Exact solution of the mean-field plus separable pairing model reexamined

Feng Pan,[1,2,*] Dan Zhou,[1] Lianrong Dai,[1] and J. P. Draayer[2]
[1]*Department of Physics, Liaoning Normal University, Dalian 116029, China*
[2]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA*
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Exact solution of the nuclear mean-field plus separable pairing model is reexamined. New auxiliary constraints for solving the Bethe ansatz equations of the model are proposed. By using these auxiliary constraints, the Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian with nondegenerate single-particle energies and nondegenerate separable pairing strengths purposed previously is verified. Since the solutions of the model with one- and two-orbit cases are known, verification of the solutions for these two special cases is made. To demonstrate structure and features of the solution, the model with three orbits in the *ds* shell is taken as a nontrivial example, of which two-pair results and the ground state of the three-pair case are provided explicitly. Since the number of equations involved increases with the number of orbits and pairs, to solve these equations for a large number of orbits and pairs seems still difficult.

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I. INTRODUCTION

Pairing has been playing an important role in many branches of physics. In nuclear physics, pairing interaction is considered one of the important types of residual interactions in a nuclear mean field to describe ground-state and low-energy spectroscopic properties of nuclei, such as binding energies, odd-even effects, single-particle occupancies, excitation spectra, and moments of inertia [1,2]. It has been shown that either spherical or deformed mean-field plus the standard (orbit-independent) pairing interaction can be solved exactly by using the Gaudin-Richardson method [3–5]. The Gaudin-Richardson equations in this case can be solved more easily by using the extended Heine-Stieltjes polynomial approach [6–9]. The deformed and spherical mean-field plus the extended pairing models have also been proposed, which can be solved more easily than the standard pairing model, especially when both the number of valence nucleon pairs and the number of single-particle orbits are large [10,11].

Exact solution of the separable pairing model with degenerate single-particle energy was proposed in Ref. [12], of which the solution is similar to that of the Gaudin-Richardson type for the standard pairing model. The separable pairing model with two nondegenerate orbits was analyzed in Ref. [13]. In Refs. [14–19], exact solution of a special family of the hyperbolic Richardson-Gaudin models was proposed, of which a special case related to the problem may also be derived based on the simple procedure shown in Ref. [20]. General nondegenerate cases were considered previously in Refs. [21,22]. However, the auxiliary constraints used in Refs. [21,22] are awkward and may be too specific, though they can be used to provide solutions of the cases presented in Ref. [20].

In this work, the Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian with nondegenerate single-particle energies and nondegenerate separable pairing strengths purposed in Refs. [21,22] is verified with the help of a set of new auxiliary constraints for solving the corresponding Bethe ansatz equations. Since solutions of the model for one- and two-orbit cases are well known, verification of the solutions for these two special cases is made. To demonstrate structure and features of the solution, the model with three orbits in the *ds* shell is taken as a nontrivial example, of which two- and three-pair solutions are provided explicitly.

II. THE MODEL AND ITS GENERAL SOLUTION

The Hamiltonian of the mean-field plus separable pairing model is given as [21,22]

$$\hat{H} = \sum_{t=1}^{p} \epsilon_{j_t} \hat{N}_{j_t} + \hat{H}_{\rm P} = \sum_{t=1}^{p} \epsilon_{j_t} \hat{N}_{j_t} - G \sum_{1\leqslant t,t'\leqslant p} c_{j_t} c_{j_{t'}} S^{+}_{j_t} S^{-}_{j_{t'}}, \tag{1}$$

where p is the total number of orbits considered, $\{\epsilon_{j_t}\}$ $(t = 1,2,\ldots,p)$ is a set of single-particle energies generated from any mean-field theory, such as those of the shell model, $\hat{N}_{j_t} = \sum_m a^{\dagger}_{j_t m} a_{j_t m}$ and $S^{+}_{j_t} = \sum_{m>0} (-1)^{j_t - m} a^{\dagger}_{j_t m} a^{\dagger}_{j_t -m}$, in which $a^{\dagger}_{j_t m_t}$ $(a_{j_t m})$ is the creation (annihilation) operator for a nucleon with angular momentum quantum number j_t and that of its projection m, and G and $\{c_{j_t}\}$ $(t = 1,2,\ldots,p)$ are the separable pairing interaction parameters, which are all assumed to be real. To avoid degeneracy, which will result in no solution from the procedure, $\epsilon_{j_t} \neq \epsilon_{j_{t'}}$ and $c_{j_t} \neq c_{j_{t'}}$ for $1 \leqslant t,t' \leqslant p$ are assumed in this work.

The set of operators $\{S^{-}_{j_t}, S^{+}_{j_t}, \hat{N}_{j_t}\}$ $(t = 1,2,\ldots,p)$, where $S^{-}_{j_t} = (S^{+}_{j_t})^{\dagger}$, generates p copies of SU(2) algebra satisfying the commutation relations

$$[\hat{N}_{j_t}/2, S^{-}_{j_{t'}}] = -\delta_{tt'} S^{-}_{j_t}, \quad [\hat{N}_{j_t}/2, S^{+}_{j_{t'}}] = \delta_{tt'} S^{+}_{j_t},$$
$$[S^{+}_{j_t}, S^{-}_{j_{t'}}] = 2\delta_{tt'} S^{0}_{j_t}, \tag{2}$$

where $S^{0}_{j_t} = (\hat{N}_{j_t} - \Omega_{j_t})/2$ with $\Omega_{j_t} = j_t + 1/2$.

*Corresponding author: daipan@dlut.edu.cn

Let

$$S^{+}(x_{\mu})=\sum_{t=1}^{p}\sum_{i=1}^{q}\frac{a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}c_{j_t}S_{j_t}^{+}=\sum_{i=1}^{q}a_{i}(x_{\mu})S^{+}(x_{\mu,i}),\quad(3)$$

where $S^{+}(x_{\mu})\equiv S^{+}(x_{\mu,1},\ldots,x_{\mu,q})$, which is frequently used to simplify the expression, depends on q variables $\{x_{\mu,1},\ldots,x_{\mu,q}\}$, and

$$S^{+}(x_{\mu,i})=\sum_{t=1}^{p}\frac{1}{c_{j_t}^{2}-x_{\mu,i}}c_{j_t}S_{j_t}^{+},\quad(4)$$

and $\{x_{\mu,i}\}$ and $\{a_i(x_\mu)\equiv a_i(x_{\mu,1},\ldots,x_{\mu,q})\}$ $(i=1,2,\ldots,q)$ are two sets of parameters to be determined for a given μ, in which $a_i(x_\mu)$ also depends on the variables $\{x_{\mu,1},\ldots,x_{\mu,q}\}$. According to the commutation relations given in Eqs. (2), we have

$$\left[\sum_{t}\epsilon_{j_t}\hat{N}_{j_t},S^{+}(x_{\mu})\right]=\sum_{i,t}\frac{2\epsilon_{j_t}a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}c_{j_t}S_{j_t}^{+},\quad(5)$$

which, now, is imposed with the following constraints:

$$\sum_{i,t}\frac{2\epsilon_{j_t}a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}c_{j_t}S_{j_t}^{+}=\sum_{t}c_{j_t}S_{j_t}^{+}+\beta(x_{\mu})S^{+}(x_{\mu}),\quad(6)$$

where $\beta(x_\mu)\equiv\beta(x_{\mu,1},\ldots,x_{\mu,q})$ may depend on $\{x_{\mu,1},\ldots,x_{\mu,q}\}$. Equation (6) can be expressed alternatively as

$$\sum_{i=1}^{q}\frac{[2\epsilon_{j_t}-\beta(x_{\mu})]a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}=1\quad\text{for } t=1,2,\ldots,p.\quad(7)$$

It is clearly shown that the pairing operator given in Eq. (3) is the same as that used in the separable pairing model [20–22] when the constraint (7) is used. Namely,

$$S^{+}(x_{\mu})=\sum_{t=1}^{p}\sum_{i=1}^{q}\frac{a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}c_{j_t}S_{j_t}^{+}=\sum_{t=1}^{p}\frac{c_{j_t}}{2\epsilon_{j_t}-\beta(x_{\mu})}S_{j_t}^{+},\quad(8)$$

where $\beta(x_\mu)\equiv\beta(x_{\mu,1},\ldots,x_{\mu,q})$ is related to eigenenergy of the model.

Practically, we can use Eq. (7) to get expressions of $a_i(x_\mu)$ with $i=1,\ldots,q$ and other $p-q\geqslant0$ relations among $\beta(x_\mu)$ and $\{x_{\mu,1},\ldots,x_{\mu,q}\}$ for a given μ. Moreover,

$$[\hat{H}_{\rm P},S^{+}(x_{\mu})]=G\sum_{t'=1}^{p}c_{j_{t'}}S_{j_{t'}}^{+}\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{2S_{j_t}^{0}a_{i}(x_{\mu})\left(c_{j_t}\right)^{2}}{c_{j_t}^{2}-x_{\mu,i}}$$
$$=G\sum_{t'=1}^{p}c_{j_{t'}}S_{j_{t'}}^{+}\Lambda_{0}(x_{\mu}),\quad(9)$$

where

$$\Lambda_{0}(x_{\mu})=\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{2S_{j_t}^{0}\left(c_{j_t}\right)^{2}a_{i}(x_{\mu})}{c_{j_t}^{2}-x_{\mu,i}}=\sum_{t=1}^{p}\frac{2S_{j_t}^{0}\left(c_{j_t}\right)^{2}}{2\epsilon_{j_t}-\beta(x_{\mu})},\quad(10)$$

and

$$S^{+}(x_{\mu},x_{\nu})\equiv S^{+}(x_{\mu,1},x_{\mu,2},\ldots,x_{\mu,q},x_{\nu,1},x_{\nu,2},\ldots,x_{\nu,q})$$
$$=[[\hat{H}_{\rm P},S^{+}(x_{\mu})],S^{+}(x_{\nu})]$$
$$=2G\sum_{t'}c_{j_{t'}}S_{j_{t'}}^{+}\sum_{ii'}a_{i}(x_{\mu})a_{i'}(x_{\nu})\times\sum_{t}\frac{\left(c_{j_t}\right)^{2}}{\left(c_{j_t}^{2}-x_{\mu,i}\right)\left(c_{j_t}^{2}-x_{\nu,i'}\right)}c_{j_t}S_{j_t}^{+}$$
$$=2G\sum_{t'}c_{j_{t'}}S_{j_{t'}}^{+}\sum_{ii'}\frac{a_{i}(x_{\mu})a_{i'}(x_{\nu})}{x_{\mu,i}-x_{\nu,i'}}\times(x_{\mu,i}S^{+}(x_{\mu,i})-x_{\nu,i'}S^{+}(x_{\nu,i'})).\quad(11)$$

For the two-pair ($k=2$) case, let

$$F(x_{1},x_{2})\equiv F(x_{1,1},x_{1,2},\ldots,x_{1,q},x_{2,1},x_{2,2},\ldots,x_{2,q})=\sum_{i'=1}^{q}\frac{a_{i'}(x_{2})x_{1,i}}{x_{1,i}-x_{2,i'}}\quad\text{for } i=1,\ldots,q.\quad(12)$$

Once Eq. (12) is solved, Eq. (11) can be rewritten as

$$S^{+}(x_{1},x_{2})=2G\sum_{t'}c_{j_{t'}}S_{j_{t'}}^{+}(F(x_{1},x_{2})S^{+}(x_{1})+F(x_{2},x_{1})S^{+}(x_{2})).\quad(13)$$

As shown in Eqs. (11)–(13), $F(x_2,x_1)$ can be obtained from $F(x_1,x_2)$ by permuting $x_{1,i}$ with $x_{2,i}$ for $i=1,\ldots,q$. However, when $k\geqslant3$, Eq. (12) is no longer valid, which is dealt with shortly.

As can be seen from Eqs. (10) and (11), $x_{\mu,i}\neq c_{j_t}^2$ for any μ, i, and t, $x_{\mu,i}\neq x_{\nu,i'}$ for given $\mu\neq\nu$ and any i and i', and $\beta(x_\mu)\neq2\epsilon_{j_t}$ for any t and μ should always be assumed to avoid divergence. In addition to the eigenequation of the model, which provides one constraint to the variable $\{x_{\mu,i}\}$ for a fixed μ, Eq. (7) and equations related to Eq. (12) for the two-pair case provide $p+q$ equations for a fixed μ, while the total number of unknowns, $\{x_{\mu,i},a_i(x_\mu)\}$, $\beta(x_\mu)$ for fixed μ, and $F(x_1,x_2)$ for the two-pair case, is $2q+2$. Thus, in order to get a unique solution to the problem, we need $q=p-1$, which is used in the following. Moreover, for a fixed μ, by removing the last equation with $t=p$ in Eq. (7), which may be used to express one of $\{x_{\mu,i}\}$ $(i=1,2,\ldots,q)$ in terms of the remaining $q-1$ variables, the remaining q equations provided by Eq. (7) may be expressed in matrix form with

$$\mathbf{B}\,\mathbf{a}=\mathbf{I},\quad(14)$$

where the vector $\mathbf{a}=(a_1(x_\mu),a_2(x_\mu),\ldots,a_q(x_\mu))^{\rm T}$, in which T denotes the matrix transposition, and $\mathbf{I}=(1,1,\ldots,1)^{\rm T}$ with q components, and

$$\mathbf{B}=\begin{pmatrix}\frac{2\epsilon_{j_1}-\beta(x_\mu)}{c_{j_1}^2-x_{\mu,1}} & \frac{2\epsilon_{j_1}-\beta(x_\mu)}{c_{j_1}^2-x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_1}-\beta(x_\mu)}{c_{j_1}^2-x_{\mu,q}}\\ \frac{2\epsilon_{j_2}-\beta(x_\mu)}{c_{j_2}^2-x_{\mu,1}} & \frac{2\epsilon_{j_2}-\beta(x_\mu)}{c_{j_2}^2-x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_2}-\beta(x_\mu)}{c_{j_2}^2-x_{\mu,q}}\\ \vdots & \vdots & \cdots & \vdots\\ \frac{2\epsilon_{j_q}-\beta(x_\mu)}{c_{j_q}^2-x_{\mu,1}} & \frac{2\epsilon_{j_q}-\beta(x_\mu)}{c_{j_q}^2-x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_q}-\beta(x_\mu)}{c_{j_q}^2-x_{\mu,q}}\end{pmatrix}.\quad(15)$$

It is obvious that Eq. (14) has a unique solution of $\mathbf{a}$ when and only when the matrix $\mathbf{B}$ is nonsingular with Det($\mathbf{B}$) $\neq0$, which requires $x_{\mu,i}\neq x_{\mu,i'}$ for any μ and $i\neq i'$.

In the following, since the formalism for even-odd systems is similar, we focus on the seniority zero cases for simplicity.

Let $|0\rangle$ be the pairing vacuum state satisfying $S^-_{j_t}|0\rangle = 0\ \forall t$. A k-pair eigenstate of Eq. (1) may be expressed as

$$|\zeta,k\rangle = S^+\big(x_1^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_k^{(\zeta)}\big)|0\rangle, \tag{16}$$

where ζ is an additional quantum number introduced to label the ζth excitation state, and the explicit operator form of $S^+(x_\mu^{(\zeta)})$ is still given by Eq. (8), which was also used in Refs. [20–22]. Using Eqs. (6), (9), and (11), we can directly check that

$$\sum_t \epsilon_{j_t}\hat{N}_{j_t}|\zeta,k\rangle = \left(\sum_t c_{j_t}S^+_{j_t} + \beta\big(x_1^{(\zeta)}\big)S^+\big(x_1^{(\zeta)}\big)\right)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_k^{(\zeta)}\big)|0\rangle + \cdots + S^+\big(x_1^{(\zeta)}\big)\cdots S^+\big(x_{k-1}^{(\zeta)}\big)\left(\sum_t c_{j_t}S^+_{j_t} + \beta\big(x_k^{(\zeta)}\big)S^+\big(x_k^{(\zeta)}\big)\right)|0\rangle \tag{17}$$

and

$$\begin{aligned}\hat{H}_{\rm P}|\zeta,k\rangle = {} & G\sum_{t'} c_{j_{t'}}S^+_{j_{t'}}\big(\bar{\Lambda}_0\big(x_1^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_k^{(\zeta)}\big) + \cdots + S^+\big(x_1^{(\zeta)}\big)\cdots S^+\big(x_{k-1}^{(\zeta)}\big)\bar{\Lambda}_0\big(x_k^{(\zeta)}\big)\big)|0\rangle \\ & + \big(S^+\big(x_1^{(\zeta)},x_2^{(\zeta)}\big)S^+\big(x_3^{(\zeta)}\big)\cdots S^+\big(x_k^{(\zeta)}\big) + S^+\big(x_1^{(\zeta)},x_3^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)S^+\big(x_4^{(\zeta)}\big)\cdots S^+\big(x_k^{(\zeta)}\big) + \cdots \\ & + S^+\big(x_1^{(\zeta)},x_k^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_{k-1}^{(\zeta)}\big) + \cdots + S^+\big(x_k^{(\zeta)},x_1^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_{k-1}^{(\zeta)}\big) \\ & + S^+\big(x_k^{(\zeta)},x_2^{(\zeta)}\big)S^+\big(x_3^{(\zeta)}\big)\cdots S^+\big(x_{k-1}^{(\zeta)}\big) + \cdots + S^+\big(x_k^{(\zeta)},x_{k-1}^{(\zeta)}\big)S^+\big(x_2^{(\zeta)}\big)\cdots S^+\big(x_{k-2}^{(\zeta)}\big)\big)|0\rangle,\end{aligned} \tag{18}$$

where

$$\bar{\Lambda}_0(x_\mu^{(\zeta)}) = -\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2 a_i(x_\mu^{(\zeta)})}{c_{j_t}^2 - x_{\mu,i}^{(\zeta)}} = -\sum_{t=1}^{p}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2}{2\epsilon_{j_t} - \beta\big(x_\mu^{(\zeta)}\big)}. \tag{19}$$

Using Eqs. (17) and (18), one can prove that the eigenequation $\hat{H}|\zeta,k\rangle = E_k^{(\zeta)}|\zeta,k\rangle$ is fulfilled if and only if

$$\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2 a_i\big(x_\mu^{(\zeta)}\big)}{c_{j_t}^2 - x_{\mu,i}^{(\zeta)}} - 2W\big(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)}\big) = \frac{1}{G}\quad \text{for } \mu = 1,2,\ldots,k, \tag{20}$$

or

$$\sum_{t=1}^{p}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2}{2\epsilon_{j_t} - \beta\big(x_\mu^{(\zeta)}\big)} - 2W\big(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)}\big) = \frac{1}{G}\quad \text{for } \mu = 1,2,\ldots,k, \tag{21}$$

where

$$W\big(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)}\big) = \sum_{\nu\neq\mu}\sum_{i'=1}^{q}\frac{a_{i'}\big(x_\mu^{(\zeta)}\big)x_{\nu,i}^{(\zeta)}}{x_{\nu,i}^{(\zeta)} - x_{\mu,i'}^{(\zeta)}}\quad \text{for } i = 1,2,\ldots,q, \tag{22}$$

of which each term for fixed ν in the sum is the same as that shown in Eq. (12). When $k = 2$, Eq. (22) becomes Eq. (12). However, every term for fixed ν in the sum of Eq. (22) depends on $\{x_{\nu,i}^{(\zeta)}\}$ with $\nu \neq \mu$, which is different from that in the Gaudin-Richardson solution of the standard pairing model [3,4], and must be considered together to be solved as shown in Eq. (22). In addition, it is obvious that $W(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)})$ for a fixed μ is symmetric with respect to any permutation among $\{x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)}\}$, which is similar to that in the Gaudin-Richardson solution of the standard pairing model. The corresponding eigenenergy is given by

$$E_k^{(\zeta)} = \sum_{\mu=1}^{k}\beta\big(x_\mu^{(\zeta)}\big). \tag{23}$$

Since $\{a_i(x_\mu^{(\zeta)})\}$ $(i = 1,2,\ldots,q)$ and $\{x_{\mu,q=p-1}^{(\zeta)}\}$ are expressed in terms of $\{\beta(x_\mu^{(\zeta)})\}$ and $\{x_{\mu,i}^{(\zeta)}\}$ $(i = 1,2,\ldots,q-1)$ according to Eq. (7), q equations given by Eq. (22) provide expressions of $x_{\mu,i}^{(\zeta)}$ $(i = 1,2,\ldots,q-1)$ and the final expression of $W(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)})$ for a fixed μ. When $p \geqslant 3$, for a fixed μ, we use Eq. (7) to get solution of $\{a_1(x_\mu^{(\zeta)}),\ldots,a_{q=p-1}(x_\mu^{(\zeta)})\}$ and $x_{\mu,p-1}^{(\zeta)}$, and use Eq. (22) to get that of $W(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)})$ for given μ and $\{x_{\mu,1}^{(\zeta)},\ldots,x_{\mu,p-2}^{(\zeta)}\}$. Meanwhile, $\beta(x_\mu^{(\zeta)})$ is determined by the Bethe ansatz equations (20) and (21). Although Eqs. (20) and (21) look quite similar to the Bethe ansatz equations for the standard pairing case, the term $W(x_\mu^{(\zeta)};x_1^{(\zeta)},\ldots,x_{\mu-1}^{(\zeta)},x_{\mu+1}^{(\zeta)},\ldots,x_k^{(\zeta)})$ involved for a fixed μ should be determined by q equations given in Eq. (22). Therefore, solutions of Eqs. (20) and (21) cannot be obtained as easily as those in the standard pairing case shown in Refs. [7,8].

III. SOME EXPLICIT EXAMPLES

In the following, we provide solutions of the model for some special cases and compare them with those known previously and display the procedure for some nontrivial cases with $p = 3$, which demonstrates that the procedure indeed works for the model with nondegenerate single-particle energies $\{\epsilon_{j_t}\}$ and nondegenerate separable pairing strengths $\{c_{j_t}\}$.

A. $p = 1$ case

For this case, we may set $a_1(x) = 1$ in Eq. (7) because it only changes the normalization factor of the eigenstates of the model. Thus, we have

$$\beta(x) = 2\epsilon_{j_1} - (c_{j_1})^2 + x. \tag{24}$$

By using Eq. (12), Eq. (20) can be written as

$$\frac{\Omega_{j_1}(c_{j_1})^2}{c_{j_1}^2 - x_\mu} + \sum_{\nu\neq\mu}\frac{2x_\nu}{x_\mu - x_\nu} = \frac{1}{G} \quad \text{for } \mu = 1,2,\ldots,k, \tag{25}$$

where the subscript $i(=1)$ is omitted with $\{x_{\mu,i=1} \equiv x_\mu\}$. For any k, up to permutations among the k components of the root, there is only one set of solution of Eq. (25). Though the root of Eq. (25) may be complex when $k \geqslant 2$, without solving Eq. (25), one can evaluate the eigenenergy according to Eq. (23) from the following procedures: Summing Eq. (25) over μ, one has

$$\sum_{\mu=1}^{k}\frac{\Omega_{j_1}(c_{j_1})^2}{c_{j_1}^2 - x_\mu} = \frac{k}{G} + k(k-1), \tag{26}$$

while multiplying Eq. (25) by x_μ and then summing over μ, one gets

$$\sum_{\mu=1}^{k}\frac{\Omega_{j_1}x_\mu(c_{j_1})^2}{c_{j_1}^2 - x_\mu} = (c_{j_1})^2\sum_{\mu=1}^{k}\frac{\Omega_{j_1}(c_{j_1})^2}{c_{j_1}^2 - x_\mu} - \Omega_{j_1}(c_{j_1})^2 k$$
$$= \frac{1}{G}\sum_{\mu=1}^{k}x_\mu. \tag{27}$$

Combining Eqs. (26) and (27), one obtains

$$\sum_{\mu=1}^{k}x_\mu = (c_{j_1})^2 G\left(k^2 - k + \frac{k}{G} - \Omega_{j_1}k\right). \tag{28}$$

However, substituting Eq. (24) into Eq. (23), we have

$$E_k^{(\zeta)} = 2\epsilon_{j_1}k - (c_{j_1})^2 k + \sum_{\mu=1}^{k}x_\mu. \tag{29}$$

Finally, substituting Eq. (28) into Eq. (29), we get

$$E_k^{(\zeta)} = 2\epsilon_{j_1}k + G(c_{j_1})^2(k^2 - k - \Omega_{j_1}k), \tag{30}$$

which is exactly the same as that given by the well-known Racah quasispin formalism for the standard pairing model with modified pairing strength $\tilde{G} = (c_{j_1})^2 G$.

B. $p = 2$ case

For this case $q = 1$. It should be assumed that $c_{j_1} \neq c_{j_2}$ and $\epsilon_{j_1} \neq \epsilon_{j_2}$. Hence, Eq. (7) provides

$$a_1(x) \equiv a = \frac{(c_{j_1})^2 - (c_{j_2})^2}{2\epsilon_{j_1} - 2\epsilon_{j_2}},$$
$$\beta(x) = \frac{2\epsilon_{j_2}(c_{j_1})^2 - 2\epsilon_{j_1}(c_{j_2})^2}{(c_{j_1})^2 - (c_{j_2})^2} + x\frac{2\epsilon_{j_1} - 2\epsilon_{j_2}}{(c_{j_1})^2 - (c_{j_2})^2}. \tag{31}$$

Equation (20) is simply given by

$$\sum_{t=1}^{p}\frac{\Omega_{j_t}(c_{j_t})^2}{c_{j_t}^2 - x_\mu^{(\zeta)}} + \sum_{\nu\neq\mu}\frac{2x_\nu^{(\zeta)}}{x_\mu^{(\zeta)} - x_\nu^{(\zeta)}} = \frac{1}{aG} \quad \text{for } \mu = 1,2,\ldots,k, \tag{32}$$

where the subscript $i(=1)$ is also omitted with $\{x_{\mu,i=1} \equiv x_\mu\}$.

Moreover, as shown in Refs. [18–20], when

$$(c_{j_t})^2 = g_1\epsilon_{j_t} + g_2 \tag{33}$$

for $t = 1,2,\ldots,p$, where g_1 and g_2 are two constants, the Hamiltonian (1) for any p in this case is exactly solvable. For the $p = 2$ case, since t can only be taken as 1 or 2, Eq. (33) provides a unique solution of the two parameters g_1 and g_2 for the $p = 2$ case with

$$g_1 = 2a, \quad g_2 = \frac{(c_{j_2})^2\epsilon_{j_1} - (c_{j_1})^2\epsilon_{j_2}}{\epsilon_{j_1} - \epsilon_{j_2}}. \tag{34}$$

As shown in Ref. [20] for the $p = 2$ case, the Bethe ansatz equations are

$$\sum_{t=1}^{p}\frac{(c_{j_t})^2\Omega_{j_t}}{2\epsilon_{j_t} - z_i^{(\zeta)}} + \sum_{l\neq i}\frac{g_1 z_l^{(\zeta)} + 2g_2}{z_i^{(\zeta)} - z_l^{(\zeta)}}$$
$$= 1/G \quad \text{for } i = 1,2,\ldots,k, \tag{35}$$

with ζth eigenenergy given by

$$E_k^{(\zeta)} = \sum_{i=1}^{k} z_i^{(\zeta)}, \tag{36}$$

of which the corresponding eigenstate is expressed as

$$|\zeta,k\rangle = S^+(z_1^{(\zeta)})\cdots S^+(z_k^{(\zeta)})|0\rangle, \tag{37}$$

where

$$S^+(z) = \sum_{t=1}^{p}\frac{1}{2\epsilon_{j_t} - z}c_{j_t}S_{j_t}^+. \tag{38}$$

By substituting $z_l = 2(x_l - g_2)/g_1$ for $l = 1,\ldots,k$ into Eqs. (35)–(37), Eqs. (35)–(37) become Eqs. (20), (23), and (16), respectively, for which the constraints given in Eq. (33) should be used. Thus, it is shown that the results for the $p = 1$ and $p = 2$ cases obtained from the procedure proposed in this work are consistent with those obtained previously.

TABLE I. The single-particle energies ϵ_{j_t} (in MeV) for the ds shell deduced from Ref. [23], the parameters $\{c_{j_t}\}$, and G (in MeV) taken from Ref. [20], where $j_1 = 5/2$, $j_2 = 1/2$, and $j_3 = 3/2$, and the overlaps $\rho(\zeta,k) = |\langle \zeta,k|\zeta,k\rangle_{\rm FMD}|$ for the $k = 2$ case and the ground state of the $k = 3$ case, where $|\zeta,k\rangle$ is the ζth k-pair excitation state given by Eq. (16), and $|\zeta,k\rangle_{\rm FMD}$ is that obtained from the full matrix diagonalization within the ds-shell subspace shown in Ref. [20].

$\epsilon_{j_1} = -3.70$	$\epsilon_{j_2} = -2.92$	$\epsilon_{j_3} = 1.90$	$c_{j_1} = 0.99583$	$c_{j_2} = -0.06334$	$c_{j_3} = 0.06562$	$G = 0.945$
$\rho(1,2) = 0.99849$	$\rho(2,2) = 0.99625$	$\rho(3,2) = 0.99884$	$\rho(4,2) = 0.99931$	$\rho(5,2) = 0.999997$	$\rho(1,3) = 0.978308$	

C. $p = 3$ case

For this case $q = 2$. $\epsilon_{j_1} \neq \epsilon_{j_2} \neq \epsilon_{j_3}$ and $c_{j_1} \neq c_{j_2} \neq c_{j_3}$ should also be assumed. To explicitly demonstrate the solutions, we take the ds shell with three orbitals $0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$, of which the single-particle energies are provided in Ref. [23], while the values of the parameters $\{c_j\}$ and the overall pairing strength G provided in Ref. [20] are used for this example, which are shown in Table I. As is known, the one-pair ($k = 1$) solution of the model for any p can be obtained easily by using $S^+(\beta)$ given in the rightmost expression of Eq. (8) directly, of which β is the only variable in the solution. Thus, the $k = 1$ trivial case is not discussed. It should be stated that condition (33), which is sufficient to be used for the $p = 2$ case, is not needed for the $p \geqslant 3$ cases according to this procedure. When $p = 3$, the nontrivial cases are those with $k \geqslant 2$.

For the $k = 2$ case, according to the procedure, we may use Eq. (7) to get $a_1(\beta(x_\mu),x_{\mu,1})$, $a_2(\beta(x_\mu),x_{\mu,1})$, and $x_{\mu,2}$, which can be expressed as

$$
\begin{aligned}
a_1(\beta(x_\mu),x_{\mu,1}) &= \frac{x_{\mu,1}^3 - 0.999995x_{\mu,1}^2 + 0.00826599x_{\mu,1} - 0.0000171316}{\beta(x_\mu)x_{\mu,1}^2 + 7.39946x_{\mu,1}^2 - 0.0087071\beta(x_\mu)x_{\mu,1} - 0.0633624x_{\mu,1} - 0.00003\,\beta(x_\mu)^2 - 0.0000425\beta(x_\mu) + 0.000804}, \quad a_2(\beta(x_\mu),x_{\mu,1})\\
&= \frac{0.00003\,\beta(x_\mu)^3 + \beta(x_\mu)^2(0.004197 - 0.98694x_{\mu,1}) + \beta(x_\mu)\big(0.00754 - 1.904x_{\mu,1} - 12.758x_{\mu,1}^2\big) - 0.09289 + 22.4168x_{\mu,1} - 59.2576x_{\mu,1}^2 - 17.6893x_{\mu,1}^3}{\beta(x_\mu)^3(-0.0044 + x_{\mu,1}) + \beta(x_\mu)^2\big(-0.038 + 7.55x_{\mu,1} + 433.7x_{\mu,1}^2 - 33206.8x_{\mu,1}^3\big) + \beta(x_\mu)\big(0.072 - 34.58x_{\mu,1} + 6365.3x_{\mu,1}^2 - 491425x_{\mu,1}^3\big) + 0.846 - 264.2x_{\mu,1} + 23353.4x_{\mu,1}^2 - 1818140x_{\mu,1}^3},\\
x_{\mu,2} &= \frac{-0.00003\beta(x_\mu)^2 - \beta(x_\mu)(0.0000425 + 0.0043536x_{\mu,1}) - 0.03168x_{\mu,1} + 0.000804}{\beta(x_\mu)(0.004354 - x_{\mu,1}) - 7.39946x_{\mu,1} + 0.03168}. \qquad (39)
\end{aligned}
$$

It should be stated that Eqs. (39) are valid for any μ and k. However, Eqs. (12) and (22) provide two sets of solutions, of which one set involves $x_{\mu,i} = x_{\mu,i'}$ for any μ and $1 \leqslant i \neq i' \leqslant 2$ for this case. This solution violates the nonsingular condition Det($\mathbf{B}$) $\neq 0$ with the matrix $\mathbf{B}$ given by Eq. (15), and should be discarded. The other set of solutions for $x_{\mu,1}$, $W(x_1;x_2) = F(x_2,x_1)$, and $W(x_2;x_1) = F(x_1,x_2)$ for this case obtained from the constraints shown in Eqs. (12) or (22) are given as

$$
\begin{aligned}
x_{1,1} &= \frac{x_{2,1}\,x_{2,2}(-x_{1,2}\,a_1(\beta(x_2),x_{2,1}) + x_{2,2}\,a_1(\beta(x_2),x_{2,1}) - x_{1,2}\,a_2(\beta(x_2),x_{2,1}) + x_{2,1}\,a_2(\beta(x_2),x_{2,1}))}{-x_{1,2}x_{2,1}a_1(\beta(x_2),x_{2,1}) + x_{2,1}x_{2,2}a_1(\beta(x_2),x_{2,1}) - x_{1,2}x_{2,2}a_2(\beta(x_2),x_{2,1}) + x_{2,1}x_{2,2}a_2(\beta(x_2),x_{2,1})},\\
x_{2,1} &= \frac{x_{1,1}\,x_{1,2}(x_{1,2}\,a_1(\beta(x_1),x_{1,1}) - x_{2,2}\,a_1(\beta(x_1),x_{1,1}) + x_{1,1}\,a_2(\beta(x_1),x_{1,1}) - x_{2,2}\,a_2(\beta(x_1),x_{1,1}))}{x_{1,1}x_{1,2}a_1(\beta(x_1),x_{1,1}) - x_{1,1}x_{2,2}a_1(\beta(x_1),x_{1,1}) + x_{1,1}x_{1,2}a_2(\beta(x_1),x_{1,1}) - x_{1,2}x_{2,2}a_2(\beta(x_1),x_{1,1})}, \qquad (40)\\
W(x_2;x_1) &= \frac{x_{1,2}\,(-x_{2,2}a_1(\beta(x_2),x_{2,1}) + x_{1,2}a_1(\beta(x_2),x_{2,1}) + x_{1,2}a_2(\beta(x_2),x_{2,1}) - x_{2,1}a_2(\beta(x_2),x_{2,1}))}{(x_{1,2} - x_{2,1})(x_{1,2} - x_{2,2})},\\
W(x_1;x_2) &= \frac{x_{2,2}(-x_{1,2}\,a_1(\beta(x_1),x_{1,1}) + x_{2,2}\,a_1(\beta(x_1),x_{1,1}) - x_{1,1}\,a_2(\beta(x_1),x_{1,1}) + x_{2,2}\,a_2(\beta(x_1),x_{1,1}))}{(x_{2,2} - x_{1,1})(x_{2,2} - x_{1,2})}. \qquad (41)
\end{aligned}
$$

It can easily be verified that $x_{\mu,1}$ or $W(x_\mu;x_\nu)$ can be obtained from $x_{\nu,1}$ or $W(x_\nu;x_\mu)$ by permuting $x_{\mu,i}$ with $x_{\nu,i}$ in the expressions. Equation (21) is simply given by

$$
\begin{aligned}
\sum_{t=1}^{3}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2}{2\epsilon_{j_t} - \beta\big(x_1^{(\zeta)}\big)} - 2W\big(x_1^{(\zeta)};x_2^{(\zeta)}\big) &= \frac{1}{G},\\
\sum_{t=1}^{3}\frac{\Omega_{j_t}\big(c_{j_t}\big)^2}{2\epsilon_{j_t} - \beta(x_2^{(\zeta)})} - 2W\big(x_2^{(\zeta)};x_1^{(\zeta)}\big) &= \frac{1}{G}. \qquad (42)
\end{aligned}
$$

By substituting Eqs. (39) into Eqs. (40) and (41), and then by substituting Eq. (41) into Eq. (42), Eqs. (40) and (42) provide four equations for $x_{1,1}^{(\zeta)}$, $x_{2,1}^{(\zeta)}$, $\beta(x_1^{(\zeta)})$, and $\beta(x_2^{(\zeta)})$, which can be solved numerically.

Table II shows all pairing excitation energies of the model in the ds shell with $k = 2$. According to Eqs. (39)–(42), we use FindRoot provided by Wolfram MATHEMATICA to search for possible roots $x_{1,1}^{(\zeta)}$, $x_{2,1}^{(\zeta)}$, $\beta(x_1^{(\zeta)})$, and $\beta(x_2^{(\zeta)})$ of Eqs. (40) and (42). Then, we use the resultants to verify whether they indeed satisfy Eqs. (40) and (42) due to the fact that the resultants are iteratively obtained by MATHEMATICA approximately, of which the accuracy cannot always be guaranteed. A better algorithm is needed when one implements the procedure to solve large p and k cases. We found that FindRoot of MATHEMATICA frequently provides one of solutions of the Bethe ansatz equations given in Eq. (21) with one of $\{x_{\mu,i}\}$ being zero ($x_{\mu,i} = 0$) or at one of the poles of Eq. (20), namely, $x_{\mu,i} = c_j^2$, with the corresponding expansion coefficient $a_i(x_\mu) = 0$, which are not solutions for our purpose and should be

TABLE II. Two-pair solutions of the spherical mean-field plus separable pairing model with parameters shown in Table I, in which the excitation energies of the model, $E^{(\zeta)}_{k=2}$ (in MeV), the corresponding expansion coefficients, and the spectral parameters are provided, where $x_{\mu,i}$ is dimensionless, the unit of $a_i(x_\mu)$ is MeV^{-1}, the unit of $\beta(x_\mu)$ is MeV, and $I = \sqrt{-1}$.

Excitation energy	$x^{(\zeta)}_{1,1}$	$x^{(\zeta)}_{1,2}$	$a_1(x^{(\zeta)}_1)$	$a_2(x^{(\zeta)}_1)$	$x^{(\zeta)}_{2,1}$	$x^{(\zeta)}_{2,2}$	$a_1(x^{(\zeta)}_2)$	$a_2(x^{(\zeta)}_2)$	$\beta(x^{(\zeta)}_1)$	$\beta(x^{(\zeta)}_2)$
$E^{(\zeta=1)}_{k=2} = -18.557$	$0.00427 - 0.008I$	$-0.0036 - 0.006I$	$0.0245 + 0.38I$	$-0.07 + 0.068I$	$-0.00393 - 0.007I$	$0.0473 - 0.0351I$	$0.0293 + 0.028I$	$0.163 - 0.12I$	$-7.216 + 2.116I$	$-11.341 - 2.116I$
$E^{(\zeta=2)}_{k=2} = -16.051$	0.002	0.262	0.0025	0.258	0.0047	0.00466	−6.509	5.8752	−10.209	−5.842
$E^{(\zeta=3)}_{k=2} = -6.421$	0.00014	0.00435	−0.081	−0.00071	0.00015	0.1538	0.0084	0.2908	3.7925	−10.213
$E^{(\zeta=4)}_{k=2} = -2.046$	0.00465	0.00388	−0.073	−0.01495	0.00862	0.00404	−0.594	−0.03658	3.794	−5.8407
$E^{(\zeta=5)}_{k=2} = 7.593$	$0.02249 - 0.001286I$	$0.04329 - 0.00004I$	$-0.079 - 0.0073I$	$-0.009 + 0.0072I$	$0.003 + 0.00077I$	$0.0043276 + 0.000057I$	$-0.073 + 0.0124I$	$-0.0149 - 0.01232I$	$3.796 + 0.0021I$	$3.797 - 0.0021I$

TABLE III. The same as Table II, but for the three-pair solution of the ground state of the model.

	$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$x_{3,1}$	$x_{3,2}$	$\beta(x_1)$	$\beta(x_2)$	$\beta(x_3)$
$E^{(\zeta=1)}_{k=3} = -25.03$	$0.0123 + 0.0114I$	$-0.0168 + 0.0158I$	$-0.0067 + 0.0096I$	$0.0168 + 0.0042I$	$-0.0061 + 0.03146I$	$0.0103 + 0.01836I$	$-6.9986 + 2.6736I$	$-5.3945 - 1.8138I$	$-12.6369 - 0.8598I$
	$-0.1658 + 0.0580I$	$0.1209 + 0.3093I$	$-0.2145 - 0.0507I$	$-0.0593 - 0.1901I$	$0.2098 - 0.2089I$	$-0.0270 + 0.1694I$			
	$a_1(x_1)$	$a_2(x_1)$	$a_1(x_2)$	$a_2(x_2)$	$a_1(x_3)$	$a_2(x_3)$			

discarded. For the results provided in Table II, Eqs. (40) and (42) are valid with errors about 1×10^{-3}. Though we cannot prove the completeness of the solutions of Eqs. (40) and (42) at present, our numerical calculation shows that there are indeed five solutions given in Table II, which are in one-to-one correspondence with those obtained by full matrix diagonalization shown in Ref. [20]. Since eigenvalues of the Hamiltonian should be real, the constraint with $\sum_{\mu=1}^{k} \beta(x_\mu)$ being real should be helpful. It should be stated that there may be many different solutions of $\{x_{\mu,i}\}$ resulting in the same set of $\beta(x_\mu)$. However, one only needs to choose one set of $\{x_{\mu,i}\}$ to get the spectral parameters $\{\beta_\mu = \beta(x_\mu)\}$ if the final results of $\{\beta_\mu\}$ up to permutations among different μ are the same because eigenstates and eigenenergies of the model only depend on $\{\beta_\mu\}$, which are all the same with any permutation among different μ. In addition, similar to that occurring in the original Richardson-Gaudin solution to the standard pairing model, there is also S_k permutation symmetry among k components of the roots. Namely, if $\{x_{1,i}, x_{2,i}, \ldots, x_{k,i}\}$ is a solution, any permutation among $\{x_{1,i}, x_{2,i}, \ldots, x_{k,i}\}$ is also a solution, among which we only need to choose one set of the solutions because they result in the same eigenstate and the corresponding eigenenergy of the model. The ground-state solution of the $k = 2$ case shown in Table II generated by Wolfram MATHEMATICA version 9.0 is provided in the Supplemental Material in Ref. [24].

Finally, we present the solution for the $k = 3$ ground state of the model, which shows general features of the procedure for cases with arbitrary p and k as well. Similar to the $k = 2$ case, the expansion coefficients $a_1(\beta(x_\mu), x_{\mu,1})$, $a_2(\beta(x_\mu), x_{\mu,1})$, and $x_{\mu,2}$ for $\mu = 1,2,3$ are still given by Eq. (39). However, Eq. (22) in this case provides rather complicated expressions of nondegenerate $x_{\mu,1}$ for $\mu = 1,2,3$, of which only equations in determining them are given. Specifically, Eq. (22) involves six equations with

$$\begin{aligned}
W(x_1;x_2,x_3) &= \frac{x_{3,1}a_1(\beta(x_1),x_1)}{x_{3,1}-x_{1,1}} + \frac{x_{2,1}a_1(\beta(x_1),x_1)}{x_{2,1}-x_{1,1}} + \frac{x_{3,1}a_2(\beta(x_1),x_1)}{x_{3,1}-x_{1,2}} + \frac{x_{2,1}a_2(\beta(x_1),x_1)}{x_{2,1}-x_{1,2}},\\
W(x_2;x_1,x_3) &= \frac{x_{1,1}a_1(\beta(x_2),x_2)}{x_{1,1}-x_{2,1}} + \frac{x_{3,1}a_1(\beta(x_2),x_2)}{x_{3,1}-x_{2,1}} + \frac{x_{1,1}a_2(\beta(x_2),x_2)}{x_{1,1}-x_{2,2}} + \frac{x_{3,1}a_2(\beta(x_2),x_2)}{x_{3,1}-x_{2,2}},\\
W(x_3;x_1,x_2) &= \frac{x_{1,1}a_1(\beta(x_3),x_3)}{x_{1,1}-x_{3,1}} + \frac{x_{2,1}a_1(\beta(x_3),x_3)}{x_{2,1}-x_{3,1}} + \frac{x_{1,1}a_2(\beta(x_3),x_3)}{x_{1,1}-x_{3,2}} + \frac{x_{2,1}a_2(\beta(x_3),x_3)}{x_{2,1}-x_{3,2}},\\
W(x_1;x_2,x_3) &= \frac{x_{3,2}a_1(\beta(x_1),x_1)}{x_{3,2}-x_{1,1}} + \frac{x_{2,2}a_1(\beta(x_1),x_1)}{x_{2,2}-x_{1,1}} + \frac{x_{3,2}a_2(\beta(x_1),x_1)}{x_{3,2}-x_{1,2}} + \frac{x_{2,2}a_2(\beta(x_1),x_1)}{x_{2,2}-x_{1,2}},\\
W(x_2;x_1,x_3) &= \frac{x_{1,2}a_1(\beta(x_2),x_2)}{x_{1,2}-x_{2,1}} + \frac{x_{3,2}a_1(\beta(x_2),x_2)}{x_{3,2}-x_{2,1}} + \frac{x_{1,2}a_2(\beta(x_2),x_2)}{x_{1,2}-x_{2,2}} + \frac{x_{3,2}a_2(\beta(x_2),x_2)}{x_{3,2}-x_{2,2}},\\
W(x_3;x_1,x_2) &= \frac{x_{1,2}a_1(\beta(x_3),x_3)}{x_{1,2}-x_{3,1}} + \frac{x_{2,2}a_1(\beta(x_3),x_3)}{x_{2,2}-x_{3,1}} + \frac{x_{1,2}a_2(\beta(x_3),x_3)}{x_{1,2}-x_{3,2}} + \frac{x_{2,2}a_2(\beta(x_3),x_3)}{x_{2,2}-x_{3,2}},
\end{aligned} \tag{43}$$

of which the first three of them may be used as the expressions of $W(x_\mu; x_\nu, x_{\nu'})$ with $\mu \neq \nu \neq \nu'$, while the last three of Eqs. (43) are used to determine $x_{\mu,1}$ ($\mu = 1,2,3$).

The Bethe ansatz equations (21) can then be expressed as

$$\begin{aligned}
\sum_{t=1}^{3} \frac{\Omega_{j_t}(c_{j_t})^2}{2\epsilon_{j_t} - \beta(x_1^{(\zeta)})} - 2W(x_1;x_2,x_3) &= \frac{1}{G},\\
\sum_{t=1}^{3} \frac{\Omega_{j_t}(c_{j_t})^2}{2\epsilon_{j_t} - \beta(x_2^{(\zeta)})} - 2W(x_2;x_1,x_3) &= \frac{1}{G},\\
\sum_{t=1}^{3} \frac{\Omega_{j_t}(c_{j_t})^2}{2\epsilon_{j_t} - \beta(x_3^{(\zeta)})} - 2W(x_3;x_1,x_2) &= \frac{1}{G}.
\end{aligned} \tag{44}$$

In solving Eqs. (44), one should keep in mind that

$$\sum_{i=1}^{2} \frac{a_i(x_\mu^{(\zeta)})}{c_{j_t}^2 - x_{\mu,i}^{(\zeta)}} = \frac{1}{2\epsilon_{j_t} - \beta(x_\mu^{(\zeta)})} \tag{45}$$

for any t should also be satisfied for $\mu = 1$, 2, and 3 as required by the constraints shown in Eq. (7), which may be used to check the final results. Similar to the $k = 2$ case, by substituting Eqs. (39) with $\mu = 1,2,3$ into Eqs. (43), and then by substituting the first three of Eqs. (43) into Eqs. (44), Eqs. (44) and the last three of Eqs. (43) provide six equations for $x_{\mu,1}^{(\zeta)}$ and $\beta(x_\mu^{(\zeta)})$ for $\mu = 1,2,3$, which may be solved numerically. For the results provided in Table III, Eqs. (43), (44), and (45) are valid with errors about 1×10^{-3}, 1×10^{-3}, and 1×10^{-14}, respectively. The results shown in Table III generated by Wolfram MATHEMATICA version 9.0 are also provided in Ref. [24].

In comparison to the exact numerical results obtained from the progressive diagonalization scheme for this case provided in Ref. [20], which is equivalent to the full matrix diagonalization within the ds-shell subspace, it is shown that the eigenenergies obtained from the procedure proposed in this work are very close to those shown in Ref. [20] with errors about 0.002 MeV for the $k = 2$ case shown in Table II, while the ground-state energy of the $k = 3$ case shown in Table III is exactly the same as that given in Ref. [20]. Since eigenstates of the model should be sensitive to the results of solution, we also calculated overlaps $\rho(\zeta,k) = |\langle \zeta,k|\zeta,k\rangle_{\rm FMD}|$, where $|\zeta,k\rangle$ is the ζth k-pair excitation state obtained in this work, and $|\zeta,k\rangle_{\rm FMD}$ is that obtained from the full matrix diagonalization within the ds-shell subspace shown in Ref. [20], which are also shown in the last row of Table I. It can be seen from Table I that the overlaps for the $k = 2$ case are all greater than 99.63%, while it is 97.83% for the $k = 3$ ground state. The overlap for the $k = 3$ ground state is not perfect mainly due to the fact that the errors in the roots of Eqs. (44) and the last three of Eqs. (43) obtained by FindRoot of MATHEMATICA seem still significant, which are difficult to be reduced with the increasing of p and k, especially when the roots are complex. Therefore, a better numerical algorithm for solving Eqs. (7), (20) or (21), and (22) is needed. Anyway, our analysis indicates that the results obtained from the procedure shown in this work are indeed reliable.

IV. CONCLUSIONS

Exact solution of the nuclear mean-field plus separable pairing model is reexamined. The suitable auxiliary constraints for solving the Bethe ansatz equations of the model are proposed. The Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian purposed in Refs. [21,22] is verified with these new auxiliary constraints. Specifically, when $p \geqslant 3$, we need to solve $p \times k$ auxiliary equations given by Eq. (7) and another $p \times k$ equations given by Eqs. (20) or (21) and Eq. (22) to get $(p-1) \times k$ variables $\{x_{\mu,1}, x_{\mu,2}, \ldots, x_{\mu,p-1}\}$ and another $(p-1) \times k$ variables $\{a_1(x_\mu), a_2(x_\mu), \ldots, a_{p-1}(x_\mu)\}$, together with k variables $\beta(x_\mu)$ and another k variables $W(x_\mu; x_1, \ldots, x_{\mu-1}, x_{\mu+1}, \ldots, x_k)$, for $\mu = 1,2,\ldots,k$. Once the k variables $\beta(x_\mu)$ $(\mu = 1,\ldots,k)$ are obtained, they can then be used to get k-pair eigenstates (16) and the corresponding eigenenergies (23). It clearly shows that the number of equations involved equals exactly the number of unknowns, which ensures the uniqueness of the solution. In addition to the solution of the model for one- and two-orbit cases, to demonstrate structure and features of the solution, the model with three orbits ($p = 3$) in the ds shell is taken as a nontrivial example, of which two-pair results and the ground state of the three-pair case are provided explicitly, which are in one-to-one correspondence to the results obtained from the full matrix diagonalization in the ds-shell subspace provided in Ref. [20]. Though only some $p = 3$ cases are presented, the formulism shown in Eqs. (7), (20) or (21), and (22) applies to any p and k as well, which seems valid for any p and k with nondegenerate single-particle energies and nondegenerate separable pairing strengths, though we cannot provide a rigorous proof of the completeness for the general case at present. Since eigenvalues of the Hamiltonian should always be real, the constraint with $\sum_{\mu=1}^{k} \beta(x_\mu)$ being real may be used in finding solutions of the model. Since the number of equations involved increases with the number of orbits and pairs, to solve these equations for a large number of orbits and pairs seems still difficult. As shown in Ref. [20], the progressive diagonalization method [25] seems more practical to solve the problem.

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[1] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, Berlin, 1980).
[2] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958); S. T. Belyaev, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **31**(11), 1 (1959).
[3] M. Gaudin, J. Phys. **37**, 1087 (1976).
[4] R. W. Richardson, Phys. Lett. **3**, 277 (1963); **5**, 82 (1963); R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964); **52**, 253 (1964).
[5] J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. **76**, 643 (2004).
[6] F. Pan, L. Bao, L. Zhai, X. Cui, and J. P. Draayer, J. Phys. A **44**, 395305 (2011).
[7] X. Guan, K. D. Launey, M. Xie, L. Bao, F. Pan, and J. P. Draayer, Phys. Rev. C **86**, 024313 (2012).
[8] X. Guan, K. D. Launey, M. Xie, L. Bao, F. Pan, and J. P. Draayer, Comput. Phys. Commun. **185**, 2714 (2014).
[9] C. Qi and T. Chen, Phys. Rev. C **92**, 051304(R) (2015).
[10] F. Pan, V. G. Gueorguiev, and J. P. Draayer, Phys. Rev. Lett. **92**, 112503 (2004).
[11] F. Pan, X. Ding, K. D. Launey, H. Li, X. Xu, and J. P. Draayer, Nucl. Phys. A **947**, 234 (2016).
[12] F. Pan, J. P. Draayer, and W. E. Ormand, Phys. Lett. B **422**, 1 (1998).
[13] A. B. Balantekin and Y. Pehlivan, Phys. Rev. C **76**, 051001(R) (2007).
[14] T. Skrypnyk, J. Math. Phys. **50**, 033504 (2009).
[15] T. Skrypnyk, Nucl. Phys. B **806**, 504 (2009).
[16] T. Skrypnyk, J. Phys. A **42**, 472004 (2009).
[17] S. M. A. Rombouts, J. Dukelsky, and G. Ortiz, Phys. Rev. B **82**, 224510 (2010).
[18] A. Birrell, P. S. Isaac, and J. Links, Inverse Probl. **28**, 035008 (2012).
[19] M. Van Raemdonck, S. De Baerdemacker, and D. Van Neck, Phys. Rev. B **89**, 155136 (2014).
[20] L. Dai, F. Pan, and J. P. Draayer, Nucl. Phys. A **957**, 51 (2017).
[21] F. Pan and J. P. Draayer, Phys. Lett. B **442**, 7 (1998).
[22] F. Pan and J. P. Draayer, Ann. Phys. (NY) **271**, 120 (1999).
[23] F. Nowacki and A. Poves, Phys. Rev. C **79**, 014310 (2009).
[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevC.95.034308 containing two files written in Wolfram MATHEMATICA version 9.0 with file name [k = 2 Ground-state.nb] for the ground state of the two-pair case and [k = 3 Ground-state.nb] for the ground state of the three-pair case.
[25] F. Pan, M. X. Xie, X. Guan, L. R. Dai, and J. P. Draayer, Phys. Rev. C **80**, 044306 (2009).