Precision analysis of electron energy spectrum and angular distribution of neutron β^- decay with polarized neutron and electron

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We give a precision analysis of the correlation coefficients of the electron energy spectrum and angular distribution of the β^- decay and radiative β^- decay of the neutron with polarized neutron and electron to order 10^{-3}. The calculation of correlation coefficients is carried out within the standard model, with contributions of order 10^{-3} caused by the weak magnetism and proton recoil taken to next-to-leading order in the large proton mass expansion, and with radiative corrections of order $\alpha/\pi \sim 10^{-3}$ calculated to leading order in the large proton mass expansion. The obtained results can be used for the planning of experiments on the search for contributions of order 10^{-4} of interactions beyond the standard model.

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I. INTRODUCTION

It is well known that the neutron β^- decay is a good laboratory for tests of the standard model (SM) [1–8]. As has been pointed out in [7,8] by example of the neutron β^- decay with a polarized neutron and unpolarized proton and electron, the weak magnetism and proton recoil corrections of order E_e/m_p, where E_e and m_p are the electron energy and proton mass, and radiative corrections of order α/π, where α is the fine-structure constant [9], define a complete set of corrections to the correlation coefficients of order 10^{-3}. These corrections provide a robust background for a search of contributions of order 10^{-4}, induced by interactions beyond the standard model (SM) [7,8]. This paper addresses a precision analysis of the neutron β^- decay with polarized neutron and electron. The aim of this paper is to give a robust background to order 10^{-3} for the experimental search for contributions of order 10^{-4}, caused by interactions beyond the SM. According to [7,8], for the realization of this aim we have to calculate in the SM the correlation coefficients of the electron energy spectrum and angular distribution of the neutron β^- decay with polarized neutron and electron by taking into account the contributions of the weak magnetism and proton recoil to next-to-leading order in the large proton mass expansion and radiative corrections of order α/π to leading order in the large proton mass expansion. These corrections make possible a meaningful search for contributions of order 10^{-4}, caused by interactions beyond the SM for most of the correlation coefficients presented here. Of course, these corrections should be meaningful if the theoretical uncertainties of the correlation coefficients, calculated to leading order in the large proton mass expansion and without radiative corrections, should be small compared to 10^{-3}. The correlation coefficients $G(E_e)$, $N(E_e)$, $Q_e(E_e)$, and $R(E_e)$ [see Eq. (3)], calculated to leading order in the large proton mass expansion and without radiative corrections, are equal to [see Eq. (8)]

$$G_0(E_e) = -1, \quad N_0(E_e) = -\frac{m_e}{E_e} A_0, \quad Q_{e0}(E_e) = -A_0,$$
$$R_0(E_e) = -\alpha \frac{m_e}{k_e} A_0, \quad A_0 = -2\frac{\lambda(1+\lambda)}{1+3\lambda^2}, \qquad (1)$$

where m_e and $k_e = \sqrt{E_e^2 - m_e^2}$ are the electron mass and its momentum, $\alpha = 1/137.036$ is the fine-structure constant [9], and λ is the axial coupling constant [1,2]. The dependence of the correlation coefficient $R_0(E_e)$ on the fine-structure constant is caused by the Coulomb distortion of the electron wave function in the Coulomb field of the proton [10,11]. The most precise, published values for the electron asymmetry from a single experiment provide values for $A_0^{(\exp)} = -0.11933(34)$ [1] and $A_0^{(\exp)} = -0.11996(58)$ [12] (see also [13]), giving $\lambda = -1.2750(9)$ and $\lambda = -1.2767(16)$, respectively, and averaging over the electron energy spectrum (see Eq. (D-59) of Ref. [7]) we obtain the following numerical values for the correlation coefficients in Eq. (1):

$$\begin{aligned}
G_0 &= \langle G_0(E_e)\rangle = -1,\\
N_0 &= \langle N_0(E_e)\rangle = 0.07825(22),\\
Q_{e0} &= \langle Q_{e0}(E_e)\rangle = 0.11933(34),\\
R_0 &= \langle R_0(E_e)\rangle = 0.000891(3) \quad [1],\\
G_0 &= \langle G_0(E_e)\rangle = -1,\\
N_0 &= \langle N_0(E_e)\rangle = 0.07866(37),\\
Q_{e0} &= \langle Q_{e0}(E_e)\rangle = 0.11996(58),\\
R_0 &= \langle R_0(E_e)\rangle = 0.000896(5) \quad [12].
\end{aligned} \qquad (2)$$

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One may see that the relative theoretical uncertainties of the correlation coefficients in Eq. (2) are of order (3–5) $\times 10^{-3}$. They are practically defined by the experimental uncertainties of the axial coupling constant. An improvement of the theoretical uncertainties of the correlation coefficients in Eq. (2) may only result through an improvement of the experimental uncertainty of the axial coupling constant λ. Hence, currently, because of the theoretical uncertainties of order (3–5) $\times 10^{-3}$, a precision analysis of the correlation coefficients $G(E_e)$, $N(E_e)$, $Q_e(E_e)$, and $R(E_e)$ to order 10^{-3} seems to be meaningful only for the correlation coefficient $G(E_e)$. Nevertheless, we calculate below the corrections, caused by the weak magnetism and proton recoil of order $1/m_p$ and radiative corrections of order α/π, to all correlation coefficients. In order to push meaningful tests of the SM to the 10^{-4} level for the correlation coefficients $N(E_e)$, $Q_e(E_e)$, and $R(E_e)$, an improvement of experimental uncertainties for the axial coupling constant is required in addition to the theoretical predictions we present in this work. We see this as an important challenge for the experimental characterization of the charged weak interaction.

The first experimental analysis of the neutron β^- decay $n \to p + e^- + \bar{\nu}_e$ with polarized neutron and electron has been undertaken by Kozela *et al.* [14,15], where the correlation coefficients $R(E_e)$ and $N(E_e)$ of the electron energy spectrum and angular distribution of the neutron β^- decay, with polarized neutron and with the electron polarized transverse to its three-momentum, were measured. The correlation coefficient $R(E_e)$ describes the correlation between the neutron polarization and the polarization and three-momentum of the electron $\vec{\xi}_n \cdot (\vec{k}_e \times \vec{\xi}_e)$, where $\vec{\xi}_n$ and $\vec{\xi}_e$ are unit polarization vectors of the neutron and electron, respectively, and $\vec{k}_e$ is the electron three-momentum. The correlation coefficient R characterizes quantitatively a T-odd and a P-odd effect, caused by violation of time reversal invariance and invariance under parity transformation. The correlation coefficient $N(E_e)$ is a quantitative characteristic of the correlation between the neutron and electron polarization $\vec{\xi}_n \cdot \vec{\xi}_e$. One may see that the experimental values $R_{\rm exp} = 0.004 \pm 0.012 \pm 0.005$ and $N_{\rm exp} = 0.067 \pm 0.011 \pm 0.004$, measured by Kozela *et al.* [15], do not contradict the predictions of the SM, given by Eq. (2), within the experimental uncertainties. The primary goal of the nTRV Collaboration through these measurements of $R(E)$ and $N(E)$ was to make a useful probe for contributions from interactions beyond the SM (see Eqs. (7)–(11) of Ref. [15]). Such tests require a search for deviations from the expected SM values for these correlations. If the precision level is to be greatly improved, the theoretical predictions for the SM values must also be refined to include contributions from the weak magnetism, proton recoil, and radiative corrections. In this way, one can produce corrections to order 10^{-3} in the correlation coefficients, and open a path to a search for traces of interactions beyond the SM to 10^{-4}.

The paper is organized as follows. In Sec. II we give the electron energy spectrum and angular distribution of the β^- decay of the neutron with polarized neutron and electron. The correlation coefficients $G(E_e)$, $N(E_e)$, $Q_e(E_e)$, and $R(E_e)$ are calculated in the SM with the contributions of the weak magnetism and proton recoil to next-to-leading order in the large proton mass expansion and with radiative corrections of order α/π, calculated to leading order in the large proton mass expansion [7]. In Sec. III we discuss some corrections of order 10^{-5} to the correlation coefficients beyond those, which are calculated in Sec. II and which have been analyzed and discussed by Wilkinson [16]. In Sec. IV we discuss the obtained results and the experimental observables. In the Appendix we calculate the photon-electron energy spectrum, the electron energy spectrum, and angular distributions of the radiative β^- decay of the neutron with polarized neutron and electron.

II. ELECTRON ENERGY SPECTRUM AND ANGULAR DISTRIBUTION

The electron energy spectrum and angular distribution of the neutron β^- decay with polarized neutron and electron takes the form [17] (see also [18] and [7])

$$\frac{d^3\lambda_n(E_e)}{dE_e d\Omega_e} = (1+3\lambda^2)\,\frac{G_F^2|V_{\rm ud}|^2}{8\pi^4}\,(E_0-E_e)^2\sqrt{E_e^2-m_e^2}\,E_e\,F(E_e,Z=1)\,\zeta(E_e)\Big\{1 + A_W(E_e)\,\frac{\vec{\xi}_n\cdot\vec{k}_e}{E_e} + G(E_e)\,\frac{\vec{\xi}_e\cdot\vec{k}_e}{E_e} + N(E_e)\,\vec{\xi}_n\cdot\vec{\xi}_e + Q_e(E_e)\,\frac{(\vec{\xi}_n\cdot\vec{k}_e)(\vec{k}_e\cdot\vec{\xi}_e)}{E_e(E_e+m_e)} + R(E_e)\,\frac{\vec{\xi}_n\cdot(\vec{k}_e\times\vec{\xi}_e)}{E_e}\Big\}, \tag{3}$$

where $G_F = 1.1664 \times 10^{-11}$ MeV^{-2} is the Fermi weak constant, $V_{\rm ud} = 0.97428(15)$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element [9], λ is a real axial coupling constant, $E_0 = (m_n^2 - m_p^2 + m_e^2)/2m_n = 1.2927$ MeV is the endpoint energy of the electron spectrum, calculated for $m_n = 939.5654$ MeV, and $m_p = 938.2720$ MeV, and $m_e = 0.5110$ MeV [9], $\vec{\xi}_n$ and $\vec{\xi}_e$ are unit polarization vectors of the neutron and electron, respectively, $F(E_e, Z=1)$ is the relativistic Fermi function [7,11]

$$F(E_e,Z=1) = \Big(1+\frac{1}{2}\gamma\Big)\,\frac{4(2r_pm_e\beta)^{2\gamma}}{\Gamma^2(3+2\gamma)}\,\frac{e^{\pi\alpha/\beta}}{(1-\beta^2)^{\gamma}} \times \Big|\Gamma\Big(1+\gamma+i\,\frac{\alpha}{\beta}\Big)\Big|^2, \tag{4}$$

where $\beta = k_e/E_e = \sqrt{E_e^2 - m_e^2}/E_e$ is the electron velocity, $\gamma = \sqrt{1-\alpha^2} - 1$, and r_p is the electric radius of the proton. In the numerical calculations we will use $r_p = 0.875$ fm [19]. The Fermi function (4) describes the contribution of the electron-proton final-state Coulomb interaction. Since it is defined by the exact solution of the Dirac equation for the electron, moving in the Coulomb field of the proton [11], it cannot introduce additional uncertainties to the approximate contributions, caused by the weak magnetism, proton recoil, and radiative corrections. We would like to emphasize that the Fermi function (4) gives a contribution to the phase space factor of the neutron of about 3.32%. The use of the approximate expression $F(E_e, Z=1) = 1 + \alpha\,\pi/\beta$ [16] diminishes the contribution of the Coulomb electron-proton final-state interaction at the level of 8.75×10^{-4}. This justifies the use of the exact Fermi function (4) for the precision analysis of the neutron β^- decay.

The correlation coefficients of the electron energy spectrum and angular distribution, Eq. (3), we calculate with the Hamiltonian of V-A weak interactions and the weak magnetism [7]

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}}\,V_{\rm ud}\left\{[\bar\psi_p(x)\gamma_\mu(1+\lambda\gamma^5)\psi_n(x)] + \frac{\kappa}{2M}\partial^\nu[\bar\psi_p(x)\sigma_{\mu\nu}\psi_n(x)]\right\}\big[\bar\psi_e(x)\gamma^\mu(1-\gamma^5)\psi_{\nu_e}(x)\big], \tag{5}$$

where $\psi_p(x)$, $\psi_n(x)$, $\psi_e(x)$, and $\psi_{\nu_e}(x)$ are the field operators of the proton, neutron, electron, and antineutrino, respectively, γ^μ, $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, and γ^5 are the Dirac matrices, $\kappa = \kappa_p - \kappa_n = 3.7058$ is the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the proton $\kappa_p = 1.7928$ and the neutron $\kappa_n = -1.9130$ and measured in nuclear magneton units [9], and $2M = m_n + m_p$ is the average nucleon mass.

The coefficients $\zeta(E_e)$ and $A_W(E_e)$ have been calculated in [3,7]. They read

$$\begin{aligned}\zeta(E_e) &= \Big(1 + \frac{\alpha}{\pi}\,g_n(E_e)\Big) + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[-2(\lambda^2 - (\kappa+1)\,\lambda)\,E_0 + (10\lambda^2 - 4(\kappa+1)\,\lambda + 2)\,E_e \\ &\quad - 2(\lambda^2 - (\kappa+1)\,\lambda)\,\frac{m_e^2}{E_e}\Big],\\ \zeta(E_e)\,A_W(E_e) &= \zeta(E_e)\Big(A(E_e) + \frac{1}{3}\,Q_n(E_e)\Big)\\ &= A_0\Big(1 + \frac{\alpha}{\pi}\,g_n(E_e) + \frac{\alpha}{\pi}\,f_n(E_e)\Big) + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[\Big\{\frac{4}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa + \frac{2}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big\}\,E_0\\ &\quad - \Big\{\frac{22}{3}\lambda^2 - \Big(\frac{10}{3}\kappa - \frac{4}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big\}\,E_e\Big],\end{aligned} \tag{6}$$

where the correlation coefficients $A(E_e)$ and $Q_n(E_e)$ are given in [3,7]. The correlation coefficient $A_W(E_e)$ without the contribution of the radiative corrections, defined by the function $f_n(E_e)$, has been calculated by Wilkinson [16]. The radiative corrections $g_n(E_e)$ and $f_n(E_e)$ (see [7]) are in analytical agreement with the radiative corrections obtained by Sirlin *et al.* [20] and Gudkov *et al.* [3], respectively (where the function $f_n(E_e)$ was calculated for the first time by Shann [21]).

Using the results obtained in [7] (see Appendix A of Ref. [7]), for other correlation coefficients in Eq. (3) we get the expressions

$$\begin{aligned}\zeta(E_e)G(E_e) &= -\Big(1 + \frac{\alpha}{\pi}\,g_n(E_e) + \frac{\alpha}{\pi}\,f_n(E_e)\Big) + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[-(10\lambda^2 - 4(\kappa+1)\,\lambda + 2)E_e + (2\lambda^2 - 2(\kappa+1)\,\lambda)\,E_0\Big],\\ \zeta(E_e)N(E_e) &= +\frac{m_e}{E_e}\Big\{-A_0\Big(1 + \frac{\alpha}{\pi}\,g_n(E_e) + \frac{\alpha}{\pi}\,h_n^{(1)}(E_e)\Big) + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[\Big(\frac{16}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{16}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_e\\ &\quad - \Big(\frac{4}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{1}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_0\Big]\Big\},\\ \zeta(E_e)Q_e(E_e) &= -A_0\Big(1 + \frac{\alpha}{\pi}\,g_n(E_e) + \frac{\alpha}{\pi}\,h_n^{(2)}(E_e)\Big) + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[\Big\{\frac{22}{3}\,\lambda^2 - \Big(\frac{10}{3}\kappa - \frac{10}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big\}\,E_e\\ &\quad - \Big\{\frac{4}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{1}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big\}E_0 + \Big(2\,\lambda^2 - (2\kappa+1)\,\lambda\Big)m_e\Big],\\ \zeta(E_e)R(E_e) &= -\alpha\,\frac{m_e}{k_e}\,A_0.\end{aligned} \tag{7}$$

The functions $f_n(E_e)$, $h_n^{(\ell)}(E_e)$ for $\ell = 1,2$ describe the radiative corrections of order α/π. They are calculated in the Appendix [see Eq. (A9)] and plotted in Fig. 1. In the electron energy region $m_e \leqslant E_e \leqslant E_0$ they vary over the regions $2.81 \times 10^{-3} \geqslant (\alpha/\pi)\,f_n(E_e) \geqslant 6.24 \times 10^{-4}$, $-6.04 \times 10^{-4} \geqslant (\alpha/\pi)\,h_n^{(1)}(E_e) \geqslant -3.37 \times 10^{-3}$ and $5.07 \times 10^{-3} \geqslant (\alpha/\pi)\,h_n^{(2)}(E_e) \geqslant 2.20 \times 10^{-3}$, respectively.

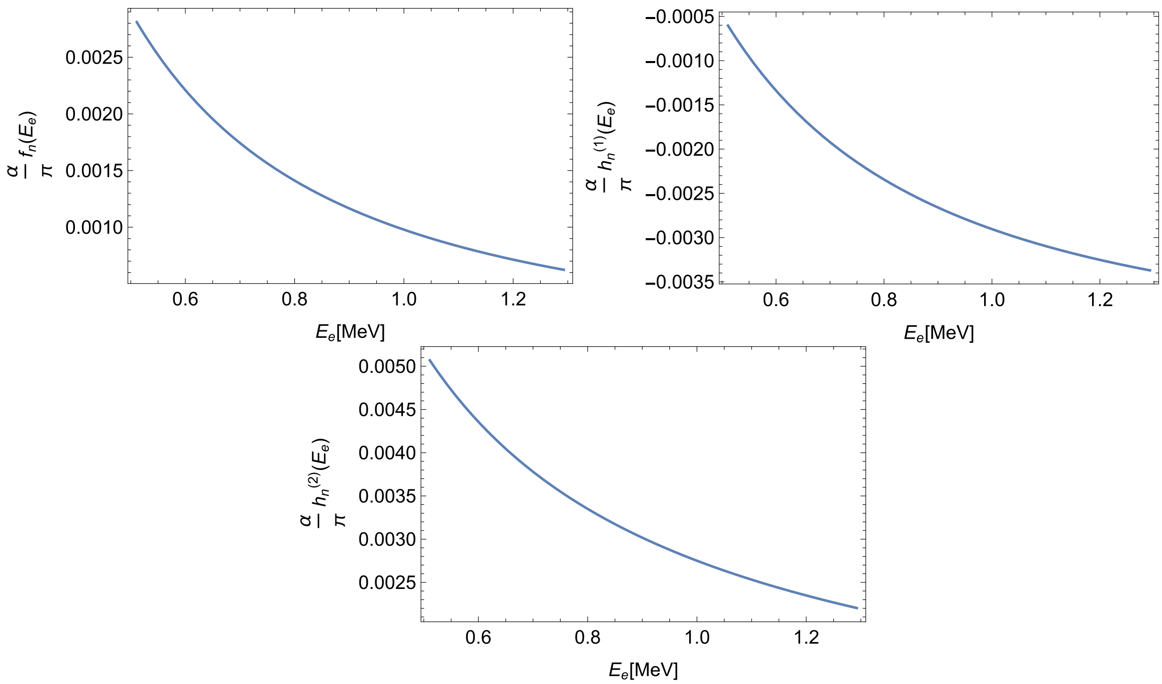

FIG. 1. Radiative corrections $(\alpha/\pi)\, f_n(E_e)$, $(\alpha/\pi)\, h_n^{(1)}(E_e)$, and $(\alpha/\pi)\, h_n^{(2)}(E_e)$ to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$ of the electron energy spectrum and angular distribution, Eq. (3).

The term proportional to the fine-structure constant α in the correlation coefficient $\zeta(E_e)R(E_e)$ is induced by the Coulomb distortion of the Dirac bispinor wave function of the electron [10,11].

Keeping the contributions of the terms of order of $1/M$ inclusively, the correlation coefficients under consideration take the form

$$
\begin{aligned}
G(E_e) &= -\Big(1 + \frac{\alpha}{\pi}\, f_n(E_e)\Big)\Big(1 + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\,(2\lambda^2 - 2(\kappa+1)\,\lambda)\,\frac{m_e^2}{E_e}\Big),\\
N(E_e) &= +\Big(1 + \frac{\alpha}{\pi}\, h_n^{(1)}(E_e)\Big)\,\frac{m_e}{E_e}\,\Big\{-A_0 + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[\Big(\frac{16}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{16}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_e\\
&\quad - \Big(\frac{4}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{1}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_0\Big]\\
&\quad - \frac{1}{M}\,\frac{A_0}{1+3\lambda^2}\Big[-(10\lambda^2 - 4(\kappa+1)\,\lambda + 2)\,E_e + (2\lambda^2 - 2(\kappa+1)\,\lambda)\Big(E_0 + \frac{m_e^2}{E_e}\Big)\Big]\Big\},\\
Q_e(E_e) &= \Big(1 + \frac{\alpha}{\pi}\, h_n^{(2)}(E_e)\Big)\Big\{-A_0 + \frac{1}{M}\,\frac{1}{1+3\lambda^2}\Big[\Big(\frac{22}{3}\,\lambda^2 - \Big(\frac{10}{3}\kappa - \frac{10}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_e\\
&\quad - \Big(\frac{4}{3}\,\lambda^2 - \Big(\frac{4}{3}\kappa - \frac{1}{3}\Big)\,\lambda - \frac{2}{3}(\kappa+1)\Big)E_0 + (2\lambda^2 - 2(\kappa+1)\,\lambda)m_e\Big]\\
&\quad - \frac{1}{M}\,\frac{A_0}{1+3\lambda^2}\Big[-(10\lambda^2 - 4(\kappa+1)\,\lambda + 2)\,E_e + (2\lambda^2 - 2(\kappa+1)\,\lambda)\Big(E_0 + \frac{m_e^2}{E_e}\Big)\Big]\Big\},\\
R(E_e) &= -\alpha\,\frac{m_e}{k_e}\,A_0,
\end{aligned}
\tag{8}
$$

where we have neglected the terms of order $(\alpha/\pi)(E_e/M) < 3 \times 10^{-6}$. The correlation coefficients in Eq. (8) are defined by a complete set of contributions to order 10^{-3}, caused by the weak magnetism and proton recoil corrections of order $1/M$ and radiative corrections of order α/π. For example, at $\lambda = -1{,}2750(9)$ and $E_0 = 1.2927$ MeV we get

$$G(E_e) = -\Big(1 + \frac{\alpha}{\pi}\, f_n(E_e)\Big)\Big(1 + 1.41\times 10^{-3}\,\frac{m_e}{E_e}\Big),$$
$$N(E_e) = -\Big(1 + \frac{\alpha}{\pi}\, h^{(1)}_n(E_e)\Big)\frac{m_e}{E_e}\,A_0\Big\{1 + \Big(-6.06\times 10^{-3} + 1.41\times 10^{-3}\,\frac{m_e}{E_e} - 1.85\times 10^{-5}\,\frac{E_e}{E_0}\Big)\Big\},$$
$$Q_e(E_e) = -\Big(1 + \frac{\alpha}{\pi}\, h^{(2)}_n(E_e)\Big)A_0\Big\{1 + \Big(-6.06\times 10^{-3} + 1.41\times 10^{-3}\,\frac{m_e}{E_e} + 2.99\times 10^{-2}\,\frac{E_e}{E_0}\Big)\Big\}, \quad (9)$$

where the correlation coefficient A_0 is factorized out of the brackets of the correlation coefficients $N(E_e)$ and $Q_e(E_e)$. The obtained results provide a robust theoretical background to order 10^{-3} for planning experiments on the search for contributions of order 10^{-4} of interactions beyond the SM. The appearance of the term of order 10^{-5} is caused by an occasional cancellation of different contributions.

III. WILKINSON'S ANALYSIS OF HIGHER ORDER CORRECTIONS

In this section we discuss the contibutions of higher order corrections, which are not calculated in Sec. II. These corrections were calculated by Wilkinson [16] and we apply them to the analysis of the correlations coefficients $G(E_e)$, $N(E)$, $Q_e(E_e)$, and $R(E_e)$, respectively. According to Wilkinson [16], the higher order corrections with respect to those calculated in Sec. II should be caused by (i) the proton recoil in the Coulomb electron-proton final-state interaction, (ii) the finite proton radius, (iii) the proton-lepton convolution, and (iv) the higher order *outer* radiative corrections.

A. Proton recoil corrections, caused by the Coulomb electron-proton final-state interaction

As has been found by Ivanov *et al.* [7], proton recoil, caused by the Coulomb electron-proton final-state interaction, leads to the following change of the Fermi function $F(E_e, Z = 1)$ (see Appendix H of Ref. [7]):

$$F(E_e, Z = 1) \to F(E_e, Z = 1)\Big(1 - \frac{\pi\alpha}{\beta}\,\frac{E_e}{M} - \frac{\pi\alpha}{\beta^3}\,\frac{E_0 - E_e}{M}\,\frac{\vec{k}_e\cdot\vec{k}_{\nu}}{E_e E_{\nu}}\Big), \quad (10)$$

where we have taken only the leading order α/M contributions. Then, $E_{\nu} = E_0 - E_e$ and $\vec{k}_{\nu}$ are the energy and three-momentum of the electron antineutrino. As has been shown in [7], the contribution of the proton recoil, caused by the final-state Coulomb electron-proton interaction(10), to the function $\zeta(E_e)$ agrees well with the result obtained by Wilkinson [16]. The corrections to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$, caused by the change of the Fermi function Eq. (10), are given by

$$\frac{\delta G(E_e)}{G(E_e)} = -\frac{\pi\alpha}{\beta}\,\frac{E_e}{M} - \frac{1}{3}\,\frac{1-\lambda^2}{1+3\lambda^2}\,\frac{\pi\alpha}{\beta^3}\,\frac{E_0 - E_e}{M}, \quad \frac{\delta N(E_e)}{N(E_e)} = -\frac{\pi\alpha}{\beta}\,\frac{E_e}{M},$$
$$\frac{\delta Q_e(E_e)}{Q_e(E_e)} = -\frac{\pi\alpha}{\beta}\,\frac{E_e}{M} - \frac{1}{3}\,\frac{1-\lambda}{1+\lambda}\,\frac{\pi\alpha}{\beta^3}\,\frac{E_0 - E_e}{M}\Big(1 + \frac{m_e}{E_e}\Big). \quad (11)$$

In the experimental electron energy region $0.761 \leqslant E_e \leqslant 0.966$ MeV the corrections to the correlation coefficients are plotted in Fig. 2. They vary in the following limits: $-2.394\times 10^{-5} \geqslant \delta G(E_e)/G(E_e) \geqslant -2.733\times 10^{-5}$, $-2.508\times 10^{-5} \geqslant \delta N(E_e)/N(E_e) \geqslant -2.779\times 10^{-5}$, and $1.220\times 10^{-4} \geqslant \delta Q_e(E_e)/Q_e(E_e) \geqslant 2.724\times 10^{-5}$, respectively.

B. Corrections caused by finite proton radius

According to Wilkinson [16], the finite proton-radius correction to the phase-space factor of the neutron β^- decay takes the form

$$L(E_e, Z = 1) = 1 + \frac{13}{60}\,\alpha^2 - \alpha\, r_p\, E_e\Big(1 - \frac{1}{2}\,\frac{m^2_e}{E^2_e}\Big)$$
$$= 1 + 1.154\times 10^{-5} - 4.183\times 10^{-5}\,\frac{E_e}{E_0} + 0.827\times 10^{-5}\,\frac{m_e}{E_e}. \quad (12)$$

The contribution of the function $L(E_e, Z = 1)$ can be absorbed by the function $\zeta(E_e)$, and through the expansions (8) may provide equal corrections to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$:

$$\frac{\delta G(E_e)}{G(E_e)} = \frac{\delta N(E_e)}{N(E_e)} = \frac{\delta Q_e(E_e)}{Q_e(E_e)} = 1 - L(E_e, Z = 1). \quad (13)$$

The contribution of the finite proton-radius corrections to the neutron lifetime is at the level of 10^{-5}.

C. Corrections caused by lepton-nucleon convolution

As has been pointed out by Wilkinson [16], the wave functions of the electron and electron antineutrino, calculated at the center of the nucleon, are not constant and may undergo a distortion in the nucleon volume that may lead to a convolution of the decay rate. Wilkinson has described such an effect by the function $C(E_e, Z = 1)$. Following Wilkinson [16] we obtain

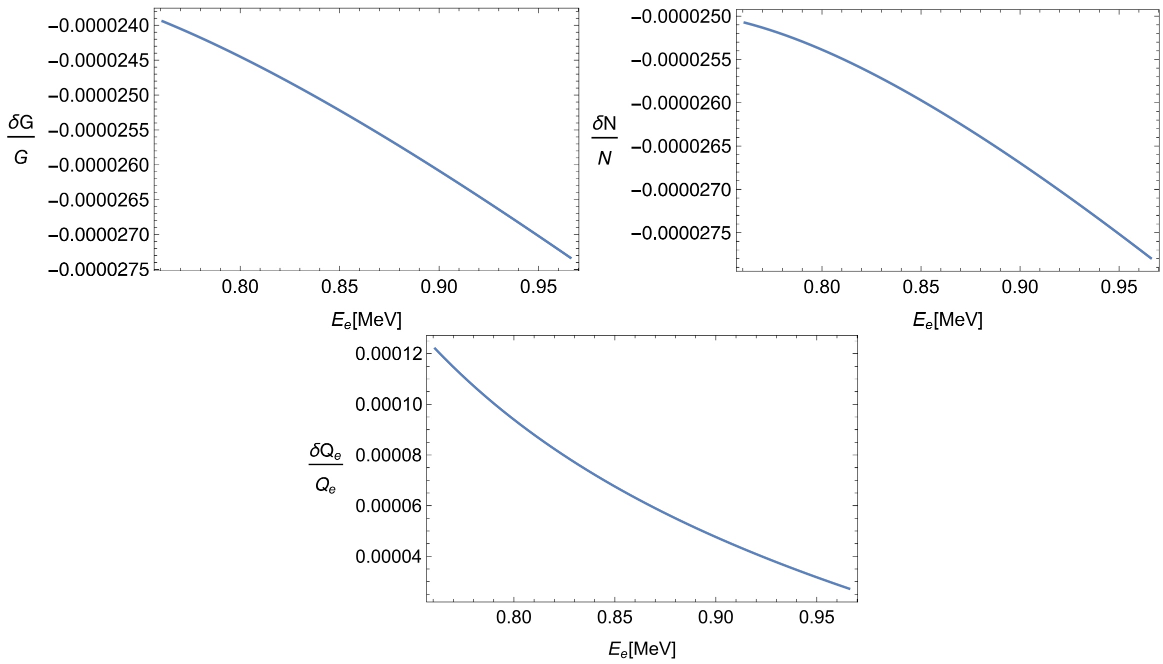

FIG. 2. Relative corrections to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$ induced by the proton recoil on the Fermi function, caused by the Coulomb electron-proton final-state interaction and calculated for the experimentally observable electron energy region $0.761 \leqslant E_e \leqslant 0.966$ MeV [7].

the function $C(E_e, Z = 1)$ in the form

$$\begin{aligned} C(E_e,Z=1) = 1 &+ \Big[\Big(-\frac{9}{20}\,\alpha^2 + \frac{1}{5}\,m_e^2 r_p^2 - \frac{1}{5}\,E_0^2 r_p^2\Big) + \Big(-\frac{1}{5}\,\alpha\, r_p\, E_0 - \frac{2}{15}\,E_0^2 r_p^2\Big)\,\frac{1-\lambda^2}{1+3\lambda^2}\Big] \\ &+ \Big[\Big(-\frac{3}{5}\,\alpha\, r_p\, E_0 + \frac{2}{5}\,E_0^2\, r_p^2\Big) + \Big(\frac{1}{5}\,\alpha\, r_p\, E_0 - \frac{2}{15}\,E_0^2\, r_p^2\Big)\,\frac{1-\lambda^2}{1+3\lambda^2}\Big]\,\frac{E_e}{E_0} + \frac{2}{15}\,m_e\, E_0\, r_p^2\,\frac{1-\lambda^2}{1+3\lambda^2}\,\frac{m_e}{E_e} \\ &+ \frac{2}{5}\Big(-1 + \frac{1}{3}\,\frac{1-\lambda^2}{1+3\lambda^2}\Big)\,E_0^2 r_p^2\,\frac{E_e^2}{E_0^2} \\ &= 1 - 2.854\times 10^{-5} - 1.238\times 10^{-5}\,\frac{E_e}{E_0} - 0.018\times 10^{-5}\,\frac{m_e}{E_e} - 1.361\times 10^{-5}\,\frac{E_e^2}{E_0^2}. \end{aligned} \quad (14)$$

Because of the expansion (8) the corrections, caused by the lepton-nucleon convolution, to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$ are equal and are given by

$$\frac{\delta G(E_e)}{G(E_e)} = \frac{\delta N(E_e)}{N(E_e)} = \frac{\delta Q_e(E_e)}{Q_e(E_e)} = 1 - C(E_e, Z=1). \quad (15)$$

The contribution of the function $C(E_e, Z = 1)$ to the neutron lifetime is at the level of 10^{-5}.

D. Higher order *outer* radiative corrections

The energy-independent radiative corrections of order $O(\alpha^2)$ and $O(\alpha^3)$ have been calculated by Wilkinson [16]. The contribution of these corrections to the phase-space factor was defined by Wilkinson as $J(Z = 1)$. Using the results, obtained by Wilkinson [16], we get $J(Z = 1) = 1 + 3.917 \times 10^{-4}$. Of course, such corrections give equal contributions to the correlation coefficients $\delta G(E_e)/G(E_e) = \delta N(E_e)/N(E_e) = \delta Q_e(E_e)/Q_e(E_e) = -3.917 \times 10^{-4}$. In principle, they should be taken into account for an experimental search of contributions of order 10^{-4} of interactions beyond the SM. The factor $J(Z = 1)$ changes the neutron lifetime by 0.3 s, which is, of course, small compared to the current experimental accuracy of the neutron lifetime $\tau_n = 880.2(1.2)$ s [22] (see also the world averaged value $\tau_n = 880.2(1.2)$ s [23]).

IV. CONCLUSION

We have calculated the correlation coefficients of the electron energy spectrum and angular distribution of the β^- decay of the neutron with polarized neutron and electron. We have performed the calculation within the SM with V-A weak interactions by taking into account the contributions of the weak magnetism and proton recoil to next-to-leading order in the large proton mass expansion and the radiative corrections of order α/π, calculated to leading order in the large proton mass expansion. Such an approximation provides a theoretical background for the analysis of contributions of order 10^{-4} of interactions beyond the SM [7,8].

The correlation coefficients $N(E_e)$ and $R(E_e)$, given by Eq. (8), averaged over the electron energy spectrum (see Eq. (D-59) of Ref. [7]) and calculated at $\lambda = -1.2750(9)$, are equal to

$$\langle N(E_e)\rangle = 0.07767(22), \quad \langle R(E_e)\rangle = 0.000891(3). \qquad (16)$$

The recent experimental data $N_{\rm exp} = 0.067 \pm 0.011 \pm 0.004$ and $R_{\rm exp} = 0.004 \pm 0.012 \pm 0.005$ [15] do not contradict the predictions of the SM within the experimental uncertainties.

Using the electron energy spectrum and angular distribution (3), we can define the rate of the β^- decay of the neutron via dependence on the neutron and electron polarizations,

$$\lambda_n(\vec{\xi}_n,\vec{\xi}_e) = \lambda_n\,(1 + \langle \bar{N}(E_e)\rangle\,\vec{\xi}_n\cdot\vec{\xi}_e), \qquad (17)$$

where λ_n is the β^--decay rate of the neutron, defining the lifetime of the neutron $\tau_n = 1/\lambda_n$, and is equal to [7]

$$\lambda_n = (1+3\lambda^2)\,\frac{G_F^2|V_{\rm ud}|^2}{2\pi^3}\,f_n(E_0, Z=1). \qquad (18)$$

The Fermi integral $f_n(E_0,Z=1)$ is given by [7]

$$f_n(E_0,Z=1) = \int_{m_e}^{E_0}(E_0-E_e)^2\sqrt{E_e^2-m_e^2}\,E_e\,F(E_e,Z=1)\Big\{\Big(1+\frac{\alpha}{\pi}\,g_n(E_e)\Big) + \frac{1}{M}\times\frac{1}{1+3\lambda^2}\Big[(10\lambda^2-4(\kappa+1)\lambda+2)\,E_e - (2\lambda^2-2(\kappa+1)\lambda)\Big(E_0+\frac{m_e^2}{E_e}\Big)\Big]\Big\}\,dE_e. \qquad (19)$$

The correlation coefficient $\bar{N}(E_e)$ is defined by the expression

$$\bar{N}(E_e) = N(E_e) + \frac{1}{3}\Big(1-\frac{m_e}{E_e}\Big)\,Q_e(E_e). \qquad (20)$$

For the experimental observation of the correlation coefficient $\bar{N}(E_e)$ we propose to analyze the asymmetry

$$P(\vec{\xi}_n,\vec{\xi}_e) = \frac{\lambda_n(\vec{\xi}_n,\vec{\xi}_e) - \lambda_n(-\vec{\xi}_n,\vec{\xi}_e)}{\lambda_n(\vec{\xi}_n,\vec{\xi}_e) + \lambda_n(-\vec{\xi}_n,\vec{\xi}_e)} = \langle\bar{N}(E_e)\rangle\,P_n P_e, \qquad (21)$$

where P_n and P_e are the neutron and electron polarizations. The asymmetry $P(\vec{\xi}_n,\vec{\xi}_e)$ can be measured for the polarized neutron and electron with parallel and antiparallel spins. Averaging $\bar{N}(E_e)$ over the electron energy spectrum (see Eq. (D-59) in Appendix D of Ref. [7]) we get $\langle\bar{N}(E_e)\rangle = 0.0911$. Thus, the theoretical prediction for asymmetry $P(\vec{\xi}_n,\vec{\xi}_e)$, obtained in the SM with the weak magnetism, proton recoil, and radiative corrections, is

$$P(\vec{\xi}_n,\vec{\xi}_e) = 0.0911\,P_n P_e. \qquad (22)$$

Our results should provide a necessary background for the measurement of the contributions of order 10^{-4} to the β^- decay of a polarized neutron with polarized electron, caused by interactions beyond the SM [7,24].

The radiative corrections to the correlation coefficients $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$ are given by the functions $f_n(E_e)$, $h_n^{(1)}(E_e)$, and $h_n^{(2)}(E_e)$, calculated in the Appendix. The photon-electron and electron energy spectra and angular distributions of the radiative β^- decay of the neutron with polarized neutron and electron, obtained in the Appendix, may be used for future experiments on the radiative β^- decay of the neutron [25].

We would like to emphasize that the radiative corrections, described by the functions $h_n^{(1)}(E_e)$ and $h_n^{(2)}(E_e)$, and the photon-electron and electron energy spectra and angular distributions of the radiative β^- decay of a neutron with polarized neutron and electron have been never calculated in the literature before.

Completing our discussion of these corrections, we would like to make two comments: (i) for predictions of 10^{-4} precision, it is apparent that the higher order outer radiative corrections, discussed in Sec. III, should be included, and (ii) for an experimental search for interactions beyond the SM, a "discovery" experiment with the required 5σ sensitivity will require experimental uncertainties of a few parts in 10^{-5}.

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APPENDIX: PHOTON-ELECTRON AND ELECTRON ENERGY SPECTRA AND ANGULAR DISTRIBUTIONS OF RADIATIVE β^- DECAY OF NEUTRON WITH POLARIZED NEUTRON AND ELECTRON

Using the results, obtained in [7], the photon-electron spectrum and angular distribution of the radiative β^- decay of the neutron with polarized neutron and electron takes the form

$$\frac{d^5\lambda_{\beta^-_c\gamma}(E_e,\omega,\vec{k}_e,\vec{n}\,)}{d\omega\, dE_e d\Omega_e d\Omega_\gamma} = (1+3\lambda^2)\,\frac{\alpha}{\pi}\,\frac{G_F^2|V_{\rm ud}|^2}{(2\pi)^5}\,\sqrt{E^2_e - m^2_e}\,E_e\,F(E_e, Z=1)\,(E_0 - E_e - \omega)^2\frac{1}{\omega}\Big(\Big[\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\Big(1 + \frac{\omega}{E_e}\Big) + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega^2}{E^2_e}\Big]$$
$$+ A_0\,\vec{\xi}_n\cdot\Big\{\Big(\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2} + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega}{E_e}\Big)\vec{\beta} + \Big[-\frac{1-\beta^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega}{E_e} + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega}{E_e}\Big(1 + \frac{\omega}{E_e}\Big)\Big]\vec{n}\Big\}$$
$$+\Big\{-\frac{m_e}{E_e}\Big[\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\zeta^0_e + \frac{(\vec{\beta}\cdot\vec{\zeta}_e) - (\vec{n}\cdot\vec{\beta}\,)(\vec{n}\cdot\vec{\zeta}_e)}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega}{E_e} + \frac{\zeta^0_e - \vec{n}\cdot\vec{\zeta}_e}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega^2}{E^2_e}\Big]$$
$$-\frac{m_e}{E_e}\,A_0\Big[\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,(\vec{\xi}_n\cdot\vec{\zeta}_e) + \frac{(\vec{\beta}\cdot\vec{\zeta}_e) - (\vec{n}\cdot\vec{\beta}\,)(\vec{n}\cdot\vec{\zeta}_e)}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,(\vec{\xi}_n\cdot\vec{n}\,)\,\frac{\omega}{E_e} + \frac{\zeta^0_e - \vec{n}\cdot\vec{\zeta}_e}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}(\vec{\xi}_n\cdot\vec{n}\,)\,\frac{\omega^2}{E^2_e}\Big]\Big\}\Big), \qquad \text{(A1)}$$

where $\beta = \sqrt{E^2_e - m^2_e}/E_e$ is the electron velocity and ω is the photon energy, the vector $\vec{n}$ is directed along the photon three-momentum, and $d\Omega_e$ and $d\Omega_\gamma$ are the elements of the solid angles of the electron and the photon, respectively. The four-vector of the electron polarization, $\zeta^\mu_e = (\zeta^0_e,\vec{\zeta}_e)$, is defined by

$$\zeta^\mu_e = (\zeta^0_e,\vec{\zeta}_e) = \Big(\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e}, \vec{\xi}_e + \frac{\vec{k}_e(\vec{k}_e\cdot\vec{\xi}_e)}{m_e(E_e + m_e)}\Big). \qquad \text{(A2)}$$

It obeys the constraints $\zeta^2_e = -1$ and $k_e\cdot\zeta_e = 0$. For the derivation of the electron energy spectrum and angular distribution, it is convenient to rewrite Eq. (A1) as follows:

$$\frac{d^5\lambda_{\beta^-_c\gamma}(E_e,\omega,\vec{k}_e,\vec{n}\,)}{d\omega\, dE_e d\Omega_e d\Omega_\gamma} = (1+3\lambda^2)\,\frac{\alpha}{\pi}\,\frac{G_F^2|V_{\rm ud}|^2}{(2\pi)^5}\,\sqrt{E^2_e - m^2_e}\,E_e\,F(E_e, Z=1)\,(E_0 - E_e - \omega)^2\,\frac{1}{\omega}\,\Big\{\Big[\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\Big(1 + \frac{\omega}{E_e}\Big) + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega^2}{E^2_e}\Big]$$
$$+ A_0\Big\{\Big[\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2} + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega}{E_e}\Big]\vec{\xi}_n\cdot\vec{\beta} + \Big[-\frac{1-\beta^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega}{E_e} + \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{\omega}{E_e}\Big(1 + \frac{\omega}{E_e}\Big)\Big]\vec{\xi}_n\cdot\vec{n}\Big\}$$
$$-\frac{m_e}{E_e}\Big\{\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e} + \frac{1}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega}{E_e}\,\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e} + \Big[\frac{\vec{n}\cdot\vec{\zeta}_e}{1 - \vec{n}\cdot\vec{\beta}} - \frac{\vec{n}\cdot\vec{\zeta}_e}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\Big]\frac{\omega}{E_e} + \frac{1}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega^2}{E^2_e}\,\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e}$$
$$-\frac{\vec{n}\cdot\vec{\zeta}_e}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega^2}{E^2_e}\Big\} - \frac{m_e}{E_e}\,A_0\Big\{\frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\vec{\xi}_n\cdot\vec{\zeta}_e + \frac{\vec{n}\cdot\vec{\xi}_n}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega}{E_e}\,\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e} + \Big[\frac{(\vec{n}\cdot\vec{\xi}_n)(\vec{n}\cdot\vec{\zeta}_e)}{1 - \vec{n}\cdot\vec{\beta}} - \frac{(\vec{n}\cdot\vec{\xi}_n)(\vec{n}\cdot\vec{\zeta}_e)}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\Big]\frac{\omega}{E_e}$$
$$+ \frac{\vec{n}\cdot\vec{\xi}_n}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega^2}{E^2_e}\,\frac{\vec{k}_e\cdot\vec{\xi}_e}{m_e} - \frac{(\vec{n}\cdot\vec{\xi}_n)(\vec{n}\cdot\vec{\zeta}_e)}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{\omega^2}{E^2_e}\Big\}\Big\}. \qquad \text{(A3)}$$

The integration over the directions of $\vec{n}$ we carry out with the following auxiliary integrals:

$$\int \frac{\beta^2 - (\vec{n}\cdot\vec{\beta}\,)^2}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{d\Omega_\gamma}{4\pi} = \frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2, \quad \int \frac{1}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{d\Omega_\gamma}{4\pi} = \frac{1}{1-\beta^2}, \quad \int \frac{1}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{d\Omega_\gamma}{4\pi} = \frac{1}{2\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big),$$
$$\int \frac{\vec{a}\cdot\vec{n}}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{d\Omega_\gamma}{4\pi} = -\frac{1}{2\beta^2}\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big) - \frac{2}{1-\beta^2}\Big](\vec{a}\cdot\vec{\beta}\,), \quad \int \frac{\vec{a}\cdot\vec{n}}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{d\Omega_\gamma}{4\pi} = \frac{1}{2\beta^2}\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big](\vec{a}\cdot\vec{\beta}\,),$$
$$\int \frac{(\vec{a}\cdot\vec{n}\,)(\vec{b}\cdot\vec{n}\,)}{(1 - \vec{n}\cdot\vec{\beta}\,)^2}\,\frac{d\Omega_\gamma}{4\pi} = \frac{1}{2}\,\frac{1}{\beta^2}\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big](\vec{a}\cdot\vec{b}\,) - \frac{1}{2}\,\frac{1}{\beta^4}\Big[\frac{3}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 4 - \frac{2}{1-\beta^2}\Big](\vec{a}\cdot\vec{\beta}\,)(\vec{b}\cdot\vec{\beta}\,),$$
$$\int \frac{(\vec{a}\cdot\vec{n}\,)(\vec{b}\cdot\vec{n}\,)}{1 - \vec{n}\cdot\vec{\beta}}\,\frac{d\Omega_\gamma}{4\pi} = -\frac{1}{4}\,\frac{1}{\beta^2}\Big[\frac{1-\beta^2}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big](\vec{a}\cdot\vec{b}\,) + \frac{1}{4}\,\frac{1}{\beta^4}\Big[\frac{3-\beta^2}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 6\Big](\vec{a}\cdot\vec{\beta}\,)(\vec{b}\cdot\vec{\beta}\,). \qquad \text{(A4)}$$

As a result the photon-electron energy spectrum and angular distribution takes the form

$$\frac{d^4\lambda_{\beta^-_c\gamma}(E_e,\omega)}{d\omega\, dE_e d\Omega_e} = (1+3\lambda^2)\,\frac{\alpha}{\pi}\,\frac{G_F^2|V_{\rm ud}|^2}{8\pi^4}\sqrt{E_e^2-m_e^2}\,E_e\,F(E_e,Z=1)\,(E_0-E_e-\omega)^2 \times \frac{1}{\omega}\Big\{\Big\{\Big(1+\frac{\omega}{E_e}+\frac{1}{2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{\omega^2}{E_e^2}\Big\}+\frac{\vec{k}_e\cdot\vec{\xi}_n}{E_e}\,A_0\Big(1+\frac{1}{\beta^2}\,\frac{\omega}{E_e}+\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big] - \frac{\vec{k}_e\cdot\vec{\xi}_e}{E_e}\Big(1+\frac{1}{\beta^2}\,\frac{\omega}{E_e}+\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]-(\vec{\xi}_n\cdot\vec{\zeta}_e)\,\frac{m_e}{E_e}\,A_0\Big\{\Big(1-\frac{3-\beta^2}{4\beta^2}\,\frac{\omega}{E_e}-\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big) \times\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{1}{2}\,\frac{\omega}{E_e}\Big\}-\frac{(\vec{\xi}_n\cdot\vec{k}_e)(\vec{k}_e\cdot\vec{\xi}_e)}{E_e^2}\,A_0\,\frac{3}{4}\,\frac{1}{\beta^2}\frac{\omega}{E_e}\Big(1+\frac{2}{3}\,\frac{\omega}{E_e}\Big)\Big\{\frac{3-\beta^2}{\beta^2}\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]-2\Big\}\Big\}. \quad \text{(A5)}$$

In terms of the irreducible scalar products, the photon-electron energy spectrum and angular distribution of the radiative β^- decay of the neutron reads

$$\frac{d^4\lambda_{\beta^-_c\gamma}(E_e,\omega)}{d\omega\, dE_e d\Omega_e} = (1+3\lambda^2)\,\frac{\alpha}{\pi}\,\frac{G_F^2|V_{\rm ud}|^2}{8\pi^4}\sqrt{E_e^2-m_e^2}\,E_e\,F(E_e,Z=1)\,(E_0-E_e-\omega)^2 \times \frac{1}{\omega}\Big\{\Big\{\Big(1+\frac{\omega}{E_e}+\frac{1}{2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{\omega^2}{E_e^2}\Big\}+\frac{\vec{k}_e\cdot\vec{\xi}_n}{E_e}\,A_0\Big(1+\frac{1}{\beta^2}\,\frac{\omega}{E_e}+\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big] - \frac{\vec{k}_e\cdot\vec{\xi}_e}{E_e}\Big(1+\frac{1}{\beta^2}\,\frac{\omega}{E_e}+\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]-(\vec{\xi}_n\cdot\vec{\xi}_e)\,\frac{m_e}{E_e}\,A_0\,\Big(\Big(1-\frac{3-\beta^2}{4\beta^2}\,\frac{\omega}{E_e}-\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big) \times\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{1}{2}\,\frac{\omega}{E_e}\Big)-\frac{(\vec{\xi}_n\cdot\vec{k}_e)(\vec{k}_e\cdot\vec{\xi}_e)}{E_e(E_e+m_e)}\,A_0\,\Big(\Big\{\Big(1-\frac{3-\beta^2}{4\beta^2}\,\frac{\omega}{E_e}-\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{1}{2}\frac{\omega}{E_e}\Big\} +(1+\sqrt{1-\beta^2})\,\frac{3}{4}\,\frac{1}{\beta^2}\frac{\omega}{E_e}\Big(1+\frac{2}{3}\,\frac{\omega}{E_e}\Big)\Big\{\frac{3-\beta^2}{\beta^2}\,\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]-2\Big\}\Big)\Big\}. \quad \text{(A6)}$$

Integrating over the photon energy over the region $\omega_{\rm min} \leqslant \omega \leqslant E_0 - E_e$ we obtain the electron energy spectrum and angular distribution

$$\frac{d^3\lambda_{\beta^-_c\gamma}(E_e)}{dE_e d\Omega_e} = (1+3\lambda^2)\,\frac{\alpha}{\pi}\,\frac{G_F^2|V_{\rm ud}|^2}{8\pi^4}\sqrt{E_e^2-m_e^2}\,E_e\,F(E_e,Z=1)\,(E_0-E_e)^2\Big\{g^{(1)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})+\frac{\vec{k}_e\cdot\vec{\xi}_n}{E_e}\,A_0\,g^{(2)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) -\frac{\vec{k}_e\cdot\vec{\xi}_e}{E_e}\,g^{(2)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})-\vec{\xi}_n\cdot\vec{\xi}_e\,\frac{m_e}{E_e}\,A_0\,g^{(3)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})-\frac{(\vec{\xi}_n\cdot\vec{k}_e)(\vec{k}_e\cdot\vec{\xi}_e)}{E_e(E_e+m_e)}\,A_0\,g^{(4)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})\Big\}, \quad \text{(A7)}$$

where the functions $g^{(i)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})$ for $i = 1,2,3,4$ are defined by the integrals

$$\begin{aligned} g^{(1)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) &= \int^{E_0-E_e}_{\omega_{\rm min}}\frac{d\omega}{\omega}\,\frac{(E_0-E_e-\omega)^2}{(E_0-E_e)^2}\,\Big\{\Big(1+\frac{\omega}{E_e}+\frac{1}{2}\frac{\omega^2}{E_e^2}\Big)\,\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{\omega^2}{E_e^2}\Big\},\\ g^{(2)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) &= \int^{E_0-E_e}_{\omega_{\rm min}}\frac{d\omega}{\omega}\,\frac{(E_0-E_e-\omega)^2}{(E_0-E_e)^2}\,\Big(1+\frac{1}{\beta^2}\frac{\omega}{E_e}+\frac{1}{2\beta^2}\frac{\omega^2}{E_e^2}\Big)\,\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big],\\ g^{(3)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) &= \int^{E_0-E_e}_{\omega_{\rm min}}\frac{d\omega}{\omega}\,\frac{(E_0-E_e-\omega)^2}{(E_0-E_e)^2}\,\Big\{\Big(1-\frac{3-\beta^2}{4\beta^2}\,\frac{\omega}{E_e}-\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{1}{2}\,\frac{\omega}{E_e}\Big\}, \quad \text{(A8)}\\ g^{(4)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) &= \int^{E_0-E_e}_{\omega_{\rm min}}\frac{d\omega}{\omega}\,\frac{(E_0-E_e-\omega)^2}{(E_0-E_e)^2}\,\Big(\Big\{\Big(1-\frac{3-\beta^2}{4\beta^2}\,\frac{\omega}{E_e}-\frac{1}{2\beta^2}\,\frac{\omega^2}{E_e^2}\Big)\Big[\frac{1}{\beta}\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]+\frac{1}{2}\frac{\omega}{E_e}\Big\}\\ &\quad +(1+\sqrt{1-\beta^2})\,\frac{3}{4}\,\frac{1}{\beta^2}\,\frac{\omega}{E_e}\Big(1+\frac{2}{3}\,\frac{\omega}{E_e}\Big)\,\Big\{\frac{3-\beta^2}{\beta^2}\,\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)-2\Big]-2\Big\}\Big). \end{aligned}$$

In terms of the functions $g^{(i)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})$, depending on the infrared cutoff $\omega_{\rm min}$, for $i = 1,2,3,4$ we determine the functions $f_n(E_e)$ and $h^{(\ell)}_n(E_e)$ for $\ell = 1,2$, which do not depend on the infrared cutoff $\omega_{\rm min}$. They are

$$\begin{aligned}
f_n(E_e) &= \lim_{\omega_{\rm min}\to 0}\big[g^{(2)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) - g^{(1)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})\big] + g_F(E_e)\,\frac{m_e}{E_e} = \frac{1}{3}\,\frac{1-\beta^2}{\beta^2}\,\frac{E_0-E_e}{E_e}\Big(1+\frac{1}{8}\,\frac{E_0-E_e}{E_e}\Big)\\
&\quad\times\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big] - \frac{1}{12}\,\frac{(E_0-E_e)^2}{E^2_e} + \frac{1-\beta^2}{2\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big),\\
h^{(1)}_n(E_e) &= \lim_{\omega_{\rm min}\to 0}\big[g^{(3)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) - g^{(1)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})\big] + g_F(E_e)\,\frac{m_e}{E_e} - g_F(E_e)\,\frac{E_e}{m_e} = -\frac{1}{4}\frac{1+\beta^2}{\beta^2}\,\frac{E_0-E_e}{E_e}\\
&\quad\times\Big(1+\frac{1}{6}\,\frac{E_0-E_e}{E_e}\Big)\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big] + \frac{1}{6}\,\frac{E_0-E_e}{E_e}\Big(1-\frac{1}{2}\,\frac{E_0-E_e}{E_e}\Big) - \frac{\beta}{2}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big),\\
h^{(2)}_n(E_e) &= \lim_{\omega_{\rm min}\to 0}\big[g^{(4)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min}) - g^{(1)}_{\beta^-_c\gamma}(E_e,\omega_{\rm min})\big] + g_F(E_e)\,\frac{m_e}{E_e} + g_F(E_e) = -\frac{1}{4}\frac{1+\beta^2}{\beta^2}\,\frac{E_0-E_e}{E_e}\\
&\quad\times\Big(1+\frac{1}{6}\,\frac{E_0-E_e}{E_e}\Big)\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big] + \frac{1}{6}\,\frac{E_0-E_e}{E_e}\Big(1-\frac{1}{2}\,\frac{E_0-E_e}{E_e}\Big) + (1+\sqrt{1-\beta^2})\\
&\quad\times\Big\{\frac{1}{4\beta^2}\,\frac{E_0-E_e}{E_e}\Big(1+\frac{1}{18}\,\frac{E_0-E_e}{E_e}\Big)\Big(\frac{3-\beta^2}{\beta^2}\Big[\frac{1}{\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big) - 2\Big] - 2\Big) + \frac{\sqrt{1-\beta^2}}{2\beta}\,\ln\Big(\frac{1+\beta}{1-\beta}\Big)\Big\}.
\end{aligned} \tag{A9}$$

For the calculation of the functions $f_n(E_e)$ and $h^{(\ell)}_n(E_e)$ for $\ell = 1,2$, defining the radiative corrections to the correlation coefficients $A_W(E_e)$, $G(E_e)$, $N(E_e)$, and $Q_e(E_e)$, respectively, we have to take into account the contribution of the virtual photon exchanges, inducing the scalar and tensor weak nucleon-lepton coupling constant (see Appendix B of Ref. [7]).

Finally we would like to note that for the calculation of the radiative corrections, defined by the functions $f_n(E_e)$ and $h^{(\ell)}_n(E_e)$ for $\ell = 1,2$, the final result does not depend on the regularization procedure. Indeed, one may use the infrared cutoff $\omega_{\rm min}$, which may be identified with the experimental threshold energy of photons, and the finite photon mass regularization (FPM) [20] (see also [3,7]). In turn the function $g_n(E_e)$ has to be calculated with the FPM regularization in order to satisfy gauge invariance and the Kinoshita-Lee-Nauenberg theorem [20] (see also [7]).

[1] H. Abele, Prog. Part. Nucl. Phys. **60**, 1 (2008).
[2] J. S. Nico, J. Phys. G: Nucl. Part. Phys. **36**, 104001 (2009).
[3] V. Gudkov, G. L. Greene, and J. R. Calarco, Phys. Rev. C **73**, 035501 (2006); V. Gudkov, *ibid.* **77**, 045502 (2008).
[4] M. Faber, A. N. Ivanov, V. A. Ivanova, J. Marton, M. Pitschmann, A. P. Serebrov, N. I. Troitskaya, and M. Wellenzohn, Phys. Rev. C **80**, 035503 (2009).
[5] T. Bhattacharya, V. Cirigliano, S. D. Cohen, A. Filipuzzi, M. González-Alonso, M. L. Graesser, R. Gupta, and H. W. Lin, Phys. Rev. D **85**, 054512 (2012).
[6] V. Cirigliano, J. Jenkins, and M. Gonzáles-Alonso, Nucl. Phys. B **830**, 95 (2010); V. Cirigliano, M. Gonzáles-Alonso, and M. L. Graesser, J. High Energy Phys. 02 (2013) 046.
[7] A. N. Ivanov, M. Pitschmann, and N. I. Troitskaya, Phys. Rev. D **88**, 073002 (2013).
[8] A. N. Ivanov, R. Höllwieser, N. I. Troitskaya, and M. Wellenzohn, Phys. Rev. D **88**, 065026 (2013).
[9] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. A **38**, 090001 (2014).
[10] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Nucl. Phys. **4**, 206 (1957); Z. Phys. **150**, 640 (1958).
[11] E. J. Konopinski, in *The Theory of Beta Radioactivity* (Clarendon, Oxford, 1966).
[12] D. Mund, B. Märkisch, M. Deissenroth, J. Krempel, M. Schumann, H. Abele, A. Petoukhov, and T. Soldner, Phys. Rev. Lett. **110**, 172502 (2013).
[13] B. Maerkisch and H. Abele, presented at 8th International Workshop on the CKM Unitarity Triangle (CKM 2014), Vienna, Austria, September 8–12, 2014, arXiv:1410.4220.
[14] A. Kozela, G. Ban, A. Bialek, K. Bodek, P. Gorel, K. Kirch, S. Kistryn, M. Kuźniak, O. Naviliat-Cuncic, J. Pulut, N. Severijns, E. Stephan, and J. Zejma, Phys. Rev. Lett. **102**, 172301 (2009).
[15] A. Kozela, G. Ban, A. Białek, K. Bodek, P. Gorel, K. Kirch, S. Kistryn, O. Naviliat-Cuncic, N. Severijns, E. Stephan, and J. Zejma, Phys. Rev. C **85**, 045501 (2012).
[16] D. H. Wilkinson, Nucl. Phys. A **377**, 474 (1982).
[17] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Phys. Rev. **106**, 517 (1957).
[18] N. Severijns, M. Beck, and O. Naviliat-Cuncic, Rev. Mod. Phys. **78**, 991 (2006).
[19] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **77**, 1 (2005); A. V. Volotka *et al.*, Eur. Phys. J. D **33**, 23 (2005).
[20] A. Sirlin, Phys. Rev. **164**, 1767 (1967); Nucl. Phys. B **71**, 29 (1974); W. J. Marciano, A. Sirlin, *ibid.* **88**, 86 (1975); Rev. Mod. Phys. **50**, 573 (1978); **50**, 905 (1978); A. García and

M. Maya, Phys. Rev. D **17**, 1376 (1978); A. Sirlin, Nucl. Phys. B **196**, 83 (1982); W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986); A. Czarnecki, W. J. Marciano, and A. Sirlin, Phys. Rev. D **70**, 093006 (2004); W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **96**, 032002 (2006); A. Czarnecki, W. J. Marciano, and A. Sirlin, *ibid.* **99**, 032003 (2007).

[21] R. T. Shann, Nuovo Cimento A **5**, 591 (1971).

[22] S. Arzumanov, L. Bondarenko, S. Chernyavsky, P. Geltenbort, V. Morozov, V. V. Nesvizhevsky, Y. Panin, and A. Strepetov, Phys. Lett. B **745**, 79 (2015).

[23] C. Partignani *et al.* (Particle Data Group), Chin. Phys. C **40**, 100001 (2016).

[24] A. Kurylov and M. J. Ramsey-Musolf, Phys. Rev. Lett. **88**, 071804 (2002); S. Profumo, M. J. Ramsey-Musolf, and S. Tulin, Phys. Rev. D **75**, 075017 (2007); M. J. Ramsey-Musolf and S. Su, Phys. Rep. **456**, 188 (2008).

[25] J. S. Nico *et al.*, Nature (London) **444**, 1059 (2006); R. L. Cooper, T. E. Chupp, M. S. Dewey, T. R. Gentile, H. P. Mumm, J. S. Nico, A. K. Thompson, B. M. Fisher, I. Kremsky, F. E. Wietfeldt, E. J. Beise, K. G. Kiriluk, J. Byrne, K. J. Coakley, and C. Fu, Phys. Rev. C **81**, 035503 (2010); R. L. Cooper *et al.*, Nucl. Instrum. Methods Phys. Res., Sect. A **611**, 219 (2009); M. J. Bales, R. Alarcon, C. D. Bass, E. J. Beise, H. Breuer, J. Byrne, T. E. Chupp, K. J. Coakley, R. L. Cooper, M. S. Dewey, S. Gardner, T. R. Gentile, D. He, H. P. Mumm, J. S. Nico, B. O'Neill, A. K. Thompson, and F. E. Wietfeldt (RDK II Collaboration), Phys. Rev. Lett. **116**, 242501 (2016).