E), \\ f(E) &= \frac{k_{\rm cap}}{k_N|r_{\rm cap}-r_{\rm tp,in,1}|}.\end{aligned} \tag{19}$$

To study the compound nucleus living for some period, we apply the idea (10) of coefficients of fusion in the internal nuclear region:

$$\begin{aligned}T^{\pm,(\rm new)}_j &\equiv T^{\pm,(\rm old)}_j(1-p_j), \\ R^{\pm,(\rm new)}_j &\equiv R^{\pm,(\rm old)}_j(1-p_j), \quad 0\leqslant p_j\leqslant 1.\end{aligned} \tag{20}$$

One can see that, with the simple potential (1) for fast complete fusion ($p_j = 1$), Eq. (20) is transformed to Eq. (18). A similar result is obtained for a potential with a barrier of arbitrarily complicated shape.

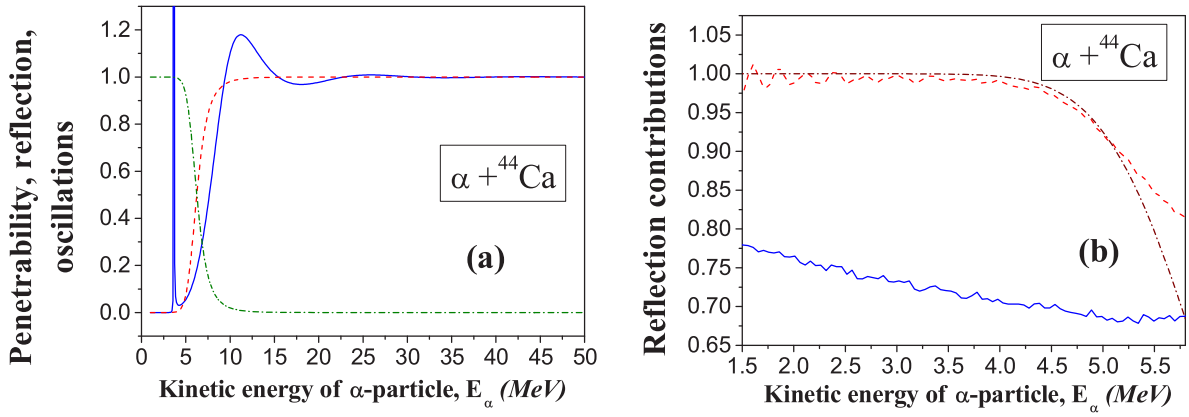

FIG. 2. (a) Penetrability $T_{\rm bar}$ (dashed red line) and reflection $R_{\rm bar}$ (dash-dotted green line) of the barrier defined in Eqs. (15), modulus of the amplitude of oscillations $|A_{\rm osc}|$ (solid blue line) as a function of the energy of the incident α particle for the reaction of $\alpha + {}^{44}$Ca at $l = 0$ (test of $T_{\rm bar} + R_{\rm bar} = 1$ is satisfied up to the first 14 digits for all energy levels considered and for all results in this paper). One can see maxima (the first one is in the under-barrier-energy region) of the amplitude of oscillation. (b) Coefficients of reflection $R_{\rm ext}$ and $R_{\rm tun}$ defined in Eqs. (15) as a function of the energy E_α of the incident α particle for the reaction of $\alpha + {}^{44}$Ca at $l = 0$. Here, the solid blue line is for the coefficient of reflection $R_{\rm ext}$, the dashed red line is for the coefficient of reflection $R_{\rm tun}$, and the dash-dotted brown line is for the full reflection $R_{\rm bar}$. One can see that, up to good accuracy, the full reflection is determined by waves propagating via the stage of the compound nucleus formation and its disintegration (i.e., by $R_{\rm tun}$). However, the potential scattering is not small and is close enough to full reflection inside the full analyzed energy region. This result contradicts a popular point of view that the reflection (in capture and decay nuclear tasks) is formed just by the internal tunneling processes inside the barrier. One can propose formula $P_{\rm ref,ps} \approx 0.75 P_{\rm ref}$ at 1.5 MeV $< E_\alpha <$ 5 MeV for quick estimates.

III. ANALYSIS

For analysis we choose the ^{44}Ca nucleus. As shown in Ref. [28], the penetrability is essentially (for the same fixed energy of the incident α particle) dependent on the internal boundary (at $r_{\rm cap}$) of the potential barrier region in calculations. Therefore, we should impose one additional condition on the determination of the barrier penetrability. In Ref. [28] we proposed a condition of minimal change of the calculated barrier penetrability at arbitrarily small variations of $r_{\rm cap}$. This condition requires that the minimum of the internal potential well should be at this point (we obtain $r_{\rm cap} = 0.44$ fm at $l = 0$ for this nucleus, in this paper we use the parametrization given in Ref. [6] and parameters of calculations are 10 000 intervals at $r_{\rm max} = 70$ fm). Thus, we use this definition of $r_{\rm cap}$ for further calculations.

In Fig. 2(a) we show the penetrability, reflection, and amplitude of oscillation as a function of the energy of the incident α particle at $l = 0$. One can see that the modulus of the amplitude of oscillations has sharp maxima while the penetrability and reflections are monotonic functions. This result is the first indication of the existence of maximally stable states of the compound nucleus that live some periods at definite energies of the incident α particle (where one level is under the barrier energy). Note that the penetrability does not provide any information about such states. For completeness, we add our calculations of the coefficients of reflection $R_{\rm ext}$ and $R_{\rm tun}$ in Fig. 2(b).

In Fig. 3(a) we present our calculations for the probability $P_{\rm cn}$ of the existence of the compound nucleus. One can see the presence of clear maxima in that dependence of the function $P_{\rm cn}$ on energy (here the first maximum of the function $P_{\rm cn}$ is in the under-barrier-energy region). These maxima should be interpreted as an indication of the most stable (i.e., lived for the longest time) states of the formed compound nucleus. Note that there are no maxima as a function of the penetrability $T_{\rm bar}$ on energy in this energy region (see Fig. 2). These dependencies have been used in the basis of the traditional calculations of the cross section of the capture of the α particle by a nucleus (for example, see Eqs. (1) and (2) in Ref. [27]). We call such states of the compound nucleus (and the corresponding energy values) "quasibound states." A unified description of the presence of these states of the compound nucleus, a prediction of the corresponding energy values and monotonic penetrabilities are advances of the method of multiple internal reflections. The clearest understanding of the presence of these quasibound states of the compound nucleus at monotonic penetrability of the barrier can be obtained from the simplest α-capture picture studied above. In particular, here one can see that the modulus $|A_{\rm osc}|$ in Eq. (6) [and corresponding sums of amplitudes in Eqs. (6)] has a clear maximum which can be larger unity. The incident wave in the external region is normalized to unity, and all these tests confirm this formalism and calculations with high accuracy. Note that accurate information about the quasibound states for above-barrier energies cannot be extracted from the interference term (there is only one clear maximum at $E_\alpha = 3.651$ MeV in the interference term at $l = 0$ in Fig. 4 in comparison with five peaks of $P_{\rm cn}$ in Fig. 2(a), the corresponding five energy values are shown in Table I).[3]

[3]We do not analyze possible very small variations of the interference term (like small peak at $E_\alpha = 6.106$ MeV in Fig. 4) in this paper; these are caused by the numeric calculations and are not connected with the quasibound states.

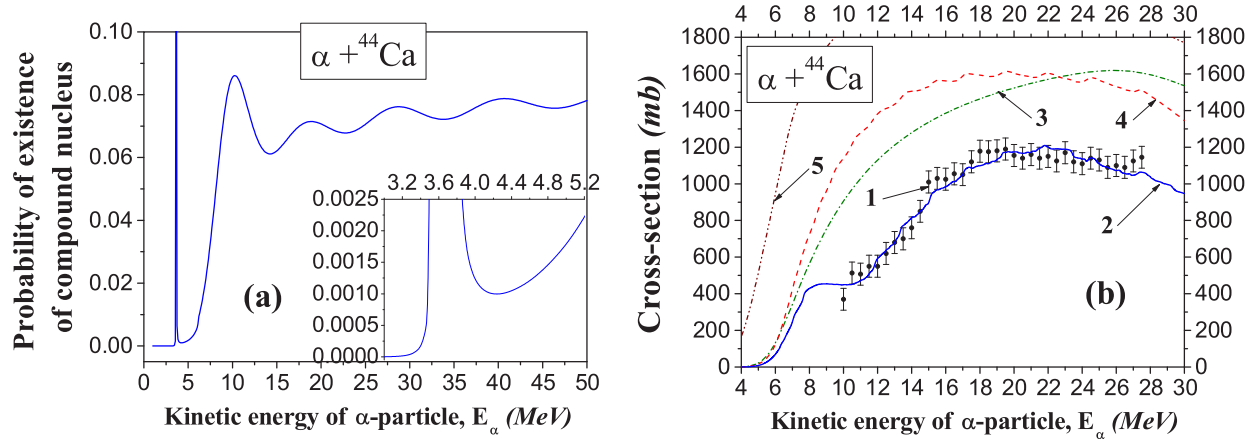

FIG. 3. (a) The probability P_{cn} of existence of the compound nucleus defined in Eqs. (17) as a function of the energy of the incident α particle for the reaction of $\alpha + {}^{44}$Ca at $l = 0$. The function P_{cn} has an oscillator behavior, with the clear presence of five maxima (here the first maximum is in the under-barrier-energy region). For under-barrier energies above this first quasibound energy level there is one minimum (at $E_{min} = 4.24$ MeV, $P_{cn} = 0.000\,986$), indicating that the nucleus becomes more transparent to penetration of the α particle. At energies closer to zero, the probability of formation of the compound nucleus decreases quickly to zero. One can see a stable picture of P_{cn} near this first quasibound energy level. Note that penetrability and reflection coefficients T_{bar} and R_{bar} have no picks near this energy (see Fig. 2). (b) Complete cross section of fusion with the included probabilities of fusion as a function of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca (parameters of calculations: 1000 intervals at $r_{max} = 70$ fm). Here, the data labeled 1 are the experimental data extracted from Ref. [27], the solid blue line 2 is a cross section defined by Eqs. (19) and (20) with included probabilities of fusion their values are given in Table II, the accuracy of agreement with experimental data is $\varepsilon_1 = 0.033\,308$, ε_1 is defined in Ref. [28], the dash-dotted green line 3 is cross section in the old definition (18), where penetrabilities are calculated by using the MIR approach, the dashed red line 4 is a cross section defined in Eqs. (19) and (20) without the coefficients of fusion, the dash-double dotted brown line 5 is a cross section in the old definition (18), where the penetrabilities are calculated by using the WKB approximation.

To estimate the fusion in the studied reaction, let us understand how closely the formula (19) provides the cross section in a comparison with the old definition (18). In this paper we use the same fusion probabilities $p_i^{(int)}$ inside the region from $r_{int,1}$ to r_{cap}, and the same fusion probabilities $p_i^{(ext)}$ inside the region from r_{cap} to $r_{int,2}$. For fast complete fusion we have $p_i^{(int)} = 1$ and $p_i^{(out)} = 0$. Such a calculated cross section and the old result are presented in Fig. 3(b) (see dashed red line 4 line and dash-dotted green line 3 line, respectively). One can see that new calculations are close enough to the previous results (they have similar shapes and have no resonances), so the new definition (19) is applicable for analysis of the fusion in this reaction. On such a basis, we now investigate the possibility of evolution of the compound nucleus and its disintegration (where the fusion probabilities are not equal to unity) and estimate the fusion via variation of the fusion probabilities.

The result of such an analysis is presented in Fig. 3(b) by the blue solid line 2. One can see that inclusion of the fusion probabilities allows us to increase agreement between theory and experimental data essentially. In Table II we present the fusion probabilities. One can see that some fusion probabilities are not equal to unity, which indicates that complete fusion in those channels is not fast. Thus, in such channels further propagation of waves without fusion (or with partial fusion) takes place inside the internal nuclear region after tunneling, i.e., the compound nucleus is formed and it evolves for some time. For such

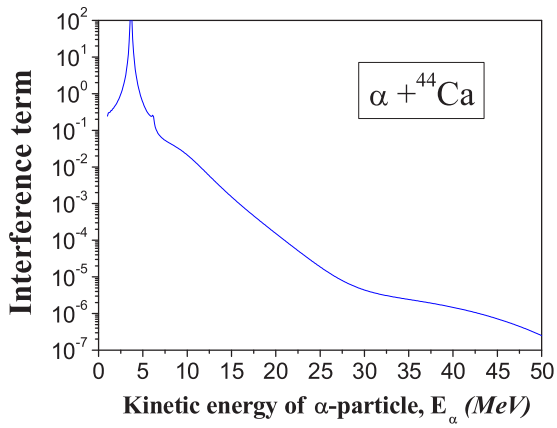

FIG. 4. Interference term P_{interf} as a function of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca at $l = 0$.

TABLE I. Predicted energy values (in MeV) for the quasibound states of the compound nucleus formed in the capture reaction of $\alpha + {}^{44}$Ca up to 50 MeV for the first l (parameters of calculations: 10000 intervals at $r_{max} = 70$ fm).

$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$	$l = 7$	$l = 14$
3.651	6.597	3.356	6.499	9.543	6.008	9.248	12.488	21.621
10.328	14.649	10.034	14.354	18.380	13.569	18.184	23.094	33.993
18.969	23.486	18.969	23.683	28.985	23.683	29.084	34.975	
28.789	34.681	28.887	35.270	41.358	35.074	41.653	49.018	
40.867	47.446	41.064	48.036		48.527			

TABLE II. Fusion probabilities for the cross section presented in Fig. 3(b) (see solid blue line 2 in that figure).

	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$	$l=8$	$l=9$	$l=10$	$l=11$	$l=12$	$l=13$	$l=14$	$l=15$	$l=16$
$p_l^{(\mathrm{int})}$	0	0	0	0	0	1.00	1.00	0	1.00	1.00	0	0	1.00	0.125	1.00	0	0
$p_l^{(\mathrm{ext})}$	1.00	1.00	1.00	0	1.00	1.00	0	0.93	0	0	0.875	0.94	0	0.875	0	0.625	1.00

channels, we find energy values for quasibound states where the compound nucleus is the most stable. In Table I we calculate the quasibound energy values for the reaction $\alpha + {}^{44}\mathrm{Ca}$.

IV. COMPARISON WITH THEORY OF QUASISTATIONARY STATES WITH COMPLEX ENERGIES

Today there is a theory of quasistationary states with complex energies [52] which allows us to determine the energies of quasistationary states in decay tasks. These quasistationary states correspond to poles of the S matrix with complex energies (for example, see Refs. [53,54]). This theory is also applied to analyze resonant states in scattering and could be used to calculate energies for the capture processes. Our comparative analysis shows that this theory gives the quasistationary states; however, these states are not the states given by our approach.[4] We choose the formalism in Ref. [53] for analysis of the theory pointed out above. As we see, the clear difference between the theory of quasistationary states with complex energies and our approach can be obtained from an analysis of two different aspects, such as the determination of the cross sections of the α capture and the determination of the states (and corresponding energy levels) for the α-nucleus interactions.

A. Determination of cross sections of α capture

In the first aspect pointed out above, we analyze the applicability of the compared approaches for the determination of the cross sections of the α capture. According to the modern models of α capture (see Refs. [1,6,28], also Refs. [25,26]), the cross section of the α capture is determined on the basis of the penetrability of the barrier. In particular, an accurate determination of the penetrability is a crucial point for a successful calculation of the cross section. Our approach provides the unified formalism to calculate the penetrabilities and probabilities of the formation of the compound nucleus. However, the theory of quasistationary states with complex energies (for example, see approach [53] for details) does not determine these characteristics. Thus, without further modifications, it cannot be used to calculate the penetrabilities and cross sections of α capture in frameworks of the modern models of α capture.

The penetrability is changed varying the space localization of the capture of the α particle by nucleus (see Ref. [28] for details and demonstrations; also Refs. [51]). This property follows directly from the definition of the penetrability in quantum mechanics. Importantly, this change of the penetrability is not small for the majority of nuclear processes (we estimate it could be up to four times for capture of the α particle by the ${}^{40,44}\mathrm{Ca}$ nuclei; for the inverse nuclear processes, such as α decays, this change is essentially larger). However, the theory of quasistationary states with complex energies ignores this point (so it is simpler and can be faster). Our formalism resolves this problem with very good accuracy (we demonstrated this point in Ref. [28] in detail with many demonstrations).

B. Determination of quantum states of α-nucleus elastic scattering

In the second aspect, we analyze the applicability of the compared approaches to determine the quantum states in α capture and scattering. As we can see, there is a clear difference between the quantum states given by the theory of quasistationary states with complex energy and our approach. This is shown from the analysis of the elastic scattering of $\alpha + {}^{44}\mathrm{Ca}$ at $l=0$ by both methods, if to use the simplest potential of form (1). However, the formalism [53] cannot directly describe such a reaction, because we should modify the asymptotic boundary conditions (5) and (6) of that paper in the form of Eqs. (1) (at $r_1 < r$) and choose real values of energy. After such a modification, unfortunately, analyzing poles or zeros of the S matrix does not give anything, because the modulus of the S matrix equals unity: $|S|=1$ (i.e., there is no zero or pole of the S matrix). One can see this from the exact analytical S matrix, which is easily obtained by our formalism ($R_0=1$):

$$S = A_R = \alpha_2^{(1)} + \sum_{i=2} \alpha_2^{(i)} = R_1^- + A_{\mathrm{osc}} T_1^- R_0 T_1^+. \qquad (21)$$

In contrast, we present calculations of the probability of existence of the compound nucleus in Fig. 1 by our approach, where one can clearly see maxim. We calculate real energies (and the wave functions) corresponding to the maximal probabilities. Note that we use the same boundary conditions imposed on the radial wave function at zero [$\chi(0)=0$] and the same normalization of the incident wave in the asymptotic [$\chi_{\mathrm{inc}}(r) = e^{-ik_2 r}$] in both approaches. Moreover, the difference between the two approaches exists as well, if

[4]The theory of quasistationary states with complex energies [52] provides quasistationary states in order to describe nonstability (i.e., nonstationarity) of formed nuclear system in scattering, also to describe nuclear system in decay, capture, etc. We introduce a new term "quasibound" for the states of the most probable existence of the compound nucleus because our approach is realized at real energies of the incident α particle, as a formal middle case between the bound and nonbound states in standard quantum mechanics (with real energies).

to use the realistic α-nucleus potential instead of the simple potential (1).

More useful information could be obtained if we include resonant and potential terms of the S matrix in the analysis of the elastic scattering above. The MIR approach clearly determines these components. For the potential (1) we have

$$S_{\rm res} = \sum_{i=2} \alpha_2^{(i)} = A_{\rm osc} T_1^- R_0 T_1^+, \quad S_{\rm pot} = \alpha_2^{(1)} = R_1^-. \quad (22)$$

The most probable formation of the compound nucleus can be characterized by maxima of the resonant component. Thus, to compare the S-matrix analysis (the formalism of Ref. [53] is based on it) and the approaches MIR (in the study of the compound nucleus), we have to compare maxima of the resonant term $S_{\rm res}$ and the probability $P_{\rm cn}$. We have

$$|S_{\rm res}|^2 = |A_{\rm osc}|^2 \left| \frac{4kk_1}{(k+k_1)^2} \right|^2,$$

$$P_{\rm cn} = |A_{\rm osc}|^2 \frac{4k}{k+k_1} \left(r_1 - \frac{\sin 2k_1 r_1}{2k_1} \right)$$

$$= |A_{\rm osc}|^2 P_{\rm loc} \frac{2k_1}{k+k_1}, \quad (23)$$

or

$$P_{\rm cn} = |S_{\rm res}|^2 P_{\rm loc} \frac{(k+k_1)^3}{8k_1 k^2}. \quad (24)$$

From these formulas, we obtain different maxima of the existence probabilities of the our compound nucleus and the resonant component of the S matrix. In Fig. 5(a) we present our calculations with such characteristics for reaction $\alpha + {}^{44}$Ca at $l = 0$ for the simple potential (1). One can see that maxima for the resonant term $S_{\rm res}$ and the probability $P_{\rm cn}$ are close, but different (see Table III). Thus, we conclude that the approach based on analysis of the resonant component of the S matrix and our approach determining the probability $P_{\rm cn}$ describe different states of the compound nucleus. In particular, energy shifts between maxima of $S_{\rm res}$ and $P_{\rm cn}$ can be determined by the factor $P_{\rm loc}(k+k_1)^3/(8k_1k^2)$. Note that the formalism of Ref. [53] does not give any explanation of the relationship between the resonant scattering and internal processes inside the well, while we provide an accurate unified formalism describing them. This is the advance of our approach, which has no alternative methods in quantum physics, at present.

TABLE III. Energies for maxima of modulus of amplitude of oscillations $A_{\rm osc}$, probability of existence of compound nucleus $P_{\rm cn}$, interference term $P_{\rm interf}$, and modulus of resonant term of S matrix, $S_{\rm res}$, for reaction $\alpha + {}^{44}$Ca at $l = 0$ for the simple potential (1). One can see that these energies for $S_{\rm res}$ are coincident with energies for maxima for $A_{\rm osc}$ but differ from energies for maxima for $P_{\rm cn}$. It is interesting to note that maximal probabilities at such energies are almost the same: $P_{\rm cn,max} = 0.0901$.

	$A_{\rm osc}$	$P_{\rm cn}$	$P_{\rm interf}$	$S_{\rm res}$
$E_{\rm max,1}$ (MeV)			0.934	
$E_{\rm max,2}$ (MeV)	6.775	6.942	6.608	6.775
$E_{\rm max,3}$ (MeV)	16.955	17.122	16.622	16.955
$E_{\rm max,4}$ (MeV)	28.638	28.805	28.137	28.638
$E_{\rm max,5}$ (MeV)	41.655	41.822	40.987	41.655

In Fig. 5(b) we present our calculations for the normalized modulus of the resonant term of the S matrix, $S_{\rm res}$, in comparison with the probability of existence of the compound

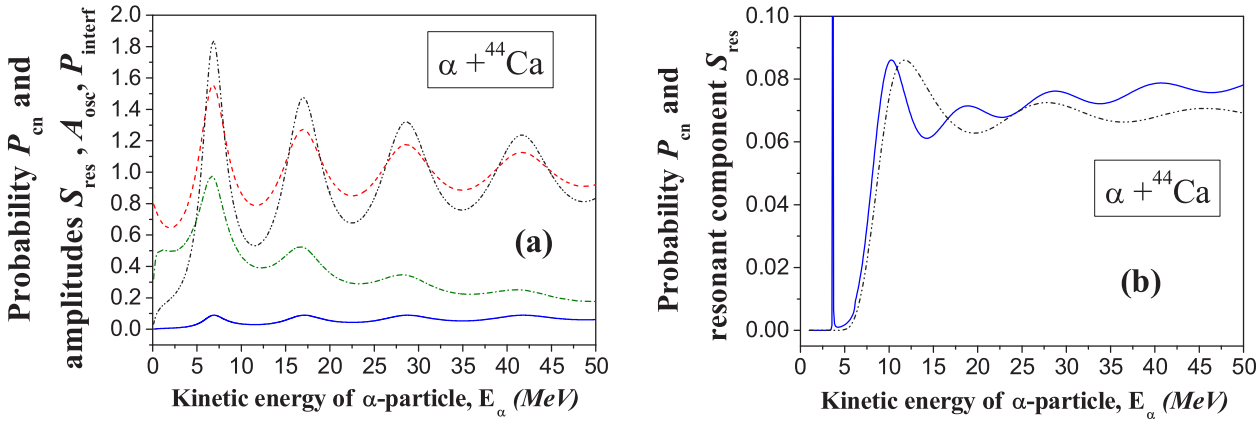

FIG. 5. (a) Modulus of resonant term of the S matrix $S_{\rm res}$ (black dash-double dotted line) in comparison with the probability of existence of the compound nucleus $P_{\rm cn}$ (blue solid line) defined by Eq. (7), modulus of the amplitude of oscillations $A_{\rm osc}$ (red dashed line) defined in Eq. (6), and interference term $P_{\rm interf}$ (green dash-dotted line) defined in Eq. (8) as a function of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca at $l = 0$ for the simple potential (1) (parameters V_1 and r_1 are taken concerning the depth of well and coordinate of the maximum of the realistic radial barrier at parametrization [6] for this reaction: we obtain $V_1 = -23.73$ MeV and $r_1 = 8.935$ fm). Values of energies for the maxima for the analyzed coefficients are presented in Table III. (b) Normalized modulus of the resonant term of the S matrix $S_{\rm res}$ (black dash-double dotted line) in comparison with the probability of existence of the compound nucleus $P_{\rm cn}$ (blue solid line) as a function of the energy of the incident α particle for reaction $\alpha + {}^{44}$Ca at $l = 0$ for the realistic potential. In calculations we define the resonant term as $S_{\rm res} = \alpha_{\rm tp,out} - A_{R,\rm ext}$, $\alpha_{\rm tp,out}$ is the amplitude of the wave function close to the external turning point. One can see that the maxima for the presented lines differ, which indicates different states characterized by the probability $P_{\rm cn}$ and resonant term $S_{\rm res}$.

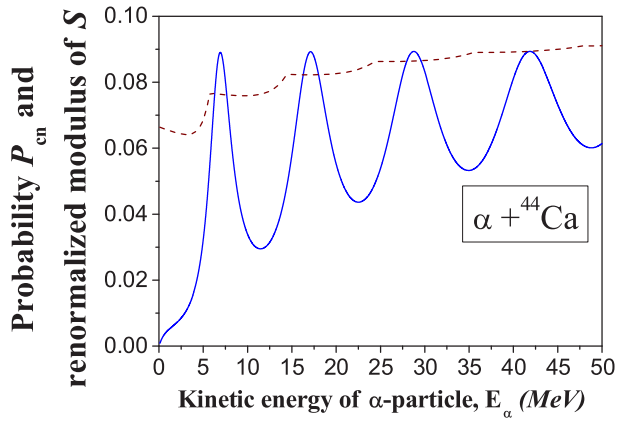

FIG. 6. Renormalized modulus of the S matrix at complex energy (brown dash-double dotted line) compared with the probability of existence of the compound nucleus P_{cn} (blue solid line) presented in Fig. (5), as a function of the real part of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca at $l = 0$ for the simple potential (1) (parameters V_1 and r_1 of the potential are taken as in the calculations of Fig. 1). One can see that the modulus of the S matrix has no zero in the studied energy region, which indicates on absence complete capture in frameworks of the formalism [53]. Minima of the modulus of the S matrix do not correspond to maxima of the probability P_{cn}. Thus, we have the different resonating energies calculated by our approach and generalization of the S-matrix approach describing states of the most probable existence of the compound nucleus.

nucleus P_{cn} for the reaction $\alpha + {}^{44}$Ca at $l = 0$ for the realistic potential. Once again, we see that maxima are different. Note that, in contrast with the WKB approach, in the fully quantum analysis the potential and resonant scattering are already dependent on an additional independent parameter defining the external boundary of the potential region with barrier (we chose it as the external turning point).

C. Determination of quantum states of α-nucleus inelastic scattering and α capture

Inclusion of the complex energies in our analysis allows us to add inelastic processes into our task. Here, the multiple internal reflection method can be easily generalized to the calculations with the complex energy of the incident α particle, because the formalism uses complex values for wave numbers k_i, amplitudes of wave function α_i and β_i, amplitudes $T_i^{\pm}$ and $R_i^{\pm}$, etc. in each potential region. However, direct application of the formalism of Ref. [53] to the studied reaction with the simplified potential (1) does not give any solution in the real energy region up to 50 MeV, which means there is no complete capture according to this approach. In Fig. 6 we present our calculations of the modulus of the S matrix with complex energy. Here, one can see that this function has no zero in the studied energy region.[5] One can generalize the formalism [53] and supposing that minima of the modulus of the S matrix correspond to states of the most probable formation of the compound nucleus. But, as one can see from this figure, these minima do not coincide with maxima of the probability of existence of the compound nucleus P_{cn}, calculated by our approach above. Such a picture clearly shows that this modification of the formalism [53] and our approach describe different states of the compound nucleus thus formed.

In Fig. 7 we present our calculations for the modulus of the S matrix and the corresponding Γ width for the realistic α-nucleus potential at complex energy of the incident α particle (we chose the real energy region up to 7 MeV). One can see that, inside the analyzed energy region the modulus of the S matrix has 10 minima. According to logic and the main positions of the theory of quasistationary states with complex energies, these minima are very close to zero and correspond to the most probable states of the possible α capture. Upon comparing this result with results given in Fig. 3(a) for the calculated probability of existence of the compound nucleus P_{cn} (we have five maxima of that function in the energy region up to 50 MeV, see Table I), we conclude that these states calculated at minima (zero) of the S matrix at complex energy of the incident α particle are essentially different from

[5]Zero of the S matrix corresponds to the boundary condition of zero outgoing wave in the asymptotic limit. This is along the main idea of he formalism of Ref. [53] adapted for the capture process.

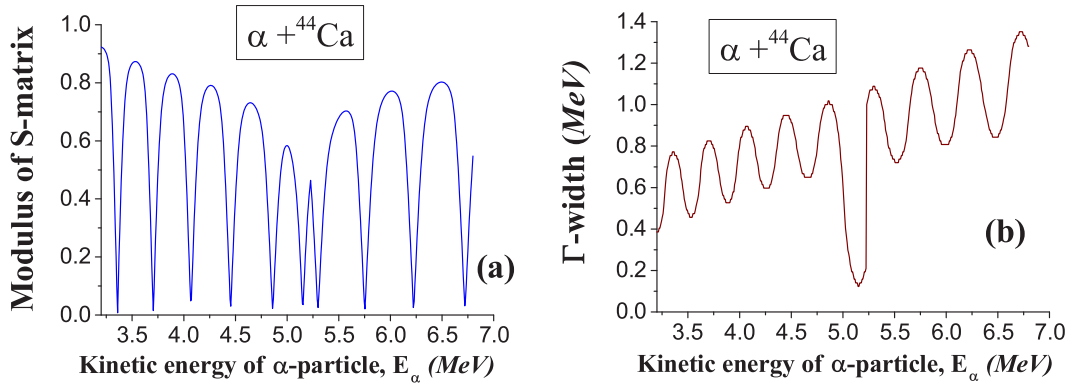

FIG. 7. (a) Modulus of S matrix at complex energy as a function of the real part of the energy of the incident α particle for the reaction $\alpha + {}^{44}$Ca at $l = 0$ for the realistic potential. (b) Γ width as a function of the real part of the energy of the incident α particle corresponding to the calculated S matrix shown in panel (a).

states given by maxima of the probability of existence of the compound nucleus calculated by our approach.

V. CONCLUSIONS

In this paper we study capture of the α particles with nuclei by the improved MIR method. We discover new most stable states (called as quasibound states) of a compound nucleus formed in this reaction. With a simple example (see Fig. 1 and explanations in caption to that figure), we explain the absence of these states in traditional calculations of α-capture cross sections as follows: The barrier penetrabilities depend monotonically on the energy of the α particle (see Fig. 2 in Ref. [28] for details). Based on such monotonic penetrabilities, full cross sections of the α capture have no peaks (see Fig. 3 in Ref. [28]). This is because traditional consideration of the α capture does not take into account the behavior of the wave function inside the internal nuclear region (the corresponding flux is not conserved) which, however, should be defined according to quantum mechanics. In terms of our analysis with improved calculations, these quasibound states appear in a complete description of evolution of the compound nucleus by including the contribution from the internal nuclear region. We describe this evolution in the internal region based on the convergence of flux in the full region. To completely describe the evolution of the compound-nuclear system, we apply and improve our previous method of the multiple internal reflections [28] (see also Ref. [51]). Advances of our method are (1) a clear picture of formation of the compound nucleus and its disintegration, (2) a detailed quantum description of compound-nucleus evolution, (3) tests of quantum mechanics (not realized in other approaches), and (4) a high accuracy of calculations (not achieved in other approaches). Another important issue of our method is that we generalize the idea of Gamow, which has widely been applied for nuclear decay and capture problems (based on tunneling through the barrier and internal oscillations inside nucleus) to our formalism. With this, we find an additional factor, which describes space distribution of the α particle inside nuclear region. However, we find no discussions of this in previous papers on the topic.

We demonstrate peculiarities of our method through the capture reaction of $\alpha + {}^{44}$Ca. In this reaction we predict quasibound energy levels (see Table I), and show that inclusion of evolution of the compound nucleus with possible fusion allows us to essentially increase the agreement between theory and experimental data. This can be seen from Fig. 3, which shows that the calculated cross section for capture of the α particles by ^{44}Ca agrees very well with experimental data. The updated data of fusion probabilities for this reaction are shown in Table II), in comparison with our previous results in Ref. [28] (see Tables 2, B.3, and F.9 in Ref. [28]).

We compared our formalism with the theory of quasistationary states with complex energies in determination of resonant states in scattering and quasistationary states in α capture (see, for example, Refs. [52–54]). We found the following:

(1) The theory of quasistationary states with complex energies could not provide calculations for the cross section of α capture according to the modern models of α capture (see Refs. [1,6,28]). Our approach provides a unified formalism to calculate the penetrabilities with the best accuracy (in order to estimate the cross sections with the modern formalism of α capture), the probabilities of existence of compound nucleus, and to estimate probabilities of fusion (see discussions in Sec. IV A).

(2) The quasistationary states (and corresponding energies) calculated for α capture by the theory of quasistationary states at complex energies differ from quasibound states given by our approach (see Fig. 7 and explanation in Sec. IV C). We add a comparative analysis between these two approaches for the $\alpha + {}^{44}$Ca scattering in Sec. IV B.

The difference could be explained by the simplest example in which a free particle moves inside the constant potential; these two approaches describe two principally different processes (for the same real energy, but nonzero Γ width). However, calculations for more complicated realistic potentials are based on such a point. The calculation time for the two approaches is similar. For example, the calculation time for the modulus of the S matrix (and the Γ width at complex energies) is around 8 s (with $N = 100$, $r_{\max} = 70$ fm, 500 intervals in real energy region, realistic barrier), and the calculation time for the probability of existence of the compound nucleus $P_{\rm cn}$ in our method is around 10 s (with the same parameters).

ACKNOWLEDGMENTS

S.P.M. thanks the Institute of Modern Physics of Chinese Academy of Sciences for its warm hospitality and support. L.-P.Z. thanks the China Scholarship Council for its support. This work has been supported by the Major State Basic Research Development Program in China (No. 2015CB856903), the National Natural Science Foundation of China (Grants No. 11575254 and No. 11175215), and the Chinese Academy of Sciences fellowships for researchers from developing countries (No. 2014FFJA0003).

[1] V. Yu. Denisov and A. A. Khudenko, α decay half-lives, α capture, and α nucleus potential, At. Data Nucl. Data Tables **95**, 815 (2009).

[2] V. Yu. Denisov, O. I. Davidovskaya, and I. Yu Sedykh, Improved parametrization of the unified model for α decay and α capture, Phys. Rev. C **92**, 014602 (2015).

[3] P. Mohr, α-nucleus potentials for the neutron-deficient p nuclei, Phys. Rev. C **61**, 045802 (2000).

[4] P. Demetriou, C. Grama, and S. Goriely, Improved global α-optical model potentials at low energies, Nucl. Phys. A **707**, 253 (2002).

[5] T. Rauscher, F.-K. Thielemann, J. Görres, and M. Wiescher, Capture of α particles by isospin-symmetric nuclei, Nucl. Phys. A **675**, 695 (2000).

[6] V. Yu. Denisov and H. Ikezoe, α-nucleus potential for α-decay and sub-barrier fusion, Phys. Rev. C **72**, 064613 (2005).

[7] V. Yu. Denisov and A. A. Khudenko, α-decay half-lives: Empirical relations, Phys. Rev. C **79**, 054614 (2009).

[8] V. Yu. Denisov and A. A. Khudenko, α-decays to ground and excited states of heavy deformed nuclei, Phys. Rev. C **80**, 034603 (2009).

[9] B. B. Back, H. Esbensen, C. L. Jiang, and K. E. Rehm, Recent developments in heavy-ion fusion reactions, Rev. Mod. Phys. **86**, 317 (2014).

[10] J. R. Birkelund, L. E. Tubbs, J. R. Huizenga, J. N. De, and D. Sperber, Heavy-ion fusion: Comparison of experimental data with classical trajectory models, Phys. Rep. **56**, 107 (1979).

[11] L. C. Vaz, J. M. Alexander, and G. R. Satchler, Fusion barriers, empirical and theoretical: Evidence for dynamic deformation in subbarrier fusion, Phys. Rep. **69**, 373 (1981).

[12] J. R. Birkelund and J. R. Huizenga, Fusion reactions between heavy nuclei, Annu. Rev. Nucl. Part. Sci. **33**, 265 (1983).

[13] M. Beckerman, Subbarrier fusion of atomic nuclei, Phys. Rep. **129**, 145 (1985).

[14] S. G. Steadman and M. J. Rhoades-Brown, Sub-barrier fusion reactions, Annu. Rev. Nucl. Part. Sci. **36**, 649 (1986).

[15] M. Beckerman, Sub-barrier fusion of two nuclei, Rep. Prog. Phys. **51**, 1047 (1988).

[16] N. Rowley, G. R. Satchler, and P. H. Stelson, On the distribution of barriers interpretation of heavy-ion fusion, Phys. Lett. B **254**, 25 (1991).

[17] R. Vandenbosch, Angular momentum distributions in subbarrier fusion reactions, Annu. Rev. Nucl. Part. Sci. **42**, 447 (1992).

[18] W. Reisdorf, Heavy-ion reactions close to the Coulomb barrier, J. Phys. G **20**, 1297 (1994).

[19] M. Dasgupta, D. J. Hinde, N. Rowley, and A. M. Stefanini, Measuring barriers to fusion, Annu. Rev. Nucl. Part. Sci. **48**, 401 (1998).

[20] A. B. Balantekin and N. Takigawa, Quantum tunneling in nuclear fusion, Rev. Mod. Phys. **70**, 77 (1998).

[21] J. F. Liang and C. Signorini, Fusion induced by radioactive ion beams, Int. J. Mod. Phys. E **14**, 1121 (2005).

[22] L. F. Canto, P. R. S. Gomes, R. Donangelo, and M. S. Hussein, Fusion and breakup of weakly bound nuclei, Phys. Rep. **424**, 1 (2006).

[23] N. Keeley, R. Raabe, N. Alamanos, and J. L. Sida, Fusion and direct reactions of halo nuclei at energies around the Coulomb barrier, Prog. Part. Nucl. Phys. **59**, 579 (2007).

[24] K. Hagino and N. Takigawa, Subbarrier fusion reactions and many-particle quantum tunneling, Prog. Theor. Phys. **128**, 1061 (2012).

[25] D. Glas and U. Mosel, On the critical distance in fusion reactions, Nucl. Phys. A **237**, 429 (1975).

[26] D. Glas and U. Mosel, Limitation on complete fusion during heavy-ion collisions, Phys. Rev. C **10**, 2620 (1974).

[27] K. A. Eberhard, Ch. Appel, R. Bangert, L. Cleemann, J. Eberth, and V. Zobel, Fusion Cross Sections for $\alpha + {}^{40,44}$Ca and the Problem of Anomalous Large-Angle Scattering, Phys. Rev. Lett. **43**(2), 107 (1979).

[28] S. P. Maydanyuk, P.-M. Zhang, and S. V. Belchikov, Quantum design using a multiple internal reflections method in a study of fusion processes in the capture of alpha-particles by nuclei, Nucl. Phys. A **940**, 89 (2015).

[29] Yu. E. Penionzhkevich and S. M. Lukyanov, Nuclei near closed shells $N = 20$ and $N = 28$, Phys. Part. Nucl. **37**, 240 (2006).

[30] J. M. D'Auria, M. J. Fluss, L. Kowalski, and J. M. Miller, Reaction cross section for low-energy alpha particles on ^{59}Co, Phys. Rev. **168**, 1224 (1968).

[31] A. R. Barnett and J. S. Lilley, Interaction of alpha particles in the lead region near the Coulomb barrier, Phys. Rev. C **9**, 2010 (1974).

[32] D. L. Hill and J. A. Wheeler, Nuclear constitution and the interpretation of fission phenomena, Phys. Rev. **89**, 1102 (1953).

[33] C. Y. Wong, Interaction Barrier in Charged-Particle Nuclear Reactions, Phys. Rev. Lett. **31**, 766 (1973).

[34] S. P. Maydanyuk and S. V. Belchikov, Method of Multiple Internal Reflections as a tool for the study of probabilities of fusion in the tasks of the capture of alpha-particles by nuclei, J. Phys. Stud. **18**, 1001 (2014).

[35] S. P. Maydanyuk, Resonant structure of the early-universe space-time, Eur. Phys. J. Plus **126**, 76 (2011).

[36] G. Gamow, Zur quantentheorie des atomkernes, Eur. Phys. J. A **51**, 204 (1928).

[37] B. Buck, A. C. Merchant, and S. M. Perez, Half-lives of favored alpha decays from nuclear ground states, At. Data Nucl. Data Tables **54**, 53 (1993).

[38] Y. A. Akovali, Review of alpha-decay data from doubly-even nuclei, Nucl. Data Sheets **84**, 1 (1998).

[39] S. B. Duarte, O. A. P. Tavares, F. Guzman, A. Dimarco *et al.*, Half-lives of favored alpha decays from nuclear ground states, At. Data Nucl. Data Tables **80**, 235 (2002).

[40] G. Audi, O. Bersillon, J. Blachot, and A. H. Wapstra, The NUBASE evaluation of nuclear and decay properties, Nucl. Phys. A **729**, 3 (2003).

[41] N. Dasgupta-Schubert and M. A. Reyes, The generalized liquid drop model alpha-decay formula: Predictability analysis and superheavy element alpha half-lives, At. Data Nucl. Data Tables **93**, 907 (2007).

[42] I. Silisteanu and A. I. Budaca, Structure and alpha-decay properties of the heaviest nuclei, At. Data Nucl. Data Tables **98**, 1096 (2012).

[43] R. G. Lovas, R. J. Liotta, A. Insolia, K. Varga, and D. S. Delion, Microscopic theory of cluster radioactivity, Phys. Rep. **294**, 265 (1998).

[44] A. Sobiczewski and K. Pomorski, Description of structure and properties of superheavy nuclei, Prog. Part. Nucl. Phys. **58**, 292 (2007).

[45] T. L. Stewart, M. W. Kermode, D. J. Beachey, N. Rowley, I. S. Grant, and A. T. Kruppa, Alpha-particle decay through a deformed barrier, Nucl. Phys. A **611**, 332 (1996).

[46] C. Xu, Z. Ren, New deformed model of α-decay half-lives with a microscopic potential, Phys. Rev. C **73**, 041301(R) (2006).

[47] R. Id Betan and W. Nazarewicz, α decay in the complex-energy shell model, Phys. Rev. C **86**, 034338 (2012).

[48] D. S. Delion and R. J. Liotta, Shell-model representation to describe alpha emission, Phys. Rev. C **87**, 041302(R) (2013).

[49] I. Silisteanu and C. I. Anghel, Alpha-decay and spontaneous fission half-lives of super-heavy nuclei around the double magic nucleus ^{270}Hs, Rom. J. Phys. **60**, 444 (2015).

[50] http://www.nndc.bnl.gov; http://www-nds.iaea.org

[51] S. P. Maydanyuk and S. V. Belchikov, Calculations of penetrability of barriers in the proton decay problem: The fully quantum approach and initial condition of decay, J. Phys. Stud. **15**(4), 4002 (2011); Problem of nuclear decay by proton emission in fully quantum consideration: Calculations of penetrability and

role of boundary condition, J. Mod. Phys. **2**, 572 (2011); V. S. Olkhovsky and S. P. Maydanyuk, Method of multiple internal reflections in description of tunneling evolution through barriers, Ukr. Phys. J. **45**, 1262 (2000); S. P. Maydanyuk, V. S. Olkhovsky, and A. K. Zaichenko, The method of multiple internal reflections in description of tunneling evolution of nonrelativistic particles and photons, J. Phys. Stud. **6**, 1 (2002); F. Cardone, S. P. Maidanyuk, R. Mignani, and V. S. Olkhovsky, Multiple internal reflections during particle and photon tunneling, Found. Phys. Lett. **19**, 441 (2006).

[52] A. I. Baz, Ya. B. Zeldovich, and A. M. Perelomov, *Rasseyanie, Reaktsii i Raspadi v Nerelyativistskoi Kvantovoi Mehanike* (Nauka, Moscow, 1971).

[53] T. Vertse, K. F. Pal, and Z. Balogh, Gamow, a program for calculating the resonant state solution of the radial Schrödinger equation in an arbitrary optical potential, Comput. Phys. Commun. **27**, 309 (1982).

[54] J. Bang and J. Zimányi, Stripping reactions leading to decaying states, Nucl. Phys. A **139**, 534 (1969); B. Gyarmati, T. Vertse, J. Zymànyi, and M. Zimànyi, Single-particle resonant-state wave functions in a simple description of isobaric analog resonances, Phys. Rev. C **1**, 1 (1970); B. Gyarmati and T. Vertse, Easy method for calculating the resonance parameters of the isobaric analogue resonance, Nucl. Phys. A **182**, 315 (1972); W. R. Coker, Gamow-state analysis of ^{54}Fe(d, n) to proton resonances in ^{55}Co, Phys. Rev. C **9**, 784 (1974); Distorted-wave Born-approximation analysis of ^{36,38}Ar(d, p) to neutron resonances in ^{37,39}Ar, **7**, 2426 (1973); G. Garcia-Calderon and R. Peierls, Resonant states and their uses, Nucl. Phys. A **265**, 443 (1976); H. R. Fiebig and A. Weiguny, Gamov states in the generator coordinate method, Z. Phys. A **279**, 275 (1976); M. Pierronne and L. Marquez, On the complex energy eigenvalue theory of alpha decay, *ibid.* **286**, 19 (1978); H. S. Bradlow, W. D. M. Rae, P. S. Fisher, N. S. Godwin, G. Proudfoot, and D. Sinclair, A DWBA analysis of heavy ion α-transfer reactions on ^{16}O, Nucl. Phys. A **314**, 171 (1979); T. Berggren, Resonance effects in the binding of a nucleon to an unbound core, *ibid.* **361**, 102 (1981); B. G. Giraud, M. V. Mihailović, R. G. Lovas, and M. A. Nagarajan, Variational approach to resonant states of many-particle systems, Ann. Phys. (N. Y.) **140**, 29 (1982); P. Salamon, Á. Baran, and T. Vertse, Distributions of the S-matrix poles in Woods–Saxon and cut-off Woods–Saxon potentials, Nucl. Phys. A **952**, 1 (2016).