Amplitude reconstruction from complete photoproduction experiments and truncated partial-wave expansions

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We compare the methods of amplitude reconstruction, for a complete experiment and a truncated partial-wave analysis, applied to the photoproduction of pseudoscalar mesons. The approach is pedagogical, showing in detail how the amplitude reconstruction (observables measured at a single energy and angle) is related to a truncated partial-wave analysis (observables measured at a single energy and a number of angles).

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I. INTRODUCTION AND MOTIVATION

A model-independent determination of amplitudes from experimental data is mathematically possible, ignoring experimental errors, if one measures a sufficient number of observables at a given energy and angle. This has been done in nucleon-nucleon scattering [1] and can be done [2,3], in principle, by using pseudoscalar-meson photoproduction data [4–6].

The complete experiment analysis (CEA) determines helicity or transversity amplitudes only up to an overall phase. This is a problem if one actually wants partial-wave amplitudes, because the undetermined phase may be different at each reconstructed energy and angle. In the analysis of pseudoscalar photoproduction data, we *do* require multipole amplitudes in order to search for resonance content, and this has led to a renewed interest [7,8] in the properties of a truncated partial-wave analysis (TPWA), as has been described by Omelaenko [9] and Grushin [10].

The number of required observables is different for the CEA and TPWA. The reason for this is obscured by the fact that very different methods have been used to derive the necessary conditions for a solution. Here, we have used several methods to clarify the connections between the two approaches. The first nontrivial example reveals many of these connections.

II. AMPLITUDES USED IN PSEUDOSCALAR-MESON PHOTOPRODUCTION

Before comparing the CEA and TPWA approaches, we review the notation used to analyze pseudoscalar photoproduction data. The multipoles and helicity amplitudes are related by [11,12]

$$H_1 = \frac{1}{\sqrt{2}} \cos\frac{\theta}{2} \sin\theta \sum_{\ell=1}^{\infty} [E_{\ell+} - M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}](P_\ell^{''} - P_{\ell+1}^{''}), \tag{1a}$$

$$H_2 = \frac{1}{\sqrt{2}} \cos\frac{\theta}{2} \sum_{\ell=0}^{\infty} [(\ell+2)E_{\ell+} + \ell M_{\ell+} + \ell E_{(\ell+1)-} - (\ell+2)M_{(\ell+1)-}](P_\ell^{'} - P_{\ell+1}^{'}), \tag{1b}$$

$$H_3 = \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} \sin\theta \sum_{\ell=1}^{\infty} [(E_{\ell+} - M_{\ell+} + E_{(\ell+1)-} + M_{(\ell+1)-}](P_\ell^{''} + P_{\ell+1}^{''}), \tag{1c}$$

$$H_4 = \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} \sum_{\ell=0}^{\infty} [(\ell+2)E_{\ell+} + \ell M_{\ell+} - \ell E_{(\ell+1)-} + (\ell+2)M_{(\ell+1)-}](P_\ell^{'} + P_{\ell+1}^{'}). \tag{1d}$$

From these one can construct the transversity amplitudes [3],

$$b_1 = \tfrac{1}{2}[(H_1 + H_4) + i(H_2 - H_3)], \tag{2a}$$

$$b_2 = \tfrac{1}{2}[(H_1 + H_4) - i(H_2 - H_3)], \tag{2b}$$

$$b_3 = \tfrac{1}{2}[(H_1 - H_4) - i(H_2 + H_3)], \tag{2c}$$

$$b_4 = \tfrac{1}{2}[(H_1 - H_4) + i(H_2 + H_3)]. \tag{2d}$$

In Table I, expressions for the observables of type S (cross section and single polarization), BT (beam-target polarization), BR (beam-recoil polarization), and TR (target-recoil polarization) are given in terms of both helicity and transversity amplitudes.

Transversity amplitudes often simplify the discussion of amplitude reconstruction, because the type-S observables determine their moduli. Another simplification is the property

$$b_2(\theta) = -b_1(-\theta) \quad \text{and} \quad b_4(\theta) = -b_3(-\theta), \tag{3}$$

which allows one to parametrize only two of the four transversity amplitudes. The form introduced by Omelaenko,

$$b_1 = c a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \alpha_i), \tag{4a}$$

$$b_3 = -c a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \beta_i), \tag{4b}$$

with $x = \tan(\theta/2)$ and L being the upper limit for ℓ, is convenient for a truncated partial-wave analysis, because the

TABLE I. Spin observables expressed in terms of helicity and transversity amplitudes. Helicity amplitudes follow Walker [11] and the SAID convention [12]. The relations in Barker, Donnachie, and Storrow [3] are adopted for observables with the replacement $N \to H_2$, $S_1 \to H_1$, $S_2 \to H_4$, $D \to H_3$. The transversity representation for $\check{H}$ corrects a typographical error in Ref. [3]. $I = (k/q)d\sigma/d\Omega$, with k and q being the photon and pion center-of-mass momenta. The checked observables $\check{O}$ are defined by $\check{O} = IO$.

Observable	Helicity representation	Transversity representation	Type
I	$\frac{1}{2}(\lvert H_1\rvert^2 + \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$\frac{1}{2}(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2)$	
$\check{\Sigma}$	$\mathrm{Re}(H_1H_4^* - H_2H_3^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2)$	S
$\check{T}$	$\mathrm{Im}(H_1H_2^* + H_3H_4^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2)$	
$\check{P}$	$-\mathrm{Im}(H_1H_3^* + H_2H_4^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2)$	
$\check{G}$	$-\mathrm{Im}(H_1H_4^* + H_2H_3^*)$	$\mathrm{Im}(b_1b_3^* + b_2b_4^*)$	
$\check{H}$	$-\mathrm{Im}(H_1H_3^* - H_2H_4^*)$	$\mathrm{Re}(b_1b_3^* - b_2b_4^*)$	BT
$\check{E}$	$\frac{1}{2}(-\lvert H_1\rvert^2 + \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$-\mathrm{Re}(b_1b_3^* + b_2b_4^*)$	
$\check{F}$	$\mathrm{Re}(H_1H_2^* + H_3H_4^*)$	$\mathrm{Im}(b_1b_3^* - b_2b_4^*)$	
$\check{O}_x$	$-\mathrm{Im}(H_1H_2^* - H_3H_4^*)$	$-\mathrm{Re}(b_1b_4^* - b_2b_3^*)$	
$\check{O}_z$	$\mathrm{Im}(H_1H_4^* - H_2H_3^*)$	$-\mathrm{Im}(b_1b_4^* + b_2b_3^*)$	BR
$\check{C}_x$	$-\mathrm{Re}(H_1H_3^* + H_2H_4^*)$	$\mathrm{Im}(b_1b_4^* - b_2b_3^*)$	
$\check{C}_z$	$\frac{1}{2}(-\lvert H_1\rvert^2 - \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$-\mathrm{Re}(b_1b_4^* + b_2b_3^*)$	
$\check{T}_x$	$\mathrm{Re}(H_1H_4^* + H_2H_3^*)$	$\mathrm{Re}(b_1b_2^* - b_3b_4^*)$	
$\check{T}_z$	$\mathrm{Re}(H_1H_2^* - H_3H_4^*)$	$\mathrm{Im}(b_1b_2^* - b_3b_4^*)$	TR
$\check{L}_x$	$-\mathrm{Re}(H_1H_3^* - H_2H_4^*)$	$\mathrm{Im}(b_1b_2^* + b_3b_4^*)$	
$\check{L}_z$	$\frac{1}{2}(\lvert H_1\rvert^2 - \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$\mathrm{Re}(b_1b_2^* + b_3b_4^*)$	

ambiguities can be linked to the conjugation of the complex roots of the above relations, with a constraint

$$\prod_{i=1}^{2L} \alpha_i = \prod_{i=1}^{2L} \beta_i. \tag{5}$$

The quantities a_{2L} and c above will be clarified in an explicit example described in Sec. III below.

In a complete experiment analysis (CEA), one attempts to determine the transversity or helicity amplitudes based on the relations in Table I and at a particular energy and angle. Barker, Donnachie, and Storrow [3] (BDS) showed how this could be done with nine well-chosen observables. For example, the case of I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, $\check{E}$, $\check{F}$, $\check{G}$, $\check{L}_x$, and $\check{L}_z$ was worked out explicitly in Ref. [3]. More recently, a counterexample to this scheme was noticed in Ref. [13], which led to the finding by Chiang and Tabakin [2] that it was possible to perform a CEA with one measurement less. In the case presented by BDS, Chiang and Tabakin demonstrated a solution with only I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, $\check{F}$, $\check{G}$, $\check{T}_x$, and $\check{L}_x$ being required.

In a truncated partial-wave analysis (TPWA), the multipole expansion of helicity or transversity amplitudes is cut off at some upper limit L. Here one finds the amplitudes, for all angles, at a particular energy. Omelaenko showed how this can be done, eliminating the root-conjugation ambiguities associated with the transversity amplitudes in Eq. (4), using an L-dependent number of angular measurements of five observables, such as I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, and $\check{F}$. As the methods of proof are very different, in the CEA and TPWA problems, it is not obvious how these results can be compared. In the following, we compare the CEA and TPWA results in such a way that the differences can be more easily understood.

III. AMPLITUDE RECONSTRUCTION

A. Trivial case: $L = 0$

It is instructive to compare methods starting with the trivial $\ell = 0$ case of a single E_{0+} multipole and build up to the case studied by Omelaenko [9] including the E_{0+}, M_{1-}, E_{1+}, and M_{1+} multipoles. If only one complex amplitude (E_{0+}) is included, from Eq. (1) we see that there are two nonzero helicity amplitudes (H_2 and H_4) which are related by a real factor. Here, we may simply measure the cross section at a single angle. While this gives only one real number, and the amplitudes are complex, the fact that observables involve only bilinear products of amplitudes, i.e., terms of the form A^*B, prevents the measurement of any overall phase associated with the amplitudes. This solves both the CEA and TPWA with the same experimental input.

B. Simplest nontrivial case: $J = 1/2$

The first nontrivial case includes the E_{0+} and M_{1-} multipoles, i.e., partial waves with $J = 1/2$. This combination again produces two nonzero helicity amplitudes (H_2 and H_4). In this case, however, the amplitudes are independent. The corresponding transversity amplitudes are given by[1]

$$b_1 = \frac{i}{\sqrt{2}}(-e^{i\theta/2}E_{0+} + e^{-i\theta/2}M_{1-}), \tag{6}$$

[1]The corresponding expression in Ref. [9] differs by an overall phase $(-i)$ and a factor $\sqrt{2}c$ which incorporates the kinematic factor of Table I, here converting $d\sigma/dt$ to I, into the definition of the transversity amplitudes.

TABLE II. Spin observables in terms of helicity amplitudes for a CEA and multipole amplitudes for a TPWA with $J = 1/2$. Here we have used $b_3 = -b_1$ and $b_4 = -b_2$.

Observable	CEA (helicity)	CEA (transversity)	TPWA	Type
I	$\frac{1}{2}(\lvert H_2\rvert^2 + \lvert H_4\rvert^2)$	$\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$(\lvert E_{0+}\rvert^2 + \lvert M_{1-}\rvert^2) - 2\cos\theta \mathrm{Re}(E_{0+}M_{1-}^*)$	
$\check{\Sigma}$	0	0	0	S
$\check{T}$	0	0	0	
$\check{P}$	$-\mathrm{Im}(H_2H_4^*)$	$\lvert b_1\rvert^2 - \lvert b_2\rvert^2$	$2\sin\theta \mathrm{Im}(E_{0+}M_{1-}^*)$	
$\check{G}$	0	0	0	
$\check{H}$	$\mathrm{Im}(H_2H_4^*)$	$-\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$-2\sin\theta \mathrm{Im}(E_{0+}M_{1-}^*)$	BT
$\check{E}$	$\frac{1}{2}(\lvert H_2\rvert^2 + \lvert H_4\rvert^2)$	$\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$(\lvert E_{0+}\rvert^2 + \lvert M_{1-}\rvert^2) - 2\cos\theta \mathrm{Re}(E_{0+}M_{1-}^*)$	
$\check{F}$	0	0	0	
$\check{O}_x$	0	0	0	
$\check{O}_z$	0	0	0	BR
$\check{C}_x$	$-\mathrm{Re}(H_2H_4^*)$	$-2\mathrm{Im}b_1b_2^*$	$\sin\theta(\lvert E_{0+}\rvert^2 - \lvert M_{1-}\rvert^2)$	
$\check{C}_z$	$\frac{1}{2}(-\lvert H_2\rvert^2 + \lvert H_4\rvert^2)$	$2\mathrm{Re}b_1b_2^*$	$2\mathrm{Re}(E_{0+}M_{1-}^*) - \cos\theta(\lvert E_{0+}\rvert^2 + \lvert M_{1-}^*\rvert^2)$	
$\check{T}_x$	0	0	0	
$\check{T}_z$	0	0	0	TR
$\check{L}_x$	$\mathrm{Re}(H_2H_4^*)$	$2\mathrm{Im}b_1b_2^*$	$-\sin\theta(\lvert E_{0+}\rvert^2 - \lvert M_{1-}\rvert^2)$	
$\check{L}_z$	$\frac{1}{2}(-\lvert H_2\rvert^2 + \lvert H_4\rvert^2)$	$2\mathrm{Re}b_1b_2^*$	$2\mathrm{Re}(E_{0+}M_{1-}^*) - \cos\theta(\lvert E_{0+}\rvert^2 + \lvert M_{1-}^*\rvert^2)$	

with $b_3 = -b_1$, and with (b_2,b_4) given by Eq. (3). In Table II,we give the observables both in terms of the helicity and transversity amplitudes (CEA) and the two included multipoles (TPWA).

Here the CEA requires four measurements at a given energy and angle. For example, I, $\check{P}$, plus either the beam-recoil sets ($\check{C}_x$ and $\check{C}_z$) or the target recoil ($\check{L}_x$ and $\check{L}_z$). The TPWA requires one fewer observable, a possible choice being I, $\check{P}$, and $\check{C}_x$ or $\check{L}_x$, compensated by a second angular measurement of the cross section.

For both the CEA and TPWA, closed expressions for the solution of the inverse problem can be obtained in this special case $J = 1/2$. It is instructive to work them out explicitly. For the quantities I, $\check{P}$, and $\check{C}_x$ in the TPWA, it is possible to parametrize the angular dependence given in Table II as

$$I = \sigma_0 + \sigma_1\cos\theta, \quad \check{P} = P_0\sin\theta, \quad \check{C}_x = C_{x0}\sin\theta, \tag{7}$$

where each coefficient carries the energy dependence of the multipoles. It is clear that to extract values for σ_0, σ_1, P_0, and C_{x0}, both spin asymmetries and the cross section are needed at the same angle, with an additional angular measurement required for the cross section.

Having obtained the four coefficients, the zeroth-order quantities of I and $\check{C}_x$ can be directly solved for the moduli of the multipoles (cf. Table II),

$$|E_{0+}| = \sqrt{\frac{\sigma_0 + C_{x0}}{2}}, \quad |M_{1-}| = \sqrt{\frac{\sigma_0 - C_{x0}}{2}}. \tag{8}$$

The relative phase $\phi_{E,M} \equiv \phi_E - \phi_M$ between the multipoles E_{0+} and M_{1-} is obtainable via the remaining two coefficients, both containing information on the real and imaginary parts of the bilinear product $E_{0+}M_{1-}^*$. The additional angular measurement for the cross section fixes the real part,

$$\mathrm{Re}(E_{0+}M_{1-}^*) = |E_{0+}||M_{1-}|\mathrm{Re}(e^{i\phi_{E,M}}) = -\tfrac{1}{2}\sigma_1, \tag{9}$$

while the imaginary part can be extracted from the single measurement of $\check{P}$,

$$\mathrm{Im}(E_{0+}M_{1-}^*) = |E_{0+}||M_{1-}|\mathrm{Im}(e^{i\phi_{E,M}}) = \tfrac{1}{2}P_0. \tag{10}$$

Together, these define the exponential of the relative phase, provided that none of the moduli vanish,

$$e^{i\phi_{E,M}} = \frac{-\sigma_1 + iP_0}{\sqrt{\sigma_0 + C_{x0}}\sqrt{\sigma_0 - C_{x0}}}. \tag{11}$$

This function can be inverted uniquely on the interval $[0,2\pi)$. Therefore, no quadrant ambiguity remains. The multipoles have been extracted up to an overall phase.

The CEA proceeds in a mathematically exactly analogous way. The observables I, $\check{P}$, $\check{C}_x$, and $\check{C}_z$ yield the moduli and relative phase $\phi_{1,2} \equiv \phi_{b_1} - \phi_{b_2}$ of the transversity amplitudes by using exactly the same calculation (cf. Table II):

$$|b_1| = \sqrt{\frac{I + \check{P}}{2}}, \quad |b_2| = \sqrt{\frac{I - \check{P}}{2}}, \tag{12}$$

$$e^{i\phi_{1,2}} = \frac{\check{C}_z - i\check{C}_x}{\sqrt{I + \check{P}}\sqrt{I - \check{P}}}. \tag{13}$$

A crucial difference, however, lies in the kinematical regions over which the CEA and TPWA operate. For a fixed energy, the CEA extracts amplitudes from observables at exactly the same angle and it is completely blind to what may happen at neighboring angles. The TPWA uses the angular distributions of the observables which, in the present case of $I(\theta)$, is linear in $\cos\theta$. One seemingly obtains a reduction from four to three observables, but this is bought at the price of having to measure angular distributions which become, for the higher truncation orders, increasingly complicated.

The difference in the nature of these analyses also becomes obvious in considering the end results they yield. The CEA returns transversity amplitudes only at a single angle, up to an energy- and angle-dependent overall phase; cf. Eqs. (12) and (13). However, from the result of the TPWA, the moduli (8), and relative phase (11) of multipoles, it is possible to infer transversity amplitudes at all angles, this time up to an energy-dependent phase.

C. Unique features of $J = 1/2$ case

It is useful to compare the special case of $J = 1/2$ with more general results for the CEA and TPWA in Refs. [2] and [9]. In Ref. [2], a complete set of eight experiments, explicitly derived and compared with the corresponding BDS case (requiring nine experiments) is $(I, \check{\Sigma}, \check{P}, \check{T}, \check{G}, \check{F}, \check{L}_x, \check{T}_x)$. Here, with a truncation to $J = 1/2$, this set becomes $(I, 0, \check{P}, 0, 0, 0, \check{L}_x, 0)$, which does not contain sufficient information, as can be seen directly from Table II. However, the older BDS set, which exchanges $\check{T}_x$ for $\check{E}$ and $\check{L}_z$, *does* constitute a complete experiment. This failure of a set of eight experiments is due to the number of zero quantities in Table II. The effect can be seen in constraint equation (4.10) employed in the derivation of Ref. [2]. Many Fierz identities listed in Ref. [2] similarly revert to zero-equals-zero relations in this special case.

Similarly, the TPWA conditions for a complete set [9], derived for a case including the E_{0+}, M_{1-}, E_{1+}, and M_{1+} multipoles, do not directly reduce to the result given here if the E_{1+} and M_{1+} multipoles are simply set to zero. In Refs. [7,9], a complete set is given as $(I, \check{P}, \check{\Sigma}, \check{T}, \check{G})$, which again is insufficient in this special case.

To understand how a truncation to $J = 1/2$ changes the result, it is instructive to repeat Omelaenko's analysis [9], which leads to the general parametrizations of Eqs. (4a) and (4b) under the constraint in Eq. (5) for all $L \geqslant 1$.

Expressing $\cos\theta$ and $\sin\theta$ in terms of $x = \tan(\theta/2)$, one can write

$$e^{i\theta} = \frac{(1+ix)^2}{1+x^2}. \tag{14}$$

Starting from the expression for b_1 in terms of multipoles given in Eq. (6), we have

$$\begin{aligned} b_1 &= \frac{i}{\sqrt{2}}(-e^{i\theta/2}E_{0+} + e^{-i\theta/2}M_{1-}) \\ &= \frac{ie^{i\theta/2}}{\sqrt{2}}\left(-E_{0+} + \frac{1-x^2-2ix}{(1+x^2)}M_{1-}\right) \\ &= \frac{-i}{\sqrt{2}}\frac{e^{i\theta/2}}{(1+x^2)}(E_{0+}+M_{1-}) \\ &\quad \times \left(x^2 + \frac{2iM_{1-}}{E_{0+}+M_{1-}}x + \frac{E_{0+}-M_{1-}}{E_{0+}+M_{1-}}\right) \\ &\equiv \frac{-i}{\sqrt{2}}\frac{e^{i\theta/2}}{(1+x^2)}a_2(x^2+\hat{a}_1x+\hat{a}_0). \end{aligned} \tag{15}$$

Note that the coefficients a_2, $\hat{a}_1$, and $\hat{a}_0$, defining the amplitude in the last step, are fully equivalent to the multipoles. Decomposing the polynomial into a product of linear factors defined by two complex roots α_1 and α_2, the Omelaenko decomposition of the amplitude b_1 is obtained as

$$b_1 = \frac{-i}{\sqrt{2}}a_2 e^{i\theta/2}\frac{\prod_{i=1}^{2}(x-\alpha_i)}{1+x^2}. \tag{16}$$

The expression for the only remaining nonredundant amplitude b_2, for $J = 1/2$, is obtained by invoking the symmetry in Eq. (3),

$$b_2(\theta) = -b_1(-\theta) = \frac{i}{\sqrt{2}}a_2 e^{-i\theta/2}\frac{\prod_{i=1}^{2}(x+\alpha_i)}{1+x^2}. \tag{17}$$

Therefore, for $J = 1/2$ there are only two α roots, no β roots and the constraint (5) no longer appears.

In view of the already-obtained results (8) and (11), the observable $\check{C}_x$ will have to be tested for its response to discrete ambiguity transformations. The full Omelaenko decomposition of this observable becomes

$$\check{C}_x = -2\,\mathrm{Im}\, b_1 b_2^* = \frac{|a_2|^2}{(1+x^2)^2}\mathrm{Im}\left[e^{i\theta}\prod_{i=1}^{2}(x-\alpha_i)(x+\alpha_i^*)\right]. \tag{18}$$

The decompositions of amplitudes b_i in terms of roots α_1 and α_2 given by Eqs. (16) and (17) facilitate a study of the discrete ambiguities of the quantities I and $\check{P}$ (as well as $\check{E}$ and $\check{H}$), since they are just linear combinations of the squared moduli $|b_1|^2$ and $|b_2|^2$ (see Table II). The ambiguities are obtained by the complex conjugation of subsets of roots, as stated below Eq. (4).

Note that the multipoles E_{0+} and M_{1-}, with an undetermined overall phase that can be arbitrarily fixed, correspond to three real numbers. For the variables (a_2,α_1,α_2) of the Omelaenko decomposition, where the phase of a_2 cannot be determined, one counts five real degrees of freedom. The general constraint equation (5), which is true for an expansion in ℓ for all $L \geqslant 1$, is missing here. So, there must be another way in which the effective number of real degrees of freedom is reduced from five to three.

One can learn more by considering the equations which relate the Omelaenko roots (α_1,α_2) to the multipoles (E_{0+},M_{1-}). Utilizing the notation of Eqs. (15) and (16), we have

$$\begin{aligned} \hat{a}_1 &\equiv -(\alpha_1+\alpha_2) = \frac{2iM_{1-}}{E_{0+}+M_{1-}}, \\ \hat{a}_0 &\equiv \alpha_1\alpha_2 = \frac{E_{0+}-M_{1-}}{E_{0+}+M_{1-}}. \end{aligned} \tag{19}$$

These relations lead to a quadratic equation with two solutions given by the roots

$$\begin{aligned} &\alpha_1^{(\mathrm{I})} = i\frac{E_{0+}-M_{1-}}{E_{0+}+M_{1-}}, \quad \alpha_2^{(\mathrm{I})} = -i \quad \text{and} \quad \alpha_1^{(\mathrm{II})} = -i, \\ &\alpha_2^{(\mathrm{II})} = i\frac{E_{0+}-M_{1-}}{E_{0+}+M_{1-}}. \end{aligned} \tag{20}$$

Both solutions remove the overcounting mentioned above. Two real degrees of freedom are always removed since one of the roots has a fixed value. Only one of the two roots depends on the multipoles.

Solutions I and II of Eq. (20) are not distinct, because both are equivalent by a simple relabeling of the roots. Taking solution I, for which α_2 is fixed to $-i$, only one discrete ambiguity remains in the Omelaenko formulation for $J = 1/2$, represented by the transformation $\alpha_1 \to \alpha_1^*$,

Using solution I, the full Omelaenko decomposition of $\check{C}_x$, Eq. (18), simplifies significantly. Again writing the exponential $e^{i\theta}$ in terms of $x = \tan(\theta/2)$ [see Eq. (14)], we have the identity

$$e^{i\theta}(x-\alpha_2)(x+\alpha_2^*) = \frac{(1+ix)^2}{1+x^2}(x+i)^2 = -(1+x^2). \tag{21}$$

The expression for $\check{C}_x$, in terms of the only nonredundant Omelaenko root, α_1, then becomes

$$\check{C}_x = \frac{-|a_2|^2}{1+x^2}\mathrm{Im}[(x-\alpha_1)(x+\alpha_1^*)]. \tag{22}$$

For the discrete symmetry, $\alpha_1 \to \alpha_1^*$, we see that expression (22) changes sign, $\check{C}_x \to -\check{C}_x$, once the ambiguity transformation is applied. Furthermore, $\check{C}_x$ generally only remains invariant at the angles $\theta = 0$ and $\theta = \pi$, where it vanishes by definition (see Table II).

Another interesting special case is found if one requires the transformation $\alpha_1 \to \alpha_1^*$ to produce no ambiguity, which can only be fulfilled for a real root. Once this condition is evaluated for the explicit form of α_1 in terms of multipoles, given in Eq. (20), one finds that $\alpha_1^* = \alpha_1$ is equivalent to $|E_{0+}| = |M_{1-}|$.

The Omelaenko decomposition of $\check{C}_x$, as well as the explicit form of this quantity written in terms of multipoles (see Table II), shows that, in this particular case, $\check{C}_x$ vanishes for all angles. Here, while the sign information associated with $\check{C}_x$ may be missing, it is not required because the discrete symmetry, which is resolved precisely by this sign, no longer exists. Also, Eqs. (8) and (11) imply that, in this special case, i.e., $|E_{0+}| = |M_{1-}|$ or equivalently $C_{x0} = 0$, the moduli of both multipoles, as well as the relative phase $\phi_{E,M}$, are determined by I and $\check{P}$ alone. This case is, however, the only situation where a solution of the inverse problem is possible with just two observables.

In summary, both the explicit inversion of the TPWA, Eqs. (8) and (11), and the study of the discrete ambiguities, according to Omelaenko's method, yield consistent results for $J = 1/2$. This has been exemplified by the solvability of the example set I, $\check{P}$, and $\check{C}_x$. The case $J = 1/2$ is special because it allows all three analyses: the CEA, TPWA, and ambiguity study, to be performed by using simple algebra. For the higher orders $L \geqslant 1$, Chiang and Tabakin [2] have published a solution for the CEA which holds apart from the special case discussed above.

An algebraic inversion of the TPWA, i.e., the extraction of the bilinear products of multipoles by an effective linearization of the problem, followed by a simple evaluation of moduli and relative phases, does not appear to be possible for $L \geqslant 1$. The only principle that carries through to the higher orders is the study of discrete ambiguities [7,9], using the expressions in Eqs. (4a), (4b), and (5).

In this way, complete sets of observables can still be proposed. However, the actual completeness of such sets should, in any case, be checked by a full solution of the inverse problem which, for the higher truncation orders, can only be done numerically.

D. Counting observables

In examining the $J = 1/2$ case, it was found that a formal solution was possible with only $\check{P}$, $\check{C}_x$, and $\check{C}_z$ (three rather than four quantities), measured at one angle, if one used the overall phase freedom to make one amplitude real and positive. This result could be understood by refining how the counting of observables is done. If a measurement, done with a fixed beam, target and detector setup, produces an "observable," then the measurement of a polarization asymmetry (spin up versus spin down) is actually two observables. These two measurements can then be combined to form both the asymmetry and the cross section. Once the cross section is known, a second asymmetry can, in principle, be determined from only one of the two possible (such as spins parallel versus antiparallel) measurements. Thus, the set ($\check{P}$, $\check{C}_x$, $\check{C}_z$) requires $2+1+1 = 4$ measurements, compared with the set (I, $\check{P}$, $\check{C}_x$, $\check{C}_z$), requiring $1+1+1+1 = 4$ measurements.

IV. COMPARING CEA AND TPWA BEYOND $J = 1/2$

In Table III, the examples discussed in detail above are generalized to higher angular-momentum cutoffs. The examples with one, two, and three multipoles show that, in the CEA and TPWA approaches, the number of measurements is the same. In cases where a TPWA is possible with all measurements at a single energy and angle, the results are directly related. Note that in the case of three multipoles, only three of the helicity or transversity amplitudes are independent. This is also true for the standard set of four multipoles (E_{0+}, M_{1-}, E_{1+}, M_{1+}) as can be most easily seen if, instead, one writes out the CGLN amplitudes,

$$F_1(\theta) = E_{0+} + 3(M_{1+} + E_{1+})\cos\theta, \tag{23a}$$

$$F_2(\theta) = 2M_{1+} + M_{1-}, \tag{23b}$$

$$F_3(\theta) = 3(E_{1+} - M_{1+}), \tag{23c}$$

$$F_4(\theta) = 0. \tag{23d}$$

With $F_4 = 0$, only three independent amplitudes can be extracted in a CEA. Consequently, also only three linear combinations of multipoles can be obtained in an experiment at a single angle.

Extending the expansion of observables, given in Eq. (8), to higher orders in $\cos\theta$ up to the highest powers for a given L, we have

$$I = \sigma_0 + \sigma_1\cos\theta + \sigma_2\cos^2\theta + \cdots + \sigma_{2L}\cos^{2L}\theta, \tag{24a}$$

$$\check{\Sigma} = \sin^2\theta(\Sigma_0 + \cdots + \Sigma_{2L-2}\cos^{2L-2}\theta), \tag{24b}$$

$$\check{T} = \sin\theta(T_0 + T_1\cos\theta + \cdots + T_{2L-1}\cos^{2L-1}\theta), \tag{24c}$$

$$\check{P} = \sin\theta(P_0 + P_1\cos\theta + \cdots + P_{2L-1}\cos^{2L-1}\theta). \tag{24d}$$

The remaining double-polarization observables ($\check{E}$, $\check{C}_x$, $\check{O}_x$, $\check{T}_z$, $\check{L}_x$) behave like I, ($\check{F}$, $\check{H}$, $\check{O}_z$, $\check{T}_x$) like $\check{T}$, $\check{G}$ like $\check{\Sigma}$, while $\check{C}_z$ and $\check{L}_z$ exhibit the highest powers up to $\cos^{2L+1}\theta$.

Table III gives examples of measurement sets involving from one to a generalized number of $4L$ multipoles using one or more angles in the TPWA. Set 1 is the trivial case and set 2 with $J = 1/2$ has already been discussed in detail. Besides the $J = 1/2$ TPWA set with the minimal number of three observables, but more than one angle, a solution also exists at one angle with four observables, which is fully equivalent to the CEA. In a set with three S, P wave multipoles, E_{0+}, E_{1+}, M_{1-}, three amplitudes are linearly independent, e.g., F_1, F_2, F_3, and both CEA and TPWA are again equivalent. Also, with TPWA at more than one angle, the number of observables can be reduced. Taking the full angular distribution, a minimal set of three polarization observables is already complete.

The next logical set is the full set four of S- and P-wave multipoles, E_{0+}, M_{1-}, E_{1+}, and M_{1+}, but this yields a surprising result. As already discussed, from Eq. (23) only three amplitudes are linearly independent, leading to a CEA, which is not sufficient to resolve all four multipoles. This is only possible by using the angular distribution of the observables, in the minimal case by measurements at a second angle. At this point, it is very interesting to note that solutions with only four observables are also possible [14]. Here we give the set of observables (I, $\check{\Sigma}$, $\check{F}$, $\check{H}$), providing a solution with no recoil measurements required. This is a very surprising result because it goes beyond the studies of Omelaenko [7,9], where unique solutions were found only with five or more observables.

A simple set with four multipoles and four independent amplitudes is set five of Table III with E_{0+}, E_{1+}, M_{1-}, and E_{2-}. In this case also $F_4 = -3E_{2-}$ is finite. For this set an equivalent set of eight observables yields unique solutions for a CEA, with four transversity amplitudes, and a TPWA, at a single angle, with four multipoles. However, taking into account the angular distribution, the number of necessary observables can be reduced to only three: I, $\check{\Sigma}$, and $\check{T}$, where no recoil measurement would be needed.

Truncating the multipole series in the total spin J (instead of angular momentum L) leads to set six with limit $J = 3/2$. This set contains six multipoles, and a CEA at one angle is certainly no longer sufficient to determine all of them. The last set seven of Table III is the full set of eight multipoles for $L = 2$ and can be generalized for any higher L. Here also the CEA is no longer related to the TPWA.

The Omelaenko method [7,9] can be applied to any given L. This method proves, in general, a unique solution is possible with five observables measured over the full angular range, i.e., at sufficient angles to determine the $\cos\theta$ or alternatively the Legendre coefficients. These are four observables from group S, the unpolarized cross section, and the three single-spin polarizations, plus one more double-polarization observable from any other group, except $\check{E}$ and $\check{H}$. The fifth observable is needed to resolve, first of all, the double ambiguity. The new solution with only four observables [14], which was found to provide a solution for set 4, has been found to solve set 7 as well, and can most likely be generalized for any higher L.

However, as discussed in Refs. [7,9], an increasing number of 4^{2L} accidental ambiguities can occur, which leads to

TABLE III. Examples of measurements at a single energy for CEA and TPWA. The number of different measurements (n), different observables (m), and different angles (k) needed for a complete analysis are given as $n(m)k$. Entries with † do not allow the comparison CEA ↔ TPWA. For cases with only one angle, the CEA and TPWA are equivalent. The number of necessary distinct angular measurements is given in brackets.

Set	Included partial waves	CEA	TPWA	Complete sets for TPWA
1	$L = 0\ (E_{0+})$	1(1)	1(1)1	$I[1]$
2	$J = 1/2\ (E_{0+}, M_{1-})$	4(4)	4(4)1	$I[1], \check{P}[1], \check{C}_x[1], \check{C}_z[1]$
			4(3)2	$I[2], \check{P}[1], \check{C}_x[1]$
3	$L = 0,1\ (E_{0+}, M_{1-}, E_{1+})$	6(6)	6(6)1	$I[1], \check{\Sigma}[1], \check{T}[1], \check{P}[1], \check{F}[1], \check{G}[1]$
			6(4)2	$I[2], \check{\Sigma}[1], \check{T}[2], \check{P}[1]$
			6(3)3	$I[3], \check{\Sigma}[1], \check{T}[2]$
4	$L = 0,1\ (E_{0+}, M_{1-}, E_{1+}, M_{1+})$	†		TPWA at one angle not possible
	full set of four S- and P-wave multipoles		8(5)2	$I[2], \check{\Sigma}[1], \check{T}[2], \check{P}[2], \check{F}[1]$
			8(4)3	$I[3], \check{\Sigma}[1], \check{F}[2], \check{H}[2]$
5	$L = 0,1,2\ (E_{0+}, M_{1-}, E_{1+}, E_{2-})$	8(8)	8(8)1	$I[1], \check{\Sigma}[1], \check{T}[1], \check{P}[1], \check{F}[1], \check{G}[1], \check{C}_x[1], \check{O}_x[1]$
			8(4)2	$I[2], \check{\Sigma}[2], \check{T}[2], \check{P}[2]$
			8(3)3	$I[3], \check{\Sigma}[2], \check{T}[3]$
6	$J \leqslant 3/2\ (E_{0+}, M_{1-}, E_{1+}, M_{1+}, E_{2-}, M_{2-})$	†		TPWA at one or two angles not possible
			12(5)3	$I[3], \check{\Sigma}[2], \check{T}[3], \check{P}[2], \check{F}[2]$
			12(4)4	$I[4], \check{\Sigma}[2], \check{F}[3], \check{H}[3]$
7	$L = 0,1,2\ (E_{0+}, \ldots, M_{2+})$	†		TPWA at one or two angles not possible
	full set of eight S-,P-,D-wave multipoles		16(6)3	$I[3], \check{\Sigma}[3], \check{T}[3], \check{P}[3], \check{F}[3], \check{G}[1]$
			16(5)4	$I[4], \check{\Sigma}[3], \check{T}[3], \check{P}[3], \check{F}[3]$
			16(4)5	$I[5], \check{\Sigma}[3], \check{F}[4], \check{H}[4]$

enormous numerical problems for $L > 2$. This problem can partly be solved by extending the set of observables. However, the accidental ambiguities depend on the dynamics of the underlying models and the physics involved, and unique solutions cannot be guaranteed in many cases, so elaborate numerical methods need to be applied. Since experimental data contain sizable statistical errors, and in most cases also systematic errors, a unique solution by this method will become increasingly difficult for larger L. Therefore, in practice, higher partial waves have to be fixed by models or if possible by theoretical constraints such as unitarity, analyticity, and fixed-t dispersion relations.

Instead of doing model applications, the results of Table III have been obtained in a more general numerical simulation procedure. The underlying multipoles numbering from two to eight were randomly chosen as complex numbers with integer values for their real and imaginary parts. From these multipoles, all observables and their coefficients were calculated and the inverse solution was searched with numerical minimization techniques by using a random search with the help of *Mathematica*. Sets 1 to 6 were quickly obtained but set 7, for $L = 2$, required a significant increase in computation time. Nevertheless, the uniqueness of the solution in terms of the squared numerical deviation is found to be of order 10^{-20}.

V. CONCLUSIONS

We have explored the CEA and TPWA, applying a number of approaches, in order to compare the information required for a complete solution. The connection is seen most easily in the first nontrivial case, $J = 1/2$, involving the interference of two multipoles or helicity and transversity amplitudes. The reduced number of observable types for a TPWA is compensated by additional angular measurements. From a physical standpoint, the appearance of θ-dependent factors in Eq. (6) is due to rotational symmetry, as contained in the rotation matrices used to construct the helicity amplitudes [11,15].

This matching of information required to determine either the multipoles or helicity and transversity amplitudes holds only when the number of independent helicity and transversity amplitudes, for a CEA, is the same as the number of multipoles used in their construction. The number of angular measurements for a TPWA grows with increasing angular-momentum cutoff, as described in Refs. [7,9]. With greater-than-four multipole amplitudes included, the TPWA and CEA problems are fundamentally different and the information required for a solution is not comparable.

Our pedagogical study of the simple $J = 1/2$ case, generalized to higher angular-momentum cutoffs, has revealed further solutions of the TPWA problem addressed by Omelaenko [9], which require only four well-selected polarization observables. These will be examined in detail in a future presentation [14].

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[1] See, for example, J. Ball *et al.*, Eur. Phys. J. C **5**, 57 (1998), and references therein.
[2] W.-T. Chiang and F. Tabakin, Phys. Rev. C **55**, 2054 (1997).
[3] I. S. Barker, A. Donnachie, and J. K. Storrow, Nucl. Phys. B **95**, 347 (1975).
[4] T. Vrancx, J. Ryckebusch, T. Van Cuyck, and P. Vancraeyveld, Phys. Rev. C **87**, 055205 (2013).
[5] J. Nys, T. Vrancx, and J. Ryckebusch, J. Phys. G **42**, 034016 (2015).
[6] A. M. Sandorfi, S. Hoblit, H. Kamano, and T.-S. H. Lee, J. Phys. G **38**, 053001 (2011).
[7] Y. Wunderlich, R. Beck, and L. Tiator, Phys. Rev. C **89**, 055203 (2014); Y. Wunderlich, Ph.D. thesis, U. of Bonn (unpublished).
[8] R. L. Workman, Phys. Rev. C **83**, 035201 (2011).
[9] A. S. Omelaenko, Sov. J. Nucl. Phys. **34**, 406 (1981).
[10] V. F. Grushin, A. A. Shikanyan, E. M. Leikin, and A. Ya. Rotvain, Yad. Fiz. **38**, 1448 (1983); V. F. Grushin, in *Photoproduction of Pions on Nucleons and Nuclei*, edited by A. A. Komar (Nova Science, New York, 1989), p. 1ff.
[11] R. L. Walker, Phys. Rev. **182**, 1729 (1969).
[12] R. A. Arndt, R. L. Workman, Z. Li, and L. D. Roper, Phys. Rev. C **42**, 1853 (1990); **42**, 1864 (1990).
[13] G. Keaton and R. Workman, Phys. Rev. C **53**, 1434 (1996).
[14] L. Tiator *et al.* (unpublished).
[15] M. Jacob and G. C. Wick, Ann. Phys. (NY) **7**, 404 (1959).