General form of the boson-fermion interaction in the interacting boson-fermion model-2

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The boson-fermion interaction in the interacting boson-fermion model-2 (IBFM-2) is derived in a systematic and general form from a quadrupole-quadrupole force using several nondegenerate levels. The boson-fermion quadrupole operator employed is obtained from the boson-fermion image of the one nucleon transfer operator which in turn can be calculated following two alternative schemes: the Otsuka-Arima-Iachello and generalized Holstein-Primakoff schemes. Four different terms (two quadrupole and two exchange) were obtained. Application of the new expressions to a single-j model is studied and analyzed.

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I. INTRODUCTION

In recent years nuclear structure studies have been present and have contributed in open questions in fundamental physics problems as the nature of the neutrino or the origin of the elusive dark matter. In both cases considerable efforts, from theoretical and experimental sides, have been made, but still new developments are necessary. The nature of the neutrino (Majorana or Dirac particle) and its absolute mass scale are being tackled through the study of the neutrinoless double β decay. In the case of the dark matter origin many candidates have been proposed to account for the unobserved matter in astronomical and cosmological studies. One of the most promising candidates are weakly interacting massive particles (WIMPs). Apart from the fact that their predicted energy density can naturally explain the dark matter density, they can interact directly with quarks, which makes them especially suitable for direct detection by scattering off nuclei. In all these cases nuclear matrix elements are involved and a reliable nuclear structure model is required for their calculation. In this context, the nuclear shell model (SM) is one of the most successful phenomenological models in describing nuclear spectroscopic data of light nuclei. However, as the number of valence nucleons increases, current computer capabilities are soon exceeded. The interacting boson model (IBM) [1] is a very good approximation of the shell model to describe the low lying collective states in many nuclei of the nuclear chart without the computer handicap of the shell model. In addition this model naturally unifies different nuclear regimes, which makes it suitable for the study of transitional nuclei and also it can be handled relatively easily. The most simple form of the IBM deals with even-even nuclei by replacing pairs of valence nucleons with bosons of angular momenta zero and 2. By coupling a fermion to the system of bosons, odd-A nuclei can be studied in this framework. This extension of the model is called the interacting boson-fermion model (IBFM) [2]. These models were used in phenomenological fits with great success when they were introduced. Soon the link with the shell model was posed and to this end extended versions of the model were presented where the isospin degrees of freedom were explicitly taken into account. The different versions were named IBM-k and IBFM-k, where $k = 1$ refers to those models where only one kind of bosons is present; $k = 2$ indicates those versions where two kinds of bosons are present, one for protons and another one for neutrons (technically they correspond to bosons with isospin $T = 1$ and $T_z = \pm 1$); $k = 3$ includes another kind of bosons with $T = 1$ and $T_z = 0$; and finally $k = 4$ includes bosons with $T = T_Z = 0$ besides those of previous versions.

The usual way to connect the shell model and the interacting boson model is through mappings between states belonging to both models and boson expansions of SM operators, the so-called Otsuka-Arima-Iachello (OAI) method [3]. Since then many papers devoted to study this connection have been published successfully deriving from the SM some IBM parameters obtained from phenomenological fits. In the IBFM the boson-fermion interaction is particularly important to describe well the coupling between the odd nucleon and the system of bosons. Scholten obtained it starting from a proton-neutron quadrupole force in the SM [4]. He used the boson image of the SM one nucleon transfer operator, which is simply the one nucleon creation operator in a particular single-particle state, to construct the boson image of the quadrupole operator and finally he coupled it to the boson quadrupole operator. In this derivation matrix elements of the one nucleon transfer operator between SM states with well defined generalized seniority are required and the number operator approximation (NOA) [5] was employed in their evaluation to simplify the calculations. Very recently this approximation was removed in the boson image of the transfer operator [6] because it produces undesirable subshell effects and it was shown that the computation of spectroscopic factors [6] and log ft [7] without it improves in realistic cases. These calculations were done using wave functions obtained using a boson-fermion interaction based in the NOA. Then the natural next step is to rederive it following Scholten's method without using the NOA. The advantage of the Scholten's method is that it provides relatively closed expressions in terms of a reduced number of parameters, in contrast to other alternative methods [8–10], where a large number of SM matrix elements are needed as an input.

In this work we derived the boson-fermion interaction in the IBFM following Scholten's method in general terms, without assuming a particular form for the boson image of the one nucleon transfer operator. This is the main goal of this work and it is explained in detail in the next section. In Sec. III we propose to use two alternative forms for the boson image of the transfer operator. The first one [6] is obtained following the OAI method without using the NOA. The second one is obtained using the generalized Holstein-Primakoff (GHP) boson expansion [11]. While the first form is suitable for spherical nuclei, the second one is more suitable for deformed nuclei. In Sec. IV we study the behavior of a single-j model using both alternatives for the transfer operator in constructing the boson-fermion interaction and compare their results. Section V summarizes the main conclusions of this work.

II. THE ONE NUCLEON TRANSFER OPERATOR AND THE BOSON-FERMION IMAGE OF THE QUADRUPOLE OPERATOR

The one nucleon transfer operator in the IBFM is obtained from the image of the SM single-nucleon creation operator in the i shell specified by the standard single-particle level quantum numbers $n_i, l_i, 1/2, j_i$, and m_i. We replace them with just one label for simplicity and denote this operator by $c^\dagger_{jm}$. This operator can be written as an infinite expansion in terms of boson operators and one fermion creation $a^\dagger_{jm}$ or annihilation $\tilde{a}_{jm}$ operator, which takes single-particle degrees of freedom into account in the IBFM. The operator $c^\dagger_{jm}$ has the general form

$$\begin{aligned} c^\dagger_{jm} &= A_j a^\dagger_{jm} + B_j (s^\dagger \tilde{a}_j)^{(j)}_m + \sum_{j'} C_{jj'} (d^\dagger \tilde{a}_{j'})^{(j)}_m \\ &\quad + \sum_{j'} D_{jj'} [(s^\dagger \tilde{d})^{(2)} a^\dagger_{j'}]^{(j)}_m + \sum_{j'} F_{jj'} [(d^\dagger \tilde{s})^{(2)} a^\dagger_{j'}]^{(j)}_m \\ &\quad + \sum_{j'L} E_{jj'L} [(d^\dagger \tilde{d})^{(L)} a^\dagger_{j'}]^{(j)}_m + G_j [(s^\dagger \tilde{s})^{(0)} \times a^\dagger_j]^{(j)}_m \\ &\quad + \cdots, \end{aligned} \tag{1}$$

where the sum in j' runs over all the orbits considered in the shell which the odd particle can occupy. For convenience, we rewrite the transfer operator in Eq. (1) in a more compact way, retaining terms up to two boson operators, which yields

$$\begin{aligned} c^\dagger_{jm} &= \mathfrak{A}_j a^\dagger_{jm} + \sum_{l,j'} \mathfrak{B}^l_{jj'} (\gamma^\dagger_l \tilde{a}_{j'})^{(j)}_m \\ &\quad + \sum_{l,l',L,j'} \mathfrak{C}^{ll'L}_{jj'} [(\gamma^\dagger_l \tilde{\gamma}_{l'})^{(L)} a^\dagger_{j'}]^{(j)}_m, \end{aligned} \tag{2}$$

where $\gamma^\dagger_l$ ($\tilde{\gamma}_l$) is a creation (annihilation) boson operator of angular momentum $l = 0,2$. The coefficients in Eq. (2) are defined as

$$\mathfrak{A}_j = A_j, \tag{3a}$$

$$\mathfrak{B}^l_{jj'} = \begin{cases} B_j \delta_{j,j'}, & l = 0 \\ C_{jj'}, & l = 2, \end{cases} \tag{3b}$$

$$\mathfrak{C}^{ll'L}_{jj'} = \begin{cases} D_{jj'} \delta_{L,2}, & l = 0, l' = 2 \\ F_{jj'} \delta_{L,2}, & l = 2, l' = 0 \\ E_{jj'L}, & l = l' = 2 \\ G_j, & l = l' = 0. \end{cases} \tag{3c}$$

The Kronecker delta $\delta_{j,j'}$ in $\mathfrak{B}^0_{j,j'}$ is written for convenience despite that it appears naturally in the angular momentum coupling, which also occurs in $\mathfrak{C}^{022}_{jj'}$ and $\mathfrak{C}^{202}_{jj'}$. The corresponding annihilation operator with good irreducible tensor character is defined as $\tilde{c}_{jm} = (-1)^{j-m} c_{j-m}$ and given by

$$\begin{aligned} \tilde{c}_{jm} &= \mathfrak{A}_j \tilde{a}_{jm} - \sum_{l,j'} \mathfrak{B}^l_{jj'} (\tilde{\gamma}_l a^\dagger_{j'})^{(j)}_m \\ &\quad + \sum_{l,l',L,j'} \mathfrak{C}^{ll'L}_{jj'} (-1)^{l+l'+L} [(\gamma^\dagger_{l'} \tilde{\gamma}_l)^{(L)} \tilde{a}_{j'}]^{(j)}_m. \end{aligned} \tag{4}$$

We are now in position to construct the quadrupole operator, as

$$q^{(2)} = \sum_{j_1,j_2} Q_{j_1 j_2} (c^\dagger_{j_1} \tilde{c}_{j_2})^{(2)}, \tag{5}$$

where

$$Q_{j_1 j_2} = -\frac{1}{\sqrt{5}} \langle j_1 || r^2 Y_2 || j_2 \rangle, \tag{6}$$

according to our phase definition of the fermion annihilation operator $\tilde{c}_{jm}$.

Substituting Eqs. (2) and (4) in Eq. (5) we obtain

$$\begin{aligned} q^{(2)} &= \sum_{j_1,j_2} Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{A}_{j_2} \left(a^\dagger_{j_1} \tilde{a}_{j_2}\right)^{(2)} - \sum_{j_1,j_2,j_2',l_2} Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{B}^{l_2}_{j_2 j_2'} \left[a^\dagger_{j_1} \left(\tilde{\gamma}_{l_2} a^\dagger_{j_2'}\right)^{(j_2)}\right]^{(2)} \\ &\quad + \sum_{\substack{j_1,j_2,j_2',\\ l_2,l_2',L_2}} Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{C}^{l_2 l_2' L_2}_{j_2 j_2'} (-1)^{l_2+l_2'+L_2} \left[a^\dagger_{j_1} \left[\left(\gamma^\dagger_{l_2'} \tilde{\gamma}_{l_2}\right)^{(L_2)} \tilde{a}_{j_2'}\right]^{(j_2)}\right]^{(2)} \\ &\quad + \sum_{j_1,j_2,j_1' l_1} Q_{j_1 j_2} \mathfrak{B}^{l_1}_{j_1 j_1'} \mathfrak{A}_{j_2} \left[\left(\gamma^\dagger_{l_1} \tilde{a}_{j_1'}\right)^{(j_1)} \tilde{a}_{j_2}\right]^{(2)} - \sum_{\substack{j_1,j_2,j_1',\\ j_2',l_1,l_2}} Q_{j_1 j_2} \mathfrak{B}^{l_1}_{j_1 j'} \mathfrak{B}^{l_2}_{j_2 j_2'} \left[\left(\gamma^\dagger_{l_1} \tilde{a}_{j_1'}\right)^{(j_1)} \left(\tilde{\gamma}_{l_2} a^\dagger_{j_2'}\right)^{(j_2)}\right]^{(2)} \\ &\quad + \sum_{\substack{j_1,j_2,j_1',j_2'\\ l_1,l_2,l_2',L_2}} Q_{j_1 j_2} \mathfrak{B}^{l_1}_{j_1 j_1'} \mathfrak{C}^{l_2 l_2' L_2}_{j_2 j_2'} (-1)^{l_2+l_2'+L_2} \left[\left[\gamma^\dagger_{l_1} \tilde{a}_{j_1'}\right]^{(j_1)} \left[\left(\gamma^\dagger_{l_2'} \tilde{\gamma}_{l_2}\right)^{(L_2)} \tilde{a}_{j_2'}\right]^{(j_2)}\right]^{(2)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{j_1,j_2,j_1'\\ l_1,l_1',L_1}} Q_{j_1j_2}\mathfrak{C}^{l_1l_1'L_1}_{j_1j_1'}\mathfrak{A}_{j_2}\big[\big[\big(\gamma^\dagger_{l_1}\tilde{\gamma}_{l_1'}\big)^{(L_1)}a^\dagger_{j_1'}\big]^{(j_1)}\tilde{a}_{j_2}\big]^{(2)}\\
&-\sum_{\substack{j_1,j_2,j_1',j_2'\\ l_1,l_1',L_1,l_2}} Q_{j_1j_2}\mathfrak{C}^{l_1l_1'L_1}_{j_1j_1'}\mathfrak{B}^{l_2}_{j_2j_2'}\big[\big[\big(\gamma^\dagger_{l_1}\tilde{\gamma}_{l_1'}\big)^{(L_1)}a^\dagger_{j_1'}\big]^{(j_1)}\big[\tilde{\gamma}_{l_2}a^\dagger_{j_2'}\big]^{(j_2)}\big]^{(2)}\\
&+\sum_{\substack{j_1,j_1',l_1,l_1',L_1\\ j_2,j_2',l_2,l_2',L_2}} Q_{j_1j_2}\mathfrak{C}^{l_1l_1'L_1}_{j_1j_1'}\mathfrak{C}^{l_2l_2'L_2}_{j_2j_2'}(-1)^{l_2+l_2'+L_2}\big[\big[\big(\gamma^\dagger_{l_1}\tilde{\gamma}_{l_1'}\big)^{(L_1)}a^\dagger_{j_1'}\big]^{(j_1)}\big[\big(\gamma^\dagger_{l_2'}\tilde{\gamma}_{l_2}\big)^{(L_2)}\tilde{a}_{j_2'}\big]^{(j_2)}\big]^{(2)}.
\end{aligned}
\tag{7}
$$

The terms 2, 4, 6, and 8 can be immediately discarded since they introduce two-particle mixing (i.e., they have two creation or annihilation fermion operators) which are beyond the IBFM-2 space. In addition, the last term produces three-body interactions in the boson part of the boson-fermion interaction. It is also not considered, because we are retaining the lowest order terms. After recoupling conveniently some of the terms and summing on the j's we obtain four contributions in the quadrupole operator:

$$q^{(2)} = q^{(2)}_{Q_1} + q^{(2)}_{Q_2} + q^{(2)}_{E_1} + q^{(2)}_{E_2}, \tag{8}$$

where $q^{(2)}_{Q_1}$ and $q^{(2)}_{Q_2}$ are related to quadrupole interaction terms, and the rest are related to exchange interaction terms. They are the following:

$$q^{(2)}_{Q_1} = \sum_{j_1j_2}\Gamma_{j_1j_2}\big(a^\dagger_{j_1}\tilde{a}_{j_2}\big)^{(2)}, \tag{9}$$

$$q^{(2)}_{Q_2} = \sum_{j_1j_2}\Gamma'_{j_1j_2}\big(a^\dagger_{j_1}\tilde{a}_{j_2}\big)^{(2)}, \tag{10}$$

$$q^{(2)}_{E_1} = \sum_{\substack{j_1,j_2\\ j_1',l_1,j_2',l_2}}\Delta^{l_1l_2}_{j_1j_2j_1'j_2'}\big[\big(\tilde{\gamma}_{l_1}a^\dagger_{j_1'}\big)^{(j_1)}\big(\gamma^\dagger_{l_2}\tilde{a}_{j_2'}\big)^{(j_2)}\big]^{(2)}, \tag{11}$$

$$
\begin{aligned}
q^{(2)}_{E_2} = &\sum_{\substack{j_1,j_2\\ j',l,l',L}}\Lambda^{ll'L}_{j_1j_2j'}\big\{\big[[(\gamma^\dagger_l\tilde{\gamma}_{l'})^{(L)}a^\dagger_{j'}]^{(j_2)}\tilde{a}_{j_1}\big]^{(2)}\\
&+(-1)^{L+l'+l+j_1-j_2}\big[a^\dagger_{j_1}[(\gamma^\dagger_{l'}\tilde{\gamma}_l)^{(L)}\tilde{a}_{j'}]^{(j_2)}\big]^{(2)}\big\},
\end{aligned}
\tag{12}
$$

where

$$\Gamma_{j_1j_2} = Q_{j_1j_2}\mathfrak{A}_{j_1}\mathfrak{A}_{j_2}, \tag{13}$$

$$\Lambda^{ll'L}_{j_1j_2j'} = Q_{j_2j_1}\mathfrak{A}_{j_1}\mathfrak{C}^{ll'L}_{j_2j'}, \tag{14}$$

$$\Delta^{l_1l_2}_{j_1j_2j_1'j_2'} = -Q_{j_1j_2}\mathfrak{B}^{l_1}_{j_1j_1'}\mathfrak{B}^{l_2}_{j_2j_2'}, \tag{15}$$

$$
\begin{aligned}
\Gamma'_{j_1j_2} &= \sum_{j,j',l} Q_{jj'}\mathfrak{B}^{l}_{jj_1}\mathfrak{B}^{l}_{j'j_2}\hat{j}\hat{j}'\\
&\quad\times(-1)^{j+j_2+l}\begin{Bmatrix} j' & j_2 & l\\ j_1 & j & 2\end{Bmatrix}\\
&= -\sum_{j,j',l}\Delta^{ll}{}_{jj'j_1j_2}\hat{j}\hat{j}'(-1)^{j+j_2+l}\begin{Bmatrix} j' & j_2 & l\\ j_1 & j & 2\end{Bmatrix}
\end{aligned}
\tag{16}
$$

and $\hat{j} = \sqrt{2j+1}$. The fifth term in Eq. (7) was recoupled to write the fermion operators in normal order. This recoupling produces the term $q^{(2)}_{Q_2}$ and also a boson-boson interaction which we do not include because we are only interested in the boson-fermion interaction. The various terms that enter in the contribution are illustrated diagrammatically in Fig. 1.

The operator $q^{(2)}$ is the most general form of the fermion quadrupole operator up to two boson operators with expressions (9)–(12). When necessary we label it with a subindex ρ to indicate that it corresponds to particles of type ρ, where $\rho = \nu$ is for neutrons and $\rho = \pi$ is for protons. Using the boson quadrupole operator $Q^{(2)}_{\rho'}$ for bosons of type ρ', the boson-fermion interaction between particles of type ρ' and ρ is then given by

$$V^{\rho'\rho}_{\mathrm{BF}} = \kappa Q^{(2)}_{\rho'}\cdot q^{(2)}_{\rho}, \tag{17}$$

where κ is the (negative) boson-fermion interaction strength. The obtained boson-fermion interaction now contains two quadrupole plus two exchange terms:

$$V^{\rho'\rho}_{\mathrm{BF}} = \kappa\Big(V^{Q_1}_{\mathrm{BF}} + V^{Q_2}_{\mathrm{BF}} + V^{E_1}_{\mathrm{BF}} + V^{E_2}_{\mathrm{BF}}\Big), \tag{18}$$

where

$$V^{Q_1}_{\mathrm{BF}} = \sum_{j_1,j_2}\Gamma_{j_1j_2}Q^{(2)}_{\rho'}\cdot\big(a^\dagger_{j_1}\tilde{a}_{j_2}\big)^{(2)}, \tag{19}$$

$$V^{Q_2}_{\mathrm{BF}} = \sum_{j_1,j_2}\Gamma'_{j_1j_2}Q^{(2)}_{\rho'}\cdot\big(a^\dagger_{j_1}\tilde{a}_{j_2}\big)^{(2)}, \tag{20}$$

$$V^{E_1}_{\mathrm{BF}} = \sum_{\substack{j_1,j_2\\ j_1',j_2'\\ l_1,l_2}}\Delta^{l_1l_2}_{j_1j_2j_1'j_2'}Q^{(2)}_{\rho'}\cdot\big[\big(\tilde{\gamma}_{l_1}a^\dagger_{j_1'}\big)^{(j_1)}\big(\gamma^\dagger_{l_2}\tilde{a}_{j_2'}\big)^{(j_2)}\big]^{(2)}, \tag{21}$$

$$
\begin{aligned}
V^{E_2}_{\mathrm{BF}} = &\sum_{\substack{j_1,j_2\\ j',l,l',L}}\Lambda^{ll'L}_{j_1j_2j'}\big\{Q^{(2)}_{\rho'}\cdot\big[\big(\mathcal{B}^{(L)}_{ll'}a^\dagger_{j'}\big)^{(j_2)}\tilde{a}_{j_1}\big]^{(2)}\\
&+(-1)^{L+l'+l+j_1-j_2}Q^{(2)}_{\rho'}\cdot\big[a^\dagger_{j_1}\big(\mathcal{B}^{(L)}_{l'l}\tilde{a}_{j'}\big)^{(j_2)}\big]^{(2)}\big\},
\end{aligned}
\tag{22}
$$

and $\mathcal{B}^{(L)}_{ll'} = (\gamma^\dagger_l\tilde{\gamma}_{l'})^{(L)}$. We can compare these expressions with those ones obtained by Scholten in the generalized seniority scheme using the NOA [4]:

$$
\begin{aligned}
V_{\mathrm{Scholten}} = &\,\kappa\sum_{jj'}Q_{jj'}(u_ju_{j'} - v_jv_{j'})Q^{(2)}_{\rho'}\cdot(a^\dagger_j\tilde{a}_{j'})^{(2)}\\
&-\kappa\sqrt{\frac{10}{N_\rho}}\sum_{jj'j''}\frac{\beta_{jj'}\beta_{j''j}}{\hat{j}}\\
&\times\big\{Q^{(2)}_{\rho'}\cdot[(d^\dagger\tilde{a}_{j''})^{(j)}(\tilde{s}a^\dagger_{j'})^{(j')}]^{(2)} + \mathrm{H.c.}\big\},
\end{aligned}
\tag{23}
$$

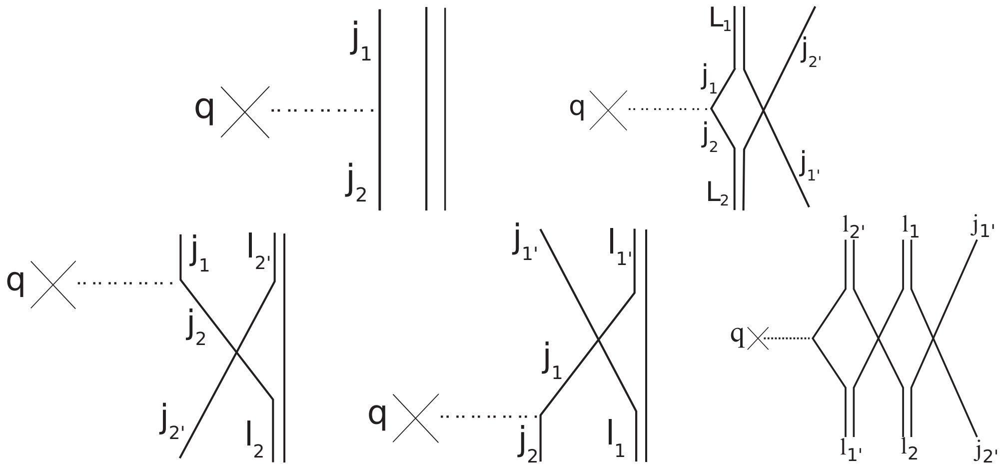

FIG. 1. Diagrammatic representation of the terms [Eqs. (9)–(12)] and the last term in Eq. (7).

where N_ρ is the number of bosons of type ρ, v_j^2 is the occupation probability of orbit j, $u_j = \sqrt{1 - v_j^2}$, and $\beta_{jj'} = Q_{jj'}(u_j v_{j'} + v_j u_{j'})$. When we rewrite it in our formalism it becomes

$$\begin{aligned} V_{\text{Scholten}} = \kappa \sum_{j_1,j_2} \bar{\Gamma}_{j_1 j_2} Q^{(2)}_{\rho'} \cdot \left(a^\dagger_{j_1} \tilde{a}_{j_2}\right)^{(2)} \\ &+ \kappa \sum_{\substack{j_1,j_2\\ j_1',j_2'}} \bar{\Delta}^{02}_{j_1 j_2 j_1' j_2'} Q^{(2)}_{\rho'} \cdot \left[\left(\tilde{s} a^\dagger_{j_1'}\right)^{(j_1)} \left(d^\dagger \tilde{a}_{j_2'}\right)^{(j_2)}\right]^{(2)} \\ &+ \kappa \sum_{\substack{j_1,j_2\\ j_1',j_2'}} \bar{\Delta}^{20}_{j_1 j_2 j_1' j_2'} Q^{(2)}_{\rho'} \cdot \left[\left(\tilde{d} a^\dagger_{j_1'}\right)^{(j_1)} \left(s^\dagger \tilde{a}_{j_2'}\right)^{(j_2)}\right]^{(2)}, \end{aligned} \tag{24}$$

where

$$\bar{\Gamma}_{j_1 j_2} = Q_{j_1 j_2}(u_{j_1} u_{j_2} - v_{j_1} v_{j_2}), \tag{25}$$

$$\bar{\Delta}^{02}_{j_1 j_2 j_1' j_2'} = \sqrt{\frac{10}{N_\rho}} (-1)^{j_1 + j_2} \frac{\beta_{j_2 j_1} \beta_{j_2' j_2}}{\hat{j}_2}, \tag{26}$$

$$\bar{\Delta}^{20}_{j_1 j_2 j_1' j_2'} = -\sqrt{\frac{10}{N_\rho}} (-1)^{j_1 + j_2} \frac{\beta_{j_1 j_2} \beta_{j_1' j_1}}{\hat{j}_1}. \tag{27}$$

We can see that Scholten's interaction includes only up to first order in the number of d boson operators, while our expressions go further and include a term with two d boson operators in $V^{E_1}_{\text{BF}}$ when $l_1 = l_2 = 2$, besides the terms in $V^{E_2}_{\text{BF}}$. These terms can be important when dealing with states of seniority $v = 3$ to account for deformation in a better way.

Below we write the exchange terms in a suitable way to program in a computer code:

$$\begin{aligned} V^{E_1}_{\text{BF}} = \sum_{\substack{j_1,j_2,L,\lambda\\ j_1',l_1,j_2',l_2}} \Delta^{l_1 l_2}_{j_1 j_2 j_1' j_2'} \sqrt{5}\, \hat{j}_1 \hat{j}_2 \hat{L} \begin{Bmatrix} l_1 & l_2 & L \\ j_1' & j_2' & \lambda \\ j_1 & j_2 & 2 \end{Bmatrix} \\ &\times (-1)^{l_1 + l_2 - L - \lambda} \left(Q^{(2)}_{\rho'} \mathcal{B}^{(L)}_{l_2 l_1}\right)^{(\lambda)} \cdot \left(a^\dagger_{j_1'} \tilde{a}_{j_2'}\right)^{(\lambda)} \\ &- \sum_{j_1,j_2} \Gamma'_{j_1 j_2} Q^{(2)}_{\rho'} \cdot \left(a^\dagger_{j_1} \tilde{a}_{j_2}\right)^{(2)}, \end{aligned} \tag{28}$$

$$\begin{aligned} V^{E_2}_{\text{BF}} = \sum_{\substack{j_1,j_2\\ j',l,l',L,\lambda}} \Lambda^{ll'L}_{j_1 j_2 j'} \hat{j}_2 \sqrt{5} \begin{Bmatrix} j_1 & j_2 & 2 \\ L & \lambda & j' \end{Bmatrix} \\ &\times \left\{ -(-1)^{l+l'} \left(Q^{(2)}_{\rho'} \mathcal{B}^{(L)}_{l'l,\rho}\right)^{(\lambda)} \cdot \left(a^\dagger_{j_1} \tilde{a}_{j'}\right)^{(\lambda)} \right. \\ &\left. + (-1)^{j'+j_1} \left(\mathcal{B}^{(L)}_{ll',\rho} Q^{(2)}_{\rho'}\right)^{(\lambda)} \cdot \left(a^\dagger_{j'} \tilde{a}_{j_1}\right)^{(\lambda)} \right\}. \end{aligned} \tag{29}$$

It is interesting to note that in writing the boson-fermion interaction in this form the quadrupole interaction $V^{Q_2}_{\text{BF}}$ can be canceled with the second term in $V^{E_1}_{\text{BF}}$ when it is recoupled to group separately the boson and fermion operator if both quadrupole Q_2 and exchange E_1 interactions have the same strength. However, in phenomenological applications they often have different strengths and the cancellation would not happen. As can be seen every term conserves both the boson and fermion number and is Hermitian.

III. V_{BF} USING OAI AND GHP

In this work we consider two different alternative schemes in order to obtain an expression for the coefficients of the one nucleon transfer operator (2). They are the OAI [3] and the GHP [11] mappings. We discuss briefly both mappings and quote the coefficients for each mapping.

A. OAI

The OAI method is based on the generalized seniority (GS) scheme in the SM. In this scheme the SM space is truncated to the SD pair space, where the states are constructed from the collective nucleon pair creation operators

$$S^\dagger = \sum_j \alpha_j \sqrt{\Omega_j} A^{\dagger(00)}_{jj}, \quad D^\dagger_\mu = \sum_{\substack{j,j'\\ j \leqslant j'}} \beta_{jj'} A^{\dagger(2\mu)}_{jj'}. \tag{30}$$

Here $\Omega_j = j + 1/2$ is half the occupation of the orbit j, and α's and β's are pair structure coefficients. The nucleon pair creation operator of total angular momentum J and projection

μ on its z axis is defined as

$$A_{jj'}^{\dagger(J\mu)} = \frac{1}{\sqrt{1+\delta_{jj'}}}(C_j^{\dagger}C_{j'}^{\dagger})_{\mu}^{(J)}, \tag{31}$$

where $C_j^{\dagger}$ is the SM single-nucleon creation operator. We do a mapping between fermion states in SD space and boson states (Marumori mapping [12]),

$$|S^{N_s}D^{N_d}aJ\rangle_{\mathrm{F}} \mapsto |s^{N_s}d^{N_d}aJ\rangle_{\mathrm{B}}, \tag{32}$$

where a accounts for all the quantum numbers necessary to identify uniquely the states. The strategy is to equate the matrix elements of any fermion operator O between fermion states to matrix elements of the boson image of the fermion operator, O^{B}, between the corresponding boson states. In the OAI method, the expressions for the coefficients from A_j to $D_{jj'}$ are given exactly in Ref. [6] using states with generalized seniority $\tilde{\nu} \leqslant 2$. $F_{jj'}$ and $E_{jj'L}$ can only be found by matrix elements where states of $\tilde{\nu} > 2$ are involved. In this mapping we obtain

$$F_{jj'} = \frac{\langle S^{N-1}D; j'||C_j^{\dagger}||S^N\rangle}{\langle s^{N-1}d; j'||[(d^{\dagger}\tilde{s})^{(2)}a_{j'}^{\dagger}]^{(j)}||s^N\rangle_{\mathrm{B}}}, \tag{33}$$

which becomes null since $C_j^{\dagger}$ connects states where $\Delta\tilde{\nu} = 1$. Meanwhile, the expression for $E_{jj'L}$ is more involved since we must orthonormalize states of $\tilde{\nu} > 2$ [13] and it is omitted in this study. Also the sum on l_1 and l_2 in Eq. (2) is restricted to the values zero and 2 without including $l_1 = l_2 = 0$, which corresponds to the terms with G_j and can be absorbed in the terms with A_j and E_{jj0} using the relation $[s^{\dagger}\tilde{s}]^{(0)} = \hat{N} - \sqrt{5}[d^{\dagger}\tilde{d}]^{(0)}$, where $\hat{N}$ is the number operator of bosons. The values of the coefficients are shown below:

$$A_j = \frac{\eta_{2N,1,j}}{\eta_{2N,0,0}}, \tag{34a}$$

$$B_j = \sqrt{N}\alpha_j\frac{\eta_{2(N-1),1,j}}{\eta_{2N,0,0}}, \tag{34b}$$

$$C_{jj'} = \frac{\sqrt{5}}{\hat{j}}\frac{\eta_{2N,2,2}^2(jj')}{\eta_{2N,2,2}\eta_{2(N-1),1,j'}}\beta_{j'j}\sqrt{1+\delta_{jj'}}, \tag{34c}$$

$$D_{jj'} = -\frac{\sqrt{5}}{\hat{j}}\frac{\eta_{2N,2,2}^2(jj')}{\eta_{2N,2,2}\eta_{2N,1,j'}}\sqrt{N}\alpha_j\beta_{j'j}\sqrt{1+\delta_{jj'}}, \tag{34d}$$

$$F_{jj'} = 0, \tag{34e}$$

where the form of the η's in Eqs. (34a)–(34d) can be found in Refs. [6,14,15] and they depend on the pair structure coefficients α_j and $\beta_{jj'}$ in Eqs. (30). They can be obtained using different methods [8,16,17]. Also it is interesting to see that Eqs. (34a)–(34d) fulfill the relation $\alpha_{j'}A_{j'}D_{jj'} + \alpha_j B_{j'}C_{jj'} = 0$.

We want to stress two points: First, all values of the coefficients and matrix elements listed above are obtained exactly without using the NOA [5]. Second, the procedure to obtain the coefficients in this way requires the calculations of the structure coefficients of the S and D collective pairs. This mapping is known to be very useful in spherical and vibrational regions, since it considers states with lower generalized seniority, where it is an approximate good quantum number [18]. Therefore, one expects to obtain good results in these cases.

B. GHP

The GHP scheme (sometimes also called Beliaev-Zelevinsky expansion) is a mapping of operators where the operator commutation relations are preserved; i.e., the operators algebra for all fermion operators is conserved. Strictly speaking, a systematic perturbation expansion is done on a small parameter, and the commutation rules must be fulfilled in each order of the expansion. The lowest order is the most important because of its physical interpretation, while the higher orders give rise to anharmonicities [19]. In this mapping the following relations are fulfilled:

$$(c_j^{\dagger}c_i^{\dagger})_{\mathrm{GHP}}^{\dagger} = (c_ic_j)_{\mathrm{GHP}}, \quad (c_j^{\dagger})_{\mathrm{GHP}}^{\dagger} = (c_j)_{\mathrm{GHP}}. \tag{35}$$

To preserve the annihilation-creation fermion operators algebra, the transfer operator has the form

$$(c_j^{\dagger})_{\mathrm{GHP}} = \sum_i a_i^{\dagger}(\sqrt{\mathbb{I} - (B^{\dagger}B)^T})_{ij} + \sum_i B_{ji}^{\dagger}a_i, \tag{36}$$

where B is the matrix of operators B_{ij}, the boson annihilation operators, which takes the place of the fermion pair C_jC_i, and T indicates matrix transposition. The square-root operator in Eq. (36) is the hallmark of GHP expansions. This square root must be expanded in a Taylor series to be used and Ref. [20] provides a procedure to speed up its convergence. When the corresponding boson operators are written as collective boson operators, the coefficients of Eq. (1) may be obtained and are given by Arias *et al.* [11]:

$$\mathfrak{A}_j = u_j\left(1 + \frac{v_j^2}{2u_j^2}\right), \tag{37a}$$

$$\mathfrak{B}_{jj'}^L = \boldsymbol{X}_{j'j}^l\frac{\hat{L}}{\hat{j}}, \tag{37b}$$

$$\mathfrak{C}_{jj'}^{ll'L} = -\frac{1}{2u_j}\sum_{j''}(-1)^{j'+j''}\boldsymbol{X}_{jj''}^l\boldsymbol{X}_{j'j''}^{l'}\frac{\hat{l}\hat{l}'\hat{L}}{\hat{j}}\begin{Bmatrix} j & j' & L \\ l' & l & j'' \end{Bmatrix}, \tag{37c}$$

where $\boldsymbol{X}_{jj'}^l$ are structure coefficients of the collective boson operators and their explicit expressions can be found in Ref. [21], and $v_j^2(u_j^2)$ is the occupation (vacancy) probability of the single-particle state j ($v_j^2 + u_j^2 = 1$).

This mapping is suitable for deformed nuclei [21,22]. Since GS is not a good quantum number and breaks down in these cases [10], in principle the GHP method is assumed to work better than the OAI method. Nevertheless, it is interesting to point out that relations between both mappings may be obtained [23].

IV. SINGLE j-SHELL CASE

To clarify the contents of the previous section in a rather simple and tractable fashion, we consider the application of the OAI and GHP mappings to study the interaction restricted to a single j shell. It is known that the single j-shell model has some deficiencies [8] since there is only one pair for each pair creation operator. Also the value of j is usually taken to be large in order to represent a large shell; therefore, there is a possibility that some uncanny effects appear because of the great value of j which do not appear in realistic cases. However, it is a simple method in order to study the boson-fermion interaction and also the role of each term of it for the different mappings. For a single j shell, we do not need to determine the value of the structure coefficients used in the OAI method since they are equal, $\alpha_j = \beta_{jj} = 1$. In the OAI mapping the values of the coefficients of the one nucleon transfer operator reduce to

$$A_j = \sqrt{\frac{\Omega_j - N}{\Omega_j}}, \tag{38}$$

$$B_j = \frac{1}{\sqrt{\Omega_j}}, \tag{39}$$

$$C_{jj} = \frac{\sqrt{10}}{\hat{j}}\sqrt{\frac{\Omega_j - N}{\Omega_j - 1}}, \tag{40}$$

$$D_{jj} = -\frac{1}{\hat{j}}\sqrt{\frac{10}{\Omega_j - 1}}. \tag{41}$$

These quantities are identical to those obtained by Scholten [4], since in the single j shell the NOA is exact. Using them, the boson-fermion interaction reads

$$\begin{aligned} V_{\mathrm{BF}} = {} & \kappa(\Gamma_{jj} + \Gamma'_{jj}) Q^{(2)}_{\rho'} \cdot (a^\dagger_j \tilde{a}_j)^{(2)} \\ & + \kappa \Lambda^{022}_{jjj} \{ Q^{(2)}_{\rho'} \cdot [[(s^\dagger \tilde{d})^{(2)} a^\dagger_j]^{(j)} \tilde{a}_j]^{(2)} \\ & + Q^{(2)}_{\rho'} \cdot [a^\dagger_j [(d^\dagger \tilde{s})^{(2)} \tilde{a}_j]^{(j)}]^{(2)} \} \\ & + \sum_{l_1, l_2} \kappa \Delta^{l_1 l_2}_{jjjj} Q^{(2)}_{\rho'} \cdot \big[\big(\tilde{\gamma}_{l_1} a^\dagger_j\big)^{(j)} \big(\gamma^\dagger_{l_2} \tilde{a}_j\big)^{(j)} \big]^{(2)}, \end{aligned} \tag{42}$$

where

$$\Gamma_{jj} = Q_{jj} \frac{\Omega_j - N}{\Omega_j}, \tag{43}$$

$$\Gamma'_{jj} = \frac{Q_{jj}}{\Omega_j} - 10 Q_{jj} \frac{\Omega_j - N}{\Omega_j - 1} \begin{Bmatrix} j & j & 2 \\ j & j & 2 \end{Bmatrix}, \tag{44}$$

$$\Lambda^{022}_{jjj} = -\frac{Q_{jj}}{\hat{j}} \sqrt{\frac{10(\Omega_j - N)}{\Omega_j(\Omega_j - 1)}}, \tag{45}$$

$$\Delta^{00}_{jjjj} = -\frac{Q_{jj}}{\Omega_j}, \tag{46}$$

$$\Delta^{02}_{jjjj} = \Delta^{20}_{jjjj} = -\frac{Q_{jj}}{\Omega_j}\sqrt{\frac{5(\Omega_j - N)}{\Omega_j - 1}}, \tag{47}$$

$$\Delta^{22}_{jjjj} = -\frac{10 Q_{jj}(\Omega_j - N)}{(2j+1)(\Omega_j - 1)}, \tag{48}$$

while Scholten's form of the interaction is

$$\begin{aligned} V_{\mathrm{Scholten}} = {} & \kappa Q_{jj} \frac{\Omega_j - 2N}{\Omega_j} Q^{(2)}_{\rho'} \cdot (a^\dagger_j \tilde{a}_j)^{(2)} \\ & - \sqrt{\frac{10}{N}} \frac{4 Q^2_{jj}}{\hat{j}} \left(\frac{\Omega_j - N}{\Omega_j}\right)\left(\frac{N}{\Omega_j}\right) \\ & \times \{ Q^{(2)}_{\rho'} \cdot [(d^\dagger \tilde{a}_j)^{(j)} (\tilde{s} a^\dagger_j)^{(j)}]^{(2)} + \mathrm{H.c.} \}. \end{aligned} \tag{49}$$

In the case of the GHP mapping, the coefficients in the boson expansion of the transfer operator are

$$A_j = \frac{2\Omega_j - N}{2\Omega_j - 2N} \sqrt{\frac{\Omega_j - N}{\Omega_j}}, \tag{50}$$

$$B_j = \frac{\sqrt{2}}{\hat{j}}, \tag{51}$$

$$C_{jj} = \frac{\sqrt{10}}{\hat{j}}, \tag{52}$$

$$D_{jj} = -\sqrt{\frac{\Omega_j}{\Omega_j - N}} \frac{\sqrt{5}}{2j+1}, \tag{53}$$

$$E_{jjL} = 5\sqrt{\frac{\Omega_j}{\Omega_j - N}} \frac{\hat{L}}{\hat{j}} \begin{Bmatrix} j & j & L \\ 2 & 2 & j \end{Bmatrix}, \tag{54}$$

$$F_{jj} = -\sqrt{\frac{\Omega_j}{\Omega_j - N}} \frac{\sqrt{5}}{2j+1}, \tag{55}$$

$$G_{jj} = -\sqrt{\frac{\Omega_j}{\Omega_j - N}} \frac{1}{2j+1}, \tag{56}$$

since $\boldsymbol{X}^L_{jj} = \sqrt{2}$, and the coefficients in the boson-fermion interaction become

$$\Gamma_{jj} = Q_{jj} \frac{(2\Omega_j - N)^2}{4\Omega_j(\Omega_j - N)}, \tag{57}$$

$$\Gamma'_{jj} = \frac{Q_{jj}}{\Omega_j} - 10 Q_{jj} \begin{Bmatrix} j & j & 2 \\ j & j & 2 \end{Bmatrix}, \tag{58}$$

$$\Lambda^{000}_{jjj} = -\frac{Q_{jj}(2\Omega_j - N)}{4\Omega_j(\Omega_j - N)}, \tag{59}$$

$$\Lambda^{022}_{jjj} = \Lambda^{202}_{jjj} = -\frac{\sqrt{5} Q_{jj}(2\Omega_j - N)}{4\Omega_j(\Omega_j - N)}, \tag{60}$$

$$\Lambda^{22L}_{jjj} = \frac{5 Q_{jj}(2\Omega_j - N)\hat{L}}{2(\Omega_j - N)\hat{j}} \begin{Bmatrix} j & j & L \\ 2 & 2 & j \end{Bmatrix}, \tag{61}$$

$$\Delta^{00}_{jjjj} = -\frac{Q_{jj}}{\Omega_j}, \tag{62}$$

$$\Delta^{02}_{jjjj} = \Delta^{20}_{jjjj} = -\frac{\sqrt{5} Q_{jj}}{\Omega_j}, \tag{63}$$

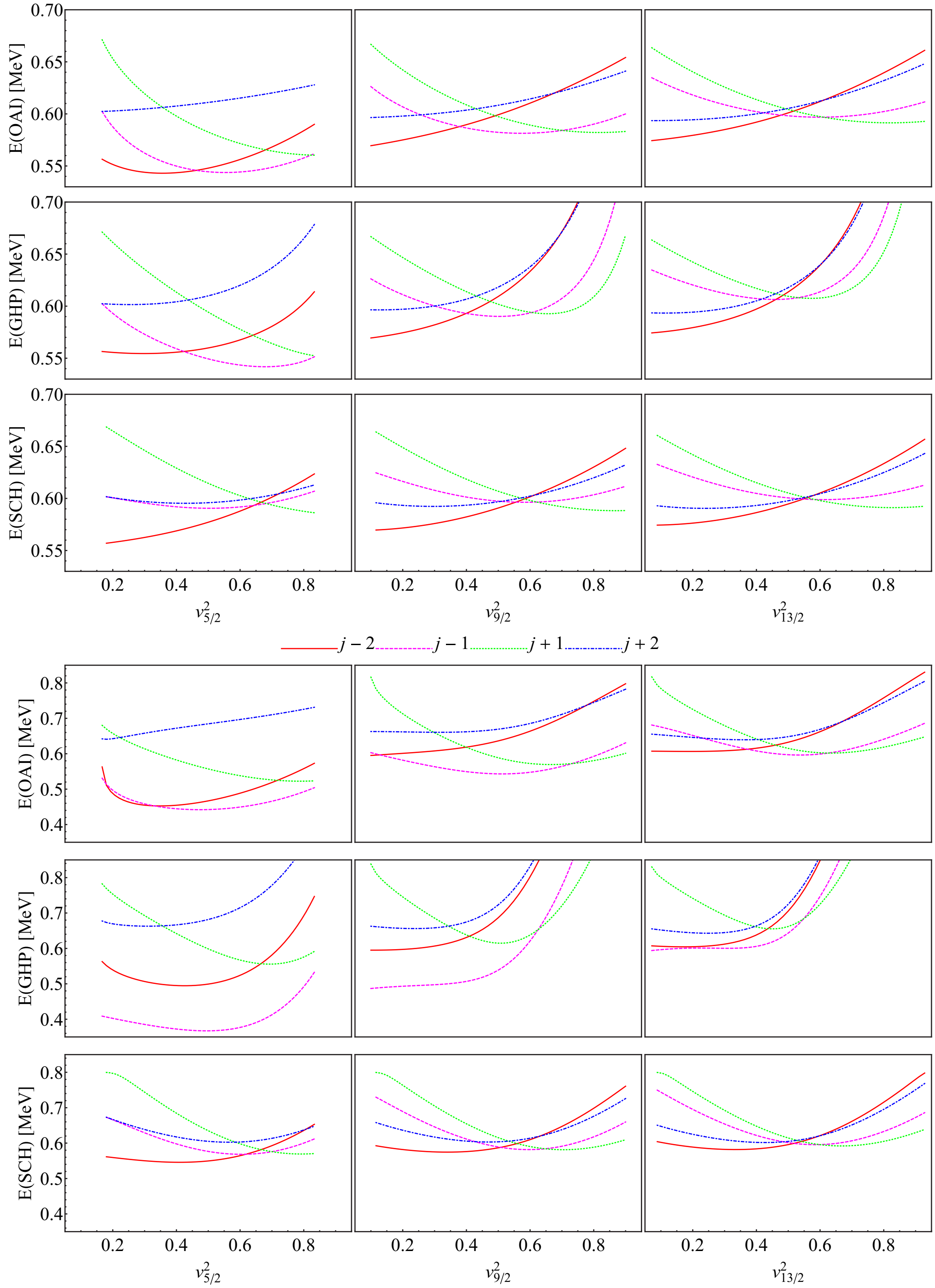

FIG. 2. Energies of a single j calculation for $j = \frac{5}{2}$ (left), $\frac{9}{2}$ (middle), and $\frac{13}{2}$ (right) with $\kappa = -0.2$ MeV (top) and $\kappa = -0.5$ MeV (bottom). Red, $J = j - 2$; magenta, $J = j - 1$; green, $J = j + 1$, and blue, $J = j + 2$. The ground state was taken as zero energy and it is not shown to visualize the energies more clearly.

$$\Delta^{22}_{jjjj} = -\frac{5Q_{jj}}{\Omega_j}. \tag{64}$$

To study the behavior of the new derived boson-fermion interaction we chose a system with an odd neutron allowed to occupy the orbit j and coupled to a core with a fixed number of proton bosons, $N_\pi = 5$, and a number of neutron bosons N_ν between zero and $j - 1/2$, which corresponds to two units less than the maximum occupancy of the orbit j. We use the

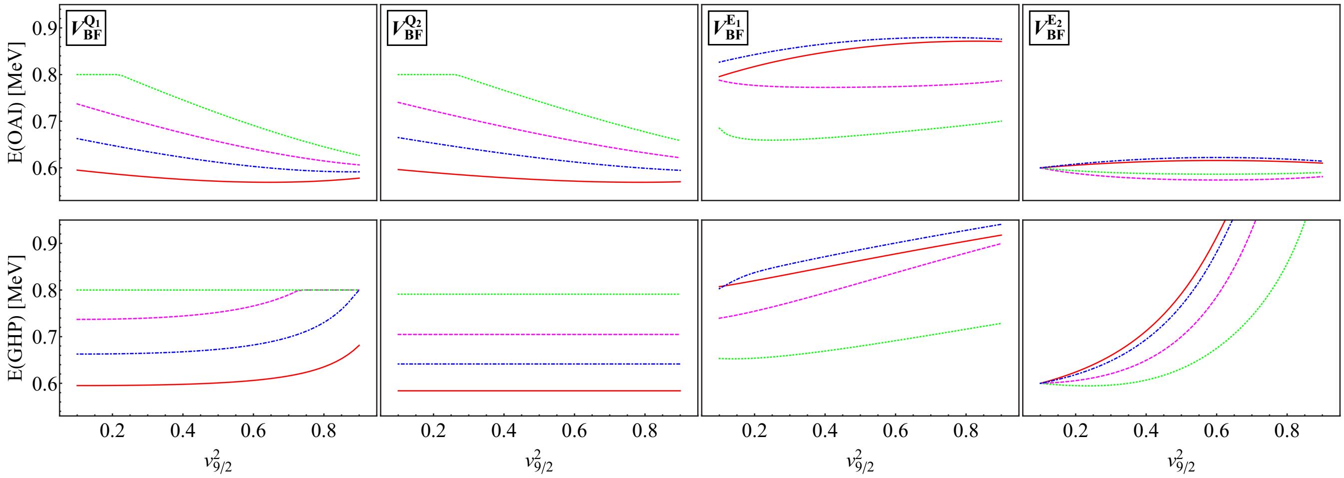

FIG. 3. Same as Fig. 2, but considering just one of the terms in the boson-fermion interaction. The calculation were done with $\kappa = -0.5$ MeV and $j = 9/2$.

simple Hamiltonian

$$H = \sum_{\rho=\pi,\nu} \epsilon_\rho \hat{n}_{d_\rho} + \kappa\big(Q_\pi^{(2)} \cdot q_\nu^{(2)}\big), \tag{65}$$

where $\chi_\pi = -\sqrt{7}/2$ and we have set the energy of the d bosons to $\epsilon_\nu = 0.8$ MeV and $\epsilon_\pi = 0.6$ MeV for neutrons and protons, respectively. In this Hamiltonian, bosons are in the vibrational limit and the fermion couples to unlike bosons. Although it is not realistic, it is used here for the purpose of understanding the difference between OAI and GHP methods. We did two different calculations: one with $\kappa = -0.2$ MeV and another with $\kappa = -0.5$ MeV, and separately for three values of $j = 5/2$, $9/2$, and $13/2$, to understand its j dependence. We have diagonalized this Hamiltonian using OAI and GHP methods in a base with states of generalized seniority $\tilde{v} \leqslant 5$ (up to two d bosons). We also included the results obtained with Scholten's interaction, which are indicated with the label SCH. The excitation energies for different values of the final angular momentum J are shown in Fig. 2 in terms of the neutron occupation probability, calculated as $v_j^2 = (2N_\nu + 1)/(2j + 1)$. The results of calculations obtained considering just one of the contributions to the boson-fermion interaction for the case $j = 9/2$ with $\kappa = -0.5$ MeV were plotted in Fig. 3. In each figure the spectrum is given for final states with $J = j \pm 1$ and $j \pm 2$ only, and relative to the state $J = j$, which always is, for the vibrational case considered here, the lowest state, i.e., the ground state.

In Fig. 2 a behavior common to all the calculations consists in that the energies of states with different values of J cross when the occupation of the j shell is close to half filled, with values for the energies below the values at the beginning ($v_j^2 \sim 0$) or at the end ($v_j^2 \sim 1$) of the shell. We can see a general trend in OAI and GHP methods, in which the energy differences among states decrease as the value of j increases and, at the same time, their excitation energies increase. This trend is more pronounced when the strength of the quadrupole interaction increases, as it can be seen when we compare the results for both values of the strength κ, since we observe that the separation between states is larger when κ increases, i.e., when the weak coupling scheme becomes less important. This effect is less pronounced using Scholten's interaction, whose results remain practically the same when $j \geqslant 9/2$. OAI and GHP methods produce results rather similar when $v_j^2 < 0.6$ for $\kappa = -0.2$ MeV, but they become different for $\kappa = -0.5$ MeV, especially for the level with $J = j - 1$. In this case we can find similar results for OAI and GHP methods only when $v_j^2 < 0.4$ and for the levels with $J = j \pm 2$ and $J = j + 1$. In contrast, Scholten's form of the interaction produces results similar to those of OAI and GHP methods just when $j = 13/2$. There is another effect present only in the GHP calculations. The energies increase abruptly when v_j^2 approaches 1. This is discussed later in the context of Fig. 3. Finally, unlike Scholten's results the so-called $J = j - 1$ anomaly appears in OAI and GHP calculations when $\kappa = -0.5$ MeV and $j = 5/2$ and $j = 9/2$. This could indicate that quadrupole-quadrupole interaction favors the presence of this anomaly.

Figure 3 shows the results obtained when considering just one of the different terms in the boson-fermion interaction for the case $j = 9/2$ and $\kappa = -0.5$ MeV. The behavior is rather different between OAI and GHP methods, except for $V_{\rm BF}^{E_1}$, where both approaches provide very similar results. In the case of the quadrupole term $V_{\rm BF}^{Q_1}$ a degeneration between the levels with $J = j \pm 1$ happens for GHP method at the end of the shell, while the levels are independent of $v_{9/2}^2$ for $V_{\rm BF}^{Q_2}$. In the OAI case both terms produce quite similar behaviors. However, we can see that the exchange term $V_{\rm BF}^{E_2}$ in the GHP case is responsible for the observed rise of the level energies when the occupancy increases. The origin of this behavior is in the presence of factors of the type $\sim u_j^{-1}$ in the coefficients of the GHP expansion in the transfer operator. Actually these factors come from the Taylor series expansion of the square-root operator about a nonzero value of the matrix element of $(B^\dagger B)^T$, as it is discussed in detail in Ref. [20]. This expansion is valid only for low values of v_j^2. For higher values and in the limit $v_j^2 \to 1$ the results obtained are spurious. Therefore, further investigation is required to obtain valid expressions in the GHP mapping when $v_j^2 \to 1$. In the OAI case the dependence is different and $V_{\mathrm{BF}}^{E_2}$ remains constant essentially and with the same values for $J = j \pm 1$ and $J = j \pm 2$. It is worthy to note that separately the different contributions to the boson-fermion interaction cannot explain the $J = j - 1$ anomaly, in contrast to the results obtained when they are combined for $\kappa = -0.5$, which nicely they reproduce it when $j < 13/2$.

V. SUMMARY AND CONCLUSIONS

In this article we derived the boson-fermion interaction in the IBFM from a quadrupole-quadrupole (q-q) force. To this end the boson-fermion quadrupole operator was obtained from the one nucleon transfer operator following two alternative approaches: OAI and GHP. The expressions obtained are suitable for the coupling between the odd nucleon and the alike bosons, which is the traditional boson-fermion interaction used in phenomenological studies in IBFM-1 and IBFM-2. We found that the interaction contains four terms: two quadrupole terms and two exchange terms. This splitting allows the introduction of four different parameters which can be fitted to study odd-A nuclei. In addition these expressions are valid also for quadrupole pairing interactions which were demonstrated as the shell model sources of the exchange interaction in IBFM in and near spherical regions of the nuclear chart [24].

We studied a single-j model with three different values of j and also for two values of the strength of the q-q force. For all cases, OAI and GHP schemes produce similar results when the shell is half or less occupied (near shell closure) for relatively small values of the strength of the quadrupole-quadrupole interaction, which indicates that for vibrational nuclei both mappings may work on the same footing. For higher values of the occupancy, the GHP method provides increasing values of excitation energies, while results from the OAI method remain stable. This behavior may change in realistic cases, where several nondegenerate orbits are present. Studies in this direction will be addressed and their results will be published elsewhere.

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