Quantum tunneling with friction

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Using the phenomenological quantum friction models introduced by P. Caldirola [Nuovo Cimento **18**, 393 (1941)] and E. Kanai [Prog. Theor. Phys. **3**, 440 (1948)], M. D. Kostin [J. Chem. Phys. **57**, 3589 (1972)], and K. Albrecht [Phys. Lett. B **56**, 127 (1975)], we study quantum tunneling of a one-dimensional potential in the presence of energy dissipation. To this end, we calculate the tunneling probability using a time-dependent wave-packet method. The friction reduces the tunneling probability. We show that the three models provide similar penetrabilities to each other, among which the Caldirola-Kanai model requires the least numerical effort. We also discuss the effect of energy dissipation on quantum tunneling in terms of barrier distributions.

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I. INTRODUCTION

In low-energy heavy-ion fusion reactions, it has been known that excitations of the colliding nuclei considerably influence the reaction dynamics; that is, fusion cross sections are largely enhanced as compared to the prediction of a simple potential model [1–4]. To take into account the excitations during reactions, the coupled-channels method has been developed. Many experimental data have been successfully accounted for with this method by including a few internal degrees of freedom which are coupled strongly to the ground state [3,5]. However, when a large number of channels are involved, the coupled-channels calculations become increasingly difficult. This is the case, e.g., fusion reactions in massive systems, in which many noncollective excitations may play an important role [6–9].

To deal with this problem, many phenomenological models based on the classical concept of friction were proposed in connection with deep inelastic heavy-ion collisions [10]. Among them, it has been found that the classical Langevin treatment works well for fusion reactions and deep inelastic collisions when the incident energy is higher than the Coulomb barrier [11–13]. When the incident energy is close to the barrier, however, the fusion reaction takes place by quantum tunneling. Hence, to apply these models to low-energy fusion reactions, a quantum mechanical extension of the friction models is essential. Notice that there are many noncollective levels in atomic nuclei not only in the high-excitation energy region but also in the low-excitation energy region [6–9], which may lead to the friction effect even at low incident energies.

Another important issue is to extend the coupled-channels approach to massive systems by taking into account the dissipation effects and to develop a quantal theory for deep inelastic collision with energy and angular momentum dissipations. Such theory would be able to describe simultaneously dissipative quantum tunneling below the Coulomb barrier and deep inelastic collision above the barrier. In that way, one may resolve a long-standing problem of surface diffuseness anomaly in heavy-ion fusion reactions; that is, an anomaly that a significantly large value for the surface diffuseness parameter in an internuclear Woods-Saxon potential has to be used to account for above barrier data of fusion cross sections [14–16]. Such theory would also provide a consistent description for deep subbarrier hindrance of fusion cross sections [4], for which the dynamics after the touching of the colliding nuclei play a crucial role [17–19].

Quantum friction has attracted lots of attention as a general problem of open quantum systems [20–27]. To date, many attempts at developing a quantum friction model have been made. They can be mainly categorized into the following two approaches. The first is to consider a system with a bath, for which the environmental bath is often simplified as, e.g., a collection of harmonic oscillators [28–31]. The second approach is to treat the couplings to the bath implicitly and introduce a phenomenological Hamiltonian with which the classical equation of motion with a frictional force is reproduced as expectation values. For this approach, Caldirola and Kanai [32,33], Kostin [34], and Albrecht [35] proposed a Hermitian Hamiltonian, whose equation of motion contains a linear frictional force, while Dekker [36] invented an approach with a non-Hermitian Hamiltonian.

In this paper, we employ the second approach and investigate quantum tunneling in the presence of friction. Even though the first approach is more microscopic, physical quantities are easier to calculate with the second approach, and thus it is easier to gain physical insight into the effect of friction on quantum tunneling. We particularly consider the three Hermitian models for quantum friction, that is, the Caldirola-Kanai, the Kostin, and the Albrecht models, to discuss the tunneling problem with quantum friction. We mention that these Hamiltonians have been applied to a tunneling problem [37–42], but a systematic study, including a comparison among the models, has yet to be carried out with respect to tunneling probabilities. In this connection, we notice that Hasse has compared the three models for a free wave-packet propagation and for a damped harmonic oscillator. He has shown that the time dependence of the width of a Gaussian wave packet varies significantly from one model to another, while all of these three models lead to the same classical equation of motion [43]. It is therefore not obvious whether the three models lead to similar penetrabilities to each other.

The paper is organized as follows. In Sec. II we briefly introduce the three quantum friction models that we employ.

In Sec. III we present our results for penetrability of a one-dimensional barrier. To compare among the three models, we first carry out a detailed study on numerical accuracy of the calculations for a free wave-packet propagation. We then discuss the energy dependence of the penetrability obtained with each of these three models. We also discuss the results in terms of barrier distribution. We finally summarize the paper in Sec. III C.

II. QUANTUM FRICTION MODELS

A. Classical equation of motion

We consider a system with a particle whose mass is m, moving in a one-dimensional space q with a potential $V(q)$ and a linear frictional force. We here consider potential scattering, and regard q as the distance between the particle and the center of the potential. The classical equation of motion for the particle reads

$$\frac{dp}{dt} + \gamma_0 p + \frac{\partial V}{\partial q}(q) = 0, \tag{1}$$

where $p = m\dot{q}$ is the kinetic momentum, the dot denoting the time derivative, and γ_0 is a friction coefficient. We have assumed that the potential depends only on q. From the classical equation of motion, Eq. (1), the time derivative of the energy $E = p^2/2m + V(q)$ reads

$$\frac{dE}{dt} = -\frac{\gamma_0}{m} p^2. \tag{2}$$

In constructing the phenomenological quantum friction models, Eqs. (1) and (2) have been used as a guiding principle; that is, it is demanded that the time dependence of the expectation values obeys the same equations as Eqs. (1) and (2) [43].

B. The Caldirola-Kanai model

In the Caldirola-Kanai model, the Hamiltonian depends explicitly on time as [32,33]

$$H = \frac{\pi^2}{2m} e^{-\gamma_0 t} + V(q) e^{\gamma_0 t}, \tag{3}$$

where π is a canonical momentum conjugate to q. The canonical quantization $[q,\pi] = i\hbar$, with $p = \pi e^{-\gamma_0 t}$, leads to the desired equations,

$$\frac{d}{dt}\langle p\rangle + \gamma_0 \langle p\rangle + \left\langle \frac{\partial V}{\partial q}(q)\right\rangle = 0, \tag{4}$$

$$\frac{d}{dt}\langle E\rangle = -\frac{\gamma_0}{m}\langle p^2\rangle. \tag{5}$$

Here, the expectation value of an operator O is denoted as $\langle O\rangle = \int dq \psi^* O \psi$ with a wave function $\psi = \psi(q,t)$. p can be regarded as the kinetic momentum operator, because the relation $\langle p\rangle = m\,(d\langle q\rangle/dt)$ holds. Notice that the kinetic momentum operator depends explicitly on time in this model.

Because the Hamiltonian (3) is Hermitian, the probability is conserved with the continuity equation of

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial q} e^{-\gamma_0 t} = 0, \tag{6}$$

where $\rho = |\psi|^2$ and $J = (\hbar/m)\mathrm{Im}(\psi^* \partial\psi/\partial q)$ are the probability density and the current, respectively, Im denoting the imaginary part.

Because the kinetic momentum operator in this model depends explicitly on time, the commutation relation between the coordinate and the physical momentum is of the form

$$[q, p(t)] = i\hbar e^{-\gamma_0 t}. \tag{7}$$

Hence, the quantum fluctuation disappears as $t \gg 1/\gamma_0$. One may consider that this unphysical feature can be neglected if one considers only a short time behavior. However, the friction is not active in that time regime, because the factor $e^{-\gamma_0 t}$ determines how much the momentum is damped, and thus the dynamics may be rather trivial there.

C. The Kostin and the Albrecht models

In the Kostin and the Albrecht models, the momentum operator is kept time independent, but a nonlinear potential W is introduced in the Hamiltonian:

$$H = \frac{p^2}{2m} + V(q) + \gamma_0 W. \tag{8}$$

In the Kostin model, the nonlinear potential is taken to be [34]

$$W_{\mathrm{Ko}} = \frac{\hbar}{2i}\left(\ln\frac{\psi}{\psi^*} - \left\langle \ln\frac{\psi}{\psi^*}\right\rangle\right), \tag{9}$$

$$= \hbar[\mathrm{Im}(\ln\psi) - \langle \mathrm{Im}(\ln\psi)\rangle], \tag{10}$$

while in the Albrecht model it is taken as [35]

$$W_{\mathrm{Al}} = \langle p\rangle(q - \langle q\rangle). \tag{11}$$

With the canonical quantization, one obtains Eq. (4) together with

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial q} = 0 \tag{12}$$

for both Hamiltonians.

The energy dissipation for the Kostin model is given by

$$\frac{d}{dt}\langle E\rangle = -\frac{\gamma_0}{m}\left\langle \left(\frac{mJ}{\rho}\right)^2\right\rangle. \tag{13}$$

Kan and Griffin rederived the Kostin Hamiltonian from a fluid dynamics point of view [22,44]. In that context, mJ/ρ in Eq. (13) is the kinetic momentum, and hence Eq. (13) is similar to Eq. (2). For the Albrecht model, however, one obtains

$$\frac{d}{dt}\langle E\rangle = -\frac{\gamma_0}{m}\langle p\rangle^2, \tag{14}$$

as is desired. Notice that the energy dissipation is proportional to $\langle p^2\rangle$ in the Caldirola-Kanai and the Kostin models [see Eqs. (5) and (13)], while it is $\langle p\rangle^2$ in the Albrecht model. In the classical limit, these quantities are the same as each other, but they may differ in quantum mechanics.

In Ref. [43], Hasse discussed a generalization of the Albrecht model and suggested a better nonlinear potential, W, which reproduces the classical reduced frequency for a damped harmonic oscillator. For simplicity, however, we consider only the Albrecht model in this paper.

D. Generalization for a collision problem

In the original models for quantum friction, the friction constant γ_0 is treated to be a constant. When considering friction in a collision problem, however, we have to introduce a friction form factor $f(q)$, because the energy dissipation occurs only during interaction. That is, the form factor $f(q)$ vanishes outside the range of the potential, $V(q)$. A naive replacement of γ_0 in the model Hamiltonians with $\gamma_0 f(q)$ does not work owing to the q dependence in the form factor. Alternatively, in this paper we consider a time-dependent friction coefficient $\gamma(t)$ which vanishes after the interaction. In the simple form of $\gamma(t) = \gamma_0 f(\langle q \rangle_t)$, the dissipation continuously occurs even after the interaction if an incident wave is equally bifurcated into transmitted and reflected waves. To avoid this undesired behavior, we choose the form

$$\gamma(t) = \gamma_0 \langle f(q) \rangle_t. \tag{15}$$

In Ref. [41], Hahn and Hasse discussed a more complex form factor but it was shown that the behavior is quite similar to the simple one.

An extension to the time-dependent friction coefficient is obvious for the nonlinear potential models, just changing γ_0 to $\gamma(t)$ in Eq. (8). For the Caldirola-Kanai model, however, the following modification is necessary [45]:

$$H = \frac{\pi^2}{2m} e^{-\int_0^t dt' \gamma(t')} + V(q) e^{\int_0^t dt' \gamma(t')}. \tag{16}$$

Here we have assumed that the initial time is $t = 0$. The uncertainty relation is now changed from Eq. (7) to

$$[q, p(t)] = i\hbar\, e^{-\int_0^t dt' \gamma(t')}. \tag{17}$$

Because of these modifications, the three Hamiltonians are now nonlinear. It means that the superposition principle is violated. We are forced to admit this undesired property, because they are inevitable in the present formalism.

III. RESULTS

To calculate the tunneling probability with the three models discussed in the previous section, we integrate the time-dependent nonlinear Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi. \tag{18}$$

In what follows, we employ the same potential as in Ref. [46], that is,

$$V(q) = V_0 e^{-\frac{q^2}{2s^2}}, \tag{19}$$

with $V_0 = 100$ MeV and $s = 3$ fm. This potential somehow simulates the ^{58}Ni + ^{58}Ni reaction, and thus we take $mc^2 = 29 \times 938$ MeV.

A. Wave-packet tunneling without friction

Before we introduce the friction, we first discuss the time-dependent approach to quantum tunneling. For the calculation of the tunneling probability for the time-dependent nonlinear Hamiltonians, the usual time-independent approach, which imposes the asymptotic plane-wave boundary condition, would not be applicable. An alternative method is to make a wave packet propagate, then observe how it bifurcates after it passes the potential region.

A wave packet is a superposition of various waves, each of which has a different energy. Hence, to obtain the tunneling probability for a certain energy, one needs either to perform the energy projection [47,48] or to broaden the spatial distribution of the wave packet so that the energy distribution becomes narrow [49]. In the former method, the tunneling probability is calculated as the ratio of the energy distribution of a transmitted wave packet to that of the incident one at a fixed energy. This method, however, is not applicable in our case, because we do not know *a priori* how much energy is lost during a collision at each energy. Therefore, we shall employ the latter approach here. To this end, it is necessary to know how narrow the energy distribution should be in the wave packet to obtain meaningful results.

To clarify the effect of finite width in the energy distribution, we take the initial wave function in the energy space $\tilde{\psi}_0(E; E_i)$ with the Gaussian form,

$$|\tilde{\psi}_0(E; E_i)|^2 = \frac{1}{\sqrt{2\pi\sigma_E^2}} e^{-\frac{(E-E_i)^2}{2\sigma_E^2}}, \tag{20}$$

where E_i and σ_E are the mean energy and the width of the energy distribution, respectively. Multiplying $\tilde{\psi}_0(E; E_i)$ by $e^{ik(q-q_0)}$ with $E = \hbar^2 k^2/(2m)$ and making its Fourier transform into the coordinate space, one obtains the initial wave function in the coordinate space, $\psi(q, t = 0; E_i)$, which is consistent with the energy distribution given by Eq. (20). Such wave function has the mean position of q_0. Notice that the energy alone does not determine the direction of propagation of the wave packet. It is determined by the interval of the integration with respect to k. We consider a propagation of the wave packet from $q_0 < 0$ towards the positive q direction, and thus we make the integration from $k = 0$ to ∞. Even though this initial wave function is somewhat different from the one used in Refs. [48,49], we find that this form is more convenient to discuss a correspondence to the time-independent solutions [see Eq. (22) below].

By integrating the time-dependent Schrödinger equation (18) from $t = 0$ to $t = t_f$, by which time the bifurcation of the wave packet has been completed, we calculate the tunneling probability $T_{\rm wp}(E_i)$ as

$$T_{\rm wp}(E_i) = \int_0^\infty dq\, |\psi(q, t_f; E_i)|^2. \tag{21}$$

In implementing the time integration, it is helpful to introduce the dimensionless time $\tau \equiv t/t_0$, by measuring the time in units of a typical time scale of the problem, t_0. For this, we take t_0 as the time taken by a free classical particle to travel some distance L, that is, $t_0 = L/\sqrt{2E_i/m}$. We choose L so that the final mean position of the transmitted wave packet at $t = t_0$ is almost independent of E_i (and the friction coefficient, γ_0) for each parameter set.

When σ_E is small enough, it is expected that $T_{\rm wp}(E_i)$ is nearly the same as the tunneling probability obtained for a certain energy E_i, $T_{\rm ex}(E_i)$. More generally, the following

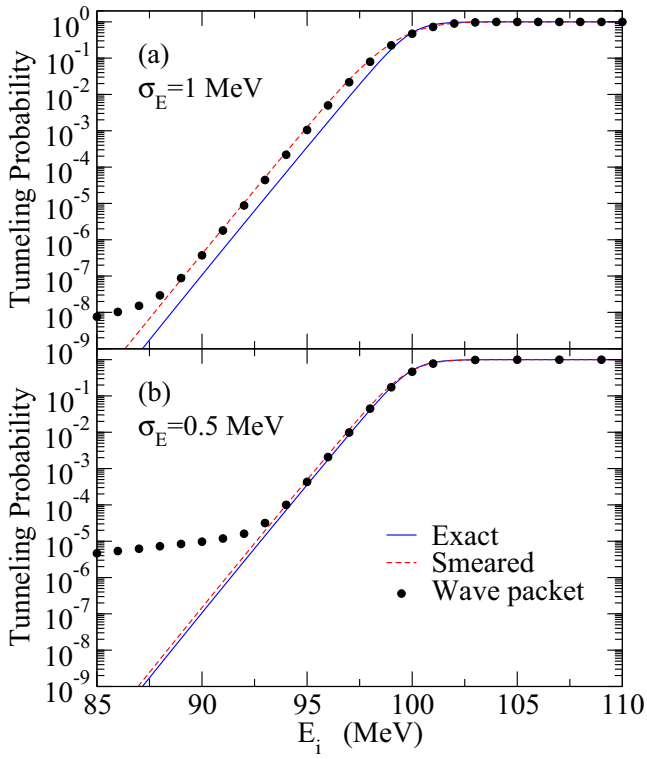

FIG. 1. Comparison of the tunneling probability of a one-dimensional potential obtained with several methods. The solid lines show the penetrability obtained with the time-independent method, while the solid circles show that with the time-dependent wave-packet method. The dashed lines denote the smeared tunneling probability, according to Eq. (22). Panel (a) is for the energy width of $\sigma_E = 1$ MeV in the wave packet, while panel (b) is for $\sigma_E = 0.5$ MeV.

relation between $T_{\rm wp}$ and $T_{\rm ex}$ should hold:

$$T_{\rm wp}(E_i) = \int_0^\infty dE |\tilde{\psi}_0(E; E_i)|^2 T_{\rm ex}(E). \tag{22}$$

Without friction, $T_{\rm ex}(E)$ can be calculated with the time-independent approach. The top panel of Fig. 1 shows the result for $\sigma_E = 1$ MeV. To solve the time-dependent Schrödinger equation with a wave packet, we employ the Crank-Nicholson method together with the tridiagonal matrix algorithm [50] with grid sizes of $\Delta\tau = 0.000\,25$ and $\Delta q = 0.01$ fm. We take a space of -100 fm $< q <$ 100 fm, and set $q_0 = -50$ fm and $L = 165$ fm. The solid line shows the penetrability obtained with the time-independent method, while the dots are obtained with the wave-packet method. The dashed line denotes the average penetrability according to Eq. (22). One can find that Eq. (22) is valid until the tunneling probability falls below 10^{-7}.

To improve the agreement between $T_{\rm wp}$ and $T_{\rm ex}$, one needs a smaller value of σ_E. The bottom panel of Fig. 1 shows the result for $\sigma_E = 0.5$ MeV. In this case, we enlarge the space to -150 fm $< q <$ 150 fm to accommodate a spatially wider wave packet. As one can see, the penetrability with the time-dependent wave-packet method is now in a good agreement with $T_{\rm ex}$ for the tunneling probability higher than 10^{-4}. We therefore use $\sigma_E = 0.5$ MeV for all the calculations shown below.

TABLE I. The dimensionless friction coefficient γ_0/c, c being the speed of light, for the weak and strong friction cases. All the values are given in units of 10^{-3}.

Strength	Caldirola-Kanai	Kostin	Albrecht
Weak ($E_{\rm loss} = 5$ MeV)	2.14	2.16	2.17
Strong ($E_{\rm loss} = 30$ MeV)	13.8	14.0	14.0

B. Free wave-packet evolution with friction

To discuss the value of a friction coefficient as well as numerical accuracy of the calculations, we next consider free wave packet in this section. As discussed in Sec. II, the potential and the corresponding friction form factor should have a similar range. In this paper we simply employ the same form for the form factor as that for the potential, Eq. (19),

$$f(q) = \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{q^2}{2s^2}}. \tag{23}$$

Here $f(q)$ is normalized so that γ_0 is interpreted as the strength of friction. Note that the dimension of γ_0 is altered from inverse time to velocity.

The friction strength γ_0 is determined based on the amount of energy loss. Because the energy loss depends on energy, we choose the barrier top energy, $E_i = 100$ MeV, as a reference. With the mean energy of the transmitted wave, E_f, the energy loss $E_{\rm loss}$ is given by $E_{\rm loss} = E_i - E_f$. We here consider weak and strong friction cases, for which $E_{\rm loss}$ is 5 and 30 MeV, respectively. These are realized when γ_0 is chosen, as listed in Table I.

Using these friction coefficients, the accuracy of integration of the time-dependent noninear Schrödinger equation is tested by checking how well the equation of motion, Eq. (4), is reproduced. The equation cannot be solved in the same way as in Sec. III A, because the matrix is no longer in a tridiagonal form owing to the nonlinearity. Instead, we carry out the numerical integration in the following way.

The discretized Schrödinger equation may be given by

$$i\hbar \frac{\psi^{n+1} - \psi^n}{\Delta t} = \frac{H^{n+1}\psi^{n+1} + H^n\psi^n}{2} \tag{24}$$

at the nth time grid. Here H is the Hamiltonian which depends on ψ. In our calculation, we simply neglect the time dependence of the Hamiltonian and obtain

$$i\hbar \frac{\psi^{n+1} - \psi^n}{\Delta t} = H^n \frac{\psi^{n+1} + \psi^n}{2}. \tag{25}$$

We integrate this equation with grid sizes of $\Delta\tau = 0.000\,25$, $\Delta q = 0.01$ fm for the Caldirola-Kanai and the Kostin models, and $\Delta\tau = 0.000\,15$, $\Delta q = 0.01$ fm for the Albrecht model.

Care must be taken in integrating the equation for the Kostin model. The nonlinear potential Eq. (10) is nothing but the phase of a wave function, and hence a naive estimation leads to discontinuities in the potential. However, one can estimate

the continuous phase by the definition [22] (see also Ref. [25])

$$\arg\psi(q,t) = \mathrm{Im}[\ln\psi(q,t)] + 2\pi(n_+ - n_-), \qquad (26)$$

where n_+ and n_- are the number of crossing the discontinuous points from π to $-\pi$ and from $-\pi$ to π, respectively. We take $q = 0$ as a reference and compute $n_+(n_-)$ by counting the point where the adjacent phase differs by less than -4 (more than 4).

Our numerical test is carried out for $E_i = 100$ MeV. We compare the following quantity with that of the no friction:

$$\mathrm{Accu} \equiv \frac{1}{\hbar}\left|\frac{d}{d\tau}\langle p\rangle + \gamma t_0\langle p\rangle\right|. \qquad (27)$$

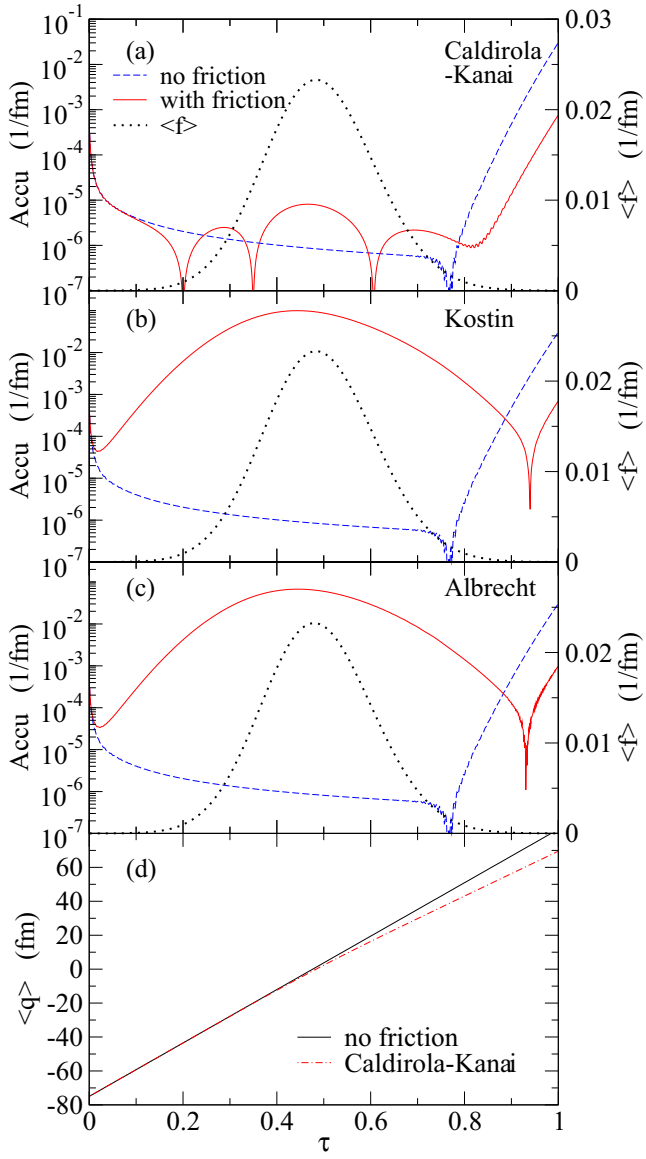

FIG. 2. (a)–(c) The numerical accuracy defined by Eq. (27) in the case of the strong friction. The dashed and the solid lines show the results without and with friction, respectively. The dotted lines are the expectation value of the form factor, $\langle f(q)\rangle$. (d) The time dependence of $\langle q\rangle$ for the Caldirola-Kanai model. All the calculations are performed with $L = 160$ fm.

Because we do not consider the potential in this section, this quantity vanishes if the equation of motion is fully satisfied. The accuracy for the strong friction case is shown in Figs. 2(a)–2(c) for the three friction models. The expectation value of the form factor, $\langle f\rangle$, is also shown to illustrate the effect of nonlinearity on numerical accuracy. The corresponding $\langle q\rangle$ as a function of τ is also shown in Fig. 2(d) for the Caldirola-Kanai model (the results for the other two models are almost the same and are not shown in the figure). We have verified that the probability is conserved within a numerical accuracy for all the calculations. It is found that the Caldirola-Kanai model can be integrated as accurately as the no friction case. In contrast, the reproduction of the equation of motion is less satisfactorily with the Kostin and the Albrecht models owing to the nonlinearity of the equations. This is expected if the nonlinearity owing to the form factor plays a much less important role as compared to the nonlinearity of the equation itself, because the nonlinearity of the Caldirola-Kanai model

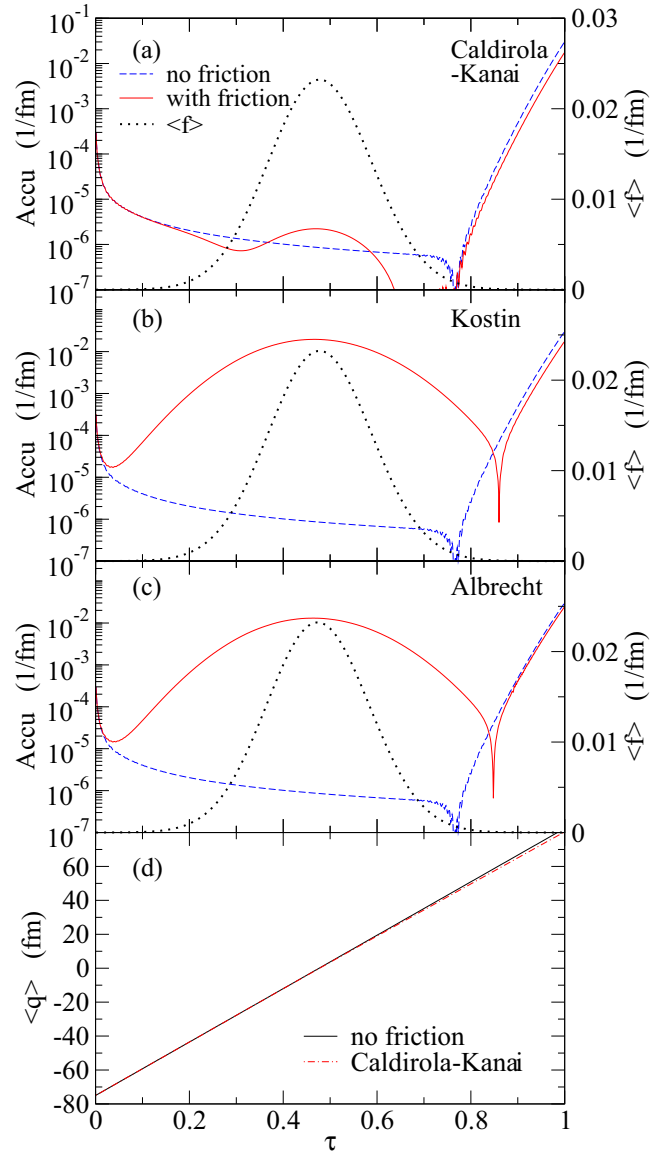

FIG. 3. Same as Fig. 2, but in the case of the weak friction.

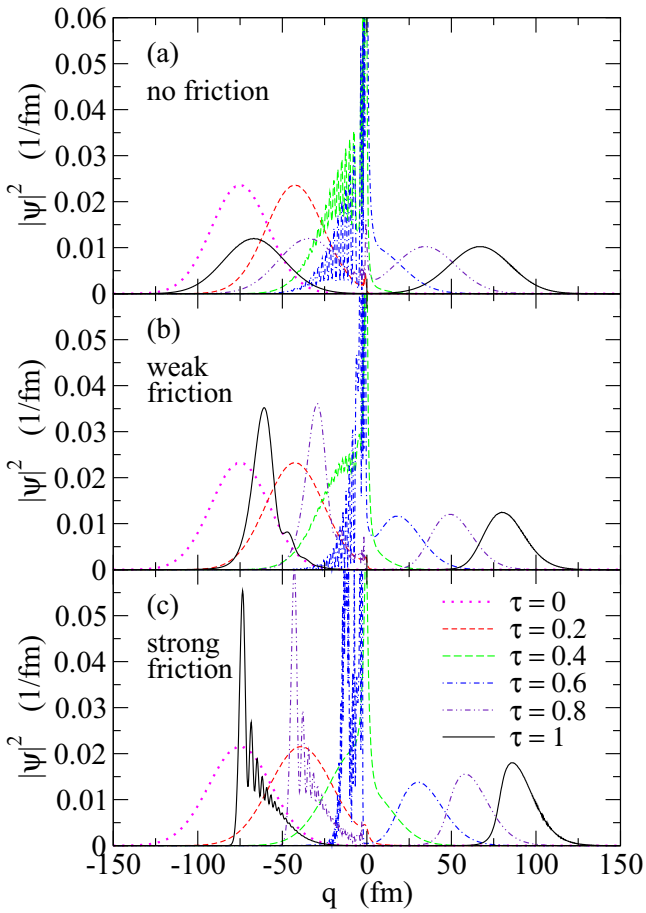

FIG. 4. The time evolution of a wave packet for a tunneling problem in the Caldirola-Kanai model. Panels (a), (b), and (c) correspond to the case of without friction, with weak friction, and with strong friction, respectively. The initial energy E_i and L are set to be $E_i = 100$ MeV and $L = 165$ fm for no friction, $E_i = 103$ MeV and $L = 165$ fm for weak friction, and $E_i = 120$ MeV and $L = 185$ fm for strong friction.

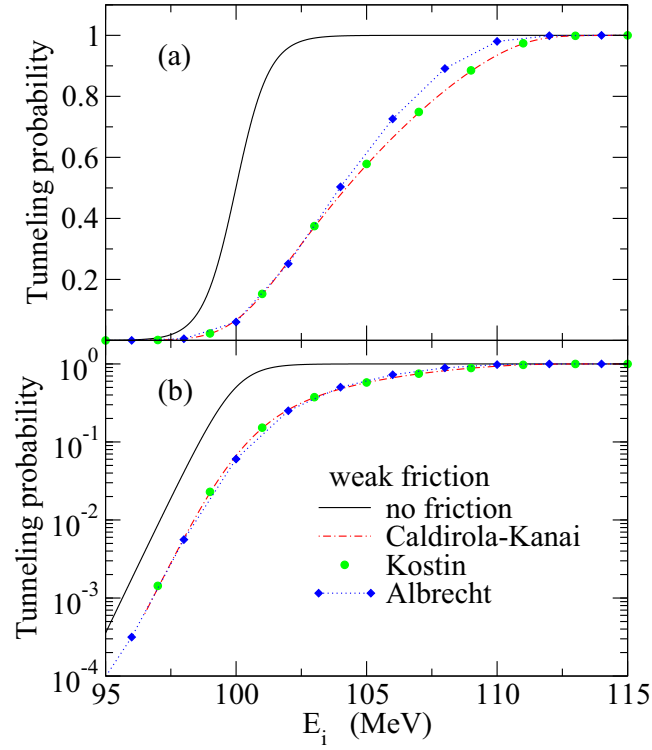

FIG. 5. The energy dependence of the tunneling probability for the Caldirola-Kanai (dot-dashed lines), the Kostin (solid circles), and the Albrecht (dotted lines with solid diamonds) models with the weak friction. Panel (a) is in the linear scale, while the panel (b) is in the logarithmic scale. The result without friction is also shown by the solid lines for a comparison.

is caused only by the friction form factor, Eq. (23). Actually, we have verified that the accuracy remains almost the same as Fig. 2 even without the form factor for all the models.

Notice that the increase of Accu at large τ is attributable to the finiteness of our space; that is, q is limited in the range of -150 fm $\leqslant q \leqslant 150$ fm. An accumulation of numerical errors is rather small, as no increase is observed when L is taken to be small enough so that the tail of the wave packet does not reach the edge of the box at $\tau = 1$ while keeping the number of step in the τ integration to be the same.

The accuracy of the nonlinear potential models does not improve even for the weak friction, as shown in Figs. 3(a)–3(c). Even though the absolute value of Accu is slightly reduced in this case, the τ dependence remains almost the same.

We should note that the damping from a bound excited state to the ground state can successfully be described with the Kostin model [22,51]. Actually, we also have verified it for a harmonic oscillator with the same grid sizes. An application to scattering problems seems more difficult with our present numerical method.

C. Quantum tunneling with friction

Let us now discuss dissipative quantum tunneling, i.e., quantum tunneling in the presence of friction. For an illustration, Fig. 4 shows the time-evolution of the wave packet for the Caldirola-Kanai model. The initial mean energy is chosen so that the transmitted wave packet has an appreciable amount. The calculations are performed with $E_i = 100$ MeV and $L = 165$ fm for no friction, $E_i = 103$ MeV and $L = 165$ fm for weak friction, and $E_i = 120$ MeV and $L = 185$ fm for strong friction. The behavior is quite similar also for the nonlinear potential models. Notice that the reflected and the transmitted wave packets at $\tau = 1$ largely deviate from a symmetric Gaussian shape in the presence of friction.

Figures 5 and 6 compare the tunneling probability as a function of energy obtained with the three friction models for the weak and the strong friction cases, respectively. We plot only the tunneling probability larger than 10^{-4}, according to the discussion in Sec. III A. One can see that the tunneling probability of the three models is nearly the same, even though there might be a possibility that the results of the Kostin and the Albrecht models suffer from numerical errors with the present setup of numerical calculations (see the discussion in Sec. III B). It is interesting to notice that the Caldirola-Kanai and the Kostin models lead to almost the same results to each other, while the result of the Albrecht model slightly deviates

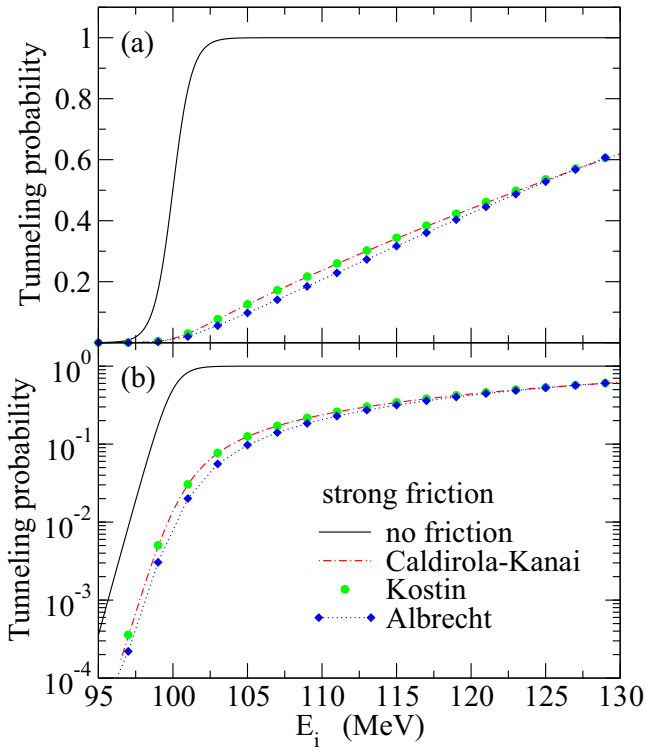

FIG. 6. Same as Fig. 5, but with the strong friction.

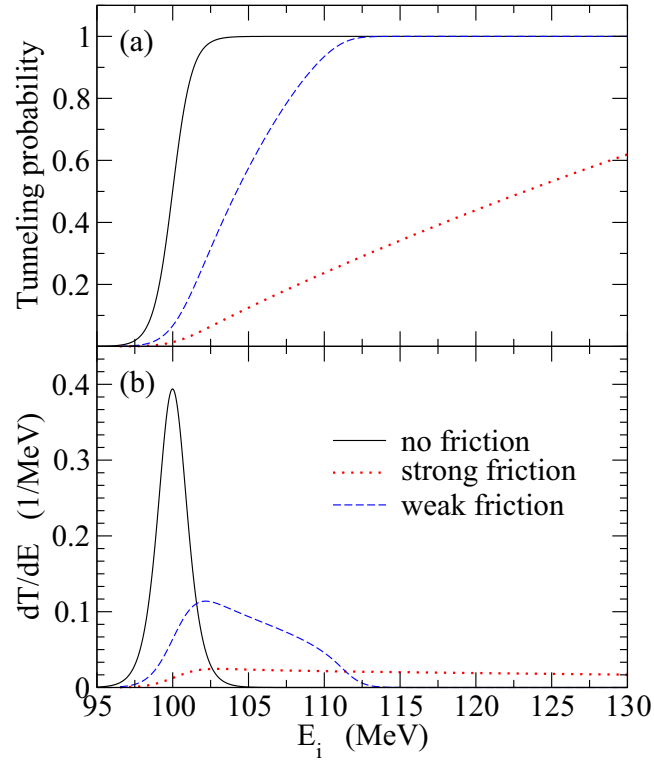

FIG. 7. (a) The penetrabilities for the Caldirola-Kanai model as a function of incident energy. The solid line corresponds to the no-friction case. The dashed and the dotted lines are for the weak and strong friction cases, respectively. (b) The corresponding barrier distribution defined as the first energy derivative of the penetrability.

from the other two models. Even though the exact cause of this different behavior is not known, a possible origin may be the fact that the energy dissipation is slightly different between the Albrecht model and the Caldirola-Kanai/Kostin models [see the discussion below Eq. (14)].

In what follows, we focus only on the result of the Caldirola-Kanai model. As can be seen in Fig. 7(a), the stronger the friction is, the lower the tunneling probability results in. This behavior is consistent with the results of Ref. [37] for a rectangular barrier unless the tunneling probability is extremely small. Reference [37] showed that the tunneling probability is not affected by friction at energies well below the barrier. Whereas the numerical accuracy has yet to be estimated to draw a conclusive conclusion concerning the role of friction in quantum tunneling at deep subbarrier energies, we simply could not confirm the result of Ref. [37] because a finite width in the wave packet prevents us to go into the deep subbarrier energy region [see Fig. 1(b)]. In Ref. [40], McCoy and Carbonell argued that the tunneling probability is either increased or decreased by fiction depending on the magnitude of the barrier height and width. We do not confirm their results either, partly because we do not include the fluctuation term in the Hamiltonian.

To gain a deeper insight into the role of friction in quantum tunneling, we next discuss a barrier distribution. In the field of heavy-ion subbarrier fusion reactions, the so-called fusion barrier distribution has been widely used in analyses of experimental data [2,52]. This quantity is defined as the second energy derivative of the product of the incident energy E and fusion cross sections $\sigma_{\rm fus}$, that is, $d^2(E\sigma_{\rm fus})/dE^2$ [53], and has provided a convenient representation to study the underlying dynamics of subbarrier fusion reactions. For the tunneling problem, this quantity corresponds to the first energy derivative of the penetrability, dT/dE [3]. The barrier distribution for the Caldirola-Kanai model is shown in Fig. 7(b). Whereas the barrier distribution shows a symmetric peak in the case of no friction, some strength is shifted towards higher energies as the strength of the friction increases and the barrier distribution becomes structured. It is interesting to notice that a similar behavior has been obtained in coupled-channels calculations for fusion in relatively heavy systems, such as ^{100}Mo + ^{100}Mo [54].

The barrier distribution indicates that the energy damping during tunneling results in a increased effective barrier, whose height is thus energy dependent and is determined by the strength of friction. This leads us to two different points of view for dissipative quantum tunneling. From one view point, the incident energy is damped by friction, while a wave packet traverses towards a fixed barrier. This can be interpreted in a different way because the effective barrier increases dynamically owing to the friction for a fixed value of incident energy. The barrier distribution shown in Fig. 7 well represents this dynamical point of view of friction.

IV. SUMMARY

We have investigated the effects of friction on quantum tunneling by applying the three friction models—the

Caldirola-Kanai, the Kostin, and the Albrecht models—to a one-dimensional tunneling problem. We have studied the energy dependence of the tunneling probability obtained as the barrier penetration rate of a wave packet, whose initial energy variance is set to be small enough. To limit a region where the dissipation is active, we have introduced the time-dependent friction coefficient. We have shown that the friction tends to prevent the wave packet from penetrating the barrier, and thus the penetrability decreases as a function of the strength of friction. We have found that the three models lead to similar penetrabilities to each other. We have also discussed the effect of friction on quantum tunneling in terms of barrier distribution and have shown that the barrier distribution becomes structured owing to friction by shifting effective barriers towards higher energies. Among the three models that we considered in this paper, we have found that the numerical accuracy can be most easily handled with the Caldirola-Kanai model.

Very recently, it has been found experimentally that heavy-ion multinucleon transfer processes in ^{16,18}O, ^{19}F + ^{208}Pb reactions populate highly excited states in the targetlike nuclei [55]. One may be able to describe such processes by extending the friction models considered in this paper to multichannel cases. We are now working towards this direction, and we will report our results in a separate paper.

Another interesting direction is to include the random force term to the quantum friction Hamiltonians, which leads to the fluctuation effect. For this purpose, a proper quantization of the fluctuation term will be needed. It would be intriguing to investigate its effect on quantum tunneling, as has been done in Ref. [31] using a system-plus-bath model. We leave it as a future work.

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