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Reexamination of proton rms radii from low-q power expansions

Ingo Sick* and Dirk Trautmann†
Departement für Physik, Universität Basel, CH-4056 Basel, Switzerland
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Several recent publications claim that the proton charge *rms* radius resulting from the analysis of electron-scattering data restricted to *low*-momentum transfer agrees with the radius determined from muonic hydrogen, in contrast to the radius resulting from analyses of the full (e,e) data set which is 0.04 fm larger. Here we show why these publications erroneously arrive at the low radii.

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Introduction. The determination of the *rms*-radius R of the proton charge distribution has recently attracted much attention. Whereas standard analyses of electron-proton scattering data yield 0.879 ± 0.009 fm [1], the Lamb shift measurement in muonic hydrogen gave 0.8409 ± 0.0004 fm [2]; this represents an $\approx 5\sigma$ discrepancy. The radii from electron scattering near 0.88 fm come from analyses that fit with excellent χ^2, the *world* cross section, and polarization transfer data up to large momentum transfer q, 5–12 fm^{-1} [3–8]. Recently, three publications [9–11] which restrict the analysis to the *low-q* data with $q_{max} = 0.7,0.9$, and 1.6 fm^{-1}, respectively, find R in the 0.84 fm neighborhood, i.e., compatible with the radius from muonic hydrogen. In this Rapid Communication, we show why these analyses, which yield values of $R \approx 0.04$ fm lower than Refs. [3–8], have led to erroneously low values.

Power-series expansion. In terms of the electric Sachs form factor $G_e(q)$ the proton charge *rms*-radius R is defined via the slope of $G_e(q^2)$ at $q^2 = 0$. It therefore seems natural to parametrize $G(q)$ in a power series,

$$G_e(q) = 1 + q^2 a_2 + q^4 a_4 + q^6 a_6 + \cdots , \tag{1}$$

where $R^2 = -6a_2$. Nonrelativistically, $a_4 = \langle r^4\rangle/120$ and $a_6 = -\langle r^6\rangle/5040$ are given by the higher moments of the charge-density distribution. The rationale behind an analysis restricted to data with *low* maximum momentum transfer q_{max}: At low enough q the terms proportional to q^{2n} with $n > 1$ (or in some cases $n > 2$) can be neglected, so a linear (quadratic) fit of the data in terms of powers of q^2 should suffice. Low-order (one-parameter) fits in terms of derived functions as, e.g., a dipole, $G(q) = 1/(1 + q^2 b_2)^2$, follow the same rationale, although these parametrizations do implicitly contain higher $q^{2n} a_{2n}$ contributions as fixed by the analytical shape of the parametrization.

Problems with expansions of the proton form factors in terms of q^{2n} have been recognized earlier [12]. Due to the peculiar shape of the proton form factor—approximately a dipole—and the peculiar shape of the corresponding charge density—approximately an exponential—the moments $\langle r^{2n}\rangle$ for $n \geqslant 2$ grow unusually fast with increasing order n. In the form factor $G(q)$ the moments $\langle r^{2n}\rangle$ are tightly coupled and give contributions of alternating signs. In an expansion with small n ($n = 1,2$) the values found for $\langle r^{2n}\rangle$ depend on the maximum n and the value of the maximum momentum transfer q_{max} employed and always yield too small $\langle r^2\rangle$. This has recently been shown by Kraus *et al.* [13] who quantitatively demonstrate the pitfalls of fits with low-order power series by analyzing pseudodata generated with known R. They show that, e.g., a linear fit in q^2 with $q_{max} = 0.7$ fm^{-1} as employed in Refs. [9,10] produces a value of R which is low by 0.04 fm.

This result of Kraus *et al.* can qualitatively be understood. When terminating the series Eq. (1) with the q^2 term, one implicitly posits $\langle r^4\rangle = 0$. As $\langle r^2\rangle \approx 0.7$ fm^2 this implies a charge density that is positive at small r (charge proton $+e$) but has a negative tail at large r; due to the larger weight in the r^4 term the tail can reduce $\langle r^4\rangle$ to 0. This negative tail of course also affects $\langle r^2\rangle$ and leads to the systematically low values of R. The same happens *mutatis mutandis* with truncations at higher order [13].

The second obvious problem with very-low q: the finite-size effect (FSE) $1 - G_e(q)$ decreases like q^2_{max}. Already at the $q \approx 0.8$ fm^{-1} of maximal sensitivity of the data to R (see below) the FSE $\approx q^2 R^2/6$ amounts to 0.09 only. The smallness of the FSE emphasizes that fits used to extract R must reach the minimal χ^2_{min} achievable, a visually good fit is not enough: A change in R of 1% corresponds to a systematic change in G_e of only 0.0015 (0.17% of G_e), a difference that is far below the resolution of typical plots of $G_e(q)$ [9–11].

The sensitivity of the data to R is shown in Fig. 1 which results from a notch test employing SOG fits, i.e., parametrizing the density as a Sum Of Gaussians, of the *world* data (for a recent reference to notch tests see Ref. [14]). When exploiting only part of the range of $q \leqslant 1.5$ fm^{-1}, one loses part of the experimental information on R; analyses which limit the data to, e.g., 0.8 fm^{-1} as performed in Refs. [9,10] then ignore half of the data sensitive to R. Restriction to a subset of the *world* data only amplifies this problem.

Contribution of higher moments. For a more detailed discussion of the problems with Eq. (1), we start from the values of $a_2, a_4, \ldots$ determined by Bernauer [15] via a power-series fit (with a χ^2 as low as a spline fit) to the Mainz data for $q_{max} = 5$ fm^{-1}. One might hope that, due to the large q_{max} and the high-order $2n = 20$ employed, the values of the lowest moments of interest here should not be affected seriously by the above-mentioned problems [12]. Figure 2 shows the percent contribution of the a_4–a_{10} terms to the FSE. Also indicated is the uncertainty in the FSE due to a (very optimistic) uncertainty of 0.2% in the experimental $G_e(q)$.

*ingo.sick@unibas.ch
†dirk.trautmann@unibas.ch

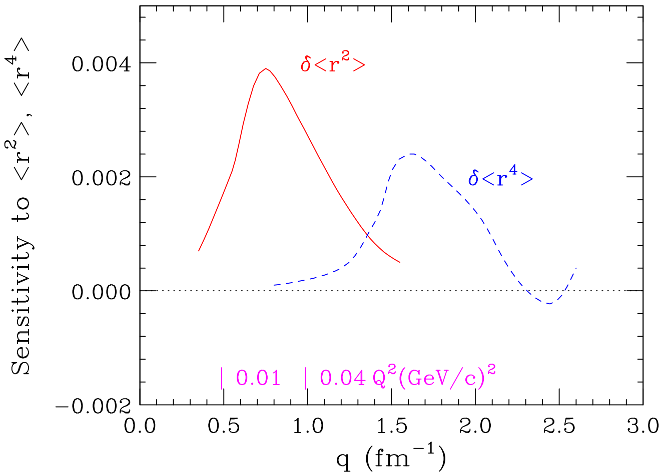

FIG. 1. Sensitivity (arbitrary units) to the moments $\langle r^2\rangle$ and $\langle r^4\rangle$ obtained from fits of the *world* data.

This figure shows several features:

(1) At the q's used in the "low-q fits" referred to above with $q_{\max} = 0.72$–0.9 fm^{-1}, the contribution of the q^4 term to the FSE $\approx q^2R^2/6$ amounts to 10–15% at the upper limit of the q range where FSE is most sensitive to R. This shows immediately and without further calculation that neglecting this contribution in a linear fit in terms of q^2 must yield a value of R^2 which is low by a comparable percentage.

(2) Even the contribution of the q^6 term is not entirely negligible (15% of the q^4 term at $q = 0.9$ fm^{-1}); when attempting to determine a_4 from a fit quadratic in q^2 a wrong value results if the contribution of the q^6 term is not accounted for.

(3) Restriction of $q_{\max}$ to extremely low values, such as to justifiably neglect the q^4 term and maintain an accuracy of 1% in R, would require $q_{\max} < 0.35$ fm^{-1}. At these values of q, the FSE is <0.015, and the typical error bars of $G_e(q)$ would yield huge uncertainties in the FSE contribution, hence R^2 (see the dashed curve).

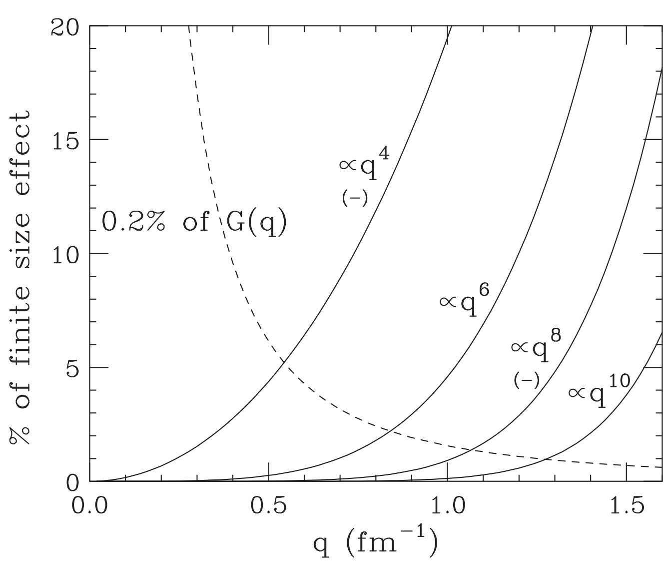

FIG. 2. The solid curves show the relative contribution (in percentages) of the q^{2n} terms to the FSE in $G_e(q)$. The dashed curve shows the relative contribution of a 0.2% uncertainty of the experimental $G_e(q)$. For comparison: The $q_{\max}$ of the fits linear in q^2 (dipole) of Refs. [9,10] ([11]) amount to 0.72, 0.90, and 1.6 fm^{-1}, respectively.

Figure 2 makes it obvious that the low-q fits of Refs. [9,10], which neglect the q^4 contribution, must find wrong values for R due to the omitted q^4 term (for a quantitative analysis see below). Figure 2 also shows without further calculation that for $q \leqslant 1.6$ fm^{-1} the information content of the data is four to five parameters (moments), which hardly can be represented correctly by a one-parameter form factor, such as employed by Horbatsch and Hessels [11] (for a quantitative discussion see below).

Higher moments from world data. As was pointed out in Ref. [12] and quantitatively demonstrated in Ref. [13] the determination of the lowest moments via a power-series fit is not very reliable and for the higher n dependent on the cutoff in n. We therefore have made an independent determination.

We use the *world* data up to the maximum momentum transfer available for G_e, 10 fm^{-1} (not including the data of Ref. [15] which show systematic differences [3]). This data set, which composes 603 cross sections and polarization transfer points, is corrected for two-photon exchange effects [16] and fitted with a Fourier transform of Laguerre functions of order 11 for both $G_e(q)$ and $G_m(q)$. Laguerre functions[1] are particularly well suited as

(1) They provide an orthonormal basis which makes multiparameter fits very efficient (even if the polynomials are not strictly orthogonal over the limited q range of the data).

(2) They have a controlled behavior at large radii r due to the $e^{-\gamma r}$ weight function, a consideration which is particularly important [20] when addressing higher moments [an aspect shared with the parametrizations of the vector dominance model (VDM)].

(3) They provide values for the moments insensitive to the cutoff in the number of terms employed; the moments $\langle r^{2n}\rangle$ are given by the lowest $2n+3$ coefficients.

The set of data can be reproduced with a χ^2 of 542 with 548 degrees of freedom when the normalizations of the individual data sets are floated. When keeping the normalizations at their measured values and *without* increasing the error bars due to systematic error of the normalizations, χ^2 amounts to 783 with 580 degrees of freedom. These χ^2 values are excellent given a set of data measured over some 50 years. The resulting values for $\langle r^4\rangle$ are $2.01 \pm 0.05(1.99)$ fm^4. The quality of the fit and the values of the moments are very close to the ones obtained using SOG [21] ($\langle r^4\rangle = 2.03$) or a VDM-type parametrization ($\langle r^4\rangle = 2.01$). We have verified that a variation of $q_{\max}$ between 7 and 12 fm^{-1} and a variation

[1] For similar expansions see Refs. [17–19].

of n between 10 and 13 changes $\langle r^4 \rangle$ by < 0.03 fm^4. Distler *et al.* [22] obtained $2.59 \pm 0.19 \pm 0.04$ from a mix of two form factor parametrizations fit separately to low-q [15] and high-q [23] data. With these preliminaries we are in the position to quantitatively discuss the recent low-q fits.

Fits to very-low-q data. Higinbotham *et al.* [10] perform a linear fit in q^2 to a subset of the data available, the form factors of Mainz80 and Saskatoon74 [24,25]. For their highest $q_{\max}$ of 0.9 fm^{-1}, which yields the result with the smallest uncertainty, they find[2] $R = 0.844 \pm 0.014$ fm. From this the authors conclude that R agrees with the value of 0.84 fm from muonic hydrogen. When repeating exactly the same analysis but adding in the q^4 and q^6 contributions using the higher moments from the fit to the high-q data, one finds a reduced χ^2 (i.e., χ^2 per degree of freedom) which is 11% smaller and a radius R of 0.899 fm. This R disagrees with the muonic value and agrees with the above-cited R's in the 0.88-fm region.

Higinbotham *et al.* also perform a fit quadratic in q^2 and found a radius of 0.873 ± 0.039 fm. This agrees with the radii in the 0.88-fm region, although, as the authors want to see it, the value is "within one σ of the muonic result". The uncertainty of ± 0.039 fm illustrates the large error bars resulting from the restriction of the analysis to a fraction of the q region sensitive to R (see Fig. 1) and the large uncertainty of $\langle r^4 \rangle$ due to the truncation in q. When using, instead of the $\langle r^4 \rangle = 1.32 \pm 0.96$ of Higinbotham *et al.*, the value of 2.01 ± 0.05 we know from the fit to the high-q data, the result for R becomes 0.901 fm with a smaller error bar of 0.010 fm.

Griffioen *et al.* [9] analyze part of the cross sections of Ref. [7] for $q < 0.72$ fm^{-1} using Eq. (1) including terms up to a_4. They use a low-q parametrization for G_m/G_e and take the shortcut of ignoring the free relative normalizations of the individual data sets.[3] They find a rms radius of 0.850 ± 0.019 fm and conclude that this value is consistent with the muonic hydrogen result of 0.84 fm.

Repeating their fit but using a_4 determined much better from the high-q fit yields a radius of 0.877 ± 0.008 fm with lower χ^2 and a significantly smaller error bar. This result agrees with the 0.88-fm-type results and disagrees with the radius from muonic hydrogen.

Griffioen *et al.* also perform fits up to order q^6 with a_4,a_6 values as given by simple models for the proton charge density (uniform, exponential, and Gaussian) which all produce the same χ^2; the resulting R values are linearly correlated with a_4. Extrapolating these values linearly to the value of a_4 given by the fit to high-q data yields $R = 0.876 \pm 0.008$ fm, again in agreement with the R's in the 0.88-fm region.

Summarizing: All the low-q fits of Refs. [9,10] yield radii in the 0.88-fm region once the higher moments of the charge density—which *are* nonzero but ignored (or poorly fixed in the low-q fits due to the truncation of the series in n of $q_{\max}$)—are properly accounted for.

Fits to not-so-low-q data. Horbatsch and Hessels [11] employ the cross sections of Ref. [7] up to a $q_{\max}$ of 1.6 fm^{-1}. They parametrize the form factors via a one-parameter dipole expression for both G_e and G_m. Their fit yields a reduced χ^2 of 1.11, and a (charge) rms-radius of $R = 0.842 \pm 0.002$ fm. From this, together with other fits which yield radii near 0.89 fm, the authors conclude that R is in the range of 0.84–0.89 fm, i.e., could be compatible with the radius from muonic hydrogen.

Figure 2 shows that for $q_{\max} = 1.6$ fm^{-1} the moments up to at least $2n = 10$ are important to get the full FSE. It is highly unlikely that the one-parameter dipole contains the mix of q^{2n} terms for $2n = 4 \cdots 10$ appropriate for the proton. Indeed, expansion of the dipole in terms of powers of q^2 shows that the numerically largest difference to the power-series fit of Ref. [15] results from the contribution of the $\langle r^4 \rangle$ term. This difference in $\langle r^4 \rangle$ alone would lead, at the $q = 0.85$ fm^{-1} of maximal sensitivity to R, to a difference ΔG_e of 0.0081 corresponding to 9.5% in the FSE, hence R^2 (causing the systematic deviations just visible in Fig. 3 of Ref. [11]). The same consideration applies to the parametrization of $G(q)$ as a (one-parameter) linear function $1 - cz$ with $z = (\sqrt{t_c - t} - \sqrt{t_c})/(\sqrt{t_c - t} + \sqrt{t_c})$ and $t = -q^2$. The lacking flexibility of the fit function, causing systematic differences between data and fit and a χ^2 larger than the one of already published fits, also affects the results from the high-q fits of Refs. [9,10].

For the fits of Horbatsch and Hessels it is not practical to correct for the effect upon R of the incorrect higher q^{2n} terms as we did above for the analyses of Refs. [9,10]; too many terms $2n = 4 \cdots 10$ would contribute. In order to demonstrate the importance of their effect we rather quote the result of a Laguerre-function fit (four terms each for G_e and G_m) to exactly the same data, yielding a lower reduced χ^2 of 1.045 and a (charge) rms radius of $R = 0.884 \pm 0.016$ fm. Due to the lacking flexibility the parametrization of Horbatsch and Hessels has a χ^2 that is higher by 50. From such a fit that is some seven σ's away from a genuine best fit, one obviously cannot get a significant value for R.

Conclusion. The moments $\langle r^{2n} \rangle$ of the proton for $n > 1$ *are* there, and they are *known* to be large. Ignoring their strong correlation with R [9–11] leads to the wrong results for the proton rms radius.

[2]Including Coulomb distortion would have increased R by $\approx$ 0.01 fm [26].

[3]Correct treatment of the normalizations of the data sets of Ref. [7], which are individually floating, would have increased the uncertainty of R by a factor of 1.6.

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