Extrapolation of scattering data to the negative-energy region

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Explicit analytic expressions are derived for the effective-range function for the case when the interaction is represented by a sum of the short-range square-well and long-range Coulomb potentials. These expressions are then transformed into forms convenient for extrapolating to the negative-energy region and obtaining the information about bound-state properties. Alternative ways of extrapolation are discussed. Analytic properties of separate terms entering these expressions for the effective-range function and the partial-wave scattering amplitude are investigated.

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I. INTRODUCTION

Asymptotic normalization coefficients (ANC) determine the asymptotics of bound-state nuclear wave functions in binary channels. ANCs are proportional to vertex constants, which determine on-shell matrix elements of virtual $a \leftrightarrow b + c$ processes and are related directly to the residue in energy of the elastic $b + c$ scattering amplitude at the pole corresponding to the bound state of nucleus a [1]. ANCs are fundamental nuclear characteristics important both in nuclear reaction and nuclear structure physics. They are used actively in analysis of nuclear reactions within various approaches. The ANCs extracted from the analysis of one process can be used to predict features of other ones. Comparing empirical values of ANCs with theoretical ones allows one to evaluate the quality of a model.

The ANC $C^{(l)}_{a\to bc}$ for the virtual decay $a \to b + c$, where l is the relative orbital angular momentum of b and c in the bound state, determines the probability of the $\{bc\}$ configuration in nucleus a at distances greater than the radius of nuclear interaction. Thus, the ANCs naturally appear in expressions for the cross sections of nuclear reactions between charged particles at low energies when, due to the Coulomb repulsion, the reactions occur at large distances. Astrophysical nuclear reactions represent the most important type of such reactions. The role of ANCs in nuclear astrophysics was first discussed in Refs. [2,3], where it was emphasized that the ANC determines the overall normalization of peripheral radiative capture reactions (see also Refs. [4,5]). Thus, the ANC method can be employed as an indirect technique in nuclear astrophysics. The ANCs can be used in evaluating the radiative width of a resonance, decaying to a bound state [6]. An instructive example of using ANCs in nuclear structure studies is the application of the ANC to determine the radii of halo nuclei [7]. Thus, it is important to know the values of ANCs.

In principle, values of ANCs can be deduced from the microscopic calculations of wave functions for corresponding nuclear systems. However, such calculations are quite involved even for few-nucleon systems [8]. The ANCs, in contrast to binding energies, cannot be directly measured. Nevertheless, there is an indirect way to determine the ANC from experiment: the ANC $C^{(l)}_{a\to bc}$ can be determined from experimental data by extrapolating, in the center-of-mass (c.m.) energy E, the partial-wave amplitude of elastic $b + c$ scattering, obtained by the phase-shift analysis, to the pole corresponding to the bound state a and lying at $E < 0$. The most natural procedure for such extrapolation is the analytic approximation of the experimental values of the effective-range function (ERF) with the subsequent continuation to the pole position. The ERF method has been successfully employed to determine the ANCs for bound (as well as resonant) nuclear states in a number of works (see, e.g., Refs. [9–11] and references therein).

The ERF is expressed in terms of scattering phase shifts. In the case of charged particles, the ERF for the short-range interaction should be modified. Such modification generates additional terms in the ERF. These terms depend only on the Coulomb interaction and may far exceed, in the absolute value, the informative part of the ERF containing the phase shifts. This fact hampers the practical procedure of the analytic continuation and affects its accuracy. It was suggested in Ref. [12] to use for the analytic continuation the quantity $\Delta_l(E)$ [which is defined below (see Sec. II)] rather than the ERF. $\Delta_l(E)$ does not contain the pure Coulomb terms. However, the validity of employing $\Delta_l(E)$ was not obvious, which resulted in some discussion since $\Delta_l(E)$, contrary to the ERF, possesses an essential singularity at $E = 0$.

In this work, instead of using the conventional parametrization of the scattering phase shift in terms of the ERF fitting parameters, we introduce the parametrization of the ERF scattering phase shift using an analytic solution of the Schrödinger equation at $E > 0$ with the adopted potential in the form of the square-well plus Coulomb interaction. In this approach our results are vigorous and obtained without any approximation. One may think that the approach is over-simplified because of the adopted square-well potential. However, it is not. To support the claim, we would like to remind to the readers a fundamental physics idea: a probe with wavelength $\lambda = 1/k$, where k is the relative momentum of the

interacting particles, is not sensitive to the structural details of the interaction potential at distances $r \ll \lambda$ [13]. Hence, in the region of very low energies, which we always cross to when extrapolating data down to the threshold, the results become insensitive to the specific shape of the used potential, whether it is Woods-Saxon, square-well, δ function, or anything else. This idea is in the foundation of the effective field theory [13]. That makes our results quite general despite using a simplified potential.

In the present work the obtained parametrization of the ERF scattering phase shift in terms of the analytic functions obtained from the solution of the Schrödinger equation at $E > 0$ is used for analytical continuation of the scattering phase shift to the region of the negative energies $E < 0$ to the bound-state pole of the scattering amplitude. Various representations of the Coulomb wave functions and corresponding expressions for the partial-wave scattering amplitudes and the ERF are considered and the most effective algorithm of analytic continuation is identified. It is shown that, although function $\Delta_l(E)$ possesses the essential singularity at $E = 0$, nevertheless, it can be analytically continued along the real axis of E to the region of negative energies.

The paper is organized as follows. Section II contains the general formalism of the elastic scattering for the superposition of the short-range and Coulomb interactions, which is necessary for the subsequent discussion. The various versions of the specific expressions of the scattering phase shifts in the case of the square-well short-range potential are considered in Sec. III. The results of calculations within the used model are presented in Sec. IV.

We use the system of units in which $\hbar = c = 1$ throughout the paper.

II. GENERAL FORMALISM OF SCATTERING IN THE PRESENCE OF THE COULOMB INTERACTION

The full amplitude of the elastic scattering of particles b and c in the presence of the Coulomb and short-range (nuclear) interactions is written as the sum of the pure Coulomb and Coulomb-nuclear amplitudes (the Coulomb interaction is taken to be repulsive):

$$f(\vec{k}) = f_C(\vec{k}) + f_{NC}(\vec{k}), \tag{1}$$

$$f_C(\vec{k}) = \sum_{l=0}^{\infty}(2l+1)\frac{\exp(2i\sigma_l)-1}{2ik}P_l(\cos\theta), \tag{2}$$

$$f_{NC}(\vec{k}) = \sum_{l=0}^{\infty}(2l+1)\exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}P_l(\cos\theta). \tag{3}$$

Here, $\vec{k}$ is the relative momentum of b and c, θ is the center-of-mass scattering angle, $\sigma_l = \arg\Gamma(l+1+i\eta)$ and δ_l are the pure Coulomb and Coulomb-nuclear phase shifts, respectively, and $\Gamma(z)$ is the Γ function.

$$\eta = Z_b Z_c e^2 \mu / k \tag{4}$$

is the Coulomb (Sommerfeld) parameter for the $b+c$ scattering state with the relative momentum related to the energy by $k = \sqrt{2\mu E}$, and μ, $Z_b e$, and $Z_c e$ are the reduced mass and the electric charges of b and c.

The behavior of the Coulomb-nuclear partial-wave amplitude $f_l = (\exp(2i\delta_l)-1)/2ik$ is irregular near $E = 0$. Therefore, one has to introduce the renormalized Coulomb-nuclear partial-wave amplitude $\tilde{f}_l$ [14–16],

$$\tilde{f}_l = \exp(2i\sigma_l)\frac{\exp(2i\delta_l)-1}{2ik}\left[\frac{l!}{\Gamma(l+1+i\eta)}\right]^2 e^{\pi\eta}. \tag{5}$$

Equation (5) can be rewritten as

$$\tilde{f}_l = \frac{\exp(2i\delta_l)-1}{2ik}C_l^{-2}(\eta), \tag{6}$$

where $C_l(\eta)$ is the Coulomb penetration factor (or Gamow factor),

$$C_l(\eta) = \left[\frac{2\pi\eta}{\exp(2\pi\eta)-1}v_l(\eta)\right]^{1/2}, \tag{7}$$

$$v_l(\eta) = \prod_{n=1}^{l}(1+\eta^2/n^2)\ (\text{for}\, l>0), \quad v_0(\eta) = 1. \tag{8}$$

The amplitude $\tilde{f}_l$ can be expressed in terms of the Coulomb-modified ERF $K_l(E)$ [14,16],

$$\tilde{f}_l = \frac{k^{2l}}{K_l(E) - 2\eta k^{2l+1}h(\eta)v_l(\eta)} \tag{9}$$

$$= \frac{1}{kC_l^2(\eta)(\cot\delta_l - i)} \tag{10}$$

$$= \frac{1}{v_l^2\Delta_l(E) - ikC_l^2(\eta)}, \tag{11}$$

where

$$K_l(E) = k^{2l+1}\left[C_l^2(\eta)(\cot\delta_l - i) + 2\eta h(k)v_l(\eta)\right], \tag{12}$$

$$h(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta), \tag{13}$$

$$\Delta_l(E) = kC_0^2(\eta)\cot\delta_l, \tag{14}$$

and $\psi(x)$ is the digamma function.

It was shown in Ref. [14] that function $K_l(E)$ defined by Eq. (12) is analytic near $E = 0$ and can be expanded into Taylor series in E. In the absence of the Coulomb interaction ($\eta = 0$) $K_l(E) = k^{2l+1}\cot\delta_l(k)$.

It should be noted that amplitude $\tilde{f}_l(E)$ possesses the essential singularity at $E = 0$. Nevertheless, analytic properties of $\tilde{f}_l$ on the real axis of the physical sheet of E are analogous to the ones of the partial wave scattering amplitude for the short-range potential, and it can be analytically continued into the negative-energy region.

If the $b+c$ system involves, in the partial wave l, the bound state a with the binding energy $\varepsilon = \varkappa^2/2\mu > 0$, then the amplitude $\tilde{f}_l$ has a pole at $E = -\varepsilon$. The residue of $\tilde{f}_l$ at

this point is expressed in terms of the ANC $C^{(l)}_{a\to bc}$ [15],

$$\operatorname{res}\tilde{f}_l(E)|_{E=-\varepsilon} = \lim_{E\to-\varepsilon}[(E+\varepsilon)\tilde{f}_l(E)] \tag{15}$$

$$= -\frac{1}{2\mu}\left[\frac{l!}{\Gamma(l+1+\eta_b)}\right]^2\left[C^{(l)}_{a\to bc}\right]^2, \tag{16}$$

where $\eta_b = Z_bZ_ce^2\mu/\varkappa$ is the Coulomb parameter for the $b+c$ bound state a.

III. PHASE SHIFTS FOR THE SUM OF THE SQUARE-WELL AND COULOMB POTENTIALS

The square-well potential is of the form

$$V(r) = \begin{cases} -V_0 & \text{if } 0 \leqslant r \leqslant R \\ 0 & \text{if } r > R \end{cases}, \tag{17}$$

where r is the distance between interacting particles, R is the radius of the square well, and $V_0 > 0$ is its depth.

The Schrödinger equation describing the system under consideration is

$$\frac{d^2u_l(r)}{dr^2} + 2\mu\left[E - \frac{l(l+1)}{2\mu r^2} - \frac{Z_bZ_ce^2}{r} - V(r)\right]u_l(r) = 0. \tag{18}$$

Let us introduce the notations: $\alpha_1 = Z_bZ_ce^2\mu$, $\eta_1 = \alpha_1/K$, $K = \sqrt{2\mu(E+V_0)}$. The solution of Eq. (18) in the inner ($r < R$) and external ($r > R$) regions are given by

$$R^{\text{in}}_l(r) = \frac{u_l(r)}{r} = \text{const}\frac{F_l(\eta_1, Kr)}{Kr}, \tag{19}$$

$$R^{\text{ext}}_l(r) = A_l[\chi^{(-)}_l(\eta,kr) - S_l\chi^{(+)}_l(\eta,kr)]/r, \tag{20}$$

$$\chi^{(\pm)}_l(\eta,kr) = G_l(\eta,kr) \pm iF_l(\eta,kr), \tag{21}$$

$$S_l = e^{2i\delta_l}. \tag{22}$$

In Eqs. (19) and (20), $F_l(\eta,\rho)$ and $G_l(\eta,\rho)$ are the regular and irregular Coulomb functions, respectively [17]. If the Coulomb interaction is turned off ($\eta = 0$), then

$$F_l(0,kr) = krj_l(kr), \quad G_l(0,kr) = -kry_l(kr), \tag{23}$$

where $j_l(x) = \sqrt{\pi/2x}J_{l+1/2}(x)$ and $y_l(x) = \sqrt{\pi/2x}Y_{l+1/2}(x)$ are spherical Bessel and Neumann functions, respectively.

Now we introduce the functions

$$\hat{F}_{l,\eta}(k,r) = F_l(\eta,kr)/kr, \tag{24}$$

$$\hat{G}_{l,\eta}(k,r) = -G_l(\eta,kr)/kr, \tag{25}$$

$$\tilde{F}_{l,\eta}(k,r) = \hat{F}_{l,\eta}(k,r)/[k^lC_l(\eta)], \tag{26}$$

$$\tilde{G}_{l,\eta}(k,r) = \hat{G}_{l,\eta}(k,r)k^{l+1}C_l(\eta), \tag{27}$$

$$\tilde{G}^{(-)}_{l,\eta}(k,r) = k^{l+1}C_l(\eta)[\hat{G}_{l,\eta}(k,r) - i\hat{F}_{l,\eta}(k,r)]$$

$$= \tilde{G}_{l,\eta}(k,r) - ik^{2l+1}C^2_l(\eta)\tilde{F}_{l,\eta}(k,r). \tag{28}$$

In Eqs. (26)–(28) the penetration factor $C_l(\eta)$ is defined by Eq. (7).

Note that $\tilde{F}_l(k,r)$ is regular at $E = k^2/2\mu = 0$, whereas $\tilde{G}_l(k,r)$ possesses the Coulomb essential singularity at $E = 0$ and behaves irregularly at $E \to -0$ [18]. As to $\tilde{G}^{(-)}_l$, it is a smooth function on the real axis of E.

The phase shifts δ_l are found from the condition of equality of logarithmic derivatives of $R^{\text{in}}_l(r)$ and $R^{\text{ext}}_l(r)$ at $r = R$:

$$\frac{1}{R^{\text{in}}_l(R)}\frac{dR^{\text{in}}_l(R)}{dR} = \frac{1}{R^{\text{ext}}_l(R)}\frac{dR^{\text{ext}}_l(R)}{dR}. \tag{29}$$

In this equation and hereafter, $d\psi(R)/dR \equiv d\psi(r)/dr|_{r=R}$.

Using Eqs. (19), (20), and (29), we get

$$\cot\delta_l = \frac{\frac{d\hat{G}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{G}_{l,\eta}(k,R)}{\frac{d\hat{F}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{F}_{l,\eta}(k,R)}. \tag{30}$$

IV. EFFECTIVE RANGE FUNCTION

According to Eqs. (12) and (30), the part of the ERF depending on phase shifts is of the form

$$k^{2l+1}C^2_l(\eta)\cot\delta_l = k^{2l+1}C^2_l(\eta)\left[\frac{\frac{d\hat{G}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{G}_{l,\eta}(k,R)}{\frac{d\hat{F}_{l,\eta}(k,R)}{dR}\hat{F}_{l,\eta_1}(K,R) - \frac{d\hat{F}_{l,\eta_1}(K,R)}{dR}\hat{F}_{l,\eta}(k,R)}\right]. \tag{31}$$

We transform Eq. (31) using the modified Coulomb functions $\tilde{F}_{l,\eta}$, and $\tilde{G}_{l,\eta}$ introduced earlier to get

$$k^{2l+1}C^2_l(\eta)\cot\delta_l = \frac{\frac{d\tilde{G}_{l,\eta}(k,R)}{dR}\tilde{F}_{l,\eta_1}(K,R) - \frac{d\tilde{F}_{l,\eta_1}(K,R)}{dR}\tilde{G}_{l,\eta}(k,R)}{\frac{d\tilde{F}_{l,\eta}(k,R)}{dR}\tilde{F}_{l,\eta_1}(K,R) - \frac{d\tilde{F}_{l,\eta_1}(K,R)}{dR}\tilde{F}_{l,\eta}(k,R)}. \tag{32}$$

The denominator in Eq. (32) does not possess the Coulomb singularities at $E > -V_0$; however, the function $\tilde{G}_l(k,R)$ is singular at $E = 0$ (see Sec. III). The singularities of $k^{2l+1}C^2_l(\eta)\cot\delta_l$ can be singled out using Eq. (28) as

$$k^{2l+1}C^2_l(\eta)\cot\delta_l = \frac{\frac{d\tilde{G}^{(-)}_{l,\eta}(k,R)}{dR}\tilde{F}_{l,\eta_1}(K,R) - \frac{d\tilde{F}_{l,\eta_1}(K,R)}{dR}\tilde{G}^{(-)}_{l,\eta}(k,R)}{\frac{d\tilde{F}_{l,\eta}(k,R)}{dR}\tilde{F}_{l,\eta_1}(K,R) - \frac{d\tilde{F}_{l,\eta_1}(K,R)}{dR}\tilde{F}_{l,\eta}(k,R)} + ik^{2l+1}C^2_l(\eta). \tag{33}$$

The first term in the right-hand side of Eq. (33) is regular at $E = 0$, whereas the second term has an essential singularity at this point.

In what follows, the specific properties of different parts of the ERF and the partial-wave scattering amplitude will be illustrated by numerical calculations applying to the $d+\alpha$ system in the S state ($l = 0$). This system involves one bound state corresponding to the ground state of the ^{6}Li nucleus. An accurate bound-state information is required, e.g., for

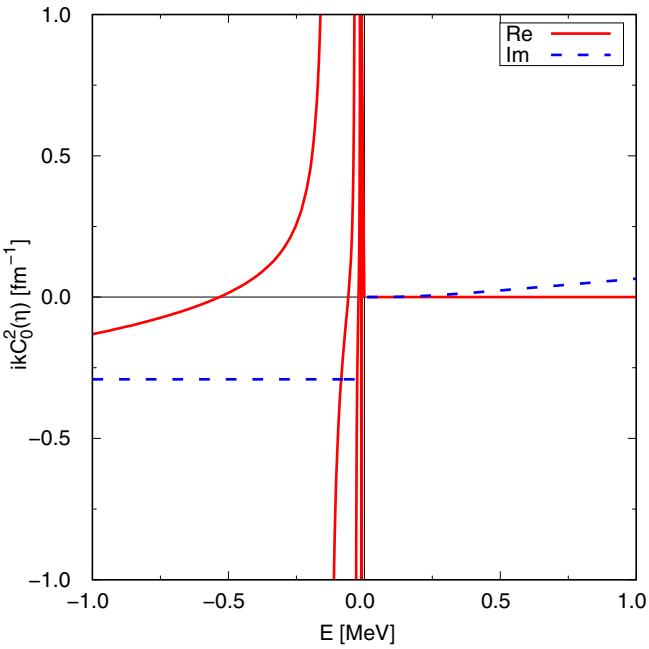

FIG. 1. Functional dependence of $ikC_0^2(\eta)$ on energy E.

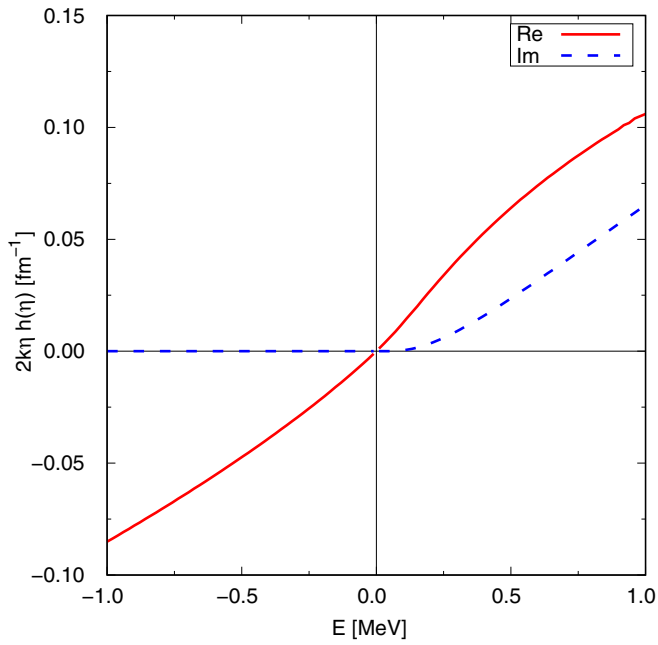

FIG. 2. Functional dependence of $2k\eta h(\eta)$ on energy E.

modeling astrophysical $\alpha + d \to {}^6\mathrm{Li} + \gamma$ radiative capture reaction [19–22]. The following parameters are used in the calculations: $m_\alpha = 3755.58$ MeV, $m_d = 877.79$ MeV, $V_0 = 7.64386$ MeV, $R = 3.73473$ fm. The values of V_0 and R were fitted to the values of the binding energy $\varepsilon = \varkappa^2/2\mu = 2.409$ MeV and the dimensionless ANC $\tilde{C}^{(0)}_{{}^6\mathrm{Li}\to\alpha d} = C^{(0)}_{{}^6\mathrm{Li}\to\alpha d}/\sqrt{2\varkappa} = 2.29$ of ^{6}Li in the $d + \alpha$ channel obtained by solving the Faddeev equations for ^{6}Li without the Coulomb interaction [9].

Consider first the features of the function $ik^{2l+1}C_l^2(\eta)$ which enters the expression for the ERF. The energy dependence of this function at $l = 0$ is presented in Fig. 1. It is seen that the imaginary part of $ikC_0^2(\eta)$ is constant at $E < 0$. This property can be rigorously proved. Making use of the definition Eq. (7) for $C_l(\eta)$ and the notations $\eta_b = \alpha_1/\varkappa$, $\alpha_1 = Z_b Z_c e^2/\mu$, one obtains at $E < 0$,

$$ikC_0^2(\eta) = \frac{2\pi i\alpha_1}{e^{-2\pi i\eta_b} - 1} = \frac{2\pi i\alpha_1 e^{\pi i\eta_b}}{e^{-\pi i\eta_b} - e^{\pi i\eta_b}} = -\pi\alpha_1 \frac{e^{\pi i\eta_b}}{\sin\pi\eta_b} = -\pi\alpha_1\cot\pi\eta_b - \pi\alpha_1 i. \quad (34)$$

It follows from Eq. (34) that $\mathrm{Re}[ikC_0^2(\eta)]$ possesses poles at $\eta_b = 1,2,\ldots$, corresponding to $\varkappa = \alpha_1/n$ $(n = 1,2,\ldots)$, and zeros at $\eta_b = 1/2,3/2,5/2\ldots$, corresponding to $\varkappa = 2\alpha_1/n$ $(n = 1,3,5,\ldots)$. Another function that appears in the expression for the ERF and helps regularize it is $2k\eta h(\eta)$. Its plot is shown in Fig. 2.

Now we demonstrate that all three expressions given by Eqs. (31), (32), and (33) can be used to calculate $k^{2l+1}C_l^2(\eta)\cot\delta_l$ and then compare the accuracy achieved by these expressions. To this end, the ERF $K_0(E)$ of Eq. (12) was calculated at $E = 5$ MeV and $E = -5$ MeV. The calculations were performed to ten significant digits. The results of the calculations are presented in Table I. The exact ERF should be real both at positive and negative energies. It is seen from the table that all three versions lead to similar results. However, as to the computational side, the versions of Eqs. (32) and (33) exploiting the modified Coulomb functions are preferable, since they lead to more accurate results at negative energies. The advantage of these versions increases further as E approaches zero. Note that the version of Eq. (33) is the most accurate out of all three versions. All numerical results for $\cot\delta_l$ discussed below are obtained using this version.

The most important part of the ERF is the function $k^{2l}v_l^2\Delta_l(E) = k^{2l+1}C_l^2(\eta)\cot\delta_l$. Its energy dependence at $l = 0$ is presented in Fig. 3. The function is complex at $E < 0$ and displays the irregularities, which are concentrated near zero from the left. It is seen that $\mathrm{Im}[kC_0^2(\eta)\cot\delta_0]$ is constant at $E < 0$. Moreover, this constant is exactly equal to $\mathrm{Im}[ikC_0^2(\eta)]$ at $E < 0$ [see Fig. 1 and Eq. (34)].

TABLE I. Effective-range functions calculated using Eqs. (31), (32), and (33).

Version	$K_0(5)$	$K_0(-5)$
Eq. (31)	$0.3120231954 - i0.14 \times 10^{-8}$	$-0.2802558905 - i0.63 \times 10^{-4}$
Eq. (32)	$0.3120231949 - i0.40 \times 10^{-9}$	$-0.2802915315 - i0.10 \times 10^{-9}$
Eq. (33)	$0.3120231949 - i0.40 \times 10^{-9}$	$-0.2802915316 + i0.14 \times 10^{-10}$

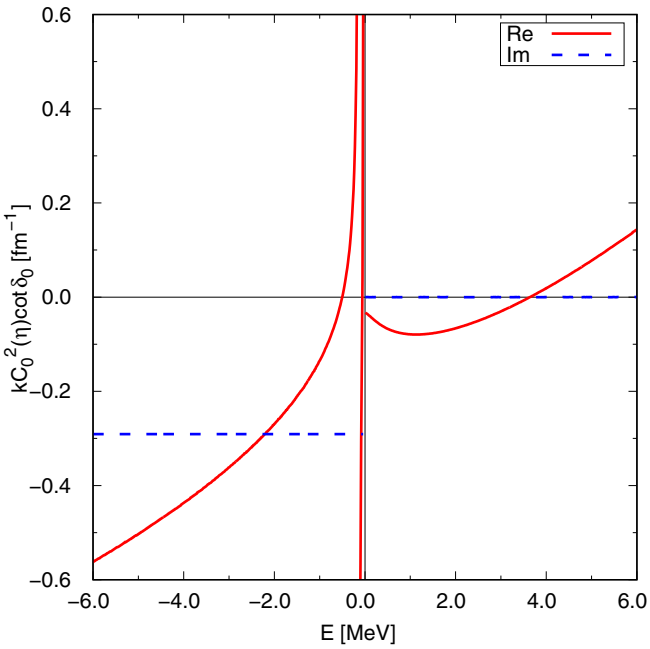

FIG. 3. Functional dependence of $kC_0^2(\eta)\cot\delta_0$ on energy E.

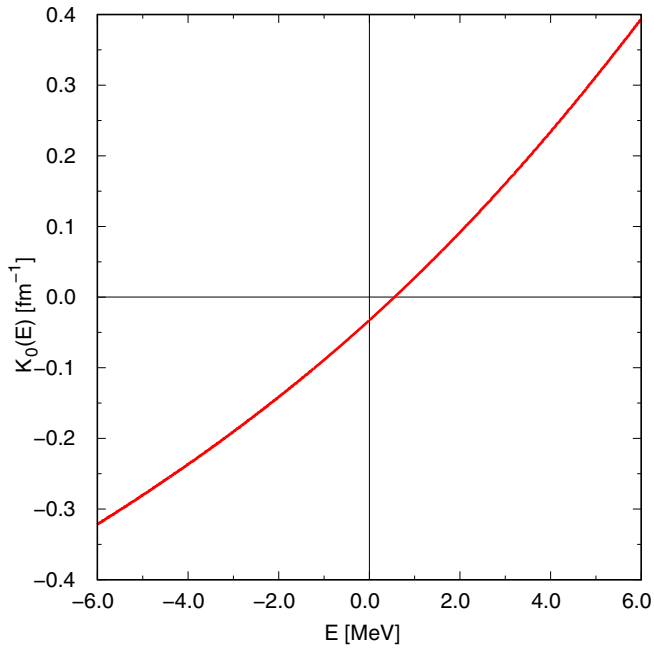

FIG. 5. Dependence of ERF $K_0(E)$ on energy E. The imaginary part is 0.

Consider now the function $D_l(E) \equiv k^{2l+1}C_l^2(\eta)(\cot\delta_l - i)$. Its plot is shown in Fig. 4 for $l = 0$. The function $D_0(E)$ is complex at positive energies and real at negative energies. Furthermore, as is seen from Figs. 1, 3, and 4, the irregularities of functions $kC_0^2(\eta)\cot\delta_0$ and $ikC_0^2(\eta)$ at $E \to -0$ disappear in the function $D_0(E)$, which is their difference. Note that $D_0(E)$ is the inverse of the renormalized partial-wave

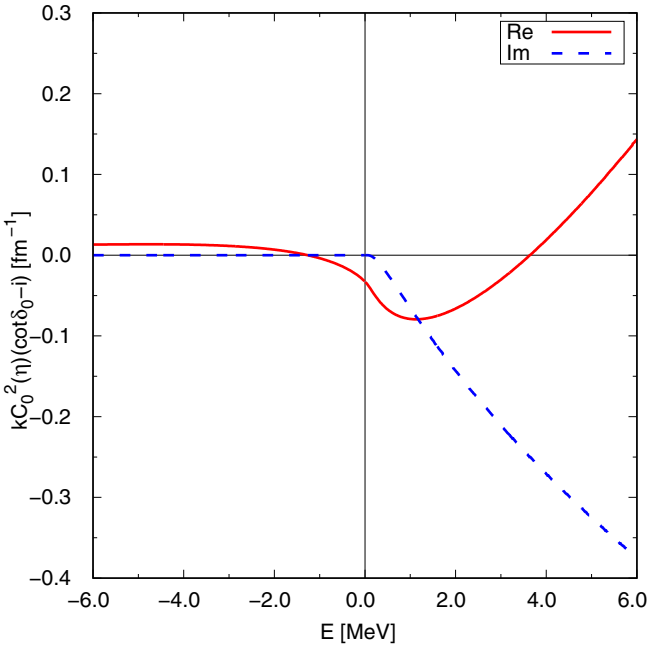

FIG. 4. Functional dependence of $kC_0^2(\eta)(\cot\delta_0 - i)$ on energy E.

Coulomb-nuclear amplitude $\tilde{f}_0$, see Eq. (9): $D_0(E) = (\tilde{f}_0)^{-1}$ [generally $D_l(E) = k^{2l}(\tilde{f}_l)^{-1}$].

Thus, the behavior of $D_0(E)$ shown in Fig. 4 corroborates the above assertion that on the real E axis the analytical properties of the amplitude $\tilde{f}_0$ are similar to those of the partial-wave amplitude of scattering from a short-range potential. Hence, $\tilde{f}_0$ can be analytically continued to the negative energy region. However, one should not forget that both $D_0(E)$ and $\tilde{f}_0(E)$ possess the essential singularity at $E = 0$.

Finally, we consider the full ERF $K_l(E)$. The energy dependence of $K_0(E)$ is displayed in Fig. 5. The ERF is real for all real values of E and is not singular at $E = 0$.

The pole of the amplitude $\tilde{f}_0$ was found using the requirement $D_0(E) \equiv kC_0^2(\eta)(\cot\delta_0 - i) = 0$. The zero of $D_0(E)$ corresponds to the point where the red line crosses the negative E-axis; see Fig. 4. The ANC $C^{(l)}_{a\to bc}$ at $l = 0$ is found using the relationship

$$\operatorname{res}\tilde{f}_0(E)|_{E=-\varepsilon} = \left[\frac{dD_0(E)}{dE}\right]^{-1}\Bigg|_{E=-\varepsilon}, \tag{35}$$

and Eq. (16). Equation (35) in turn follows from Eqs. (11) and (15). For the $d+\alpha$ system we obtain the following values of the binding energy ε and the dimensionless ANC $\tilde{C}^{(l)}_{a\to bc}$: $\varepsilon = 1.268$ MeV, $\tilde{C}^{(0)}_{^6\mathrm{Li}\to\alpha d} = C^{(0)}_{^6\mathrm{Li}\to\alpha d}/\sqrt{2\varkappa} = 2.683$. These values coincide with the values of ε and $\tilde{C}^{(0)}_{^6\mathrm{Li}\to\alpha d}$ obtained from direct solution of the Schrödinger Eq. (18) for the bound state of the $d+\alpha$ system. This agreement confirms the validity of the employed procedure of the analytic continuation of the scattering characteristics from positive to negative energies.

As mentioned earlier, if the Coulomb interaction is turned off, then $\varepsilon = 2.409$ MeV and $\tilde{C}^{(0)}_{^6\mathrm{Li}\to\alpha d} = 2.29$. Meanwhile,

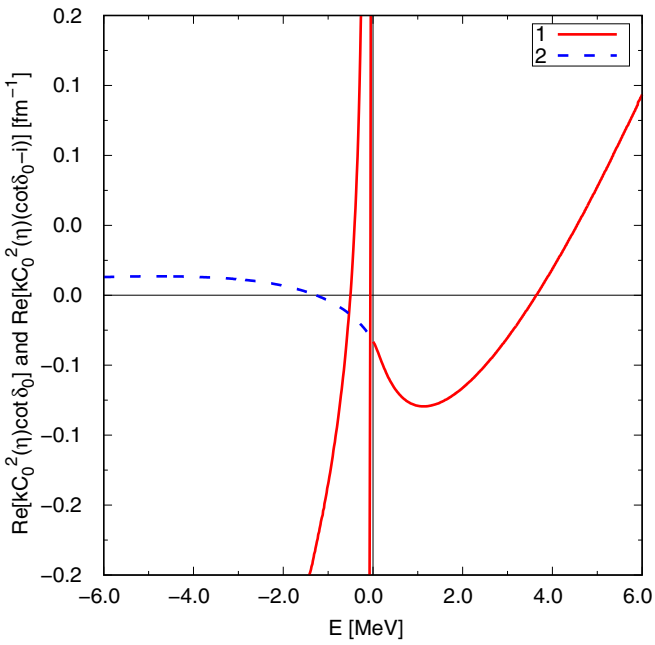

FIG. 6. Functional dependence of $\mathrm{Re}[kC_0^2(\eta)\cot\delta_0]$ (line 1) and $\mathrm{Re}[kC_0^2(\eta)(\cot\delta_0 - i)]$ (line 2) on energy E. For $E > 0$ they coincide.

the experimental value of ε is 1.47 MeV. The analytic approximation of the experimental $d+{}^4$He scattering phase shifts with subsequent continuation to negative energies results in $\tilde{C}^{(0)}_{^6\mathrm{Li}\to\alpha d} = 2.93$ [9].

Recently, in several works, experimental data on the elastic scattering have been used to get the information about bound states. The following procedure is usually used (see, e.g., Refs. [10,11]). The values of the ERF $K_l(E)$ at $E > 0$ obtained by the phase-shift analysis of experimental data are approximated by some analytic function, say, by several first terms of the effective-range expansion or by the Padé approximant. The approximating function obtained in such a way, as well as the exact function $K_l(E)$ do not possess singularities at $E = 0$. Hence, it could be continued analytically to the negative energy region. After that, using Eq. (9) one can find the location of the pole of the amplitude $\tilde{f}_l$ corresponding to the bound state and the residue of $\tilde{f}_l$ at that pole in terms of which the value of the ANC is expressed.

A somewhat different procedure was used in Ref. [12] in order to obtain the information about the bound states of the ^{16}O nucleus from the data on the elastic α-^{12}C scattering: instead of $K_l(E)$, function $\Delta_l(E)$ is approximated and continued to the region $E < 0$. Our results presented in Fig. 3 show that $\Delta_l(E)$ is a smooth function of E at $E > 0$; however, its behavior is irregular near $E = 0$ at $E < 0$. On the other hand, at $E > 0$ function $\Delta_l(E)$ coincides with the quantity $\mathrm{Re}[D_l(E)]$. As is seen from Figs. 4 and 6, $\mathrm{Re}[D_l(E)]$ is a smooth function of E both at $E > 0$ and at $E < 0$. Therefore, $\mathrm{Re}[D_l(E)]$ can be considered as the extrapolation of the function $\Delta_l(E)$ defined at $E > 0$ to the negative-energy region. This circumstance may serve as the justification of the procedure suggested in Ref. [12].

V. CONCLUSIONS

In the present paper the explicit analytic expressions have been derived for the ERF and the partial-wave scattering amplitude in the case of the interaction given by the sum of a short-range square-well and the Coulomb potentials. These expressions have been transformed into the forms convenient for the analytic continuation to the negative-energy region. The analytic properties of separate terms entering the expressions for the ERF and the scattering amplitude have been investigated.

It is demonstrated that function $\Delta_l(E)$ suggested in Ref. [12] can be used to obtain information about bound-state properties. In spite of having the essential singularity at $E = 0$, function $\Delta_l(E)$ can be analytically continued from the positive to the negative energy region along the real E axis. For instance, function $f(z) = \exp(-1/z^2)$ provides an example of such functions. Indeed, $f(z)$ possesses the essential singularity at $E = 0$. Nevertheless, the function itself and all its derivatives are smooth functions on the real axis, including $z = 0$. Using $\Delta_l(E)$ rather than the ERF $K_l(E)$ might be preferable, since $K_l(E)$, in contrast to $\Delta_l(E)$, contains a pure Coulomb term that may far exceed, in the absolute value, the term containing the information about the phase shifts.

Note that all qualitative results obtained in the present work do not depend on the specific values of the parameters of the potential used in the numerical calculations. Moreover, though all calculations were performed for $l = 0$, the inferences made should be valid for arbitrary l.

It is interesting to note that if the Coulomb interaction is turned off ($\eta = 0$), then $\mathrm{Re}[D_l(E)]$ evidently loses the Coulomb essential singularity but acquires a square-root type singularity at $E = 0$, which is absent at $\eta \neq 0$. This singularity corresponds to the normal threshold of the scattering amplitude. It follows from our calculations that, if one substitutes $\gamma Z_b Z_c$ for $Z_b Z_c$ in Eq. (4), then at $\gamma \to 0$ the first derivative of $\mathrm{Re}[D_0(E)]$ tends to $-\infty$. One may conclude that the behavior of $\mathrm{Re}[D_l(E)]$ in the vicinity of $E = 0$ is more smoother at larger values of η, that is, at larger values of $Z_b Z_c \mu$. It means that the procedure suggested in Ref. [12] is more effective for heavier nuclei.

Concluding, we emphasize once again that despite using the square-well potential (which allows to solve the Schrödinger equation analytically), our results are quite general because at very low energies for which $\lambda = 1/k \gg r$ the scattering phase shifts are not sensitive to the details of the adopted potential. This idea is in the foundation of effective field theory. That is why we believe that our results can be useful for researchers working in the effective field theory [23–25] as well, since they fit the elastic scattering data at positive energies, and here we investigate a possibility of extrapolating the data to the bound-state pole.

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