Kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion

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We develop a set of kinetic equations for hydrodynamic fluctuations which are equivalent to nonlinear hydrodynamics with noise. The hydrokinetic equations can be coupled to existing second-order hydrodynamic codes to incorporate the physics of these fluctuations. We first show that the kinetic response precisely reproduces the renormalization of the shear viscosity and the fractional power ($\propto \omega^{3/2}$) which characterizes equilibrium correlators of energy and momentum for a static fluid. Then we use the hydrokinetic equations to analyze thermal fluctuations for a Bjorken expansion, evaluating the contribution of thermal noise from the earliest moments and at late times. In the Bjorken case, the solution to the kinetic equations determines the coefficient of the first fractional power of the gradient expansion ($\propto 1/(\tau T)^{3/2}$) for the expanding system. Numerically, we find that the contribution to the longitudinal pressure from hydrodynamic fluctuations is larger than second-order hydrodynamics for typical medium parameters used to simulate heavy ion collisions.

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I. INTRODUCTION

A. Overview

The purpose of the current paper is to develop a set of kinetic equations for hydrodynamic fluctuations and to use these kinetic equations to study corrections to Bjorken flow arising from thermal fluctuations. The specific test case of Bjorken flow (which is a hydrodynamic model for the longitudinal expansion of a nucleus-nucleus collision [1]) is motivated by the experimental program of ultrarelativistic heavy-ion collisions at Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Detailed measurements of two-particle correlation functions have provided overwhelming evidence that the evolution of the excited nuclear material is remarkably well described by the hydrodynamics of the quark gluon plasma (QGP) with a small shear viscosity to entropy ratio of order $\eta/s \sim 2/4\pi$ [2,3]. The typical relaxation times of the plasma, while short enough to support hydrodynamics, are not vastly smaller than the inverse expansion rates of the collision. For this reason, the gradient expansion underlying the hydrodynamic formalism has been extended to include first- and second-order viscous corrections [4], and these corrections systematically improve the agreement between hydrodynamic simulations and measured two-particle correlations [2]. Additional corrections, which have not been systematically included, arise from thermal fluctuations of the local energy and momentum densities and could be significant in nucleus-nucleus collision where only ~20 000 particles are produced. This has prompted a keen practical interest in the heavy ion community in simulating relativistic hydrodynamics with stochastic noise [5–11]. In a nonrelativistic context, such simulations have reached a fairly mature state [12–14]. For a static fluid, thermal fluctuations give rise through the nonlinearities of the equations of motion to fractional powers in the fluid response function at small frequency, $G_R(\omega) \propto \omega^{3/2}$. Indeed, the long-time tails first observed in molecular-dynamics simulations [15–17] are a consequence of this nonanalytic $\omega^{3/2}$ behavior. For Bjorken flow, the same nonlinear stochastic physics leads to fractional powers in the gradient expansion for the longitudinal pressure of the fluid. One of the goals of this paper is to compute the coefficient of the first fractional power in this expansion.

The measured two-particle correlations in heavy ion collisions reflect both the fluctuations in the initial conditions and thermal fluctuations. Thermal fluctuations are believed to be a small (but conceptually important) correction to nonfluctuating hydrodynamics [6–8]. In addition, thermal fluctuations can become significant close to the QCD critical point [9,18] and in smaller colliding systems such as proton-nucleus and proton-proton collisions [7], which show remarkable signs of collectivity [19].

In the current paper, rather than simulating nonlinear fluctuating hydrodynamics directly, we will reformulate fluctuating hydrodynamics as nonfluctuating hydrodynamics (describing a long wavelength background) coupled to a set of kinetic equations describing the phase space distribution of short wavelength hydrodynamic fluctuations. For Bjorken flow this set of equations can be solved to determine the first fractional powers in the gradient expansion.

B. Hydrodynamics with noise and fractional powers in the gradient expansion

At finite temperature, real-time dynamics in each regime of scales has an efficient and systematic description by an

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effective theory [20]. Hydrodynamics is a long-wavelength effective theory which describes the evolution of conserved quantities by organizing corrections in powers of gradients. For the hydrodynamic expansion to apply, we require frequencies under consideration to be small compared to the microscopic relaxation rates

$$\epsilon \equiv \frac{\omega\eta}{(e+p)c_s^2} \ll 1, \tag{1}$$

where we have estimated the microscopic relaxation time with the hydrodynamic parameters, $\tau_R \equiv \eta/(e+p)c_s^2$ [21], and for later convenience defined $\epsilon \equiv \omega\tau_R$.

For definiteness, we follow precedent [4,22,23] and consider a conformal neutral fluid driven from equilibrium by a small metric perturbation $h_{xy}(\omega)$ of frequency ω. Within the framework of linear response (see Sec. II and Ref. [24] for further details), the stress tensor at low frequency takes the form

$$\delta T^{xy} = -h_{xy}(\omega)\left[p - i\omega\eta + \left(\eta\tau_\pi - \frac{\kappa}{2}\right)\omega^2\right]. \tag{2}$$

The first term is the prediction of ideal hydrodynamics $\delta T^{xy} = -p h_{xy}$; the middle term is the prediction of first-order viscous hydrodynamics [22], where η is the shear viscosity; finally, the last term is the prediction of second-order hydrodynamics, where τ_π and κ are the associated second-order parameters [4].

In writing Eq. (2) we have neglected additional contributions stemming from fluctuations which will be described below. Thermal fluctuations can be incorporated into the hydrodynamic description by including stochastic terms into the equations of motion [25–27]

$$d_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} + T^{\mu\nu}_{\rm visc.} + S^{\mu\nu}, \tag{3}$$

where variance of the noise, $\langle S^{\mu\nu}S^{\rho\sigma}\rangle \sim 2T\eta\delta(t-t')$, is determined by the fluctuation dissipation theorem at temperature T and introduces no new parameters into the effective theory.[1] After including these stochastic terms, the correlators of momentum and energy evolve to their equilibrium values in the absence of the external force, $h_{xy}(\omega)$. Specifically, the equilibrium two-point functions of the energy and momentum densities, $\delta e(t,\boldsymbol{x}) \equiv T^{00}(t,\boldsymbol{x}) - \langle T^{00}\rangle$ and $g^i(t,\boldsymbol{x}) \equiv T^{0i}$ respectively, approach the textbook result [25]

$$\langle \delta e(t,\boldsymbol{k})\delta e(t,-\boldsymbol{k}')\rangle = \frac{(e+p)T}{c_s^2}\,(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \tag{4a}$$

$$\langle g^i(t,\boldsymbol{k})g^j(t,-\boldsymbol{k}')\rangle = (e+p)T\;\delta^{ij}(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \tag{4b}$$

where c_s is the speed of sound and $\delta e(t,\boldsymbol{k})$ notates the spatial Fourier transform of $\delta e(t,\boldsymbol{x})$. In the presence of an external force or a nontrivial expansion, these correlations are driven away from equilibrium. The purpose of hydrodynamics with noise is to describe in detail these deviations from equilibrium.

Due to the nonlinear character of hydrodynamics, the thermal fluctuations change the evolution of the system. Indeed, a diagrammatic analysis of the hydrodynamic response at one-loop order shows that the stress in the presence of a weak external field (or the retarded Green's function) is

$$\langle T^{xy}(\omega)\rangle = -h_{xy}(\omega)\Bigg\{p - i\omega\eta + (i+1)\frac{\left[7+\left(\frac{3}{2}\right)^{3/2}\right]}{240\pi} \times T\left(\frac{\omega}{\gamma_\eta}\right)^{3/2} + \mathcal{O}(\omega^2)\Bigg\}, \tag{5}$$

where p, e, and η are renormalized physical quantities (see Secs. II A and II B for further discussion of the renormalization), and

$$\gamma_\eta \equiv \frac{\eta}{e+p}, \tag{6}$$

is the momentum diffusion coefficient [23,28]. As emphasized and estimated previously, the fractional order $\omega^{3/2}$ is parametrically larger than second-order hydrodynamics [23]. However, the coefficient of the $\omega^{3/2}$ terms is vanishingly small in weakly coupled theories and in strongly coupled theories at large N_c, and therefore second-order hydrodynamics may be an effective approximation scheme except at very small frequencies. In the context of holography, the $\omega^{3/2}$ term can only be determined by performing a one-loop calculation in the bulk [29].

In the current paper we will rederive Eq. (5) using a kinetic description of short-wavelength hydrodynamic fluctuations. For an external driving frequency of order ω, we identify an important length scale set by equating the damping rate and the external frequency

$$\gamma_\eta k_*^2 \sim \omega\,, \quad k_* \sim \left(\frac{\omega}{\gamma_\eta}\right)^{1/2}. \tag{7}$$

We will refer to the k_* as the *dissipative scale* below (see also Ref. [29]). Modes with wave numbers significantly larger than the dissipative scale, $k \gg k_*$, are damped and re-excited by the noise on a time scale which is short compared to period $2\pi/\omega$, and this rapid competition leads to the equilibration of these shorter wavelengths; i.e., their equal time correlation functions are given by Eq. (4). By contrast, modes with wave numbers of order $k \sim k_*$ have equal time correlation functions which deviate from the equilibrium expectation values.

It is notable that the wave numbers of interest k_* are large compared to ω/c_s, but still small compared to microscopic wave numbers of order the inverse mean free path.[2] Estimating the mean free path as $\ell_{\rm mfp} = c_s\tau_R$, we see that the strong

[1] We follow a standard notation for hydrodynamics summarized in Ref. [21]. d_μ notates a covariant derivative using the "mostly plus" metric convention. $T^{\mu\nu}_{\rm ideal} = (e+p)u^\mu u^\nu + p g^{\mu\nu}$ and $T^{\mu\nu}_{\rm visc} = -\eta\sigma^{\mu\nu}$, where $\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}(d_\rho u_\sigma + d_\sigma u_\rho - \frac{2}{3}g_{\rho\sigma}d_\gamma u^\gamma)$, with $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. The noise correlator is fully specified in Eq. (15) of Sec. II.

[2] The effect of second-order hydrodynamics is suppressed compared to the first-order hydrodynamics as long as the derivative expansion works, i.e., $k \ll 1/\ell_{\rm mfp}$. The causal property of the second-order hydrodynamics is gained by modifying the dispersions at $k \sim 1/\ell_{\rm mfp}$.

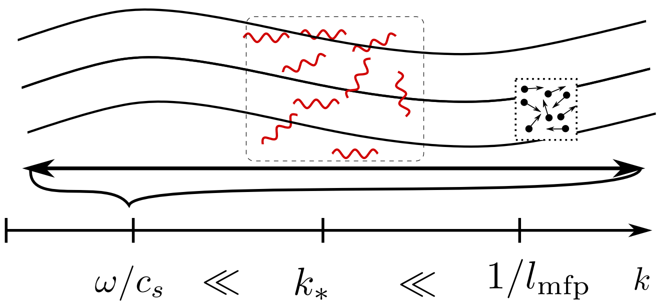

FIG. 1. The hydrokinetic description of noise is based on the separation of scales between the long wavelength hydrodynamic background (with $k \sim \omega/c_s$) and shorter wavelength hydrodynamic fluctuations (with $k \sim k_* \equiv \sqrt{\omega/\gamma_\eta}$). The wavelengths of the hydrodynamic fluctuations are still much longer than microscopic mean free path. The hydrodynamic fluctuations are driven out of equilibrium by the expanding background, and this deviation is the origin of the long-time tail correction to the stress tensor.

inequalities

$$\frac{\omega}{c_s} \ll k_* \ll \frac{1}{\ell_{\rm mfp}} \tag{8}$$

can be written as

$$\frac{\omega}{c_s} \ll \frac{\omega}{c_s}\frac{1}{\sqrt{\epsilon}} \ll \frac{\omega}{c_s}\frac{1}{\epsilon}, \tag{9}$$

and thus hold whenever hydrodynamics is applicable, $\epsilon \ll 1$. The scale separation illustrated in Fig. 1 can be used to set up an approximation scheme where modes of order k_* on a soft ($k \sim \omega/c_s$) background are treated with a kinetic or Wentzel-Kramers-Brillouin (WKB) type approximation scheme. We will develop the appropriate kinetic equations in Sec. II. These kinetic equations can be solved and used to determine how the two point functions of energy and momentum with wave numbers of order k_* deviate from equilibrium when driven by an external perturbation. The $\omega^{3/2}$ term in Eq. (5) roughly represents the contribution of $\int k^2 dk \sim k_*^3$ slightly out of equilibrium hydrokinetic modes per volume, with each mode contributing $\frac{1}{2}T$ of energy to the stress tensor. Note that the contribution to the stress tensor of modes outside of the kinetic regime $k \ll k_*$ is suppressed by phase space.

Similar kinetic equations can be derived for much more general flows. We will establish the appropriate kinetic equations for a Bjorken expansion [1], which is a useful model for the early stages of a heavy ion collision. The ideal, first-order, and second-order terms in the gradient expansions have been given in Refs. [1,30], and [4,31] respectively. For a conformal (nonfluctuating) fluid the longitudinal pressure during a Bjorken expansion takes the form

$$\tau^2 T^{\eta\eta} = p - \frac{4}{3}\frac{\eta}{\tau} + \frac{8}{9\tau^2}(\lambda_1 - \eta\tau_\pi) + \cdots . \tag{10}$$

The expansion rate is $\partial_\mu u^\mu = 1/\tau$, and each higher term in the gradient expansion is suppressed by an integer power of $1/\tau T$. For Bjorken flow the expansion rate plays the role of frequency, and the distribution of sound modes are characterized by a dissipative scale analogous to Eq. (7) of order[3]

$$k_* \sim \frac{1}{(\gamma_\eta \tau)^{1/2}}. \tag{11}$$

At this scale the viscous damping rate balances the expansion rate. These hydrodynamic modes satisfy the inequality

$$\frac{1}{c_s\tau} \ll k_* \ll \frac{1}{\ell_{\rm mfp}}, \tag{12}$$

and this strong set of inequalities can be used to determine a kinetic equation for hydrodynamic modes of order k_*. The equal time correlation functions for wave numbers of this order deviate from their equilibrium form in Eq. (4), and the kinetic equations precisely determine the functional dependence of this deviation. Finally, these modes contribute to the longitudinal pressure and determine first fractional power in the longitudinal pressure of a conformal fluid [analogous to Eq. (5)]. In Sec. II we will establish that this nonlinear correction to the longitudinal component of the stress tensor is

$$\frac{\langle \tau^2 T^{\eta\eta}\rangle}{e+p} = \left[\frac{p}{e+p} - \frac{4}{3}\frac{\gamma_\eta}{\tau} + \frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}} + \mathcal{O}\left(\frac{1}{(\tau T)^2}\right)\right]. \tag{13}$$

Noise also contributes to transverse momentum fluctuations, and this contributes at quadratic order to $\langle T^{\tau\tau}\rangle$ as we discuss in Sec. III. Thus, a complete description of a Bjorken expansion with noise must also re-examine the relationship between the background energy density e and the one-point function $\langle T^{\tau\tau}\rangle$.

An outline of the paper is as follows. In Sec. II we consider a static fluid perturbed by an external gravitational perturbation. The purpose of this section is to introduce the kinetic equations and to reproduce the results of the diagrammatic analysis of Refs. [23,28] using the hydrokinetic theory adopted here. In Sec. III B we linearize the hydrodynamic equations of motion to determine the appropriate kinetic equations for a Bjorken expansion. In Secs. III C and III D we determine the solutions to the kinetic theory and use these solutions to evaluate the contribution of hydrodynamic modes to the stress tensor. We give an intuitive physical interpretation of the main results of the paper in Sec. III E. Finally we conclude with results and discussion in Sec. IV.

II. HYDRODYNAMIC FLUCTUATIONS IN A STATIC FLUID

We will first derive the kinetic equations for hydrodynamic fluctuations in homogeneous flat space in Sec. II A. The purpose here is to introduce notation and to discuss the kinetic approximations in the simplest context. Then in Sec. II B

[3]The quantities $k_*(\tau)$, $\gamma_\eta(\tau)$, $s(\tau)$, ... are all functions of time for a Bjorken expansion, e.g., for a conformal equation of state and an ideal expansion, $k_*(\tau) \propto \tau^{-2/3}$, $\gamma_\eta \propto \tau^{1/3}$, $s(\tau) \propto \tau^{-1}$, etc. Throughout the paper $k_*, \gamma_\eta, s, \ldots$ (without a time argument) will denote the physical quantity at the final time of consideration. The explicit time argument will be used when needed, e.g., $k_*(\tau') = k_*(\tau/\tau')^{-2/3}$.

we will perturb the system with a gravitational field and derive the appropriate kinetic theory in this case. We then use this hydrokinetic theory to reproduce the results of loop calculations [23,28] for the renormalization of the shear viscosity and the long time tails which characterize the hydrodynamic response due to nonlinear noise effects.

A. Relaxation equations for hydrodynamic fluctuations

To illustrate the approximations that follow and to introduce notation, we first will derive kinetic equations for the two point functions for energy and momentum density perturbations around a static homogeneous background. The basics of the techniques adopted in our analysis is reviewed in Refs. [32,33]. The (bare) background quantities of the hydrodynamic effective theory, such as the energy density, pressure, and shear viscosity [$e_0(\Lambda)$, $p_0(\Lambda)$, and $\eta_0(\Lambda)$ respectively] are calculated by integrating out fluctuations above a scale Λ, i.e., by excluding the contributions of hydrodynamic fluctuations with wave number $k < \Lambda$ to the stress tensor. This is important because modes with $k < \Lambda$ will not be in equilibrium when the system is perturbed by a driving force. The relation between the bare parameters and the physical quantities (which may be computed in infinite volume with lattice QCD for instance) is discussed in Sec. II B and in Ref. [23], where $\eta_0(\Lambda)$ is referred to as $\eta_{\rm cl}(p_{\rm max})$.

To derive a relaxation equation for the two point functions we linearize the equations of stochastic hydrodynamics and study the eigenmodes of the system. The correlations between eigenmodes with vastly different frequencies are neglected in a kinetic (or coarse graining) approximation. For the constant background $e_0 = $ const, and to linear order in field perturbations and stochastic fluctuations, the equations of motion [Eq. (3)] become

$$\partial_t \delta e + i k_i g^i = 0, \tag{14a}$$

$$\partial_t g_i + i k_i \delta p + \gamma_\eta k^2 g_i + \tfrac{1}{3}\gamma_\eta k_i k_j g^j = -\xi_i, \tag{14b}$$

where $\gamma_\eta \equiv \eta_0/(e_0+p_0)$ is computed with bare quantities, and $-\xi_i$ is the stochastic force, $-ik_j S^j{}_i(t,\boldsymbol{k})$. Here $S^j{}_i(t,\boldsymbol{k})$ are spatial components of the noise tensor with equilibrium correlation given by [26]

$$\begin{aligned}\langle S^{\mu\nu}(t_1,\boldsymbol{k}) S^{\alpha\beta}(t_2,-\boldsymbol{k}')\rangle &= 2T\eta_0\big[(\Delta^{\mu\alpha}\Delta^{\nu\beta}+\Delta^{\mu\beta}\Delta^{\nu\alpha}) - \tfrac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\big] \\ &\quad \times (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta(t_1-t_2).\end{aligned} \tag{15}$$

It is convenient to combine Eq. (14) into a single matrix equation for an amalgamated field $\phi_a = (c_s\delta e, g_j)$

$$\partial_t \phi_a(t,\boldsymbol{k}) = -i\mathcal{L}_{ab}\phi_b - \mathcal{D}_{ab}\phi_b - \xi_a, \tag{16}$$

where ideal and dissipative terms are

$$\mathcal{L}_{ab} = \begin{pmatrix} 0 & c_s k_j \\ c_s k_i & 0 \end{pmatrix}, \quad \mathcal{D}_{ab} = \gamma_\eta \begin{pmatrix} 0 & 0 \\ 0 & k^2\delta_{ij} + \frac{1}{3}k_i k_j \end{pmatrix}, \tag{17}$$

and the stochastic noise ξ_a satisfies correlation equation

$$\begin{aligned}\langle \xi_a(t_1,\boldsymbol{k})\xi_b(t_2,-\boldsymbol{k}')\rangle &= 2T(e_0+p_0)\mathcal{D}_{ab}(2\pi)^3 \times \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta(t_1-t_2).\end{aligned} \tag{18}$$

At the dissipative scale, the acoustic matrix $\mathcal{L} \sim c_s k_*$ originating from ideal equations of motion dominates over the competing dissipation $\mathcal{D}$ and fluctuation ξ_a terms. $\mathcal{L}_{ab}$ has four eigenmodes: two longitudinal sound modes with $\lambda_\pm = \pm c_s|\boldsymbol{k}|$ and two transverse zero modes ($\lambda_{T_1} = \lambda_{T_2} = 0$). Since $\mathcal{L}$ drives evolution of ϕ_a, it will be convenient to analyze the dynamics in terms of eigenmodes of $\mathcal{L}_{ab}$:[4]

$$(e_\pm)_a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm\hat{k} \end{pmatrix}, \quad (e_{T_1})_a = \begin{pmatrix} 0 \\ \vec{T}_1 \end{pmatrix}, \quad (e_{T_2})_a = \begin{pmatrix} 0 \\ \vec{T}_2 \end{pmatrix}, \tag{19}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and $\vec{T}_1$ and $\vec{T}_2$ are two orthonormal spatial vectors perpendicular to $\hat{\boldsymbol{k}}$:

$$\hat{k} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{20a}$$

$$\vec{T}_1 = (-\sin\varphi, \cos\varphi, 0), \tag{20b}$$

$$\vec{T}_2 = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta). \tag{20c}$$

Now we will derive a relaxation equation for the two-point correlation function of hydrodynamic fluctuations by defining a density matrix $N_{ab}(t,\boldsymbol{k})$

$$\langle \phi_a(t,\boldsymbol{k})\phi_b(t,-\boldsymbol{k}')\rangle \equiv N_{ab}(t,\boldsymbol{k})(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}') \tag{21}$$

and analyzing the time evolution of $N_{ab}(t,\boldsymbol{k})$.

The analysis is most transparent in the eigenbasis, $\phi_A \equiv \phi_a(e_A)_a$ with $A = +,-,T_1,T_2$, and below we will determine the equation of motion for $N_{AB} \equiv \langle\phi_A\phi_B\rangle$, where $A,B = +,-,T_1,T_2$. We note that the positive and negative sound modes ϕ_+ and ϕ_- are related since the hydrodynamic fields are real, $\phi_-^*(\boldsymbol{k},t) = \phi_+(-\boldsymbol{k},t)$.

Using the equations of motion for ϕ_A we calculate the infinitesimal change of $N_{AB}(t+dt) - N_{AB}(t)$ and use the equal time correlator for the noise [Eq. (18)] to find a differential equation for N_{AB}:

$$\partial_t N = -i[\mathcal{L},N] - \{\mathcal{D},N\} + 2T(e_0+p_0)\mathcal{D}, \tag{22}$$

where $[X,Y] \equiv XY - YX$, $\{X,Y\} \equiv XY + YX$, and $[\mathcal{L},N]_{AB} = (\lambda_A - \lambda_B)N_{AB}$. We are interested in the evolution of two-point correlation functions over time scales much larger than acoustic oscillations, $\Delta t \gg 1/(c_s k_*)$. On these time scales the off-diagonal matrix elements of the density matrix, N_{+T_1}, for example, rapidly oscillate reflecting the large difference in eigenvalues, $\lambda_+ - \lambda_{T_1} \sim c_s k_*$. In a coarse-graining approximation the contributions of these off-diagonal matrix elements to physical quantities can be neglected when averaged over times long compared to $1/(c_s k_*)$. This reasoning does not apply to the diffusive modes $A,B = T_1,T_2$ where both eigenvalues are zero, but rotational symmetry in the transverse xy-plane requires $N_{T_1T_2}$ to vanish.[5]

[4]Another reason why analysis in terms of eigenmodes of $\mathcal{L}_{ab}$ is convenient is that they form a real and orthonormal basis and the projection onto each mode is easily handled.

[5]Rotational symmetry in the transverse xy plane requires that $\langle g_i g_j\rangle \sim A\delta_{ij} + B\hat{k}_i\hat{k}_j$, where $i,j = x,y$. Such a tensor structure has vanishing T_1T_2 projection.

With these approximations, the nontrivial relaxation equations of two-point correlation functions in Eq. (22) are

$$\partial_t N_{\pm\pm}(t,\boldsymbol{k}) = -\tfrac{4}{3}\gamma_\eta k^2(N_{\pm\pm} - N_0), \tag{23a}$$

$$\partial_t N_{T_1T_1}(t,\boldsymbol{k}) = -2\gamma_\eta k^2(N_{T_1T_1} - N_0), \tag{23b}$$

$$\partial_t N_{T_2T_2}(t,\boldsymbol{k}) = -2\gamma_\eta k^2(N_{T_2T_2} - N_0), \tag{23c}$$

where

$$N_0 = T(e_0 + p_0) \tag{24}$$

is the equilibrium value for N_{AA} [cf. Eq. (4)]. In the absence of external perturbations, two-point correlation functions relax to their equilibrium values. The next step towards general kinetic equations is to study how equal time correlations are driven out of equilibrium by the presence of external fields.

B. Linear response to gravitational perturbations

In this section we will study the evolution of two-point energy and momentum correlators in the presence of a time-varying gravitational field. We determine the kinetic equations in the time-dependent background and use these equations to reproduce the modifications of the retarded Green's function [Eq. (5)] due to thermal fluctuations, which were previously found by a one-loop calculation [23,28].

A straightforward way of introducing an external source to equations of motion is to study fluctuating hydrodynamics in the presence of a small metric perturbation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The Green's function records the response of $T^{\mu\nu}$ to the metric perturbation

$$\delta\langle T^{\mu\nu}(\omega)\rangle = -\tfrac{1}{2}G_R^{\mu\nu,\alpha\beta}(\omega)h_{\alpha\beta}(\omega). \tag{25}$$

For a constant homogeneous background with time-dependent metric perturbation $h_{ij}(t)$, symmetry constrains the form of the retarded Green's function

$$G_R^{ij,kl}(\omega) = \mathring{G}_R(\omega)\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \tfrac{2}{3}\delta^{ij}\delta^{kl}\right) + \overline{G}_R(\omega)\,\delta^{ij}\delta^{kl}, \tag{26}$$

and therefore we can obtain the Green's function in Eq. (5), i.e., $\mathring{G}_R(\omega)$, by considering a diagonal traceless metric perturbation, $h_{ij}(t) = h(t)\,\mathrm{diag}\,(1,1,-2)$.

In the presence of metric perturbations and thermal fluctuations, the energy momentum tensor is

$$\delta\langle T^{ij}(t)\rangle = -p_0 h^{ij} - \eta_0\partial_t h^{ij} + \frac{\langle g^i(t,\boldsymbol{x})g^j(t,\boldsymbol{x})\rangle}{e_0 + p_0}, \tag{27}$$

where the nonlinear term stems from the constitutive relation of ideal hydrodynamics, $T^{ij} = p_0\delta^{ij} + (e_0 + p_0)u^i u^j$. The averaged squared momentum, $\langle g^i(t,\boldsymbol{x})g^j(t,\boldsymbol{x})\rangle$, is related to the two-point functions of g^i in $\boldsymbol{k}$ space as

$$\langle g^i(t,\boldsymbol{x})g^j(t,\boldsymbol{x})\rangle = \int \frac{d^3k}{(2\pi)^3}N^{ij}(t,\boldsymbol{k}). \tag{28}$$

In this integral, the equilibrium value of N^{ij} and its first viscous correction will renormalize p_0 and η_0 (see below), while the finite remainder will determine the first fractional power in the stress tensor correlator $\propto \omega^{3/2}$.

Studying the hydrodynamic equations in Eq. (3) and neglecting metric perturbations of the dissipative terms, we find that the linearized equations of motion are identical to flat background Eq. (14), but now there is a difference between covariant and contravariant indices

$$\partial_t\delta e + ik_i g^i = 0, \tag{29a}$$

$$\partial_t g_i + ik_i\delta p + \gamma_\eta k^2 g_i + \tfrac{1}{3}\gamma_\eta k_i k_j g^j = -\xi_i. \tag{29b}$$

To avoid this complication, we use a vielbein formalism and scale the spatial components of momentum and wave number by $\sqrt{g_{ij}}$; i.e., g^i and k_j are replaced by

$$G_{\hat{\imath}} = \left(1 + \tfrac{1}{2}h_{ij}\right)g^j, \tag{30a}$$

$$K_{\hat{\imath}} = \left(1 - \tfrac{1}{2}h^{ij}\right)k_j, \tag{30b}$$

where now the position of hatted indices is unimportant. Analogously to Eq. (16), we obtain a matrix equation for $\phi_a = (c_s\delta e, G_{\hat{\imath}})$

$$\partial_t\phi_a(t,\boldsymbol{k}) = -i\mathcal{L}_{ab}\phi_b - \mathcal{D}_{ab}\phi_b - \xi_a - \mathcal{P}_{ab}\phi_b, \tag{31}$$

with an additional metric dependent source term

$$\mathcal{P}_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \tfrac{1}{2}\partial_0 h_{\hat{\imath}\hat{\jmath}} \end{pmatrix}, \tag{32}$$

which drives the hydrodynamic fluctuations away from equilibrium. The eigenbasis of $\mathcal{L}$ [see Eq. (19)] is now defined with respect to the time-dependent vector $\vec{K}(t)$ but remains orthonormal at all times. Furthermore, the metric perturbation preserves rotational symmetry in the transverse xy plane, and this guarantees that the T_1 and T_2 modes are not mixed by the time-dependent perturbation. Thus, the only nontrivial diagonal components of the symmetrized energy and momentum two-point functions are

$$\partial_t N_{\pm\pm} = -\tfrac{4}{3}\gamma_\eta K^2(N_{\pm\pm} - N_0) - \tfrac{1}{2}\partial_t h\,(\sin^2\theta_K - 2\cos^2\theta_K)N_{\pm\pm}, \tag{33a}$$

$$\partial_t N_{T_1T_1} = -2\gamma_\eta K^2(N_{T_1T_1} - N_0) - \partial_t h\, N_{T_1T_1}, \tag{33b}$$

$$\partial_t N_{T_2T_2} = -2\gamma_\eta K^2(N_{T_2T_2} - N_0) - \partial_t h\,(\cos^2\theta_K - 2\sin^2\theta_K)N_{T_2T_2}. \tag{33c}$$

We can find a perturbative solution to these equations for a small periodic metric perturbation, e.g.,

$$N_{T_2T_2}(\omega,\boldsymbol{k}) \simeq N_0\left[2\pi\delta(\omega) + \frac{i\omega h(\omega)(\cos^2\theta_K - 2\sin^2\theta_K)}{-i\omega + 2\gamma_\eta K^2}\right]. \tag{34}$$

To find the correction to the energy momentum tensor due to the nonlinear momentum fluctuations in Eq. (27), we need to perform the k space integral in Eq. (28)

$$\langle\phi_a(x)\phi_b(x)\rangle = \int\frac{d^3K}{(2\pi)^3}N_{ab}(t,\boldsymbol{k}), = \int\frac{K^2dK\,d\cos\theta_K\,d\varphi_K}{(2\pi)^3}(e_A)_a N_{AB}(t,\boldsymbol{k})(e_B)_b. \tag{35}$$

Note that care should be taken when transforming the zeroth-order value $N_{AA} = N_0$ to original unhatted basis as it produces terms linear in metric perturbation. The modification of the response function $\mathring{G}_R(\omega)$ due to the momentum fluctuations [i.e., the last term in Eq. (27)] is

$$\mathring{G}_R(\omega) = -\frac{1}{6}(\delta T^{xx} + \delta T^{yy} - 2\delta T^{zz})/h(\omega),$$
$$\supset -\frac{T}{6}\int \frac{d^3K}{(2\pi)^3}\left[-6 + i\omega\frac{(\sin^2\theta_K - 2\cos^2\theta_K)^2}{-i\omega + \frac{4}{3}\gamma_\eta K^2} + i\omega\frac{1 + (\cos^2\theta_K - 2\sin^2\theta_K)^2}{-i\omega + 2\gamma_\eta K^2}\right]. \tag{36}$$

Performing K-space integral with ultraviolet cutoff, $K_{\max} = \Lambda$, and adding the remaining terms in Eq. (27), we find

$$\mathring{G}_R(\omega) = \left(p_0 + \frac{\Lambda^3}{6\pi^2}T\right) - i\left(\eta_0 + \frac{\Lambda}{\gamma_\eta}\frac{17}{120\pi^2}T\right)\omega + (1+i)\frac{1}{\gamma_\eta^{3/2}}\frac{\left(\frac{3}{2}\right)^{3/2} + 7}{240\pi}T\omega^{3/2}, \tag{37}$$

in agreement with previous work [23,28]. The first two terms in Eq. (37) are the renormalized pressure [$p \equiv p_0(\Lambda) + O(T\Lambda^3)$] and shear viscosity [$\eta \equiv \eta_0(\Lambda) + O(\Lambda T^2)$] as discussed previously [23]. In general, $\Lambda \ll 1/\ell_{\rm mfp} \lesssim T$ holds and the renormalization only slightly shifts the quantities in the thermodynamic limit ($\Lambda \to 0$). Further discussion of the renormalization of these quantities is given in the next section when the expanding case is presented.

The last term is the finite nonlinear modification of the medium response and agrees with loop calculations in equilibrium. The kinetic approach outlined in this section has the advantage that it can be readily applied to more general backgrounds, and we will exploit this advantage to calculate the analogous correction for a Bjorken expansion in the next section. In contrast to the linear response described here, the deviation from equilibrium in the expanding case is of order unity. Consequently, computing the first fractional power in an expanding system with the diagrammatic formalism would require an extensive resummation, which would invariably reproduce kinetic calculation described in the next section [34].

III. HYDRODYNAMIC FLUCTUATIONS FOR A BJORKEN EXPANSION

In this section we will derive the kinetic evolution equations for hydrodynamic fluctuations during a Bjorken expansion. We consider a neutral conformal fluid, for which $c_s^2 = 1/3, \zeta = 0$, and $\mu_{\rm B} = 0$. In Bjorken coordinates the energy and momentum conservation laws are

$$\partial_\mu T^{\mu\nu} + \frac{1}{\tau}T^{\tau\nu} + \Gamma^\nu_{\mu\beta}T^{\mu\beta} = 0, \tag{38}$$

with $\Gamma^\tau_{\eta\eta} = \tau$ and $\Gamma^\eta_{\tau\eta} = \Gamma^\eta_{\eta\tau} = 1/\tau$ [21]. For hydrodynamics without noise the background flow fields are independent of transverse coordinates and rapidity and satisfy

$$\frac{d(\tau T^{\tau\tau})}{d\tau} = -\tau^2 T^{\eta\eta}, \tag{39}$$

$$\frac{d(\tau T^{\tau i})}{d\tau} = 0, \tag{40}$$

where roman indices, $i, j \ldots$, run over transverse coordinates x, y. The transverse momentum $T^{\tau i}$ is constant and can be chosen to be zero. In hydrodynamics $T^{\tau\tau}$ and $\tau^2 T^{\eta\eta}$ are related by constitutive equations

$$T^{\tau\tau} = e, \tag{41}$$

$$\tau^2 T^{\eta\eta} = c_s^2 e - \frac{4\eta}{3\tau}. \tag{42}$$

Note that in $\tau^2 T^{\eta\eta}$ the viscous correction is of order $\epsilon = \eta/(e+p)\tau \ll 1$ smaller than the ideal part, and the solution is approximately $e(\tau) = e(\tau_0)(\tau_0/\tau)^{1+c_s^2}[1 + \mathcal{O}(\epsilon)]$.

We will consider the evolution of linearized fluctuations on top of this background. The effect of these fluctuations on the background evolution can then be included as a correction after the two-point functions are known, i.e.,

$$\frac{d\langle\!\langle T^{\tau\tau}\rangle\!\rangle}{d\tau} = -\frac{\langle\!\langle T^{\tau\tau}\rangle\!\rangle + \langle\!\langle \tau^2 T^{\eta\eta}\rangle\!\rangle}{\tau}, \tag{43}$$

where the constitutive relations take the form

$$\langle\!\langle T^{\tau\tau}\rangle\!\rangle = e + \frac{\langle\!\langle \vec{G}^2\rangle\!\rangle}{e+p}, \tag{44}$$

$$\langle\!\langle \tau^2 T^{\eta\eta}\rangle\!\rangle = c_s^2 e - \frac{4\eta}{3\tau} + \frac{\langle\!\langle (G^{\hat{z}})^2\rangle\!\rangle}{e+p}. \tag{45}$$

Here and below $e(\tau)$ is the average rest frame energy density;[6] $\vec{G}$ is the momentum density $\vec{G} = (T^{\tau x}, T^{\tau y}, \tau T^{\tau\eta})$; and all quantities are renormalized as explained more completely below.

There are two sorts of fluctuations to consider: fluctuations in the initial conditions (which are long range in rapidity) and hydrodynamic fluctuations stemming from thermal noise (which are short range in rapidity). The averages over the initial conditions and noise are denoted with $\langle\cdots\rangle_{\tau_0}$ and $\langle\cdots\rangle$ respectively, while the average over both fluctuations is denoted with the double brackets $\langle\!\langle\cdots\rangle\!\rangle$. Since the transverse momentum per rapidity is conserved for boost-invariant fields, approximately boost-invariant initial fluctuations in $\tau T^{\tau i}$ remain important at late times. In Sec. III A we study initial transverse momentum fluctuations, while in remainder of the paper we complete our study of thermal fluctuations during a Bjorken expansion.

[6] $e(\tau)$ notates the average *rest frame* energy density and does not fluctuate; $\langle\!\langle T^{\tau\tau}\rangle\!\rangle$ is the average energy density. In general, the rest frame energy density $e + \delta e$ in a finite volume would be estimated from sample estimate of $T^{\tau\tau}$ and $\vec{G}$ through the (ideal) constitutive equations, $e + \delta e \simeq T^{\tau\tau} - \frac{\vec{G}^2}{(1+c_s^2)T^{\tau\tau}}$. Thus e is given by Eq. (44), and $\delta e \simeq \delta T^{\tau\tau} - \delta(\vec{G}^2/T^{\tau\tau})/(1+c_s^2) \simeq \delta T^{\tau\tau}$.

A. A Bjorken expansion with initial transverse momentum fluctuations

After the initial passage of two large nuclei in a specific event, each rapidity interval contains a finite amount of transverse momentum, although the event-averaged transverse momentum per rapidity is zero. This initial transverse momentum is spread over a large rapidity range by the subsequent rescatterings in the initial state. Ultimately, this dynamical process can be described by (transverse) momentum diffusion in rapidity and can be modeled with hydrodynamics and noise; see Sec. III D. Here we will determine how long-range transverse momentum fluctuations in the initial state influence the evolution of the background energy density at late times.

As a model for the initial conditions in the x,y plane, we take Gaussian statistics for the initial transverse momentum fluctuations

$$\left\langle \tau_0 g_\perp^i(\tau_0,\vec{x}_\perp)\, \tau_0 g_\perp^j(\tau_0,\vec{y}_\perp)\right\rangle_{\tau_0} = \chi_{\tau_0}^{gg}\delta^{ij}\delta^2(\vec{x}_\perp - \vec{y}_\perp), \tag{46}$$

where $g_\perp^i(\tau,\vec{x}_\perp) \equiv T^{\tau i}$ is approximately independent of rapidity, so that each (large) rapidity interval is approximately boost invariant. Integrating over the transverse area $\mathcal{A}$, the total transverse momentum per rapidity,

$$\frac{dp^x}{d\eta} \equiv \int_{\mathcal{A}} d^2x_\perp\; \tau_0 g_\perp^x(\tau_0,\vec{x}_\perp), \tag{47}$$

fluctuates from event to event with a scaled variance of

$$\chi_{\tau_0}^{gg} \equiv \left\langle \frac{1}{\mathcal{A}}\left(\frac{dp^x}{d\eta}\right)^2\right\rangle_{\tau_0}. \tag{48}$$

To find out how this fluctuating initial condition changes the evolution of the system, we linearize the equations of motion of viscous hydrodynamics and Fourier transform with respect to the transverse coordinates

$$\vec{g}_\perp(\tau,\vec{k}_\perp) \equiv \int d^2x_\perp e^{i\vec{k}_\perp\cdot\vec{x}_\perp}\,\vec{g}_\perp(\tau,\vec{x}_\perp). \tag{49}$$

The full equations of motion are given in the next section; see Eq. (56). Decomposing the transverse momentum fluctuation into longitudinal and transverse pieces

$$g_\perp^i(\tau,\vec{k}_\perp) = g_L^i(\vec{k}_\perp) \;+\; g_T^i(\vec{k}_\perp), \tag{50}$$

with $\hat{k}_\perp^i g_T^i = 0$ and $g_L^i = \hat{k}_\perp^i\hat{k}_\perp^j g_\perp^j$, we find that the transverse piece obeys a two-dimensional diffusion equation

$$\partial_\tau\left(\tau g_T^i\right) + \gamma_\eta k_\perp^2\left(\tau g_T^i\right) = 0, \tag{51}$$

with initial conditions specified by Eq. (46)

$$\begin{aligned}&\left\langle \tau_0 g_T^i(\tau_0,\vec{k}_\perp)\,\tau_0 g_T^j(\tau_0,-\vec{k}'_\perp)\right\rangle_{\tau_0}\\ &\quad= \chi_{\tau_0}^{gg}\,(\delta^{ij} - \hat{k}^i\hat{k}^j)\,(2\pi)^2\delta^2(\vec{k}_\perp - \vec{k}'_\perp).\end{aligned} \tag{52}$$

Solving the diffusion equation with a time-dependent diffusion constant $\gamma_\eta \propto \tau^{c_s^2}$, we see that the variance at a specified space time point due to the fluctuating initial conditions is[7]

$$\left\langle \tau g_\perp^i(\tau,\vec{x}_\perp)\tau g_\perp^j(\tau,\vec{x}_\perp)\right\rangle_{\tau_0} = \delta^{ij}\,\frac{\chi_{\tau_0}^{gg}}{12\pi\gamma_\eta\tau}. \tag{53}$$

Thus, we see that a fluctuating initial conditions contributes quadratically to the average stress tensor

$$\frac{\langle\tau^2 T^{\eta\eta}\rangle_{\tau_0}}{e+p} = \frac{p}{e+p} - \frac{4\gamma_\eta}{3\tau}, \tag{54a}$$

$$\frac{\langle T^{xx}\rangle_{\tau_0}}{e+p} = \frac{p}{e+p} + \frac{2\gamma_\eta}{3\tau} + \left[\frac{\chi_{\tau_0}^{gg}}{\tau^2(e+p)^2}\right]\frac{1}{12\pi\gamma_\eta\tau}, \tag{54b}$$

$$\langle T^{yy}\rangle_{\tau_0} = \langle T^{xx}\rangle_{\tau_0}, \tag{54c}$$

$$\langle T^{\tau\tau}\rangle_{\tau_0} = \langle T^{xx}\rangle_{\tau_0} + \langle T^{yy}\rangle_{\tau_0} + \langle\tau^2 T^{\eta\eta}\rangle_{\tau_0}, \tag{54d}$$

where $p = c_s^2 e$.

B. Kinetic equations of hydrodynamic fluctuations

To derive the kinetic equations we will follow the strategy of Sec. II A, and expand all fluctuations in Fourier modes conjugate to transverse coordinates and rapidity, e.g.,

$$\delta e(\tau,\boldsymbol{k}) \equiv \int d\eta\, d^2x_\perp\, e^{i\vec{k}_\perp\cdot\vec{x}_\perp + i\kappa\eta}\,\delta e(\tau,x_\perp,\eta). \tag{55}$$

The linearized equations of motion of all hydrodynamic fields around the Bjorken background read

$$0 = \left(\frac{\partial}{\partial\tau} + \frac{1+c_s^2}{\tau}\right)\delta e + i\vec{k}_\perp\cdot\vec{g}_\perp + i\kappa g^\eta + \xi^\tau, \tag{56a}$$

$$\begin{aligned}\vec{0}_\perp = &\left(\frac{\partial}{\partial\tau} + \frac{1}{\tau}\right)\vec{g}_\perp + c_s^2 i\vec{k}_\perp\delta e + \gamma_\eta\left(k_\perp^2 + \frac{\kappa^2}{\tau^2}\right)\vec{g}_\perp\\ &+ \frac{1}{3}\gamma_\eta\vec{k}_\perp(\vec{k}_\perp\cdot\vec{g}_\perp + \kappa g^\eta) + \vec{\xi}_\perp,\end{aligned} \tag{56b}$$

$$\begin{aligned}0 = &\left(\frac{\partial}{\partial\tau} + \frac{3}{\tau}\right)g^\eta + \frac{c_s^2 i\kappa}{\tau^2}\delta e + \gamma_\eta\left(k_\perp^2 + \frac{\kappa^2}{\tau^2}\right)g^\eta\\ &+ \frac{1}{3\tau^2}\gamma_\eta\kappa(\vec{k}_\perp\cdot\vec{g}_\perp + \kappa g^\eta) + \xi^\eta,\end{aligned} \tag{56c}$$

where $(g_\perp^x, g_\perp^y, g^\eta) = (T^{\tau x}, T^{\tau y}, T^{\tau\eta})$. As in Secs. II A and II B the hydrodynamic parameters in these equations (such as γ_η) are constructed from the bare parameters, $e_0(\Lambda)$, $p_0(\Lambda)$, $\eta_0(\Lambda)$ and evolve according to ideal hydrodynamics, $e_0(\tau) = e_0(\tau_0)(\tau_0/\tau)^{1+c_s^2}$. We also neglected variation in viscosity $\delta\eta/\tau \ll \delta p, \delta e$, which is smaller by a factor $\epsilon = \eta_0/[(e_0 + p_0)c_s^2\tau] \ll 1$ for conformal fluid. Note also that the temporal noise component ξ^τ is smaller than $\xi^{i_\perp}$ and $\tau\xi^\eta$ by a factor $1/(k_*\tau) \sim \epsilon^{1/2}$ and the former can be neglected.

Following the procedure outlined in Sec. II we rewrite Eqs. (56) in a compact matrix notation. We define $\vec{G} = (G^{\hat{x}}, G^{\hat{y}}, G^{\hat{z}}) \equiv (\vec{g}_\perp, \tau g^\eta)$ and $\vec{K} = (K_{\hat{x}}, K_{\hat{y}}, K_{\hat{z}}) \equiv (\vec{k}_\perp, \kappa/\tau)$, so that equation of motion for $\phi_a \equiv (c_s\delta e, \vec{G})$ is

$$\partial_\tau\phi_a(\tau,\boldsymbol{k}) = -i\mathcal{L}_{ab}\phi_b - \mathcal{D}_{ab}\phi_b - \xi_a - \mathcal{P}_{ab}\phi_b, \tag{57}$$

[7] Here we are neglecting the longitudinal contribution, $\langle g_L g_L\rangle$, which decreases more rapidly than $1/\tau$ at late times.

$$\mathcal{L} = \begin{pmatrix} 0 & c_s\vec{K} \\ c_s\vec{K} & 0 \end{pmatrix},$$
$$\mathcal{D} = \gamma_\eta \begin{pmatrix} 0 & 0 \\ 0 & K^2\delta_{\hat{i}\hat{j}} + \frac{1}{3}K_{\hat{i}}K_{\hat{j}} \end{pmatrix}, \tag{58}$$
$$\mathcal{P} = \frac{1}{\tau}\begin{pmatrix} 1+c_s^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix}, \tag{59}$$

with noise correlator

$$\langle \xi_a(\tau,\boldsymbol{k})\xi_b(\tau',-\boldsymbol{k})\rangle = \frac{2T(e_0+p_0)}{\tau}\mathcal{D}_{ab}(2\pi)^3 \times \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta(\tau-\tau'). \tag{60}$$

Here $\delta^3(\boldsymbol{k}-\boldsymbol{k}') \equiv \delta^2(\vec{k}_\perp - \vec{k}'_\perp)\delta(\kappa-\kappa')$ and the factor of $1/\tau$ stems from the Jacobian of the coordinate system $\delta^4(x-x')/\sqrt{g(x)}$.

The kinetic equation for the two-point functions

$$\langle \phi_a(\tau,\boldsymbol{k})\phi_b(\tau,-\boldsymbol{k}')\rangle \equiv N_{ab}(\tau,\boldsymbol{k})(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}') \tag{61}$$

is obtained similarly to Sec. II

$$\partial_\tau N(\tau,\boldsymbol{k}) = -i[\mathcal{L},N] - \{\mathcal{D},N\} + \frac{2T(e_0+p_0)}{\tau}\mathcal{D} - \{\mathcal{P},N\}. \tag{62}$$

The eigenvectors of $\mathcal{L}$ are of the same form as before, Eq. (19),

$$(e_\pm)_a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm\hat{K} \end{pmatrix}, \quad (e_{T_1})_a = \begin{pmatrix} 0 \\ \vec{T}_1 \end{pmatrix}, \quad (e_{T_2})_a = \begin{pmatrix} 0 \\ \vec{T}_2 \end{pmatrix}. \tag{63}$$

However, now the wave number vector $\vec{K}$ is time dependent:

$$\hat{K} \equiv \frac{(\vec{k}_\perp,\kappa/\tau)}{\sqrt{k_\perp^2 + (\kappa/\tau)^2}} \tag{64a}$$
$$\equiv (\sin\theta_K\cos\varphi_K, \sin\theta_K\sin\varphi_K, \cos\theta_K). \tag{64b}$$

The azimuthal angle φ_K is independent of time due to the residual rotational symmetry of the background in the xy plane. Following the same arguments as in Sec. II, we arrive at the kinetic equations for diagonal components,

$$\partial_\tau N_{\pm\pm} = -\frac{4}{3}\gamma_\eta K^2\left[N_{\pm\pm} - \frac{T(e_0+p_0)}{\tau}\right] - \frac{1}{\tau}\left(2+c_s^2+\cos^2\theta_K\right)N_{\pm\pm}, \tag{65a}$$
$$\partial_\tau N_{T_1T_1} = -2\gamma_\eta K^2\left[N_{T_1T_1} - \frac{T(e_0+p_0)}{\tau}\right] - \frac{2}{\tau}N_{T_1T_1}, \tag{65b}$$
$$\partial_\tau N_{T_2T_2} = -2\gamma_\eta K^2\left[N_{T_2T_2} - \frac{T(e_0+p_0)}{\tau}\right] - \frac{2}{\tau}(1+\sin^2\theta_K)N_{T_2T_2}. \tag{65c}$$

The first terms on the right-hand side describe relaxation of N_{AA} toward local equilibrium $T(e_0+p_0)/\tau$, and the second terms drive N_{AA} out of equilibrium through the interaction with the background flow.

We derived these equations relying on the scale separation given in Eq. (12). The off-diagonal components between gapped modes (such as between the $\pm$ and T_1 and T_2 modes) are ignored because they rapidly rotate, as discussed in Sec. II A. Note that the transverse mode ϕ_{T_1} is so chosen that it does not mix with the other modes. This is possible because of the residual rotational symmetry in the xy plane in the Bjorken expansion. Therefore, the kinetic equation for $N_{T_1T_1}$, Eq. (65b), holds without the scale separation in Eq. (12) and is applicable for all wave numbers k from to zero to $1/\ell_{\rm mfp}$.

C. Nonlinear fluctuations in the energy momentum tensor

Now let us investigate the solution of the kinetic equations close to the cutoff and isolate the ultraviolet divergent contribution. Solving Eq. (65) in series of $1/(\gamma_\eta K^2\tau)$ we obtain an asymptotic solution for large K/k_*

$$\frac{N_{\pm\pm}(\tau,\boldsymbol{k})}{T(e_0+p_0)/\tau} = 1 + \frac{c_s^2 - \cos^2\theta_K}{\frac{4}{3}\gamma_\eta K^2\tau} + \cdots, \tag{66a}$$
$$\frac{N_{T_1T_1}(\tau,\boldsymbol{k})}{T(e_0+p_0)/\tau} = 1 + \frac{c_s^2}{\gamma_\eta K^2\tau} + \cdots, \tag{66b}$$
$$\frac{N_{T_2T_2}(\tau,\boldsymbol{k})}{T(e_0+p_0)/\tau} = 1 + \frac{c_s^2 - \sin^2\theta_K}{\gamma_\eta K^2\tau} + \cdots, \tag{66c}$$

where we used $\partial_\tau[T(e_0+p_0)] \simeq -(1+2c_s^2)[T(e_0+p_0)]/\tau$, which is adequate for the desired accuracy of the present analysis. For a given $K^2\gamma_\eta\tau = (K/k_*)^2$ and θ_K at final time τ, we can solve Eq. (65) numerically and find a steady-state solution at late time $\tau \gg \tau_0$. We compare this steady-state solution to the asymptotic form Eq. (66) in Fig. 2.

Equation (66) is analogous to the ideal and first viscous correction to the thermal distribution function, $f_0 + \delta f$, which are used in heavy ion phenomenology and in determining the shear viscosity [21]. At large K/k_* the distribution N_{AA} attains its equilibrium value, $T(e_0+p_0)/\tau$, up to viscous corrections of order τ_R/τ, where τ_R is a typical relaxation time for a mode of momentum K, $\tau_R \sim 1/\gamma_\eta K^2$.

The energy-momentum tensor averaged over fluctuations is given by

$$\langle T^{\tau\tau}\rangle = e_0 + \frac{\langle \vec{G}^2\rangle}{e_0+p_0}, \tag{67a}$$
$$\langle T^{xx}\rangle = p_0 + \frac{2\eta_0}{3\tau} + \frac{\langle (G^{\hat{x}})^2\rangle}{e_0+p_0}, \tag{67b}$$
$$\langle T^{yy}\rangle = p_0 + \frac{2\eta_0}{3\tau} + \frac{\langle (G^{\hat{y}})^2\rangle}{e_0+p_0}, \tag{67c}$$
$$\langle \tau^2 T^{\eta\eta}\rangle = p_0 - \frac{4\eta_0}{3\tau} + \frac{\langle (G^{\hat{z}})^2\rangle}{e_0+p_0}. \tag{67d}$$

Calculating $N_{ab}(\tau,\boldsymbol{k}) = [(e_A)_a N_{AB}(e_B)_b]$ from the kinetic theory, we determine $\langle\phi_a(\tau,\boldsymbol{k})\phi_b(\tau,-\boldsymbol{k})\rangle$ with $\phi_a = (c_s\delta e, \vec{G})$

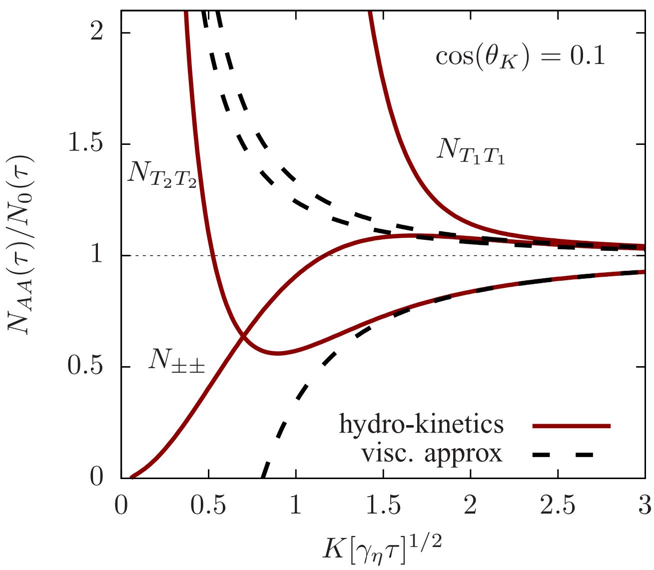

FIG. 2. Steady-state solutions of Eq. (65) for the two-point energy-momentum correlation functions during a Bjorken expansion at late times, $\tau \gg \tau_0$. The correlations are plotted as a function of $K[\gamma_\eta \tau]^{1/2}$ for final time angle $\cos\theta_K = 0.1$. For comparison, leading-order viscous solutions in $1/(\gamma_\eta K^2 \tau)$ are also shown, Eq. (66). The differences of the steady-state solutions from their asymptotic forms induce finite corrections to energy-momentum tensor, Eq. (74).

in Fourier space, yielding

$$\langle \phi_a(x)\phi_b(x)\rangle = \int \frac{d^2k_\perp d\kappa}{(2\pi)^3} N_{ab}(\tau,\boldsymbol{k}),$$
$$= \tau \int \frac{K^2 dK d\cos\theta_K d\varphi_K}{(2\pi)^3}(e_A)_a N_{AB}(\tau,\boldsymbol{k})(e_B)_b. \quad (68)$$

Note that the momentum integral is done in final time variables, $\vec{K}(\tau)$. As shown below, the integration in Fourier space is divergent in the ultraviolet. Therefore, we regulate the integral by introducing a cutoff at $|\vec{K}| \sim \Lambda$. In turn, the background quantities such as e_0 and η_0 must be renormalized and depend on Λ so that the total result is independent of Λ. The choice of Λ is arbitrary as long as $k_* \ll \Lambda \ll 1/\ell_{\rm mfp}$ so that the nonlinear contribution with $|\vec{K}| \sim \Lambda$ is independent of the background flow.

The integration in Eq. (68) includes the soft fluctuations for which the kinetic equation may not be applicable. However, this contribution is suppressed by phase space and the kinetic result can be extrapolated into this regime with negligible errors.

Combining Eqs. (66), (67), and (68), the energy momentum tensor is obtained as

$$\langle T^{\tau\tau}\rangle = e_0 + 3T\int_0^\Lambda \frac{K^2 dK}{2\pi^2} + \Delta T^{\tau\tau}, \quad (69a)$$

$$\langle T^{xx}\rangle = p_0 + \frac{2\eta_0}{3\tau} + T\int_0^\Lambda \frac{K^2 dK}{2\pi^2} + \frac{17}{90}\frac{T(e_0+p_0)}{\eta_0\tau}\int_0^\Lambda \frac{dK}{2\pi^2} + \Delta T^{xx}, \quad (69b)$$

$$\langle T^{yy}\rangle = p_0 + \frac{2\eta_0}{3\tau} + T\int_0^\Lambda \frac{K^2 dK}{2\pi^2} + \frac{17}{90}\frac{T(e_0+p_0)}{\eta_0\tau}\int_0^\Lambda \frac{dK}{2\pi^2} + \Delta T^{yy}, \quad (69c)$$

$$\langle \tau^2 T^{\eta\eta}\rangle = p_0 - \frac{4\eta_0}{3\tau} + T\int_0^\Lambda \frac{K^2 dK}{2\pi^2} - \frac{17}{45}\frac{T(e_0+p_0)}{\eta_0\tau}\int_0^\Lambda \frac{dK}{2\pi^2} + \tau^2\Delta T^{\eta\eta}, \quad (69d)$$

where the finite contributions $\Delta T^{\tau\tau}$, ΔT^{xx}, ΔT^{yy}, and $\tau^2\Delta T^{\eta\eta}$ are discussed in the next section. By comparing terms with the same explicit τ dependence, the ultraviolet divergences are absorbed into the renormalized hydrodynamic variables

$$e = e_0(\Lambda) + \frac{T\Lambda^3}{2\pi^2}, \quad (70a)$$

$$p = p_0(\Lambda) + \frac{T\Lambda^3}{6\pi^2}, \quad (70b)$$

$$\eta = \eta_0(\Lambda) + \frac{17\Lambda}{120\pi^2}\frac{T[e_0(\Lambda)+p_0(\Lambda)]}{\eta_0(\Lambda)}. \quad (70c)$$

Note that we do not assign a cutoff dependence to the temperature. The coefficients of the cubic and linear renormalizations of the pressure and shear viscosity are independent of the background expansion and match the static fluid results of Sec. II B. Here e, p, and η are physical quantities at a given temperature T in an infinite volume. Using the physical quantities, the energy-momentum tensor is given as

$$\langle T^{\tau\tau}(\tau)\rangle = e + \Delta T^{\tau\tau}, \quad (71a)$$

$$\langle T^{xx}(\tau)\rangle = p + \frac{2\eta}{3\tau} + \Delta T^{xx}, \quad (71b)$$

$$\langle T^{yy}(\tau)\rangle = p + \frac{2\eta}{3\tau} + \Delta T^{yy}, \quad (71c)$$

$$\langle \tau^2 T^{\eta\eta}(\tau)\rangle = p - \frac{4\eta}{3\tau} + \tau^2\Delta T^{\eta\eta}. \quad (71d)$$

If the two-point functions of the fluctuations were completely determined by the first two terms in Eq. (66), their contributions would be completely absorbed by the renormalization of the background flow parameters such as $p_0(\Lambda)$ and $\eta_0(\Lambda)$. However, the kinetic equations yield residual contributions, since the full solution deviates from its asymptotic form for $K \sim k_*$ as seen from Fig. 2. The purpose of hydrodynamics with noise is to capture this contribution.

Physically, the parameters $e_0(\Lambda)$, $p_0(\Lambda)$, and $\eta_0(\Lambda)$ in fluctuating hydrodynamics reflect the equilibrium properties of modes above a cutoff Λ, which have been already integrated out. Equivalently, these parameters are determined by modes contained in a cell of size $a \sim 2\pi/\Lambda$. For example, $p_0(\Lambda)$ is the partial pressure from equilibrated modes above the cutoff (inside a cell), while the partial pressure from the modes below the cutoff (larger than a cell size) is determined dynamically with fluctuating hydrodynamics. The second terms on the right-hand sides of Eq. (70) are the contributions to each quantity from the modes below the cutoff, when all of these long-wavelength modes are in perfect equilibrium in infinite volume.

D. Out-of-equilibrium noise contributions to energy momentum tensor

In this section we determine the residual contributions to the energy momentum tensor, $\Delta T^{\mu\nu}$, in Eq. (71) after the hydrodynamic parameters have been renormalized. We evaluate the precise numerical factors of the long-time tail terms for a Bjorken expansion (which is the main result of this paper) and identify additional contributions from the noise at early times. The mathematical procedure is somewhat involved, so here we outline the calculation and present results, delegating the technical details to the Appendix.

To find the full out-of-equilibrium correlators, we need to solve Eq. (65), which can be written in the following general form,

$$\partial_\tau N_{AA}(\tau,\boldsymbol{k}) = f(\tau,\boldsymbol{k})N_{AA}(\tau,\boldsymbol{k}) + g(\tau,\boldsymbol{k}), \tag{72}$$

where $f(\tau,\boldsymbol{k})$ has contributions from both the dissipative and external forcing terms, and $g(\tau,\boldsymbol{k})$ is the inhomogeneous term coming from the equilibrium correlation functions. A formal solution of Eq. (72) is given by

$$N_{AA}(\tau,\boldsymbol{k}) = N_{AA}(\tau_0,\boldsymbol{k})e^{\int_{\tau_0}^{\tau} d\tau' f(\tau',\boldsymbol{k})} + \int_{\tau_0}^{\tau} d\tau'' g(\tau'',\boldsymbol{k}) e^{\int_{\tau_0}^{\tau''} d\tau' f(\tau',\boldsymbol{k})}. \tag{73}$$

The first term describes the evolution of the initial correlation density matrix $N_{AA}(\tau_0,\boldsymbol{k})$ to final time τ. The second term in Eq. (73) is the contribution from thermal fluctuations. As we will see, only the $N_{T_1T_1}$ contribution is sensitive to the initial conditions and the thermal fluctuations at early times. For the T_1T_1 correlator we will take the initial conditions described by Eq. (52) in Sec. III A, $\tau_0^2 N_{T_1,T_1}(\tau_0,\boldsymbol{k}) = \chi^{gg}_{\tau_0} 2\pi\delta(\kappa)$.

Substituting the formal solution for N_{AA} in Eqs. (67) and (68), we can determine the stress tensor at time $\tau \gg \tau_0$. The integral $\int d^3k$ in Eq. (68) diverges, but after subtracting Λ^3 and Λ divergences discussed in the previous section, we finally obtain the finite correction to energy momentum tensor $\Delta T^{\mu\nu}$. Writing Eq. (71) in full gives

$$\frac{\langle\langle \tau^2 T^{\eta\eta}(\tau)\rangle\rangle}{e+p} = \frac{p}{e+p} - \frac{4\gamma_\eta}{3\tau} + \frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}, \tag{74a}$$

$$\frac{\langle\langle T^{xx}(\tau)\rangle\rangle}{e+p} = \frac{p}{e+p} + \frac{2\gamma_\eta}{3\tau} + \left[\frac{\chi^{gg}_{\tau_0}+\delta\chi^{gg}_{\tau_0}}{\tau^2(e+p)^2}\right]\frac{1}{(12\pi\gamma_\eta\tau)} - \frac{0.273836}{s\,(4\pi\gamma_\eta\tau)^{3/2}}, \tag{74b}$$

$$\langle\langle T^{yy}\rangle\rangle = \langle\langle T^{xx}\rangle\rangle, \tag{74c}$$

$$\langle\langle T^{\tau\tau}\rangle\rangle = \langle\langle T^{xx}\rangle\rangle + \langle\langle T^{yy}\rangle\rangle + \langle\langle \tau^2 T^{\eta\eta}\rangle\rangle. \tag{74d}$$

The coefficients 1.08318 and −0.273836 of the long-time tails, $1/(\gamma_\eta\tau)^{3/2}$, are obtained by numerical integration as explained in the Appendix. The term $\chi^{gg}_{\tau_0} + \delta\chi^{gg}_{\tau_0}$ records the initial variance in transverse momentum in a given rapidity slice [see Eqs. (46) and (48)] together with the thermal contribution

$$\chi^{gg}_{\tau_0} + \delta\chi^{gg}_{\tau_0} = \left\langle \frac{1}{\mathcal{A}}\left(\frac{dp^x}{d\eta}\right)^2\right\rangle_{\tau_0} + \left(\frac{T(e+p)\tau_0}{\sqrt{12\pi\gamma_\eta/\tau_0}}\right)_{\tau_0}, \tag{75}$$

where the brackets $(\cdots)_{\tau_0}$ indicate that all contained quantities are to be evaluated at the initial time, τ_0. We will provide an intuitive discussion of the result in the next section.

E. Qualitative discussion of Eq. (74)

1. Long-time tails: $1/(\gamma_\eta\tau)^{3/2}$

Examining Eq. (74), we see two groups of terms. The first group is proportional to $1/(\gamma_\eta\tau)^{3/2}$ and is independent of initial conditions. By contrast, the second group is proportional to $1/(\gamma_\eta\tau)$ and depends on the initial transverse momentum fluctuations through the parameter $\chi^{gg}_{\tau_0} + \delta\chi^{gg}_{\tau_0}$ (see Sec. III A). We will first describe the terms proportional to the fractional power $1/(\gamma_\eta\tau)^{3/2}$, known as the long-time tails.

Squared fluctuations in equilibrium are of order $\langle\delta e(\vec{x})\delta e(\vec{y})\rangle_{\rm eq}/e^2 \sim \langle v^i(\vec{x})v^j(\vec{y})\rangle_{\rm eq} \sim s^{-1}\delta(\vec{x}-\vec{y})$, where s is the entropy density [see Eq. (4)]. Thus a fluctuation with wave number k is suppressed by $\sqrt{k^3/s}$. The suppression factor k^3/s is roughly the inverse of the degrees of freedom inside a box of volume $\Delta V \sim (1/k)^3$, which must be a huge number for local thermodynamics to apply. This is why the linear analysis of the hydrodynamic fluctuations is justified.

The energy momentum tensor in viscous hydrodynamics is expanded in powers of gradients, leading to corrections in powers of $\epsilon \equiv \eta/(e+p)\tau \ll 1$. In addition, as discussed in Sec. I B, the fluctuations with wave number of order $|\vec{K}| \sim k_* \sim 1/(\gamma_\eta\tau)^{1/2}$ dominate the nonlinear noise correction to the stress tensor, which is suppressed by $s\Delta V \equiv s/k_*^3 \gg 1$. This correction to the longitudinal pressure reflects the equipartition of energy, with $\frac{1}{2}T$ of energy per mode, and the number of nonequilibrium modes per volume $\sim k_*^3$. To summarize, the reasoning in this paragraph leads to the following parametric estimate for the longitudinal stress:

$$\frac{\langle\tau^2 T^{\eta\eta}\rangle}{e+p} \sim \left[\frac{1}{4} + \frac{\eta}{(e+p)\tau} + \frac{1}{s(\gamma_\eta\tau)^{3/2}} + \cdots\right], \tag{76}$$

which is reflected by Eq. (74).

2. Transverse momentum diffusion in rapidity: $1/\gamma_\eta\tau$

Additional corrections to the stress in Eq. (74) decrease as $1/\gamma_\eta\tau$, in contrast to the long-time tails. As described in Sec. III A, long-range (in rapidity) initial transverse momentum fluctuations correct the mean transverse pressures, T^{xx} and T^{yy}, by a term proportional to $\chi^{gg}_{\tau_0}/\gamma_\eta\tau$ [see Eq. (74b)]. Hydrodynamic noise in the initial state adjusts this correction by adding to the long-range fluctuations of transverse momentum [see Eqs. (74b) and (75)]. The goal of this section is to explain this process qualitatively and to quantitatively explain the adjustment, $\chi^{gg}_{\tau_0} \to \chi^{gg}_{\tau_0} + \delta\chi^{gg}_{\tau_0}$.

Formally, the $N_{T_1T_1}$ correlation function is sensitive to the noise at the initial time τ_0, which arises from a restricted region of $\vec{K}$-space integration, $k_\perp \sim k_*$ and $\kappa/\tau \sim k_*(\tau_0)(\tau_0/\tau) \sim k_*(\tau_0/\tau)^{1/3} \ll k_*$. In this region the longitudinal momentum

κ/τ_0 reflects the dissipative scale $k_*(\tau_0)$ at the initial time τ_0, while the transverse momenta reflect the dissipative scale at final time τ.

The dynamics in this phase space region is the following. During the initial moments, thermal fluctuations lead to a local fluctuation of transverse momentum in a given rapidity slice for each cell in the transverse plane:

$$\left\langle \left(\tau_0 \Delta g_\perp^x\right)^2 \right\rangle \sim \left[\frac{T\,(e+p)\,\tau_0}{\Delta\eta\,(\Delta x_\perp)^2}\right]_{\tau_0}. \tag{77}$$

Here (as before) the brackets $(\cdots)_{\tau_0}$ indicate that all contained quantities should be evaluated at τ_0. During an initial time of order τ_0, the momentum per rapidity diffuses to a finite longitudinal width [5] (see below)

$$\Delta\eta \to \sigma_\eta(\tau_0) \equiv \sqrt{6\gamma_\eta(\tau_0)/\tau_0}. \tag{78}$$

The process is diffusive because the transverse momentum per rapidity is conserved. The rapidity width is finite because the longitudinal expansion shuts off the diffusion process. $\sigma_\eta(\tau_0)$ is broader than the rapidity width of subsequent interest, which is of order $\sigma_\eta(\tau)$. Thus, after an initial transient, the transverse momentum per rapidity may be considered approximately constant in time and rapidity, though localized in this transverse plane

$$\left\langle \left(\tau \Delta g_\perp^x\right)^2 \right\rangle \sim \left[\frac{T\,(e+p)\,\tau_0}{\sqrt{\gamma_\eta/\tau_0}}\right]_{\tau_0} \frac{1}{(\Delta x_\perp)^2}. \tag{79}$$

At much later times these transverse momentum fluctuations diffuse transversely (as described in Sec. III A), leading to a correction of order

$$\frac{\langle T^{xx}\rangle}{e+p} \sim \frac{1}{\tau^2(e+p)^2}\left[\frac{T(e+p)\tau_0}{\sqrt{\gamma_\eta/\tau_0}}\right]_{\tau_0} \frac{1}{\gamma_\eta \tau}, \tag{80}$$

which qualitatively reproduces the correction in Eq. (74b).

Now we will briefly sketch this reasoning with equations. At the early time moments $\tau \sim \tau_0$, the wave vector is predominantly longitudinal $\vec{K} \simeq (\vec{0}_\perp, \kappa/\tau)$ and the transverse momentum correlator

$$\langle g_\perp^i(\tau,\boldsymbol{k}) g_\perp^j(\tau,-\boldsymbol{k}')\rangle \equiv N^{ij}(\boldsymbol{k},\tau)(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}') \tag{81}$$

can be reconstructed from $N_{T_1T_1}$ and $N_{T_2T_2}$

$$N^{ij}(\tau,\boldsymbol{k}) = \sum_{A\in T_1,T_2} e_A^i e_A^j N_{AA}(\tau,\boldsymbol{k}), \tag{82}$$

since $\vec{T}_1$ and $\vec{T}_2$ form a basis for the transverse plane. In this limit, the equations of motion for $N_{T_1T_1}$ and $N_{T_2T_2}$ [see Eqs. (65b) and (65c)] are the same, and N^{ij} satisfies a one-dimensional diffusion equation with a source at early times

$$\left[\partial_\tau + 2\gamma_\eta\left(\frac{\kappa}{\tau}\right)^2\right](\tau^2 N^{ij}) = 2\gamma_\eta\left(\frac{\kappa}{\tau}\right)^2 T(e+p)\tau\delta^{ij}. \tag{83}$$

The left-hand side of Eq. (83) represents the diffusion of transverse momentum in rapidity, while the right-hand side represents the thermal transverse momentum fluctuations at the earliest moments, which act as a source. The source for the fluctuations, $2T\eta\,(\kappa/\tau)^2$, is a rapidly decreasing function of time and is dominant for times of order τ_0.

The Green's function propagating data from τ' to τ for the left-hand side of Eq. (83) is

$$G^{ij}(\tau\eta\vec{x}_\perp|\tau'\eta'\vec{x}'_\perp) = \frac{e^{-(\eta-\eta')^2/(12\gamma_\eta(\tau')/\tau')}}{\sqrt{12\pi\gamma_\eta(\tau')/\tau'}}\delta^{ij}\delta^2(\vec{x}_\perp - \vec{x}'_\perp), \tag{84}$$

for $\tau \gg \tau'$. Thus, a fluctuation localized in rapidity at time τ_0 will diffuse to a finite rapidity width of $\sigma_\eta(\tau_0) = \sqrt{6\gamma_\eta(\tau_0)/\tau_0}$ at late times[8] [5,35]. This is a small rapidity width in absolute units (since $\gamma_\eta(\tau_0)/\tau_0 \ll 1$ when hydrodynamics is a good approximation), but much broader than the rapidity width of interest at the final time, $\gamma_\eta(\tau_0)/\tau_0 \gg \gamma_\eta(\tau)/\tau$.

Returning to Eq. (83), we solve the equation and determine the transverse momentum correlation function (in the same rapidity slice) at an intermediate time τ', which is large compared to τ_0 but much much less than the final time τ, $\tau_0 \ll \tau' \ll \tau$

$$\tau'^2\left\langle g_\perp^i(\tau',\eta,\vec{x}_\perp) g_\perp^j(\tau',\eta,\vec{y}_\perp)\right\rangle = \int \frac{d\kappa\, d^2k_\perp}{(2\pi)^3} e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{y}_\perp)}\,\tau'^2 N^{ij}(\tau',\kappa). \tag{86}$$

Implementing these steps, we find

$$\tau'^2\left\langle g_\perp^i(\tau',\eta,\vec{x}_\perp) g_\perp^j(\tau',\eta,\vec{y}_\perp)\right\rangle = \left[\frac{T(e+p)\tau_0}{\sqrt{12\pi\gamma_\eta/\tau_0}}\right]_{\tau_0}\delta^{ij}\delta^2(\vec{x}_\perp - \vec{y}_\perp). \tag{87}$$

This has the same form as the initial conditions described in Sec. III A, and fluctuations at the earliest moments simply increase the variance of long-range transverse momentum fluctuations by a constant amount,

$$\delta\chi_{\tau_0}^{gg} = \left[\frac{T(e+p)\tau_0}{\sqrt{12\pi\gamma_\eta/\tau_0}}\right]_{\tau_0}, \tag{88}$$

reproducing Eq. (75). In a sense, this constant shift simply finalizes the thermalization process described at the start of Sec. III A. The correction $\delta\chi_{\tau_0}^{gg}$ scales as $\tau_0^{-1/3}$ and is therefore small compared to the first term in Eq. (75) if τ_0 is large compared to a typical thermalization time.

IV. RESULTS AND DISCUSSION

In this paper we determined a set of kinetic equations which describe the evolution of hydrodynamic fluctuations during

[8] In Refs. [5,35] the authors consider an initial distribution which is Gaussian in rapidity of width σ_0. During the expansion, the width is broadened by the diffusion process

$$\sigma_0^2 \to \sigma_0^2 + 6\frac{\gamma_\eta(\tau_0)}{\tau_0}. \tag{85}$$

These authors considered constant $\eta/(e+p)$ and found a factor of 4 rather than 6 in Eq. (85).

a Bjorken expansion. We used these equations to find the first fractional power correction to the longitudinal pressure, $\propto 1/(\tau T)^{3/2}$, at late times. The evolution equations can be extended to much more general flows and ultimately coupled to existing hydrodynamic codes.

The kinetic equations for hydrodynamic fluctuations are a WKB (or rotating wave) type approximation of the full stochastic hydrodynamic evolution equations. This approximation is justified because the relevant hydrodynamic modes have wave numbers of order

$$k_* \sim \sqrt{\frac{e+p}{\eta\tau}}, \tag{89}$$

which is large compared to the inverse expansion rate, $1/\tau$. For example, the kinetic equation for the sound mode with wave number $\vec{K} = (\vec{k}_\perp, \kappa/\tau)$ interacting with the Bjorken background takes the form of a relaxation-type equation

$$\partial_\tau N_{++}(\tau,\boldsymbol{k}) = -\frac{4}{3}\gamma_\eta K^2\left[N_{++} - \frac{T(e_0+p_0)}{\tau}\right] - \frac{1}{\tau}\left(2 + c_s^2 + \cos\theta_K\right)N_{++}. \tag{90}$$

$N_{++}(\tau,\boldsymbol{k})$ are short wavelength (symmetrized) two-point functions of conserved stress tensor components, $\phi_+ \equiv (c_s\delta e + \hat{K}\cdot\vec{G})/\sqrt{2}$ in an evolving Bjorken hydrodynamic background [see Sec. III and Eq. (65) for the remaining modes]. At high wave numbers $K \gg k_*$, the distribution function N_{++} reaches its equilibrium form $T(e_0+p_0)/\tau$, up to first viscous corrections which may be found by solving Eq. (90) order by order at large K/k_* [see Eq. (66a)]. This asymptotic form is responsible for the renormalization of the pressure and shear viscosity. For wave numbers of order k_* the hydrodynamic fluctuations are not in equilibrium at all, but reach a nonequilibrium steady state at late times. A graph of this nonequilibrium steady state is given in Fig. 2.

The deviation of hydrodynamic fluctuations from equilibrium has consequences for the evolution of the system. Indeed, the longitudinal pressure $\tau^2 T^{\eta\eta}$ receives a correction from the unequilibrated modes

$$\frac{\langle\tau^2 T^{\eta\eta}\rangle}{e+p} = \left[\frac{p}{e+p} - \frac{4}{3}\frac{\gamma_\eta}{\tau} + \frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}} + \frac{(\lambda_1-\eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}\right], \tag{91}$$

where we have repeated Eq. (74a) for convenience. The correction to the pressure $\sim T/(\gamma_\eta\tau)^{3/2}$ is of order $\sim T k_*^3$, reflecting the number of modes of order k_* and the energy per mode, $\frac{1}{2}T$. In contrast to all previous analyses of long-time tails [23,28], the hydrodynamic fluctuations in the expanding case are not close to equilibrium, and a one-loop expansion around equilibrium is not an appropriate approximation scheme. Our kinetic description effectively resums all diagrams contributing at the same order in the presence of expansion [34].

Formally, the noise correction is lower order than the correction due to second-order hydrodynamics, which is proportional to a particular combination of second order parameters, $\lambda_1 - \eta\tau_\pi$. To quantify the importance of thermal fluctuations in practice, we take representative numbers for the entropy from the lattice [36,37], estimates for the second-order hydrodynamic coefficients based on weakly and strongly coupled plasmas [4,31,38], and an estimate for τT at $\tau \sim 3.5$ fm based on hydrodynamic simulations[9]

$$\frac{T^3}{s} \simeq \frac{1}{13.5}, \tag{92a}$$

$$\frac{(\lambda_1-\eta\tau_\pi)}{e+p} \simeq -0.8\left(\frac{\eta}{e+p}\right)^2, \tag{92b}$$

$$\tau T \simeq 4.5. \tag{92c}$$

Then, for $\eta/s \simeq 1/4\pi$, Eq. (91) evaluates to

$$\frac{\langle\tau^2 T^{\eta\eta}\rangle}{e+p} = \frac{1}{4}\left[1. - 0.092\left(\frac{4.5}{\tau T}\right) + 0.034\left(\frac{4.5}{\tau T}\right)^{3/2} - 0.00085\left(\frac{4.5}{\tau T}\right)^2\right], \tag{93}$$

while for $\eta/s = 2/4\pi$, we find

$$\frac{\langle\tau^2 T^{\eta\eta}\rangle}{e+p} = \frac{1}{4}\left[1. - 0.185\left(\frac{4.5}{\tau T}\right) + 0.013\left(\frac{4.5}{\tau T}\right)^{3/2} - 0.0034\left(\frac{4.5}{\tau T}\right)^2\right]. \tag{94}$$

For the smaller shear viscosity, Eq. (93), the nonlinear noise contribution completely dominates over the second-order hydro contribution. For the larger shear viscosity, Eq. (94), the noise remains three times larger than second-order hydro, but this contribution is only $\sim$10% of the first-order viscous term. Finally, for $\eta/s \sim 3/4\pi$ the noise and second-order hydro contributions become comparable.

The evolution of the average energy density of the system obeys

$$\frac{d\langle\langle T^{\tau\tau}\rangle\rangle}{d\tau} = -\frac{\langle\langle T^{\tau\tau}\rangle\rangle + \langle\langle \tau^2 T^{\eta\eta}\rangle\rangle}{\tau}, \tag{95}$$

where the double brackets notate an average over (long range in rapidity) initial conditions and thermal noise.[10] To close the system of equations, the relationship between average energy density $\langle\langle T^{\tau\tau}\rangle\rangle$ and the average rest frame energy density $e(\tau)$ must be specified, and this relation is given in Eq. (74). $T^{\tau\tau}$, T^{xx}, and T^{yy} are sensitive to hydrodynamic noise at the earliest moments in addition to the long-time tails. In these cases thermal noise in the initial state adds to the long-range rapidity correlation functions of transverse momentum, which

[9]We take an estimate for the (approximately constant) average entropy in the transverse plane from a recent LHC simulation for PbPb collisions at $\sqrt{s} = 2.76$ TeV/nucleon, $\langle\tau_o s(\tau_o)\rangle \simeq 4.0\,\mathrm{GeV}^2$ [39]. We take a time of $\tau \sim 3.5$ fm (which is the time at which the elliptic flow develops [21]), where $T \simeq 250$ MeV.

[10]The longitudinal pressure in Eq. (91) is independent of fluctuations in the initial conditions at late times. Thus, only the average over the noise is relevant in this case, $\langle\langle\tau^2 T^{\eta\eta}\rangle\rangle = \langle\tau^2 T^{\eta\eta}\rangle$.

are already present without noise. This result is encapsulated by Eq. (75) and is discussed in Secs. III A and III E 2.

Although the analysis of hydrodynamic fluctuations in this paper was limited to conformal neutral fluids and a Bjorken expansion, the techniques developed here can be applied to much more general flows. A next step is to generalize the kinetic equations in Eq. (65) to an arbitrary expansion and to couple such generalized equations to existing second-order hydrodynamic codes. In addition, it will be phenomenologically important to extend this work to nonconformal systems with net baryon number. Near the QCD critical point the noise will continue to grow without bound, leading to a critical renormalization of the bulk viscosity. In an expanding system these fluctuations will not be fully equilibrated. We believe the formalism set up in this paper provides the first steps towards quantitatively analyzing this rich dynamical regime.

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APPENDIX: COMPUTATION OF FINITE RESIDUAL CONTRIBUTIONS

In this Appendix we provide the details of the computation sketched in Sec. III D for the residual out-of-equilibrium noise contribution to the energy momentum tensor for a Bjorken background. Let us scale the correlation density matrix by the equilibrium value:

$$R_{AA}(\tau,\boldsymbol{k}) \equiv \frac{N_{AA}(\tau,\boldsymbol{k})}{T(e_0+p_0)/\tau}. \tag{A1}$$

The kinetic equations of motion Eq. (65) written for relative density matrix R_{AA} are

$$\partial_\tau R_{\pm\pm} = -\frac{4}{3}\gamma_\eta K^2(R_{\pm\pm}-1) + \frac{c_s^2-\cos^2\theta_K}{\tau}R_{\pm\pm}, \tag{A2}$$

$$\partial_\tau R_{T_1T_1} = -2\gamma_\eta K^2\left(R_{T_1T_1}-1\right) + \frac{2c_s^2}{\tau}R_{T_1T_1}, \tag{A3}$$

$$\partial_\tau R_{T_2T_2} = -2\gamma_\eta K^2\left(R_{T_2T_2}-1\right) + \frac{2\left(c_s^2-\sin^2\theta_K\right)}{\tau}R_{T_2T_2}. \tag{A4}$$

Using dimensionless variables $t \equiv \tau'/\tau$ and $\vec{r} \equiv \vec{K}/k_*$ with $\vec{K} = (\vec{k}_\perp, \kappa/\tau)$ and $k_* = 1/(\gamma_\eta\tau)^{1/2}$ defined at τ, the Green's functions for the homogeneous parts are

$$G_{\pm\pm}(\tau',\tau;\boldsymbol{k}) = \frac{1}{t^{c_s^2}}\frac{1}{\sqrt{A(t,\theta_K)}}\exp\left[-\frac{4}{3}r^2B(t,\theta_K)\right], \tag{A5}$$

$$G_{T_1T_1}(\tau',\tau;\boldsymbol{k}) = \frac{1}{t^{2c_s^2}}\exp[-2r^2B(t,\theta_K)], \tag{A6}$$

$$G_{T_2T_2}(\tau',\tau;\boldsymbol{k}) = t^{2-2c_s^2}A(t,\theta_K)\exp[-2r^2B(t,\theta_K)], \tag{A7}$$

where

$$A(t,\theta_K) \equiv \sin^2\theta_K + \frac{\cos^2\theta_K}{t^2}, \tag{A8}$$

$$B(t,\theta_K) \equiv \frac{\sin^2\theta_K}{1+c_s^2}\left(1-t^{1+c_s^2}\right) + \frac{\cos^2\theta_K}{1-c_s^2}\left(\frac{1}{t^{1-c_s^2}}-1\right). \tag{A9}$$

With these Green's functions, R_{AA} due to thermal fluctuations (in contrast to initial fluctuations discussed in Sec. III A) is given by

$$R_{++}(\tau,\boldsymbol{k}) = \int_{\tau_0}^{\tau} d\tau' \frac{4}{3}\gamma_\eta(\tau')\left(k_\perp^2 + \frac{\kappa^2}{\tau'^2}\right)G_{++}(\tau',\tau;\boldsymbol{k}), \tag{A10}$$

and similarly for the other modes (change 4/3 to 2 for the transverse modes). Since the asymptotic solution of R_{AA} for large K is known, we define the remainder of R_{AA} as

$$R^{(r)}_{\pm\pm}(\tau,\boldsymbol{k}) \equiv R_{\pm\pm}(\tau,\boldsymbol{k}) - \left(1 + \frac{c_s^2-\cos^2\theta_K}{\frac{4}{3}\gamma_\eta K^2\tau}\right), \tag{A11}$$

$$R^{(r)}_{T_1T_1}(\tau,\boldsymbol{k}) \equiv R_{T_1T_1}(\tau,\boldsymbol{k}) - \left(1 + \frac{c_s^2}{\gamma_\eta K^2\tau}\right), \tag{A12}$$

$$R^{(r)}_{T_2T_2}(\tau,\boldsymbol{k}) \equiv R_{T_2T_2}(\tau,\boldsymbol{k}) - \left(1 + \frac{c_s^2-\sin^2\theta_K}{\gamma_\eta K^2\tau}\right). \tag{A13}$$

Using $R^{(r)}_{AA}$ the residual contribution to the energy-momentum tensor is calculated from Eq. (67a) and Eq. (68) as

$$\Delta T^{xx} = T\int\frac{d^3K}{(2\pi)^3}\begin{bmatrix}\frac{R^{(r)}_{++}+R^{(r)}_{--}}{2}\sin^2\theta_K\cos^2\varphi_K\\ +R^{(r)}_{T_1T_1}\sin^2\varphi_K\\ +R^{(r)}_{T_2T_2}\cos^2\theta_K\cos^2\varphi_K\end{bmatrix}, \tag{A14}$$

$$\Delta T^{yy} = T\int\frac{d^3K}{(2\pi)^3}\begin{bmatrix}\frac{R^{(r)}_{++}+R^{(r)}_{--}}{2}\sin^2\theta_K\sin^2\varphi_K\\ +R^{(r)}_{T_1T_1}\cos^2\varphi_K\\ +R^{(r)}_{T_2T_2}\cos^2\theta_K\sin^2\varphi_K\end{bmatrix}, \tag{A15}$$

$$\tau^2\Delta T^{\eta\eta} = T\int\frac{d^3K}{(2\pi)^3}\begin{bmatrix}\frac{R^{(r)}_{++}+R^{(r)}_{--}}{2}\cos^2\theta_K\\ +R^{(r)}_{T_2T_2}\sin^2\theta_K\end{bmatrix}, \tag{A16}$$

$$\Delta T^{\tau\tau} = \Delta T^{xx} + \Delta T^{yy} + \tau^2\Delta T^{\eta\eta}. \tag{A17}$$

Substituting the subtracted solution $R^{(r)}_{AA}$ into (A14)–(A17) and performing r integration with a Gaussian cutoff $\exp[-r^2k_*^2/\Lambda^2]$, we get

$$\frac{[\tau^2\Delta T^{\eta\eta}(\tau)]}{T(\tau)k_*^3} = \frac{3\sqrt{\pi}}{8}\int_{-1}^{1}\frac{d(\cos\theta_K)}{4\pi^2}\int_{\tau_0/\tau\to 0}^{1}dt\left\{\frac{\frac{4}{3}\cos^2\theta_K\sqrt{A(t,\theta_K)}}{\left[\frac{4}{3}B(t,\theta_K)+k_*^2/\Lambda^2\right]^{5/2}} + \frac{2t^{2-c_{s0}^2}\sin^2\theta_K A(t,\theta_K)^2}{\left[2B(t,\theta_K)+k_*^2/\Lambda^2\right]^{5/2}}\right\} - [\mathcal{O}(\Lambda^3)+\mathcal{O}(\Lambda)], \tag{A18}$$

$$\frac{[\Delta T^{xx}(\tau)+\Delta T^{yy}(\tau)]}{T(\tau)k_*^3} = \frac{3\sqrt{\pi}}{8}\int_{-1}^{1}\frac{d(\cos\theta_K)}{4\pi^2}\int_{\tau_0/\tau}^{1}dt\left\{\frac{\frac{4}{3}\sin^2\theta_K\sqrt{A(t,\theta_K)}}{\left[\frac{4}{3}B(t,\theta_K)+k_*^2/\Lambda^2\right]^{5/2}} + \frac{2t^{2-c_s^2}\cos^2\theta_K A(t,\theta_K)^2+2t^{-c_s^2}A(t,\theta_K)}{[2B(t,\theta_K)+k_*^2/\Lambda^2]^{5/2}}\right\} - [\mathcal{O}(\Lambda^3)+\mathcal{O}(\Lambda)]. \quad \text{(A19)}$$

The ultraviolet divergent terms $\mathcal{O}(\Lambda^3,\Lambda)$ are from the asymptotic form of R_{AA} at large K in (A11)–(A13). Near $t=1$, $B(t,\theta_K)\simeq 1-t$ and the cutoff Λ regulates the divergence in time integral. To isolate the divergences, we perform the partial integration twice and pick up cubic and linear divergences from the surface terms at $t=1$. The resultant divergences are precisely canceled by $\mathcal{O}(\Lambda^3,\Lambda)$ terms.

After subtracting the ultraviolet divergences at $t=1$ and doing $\cos\theta_K$ integral analytically, the remaining time integration has to be done numerically. $R^{(r)}_{T_1T_1}$ mode contribution to T^{xx} and T^{yy} is divergent in the limit $\tau\gg\tau_0$. Since the analytic behavior of the integrand around $t\sim 0$ is known, we can explicitly subtract the part sensitive to early times from the integrand to extract remaining finite pieces for $R^{(r)}_{T_1T_1}$ mode. Numerical integration results necessary to find finite stress tensor corrections in Eqs. (A14)–(A16) are summarized in Table I. Summing contributions from the different modes to the longitudinal and transverse components of energy momentum tensor gives the numerical coefficients 1.08318 and −0.273836 as seen in Eq. (74).

TABLE I. Numerical values of finite pieces of regularized $R^{(r)}_{AA}$ integrals for energy momentum tensor corrections. For the special case of $\int d^3r R^{(r)}_{T_1T_1}$ the remaining one-dimensional time integral can be done analytically.

$R^{(r)}_{AA}$	$(4\pi)^{-3/2}\int d^3r R^{(r)}_{AA}$	$(4\pi)^{-3/2}\int d^3r \cos^2\theta_K R^{(r)}_{AA}$
$R^{(r)}_{\pm\pm}$	−0.439511	0.021281
$R^{(r)}_{T_1T_1}$	$-\frac{\pi}{3\sqrt{6}}\approx -0.427517$	−0.467513
$R^{(r)}_{T_2T_2}$	1.402539	0.340636

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