Sum rule study for double Gamow-Teller states

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(Received 21 September 2016; published 27 December 2016)

We study the sum rules of double Gamow-Teller (DGT) excitations through double spin-isospin operator $(\sigma t_-)^2$. In general, 2^+ states in the grand-daughter nuclei have dominant transition strength in DGT excitations and 0^+ states are weak, except in $T = 1$ mother nuclei in which 0^+ strength is competitive with 2^+ strength. The isospin selection of DGT is also discussed among five possible isospin states in grand-daughter nuclei. A possibility to extract the unit cross section for the DGT transition strength is pointed out in the $(\sigma t_-)^2$ excitation of double isobaric analog state (DIAS) in $T = 1$ nuclei.

Double charge exchange excitations (DCX) induced by heavy ion beams at intermediate energies [1,2] attract a lot of interest in relations with new collective excitations such as double Gamow-Teller giant resonance (DGTR) and also double β decay matrix elements. In the 1980s, the double charge exchange reactions (DCX) were performed by using pion beams, i.e., (π^+,π^-) and (π^-,π^+) reactions. Through these experimental studies, the double isobaric analog states (DIAS), the dipole giant resonance (GDR) built on the IAS, and the double dipole resonance states (DGDR) are identified [3–5]. However, DGTR were not found in the pion double charge exchange spectra. In the middle of 1990s, the heavy-ion double charge exchange experiments were performed at energies of 76 and 100 MeV/u with a hope that DGTR might be observed in the ^{24}Mg(^{18}O, ^{18}Ne)^{24}Ne reaction [6]. However, no clear evidence of DGTR was found in these reactions. This is mainly because the $(^{18}$O, ^{18}Ne) reaction is a $(2n,2p)$ type DCX one and even the single GT τ_+ resonance by (n,p) reaction is weak in a $N = Z$ nucleus ^{24}Mg. Other reasons were low intensity of beams and nonoptimal energy to excite the spin-isospin responses. A new research program based on a new DCX reaction $[^{12}$C, ^{12}Be(0_2^+)] is planned at RIKEN RIBF facility with high intensity heavy ion beams at the optimal energy of $E_{\text{lab}} = 250$ MeV/u to excite the spin-isospin response [7]. A big advantage of this reaction is based on the fact that it is a $(2p,2n)$ type DCX reaction and one can use neutron-rich target to excite DGT strength. Although many theoretical efforts have been made for studies of double β decays, DGT strengths corresponding to the double β decays are still too small to be identified in the reaction experiments. Minimum-biased theoretical prediction based on sum rules will provide a robust and global view of the DGTR and can be a good guideline for the future experimental studies.

In this paper, we present useful formulas to analyze the DGT strength using several sum rules for the double spin-isospin operator $(\sigma t_-)^2$. We study also DIAS excited through the double Fermi transition operator $(t_-)^2$ and double GT operator $(\sigma t_-)^2$. The DIAS state might be a sharp peak close to DGT peaks and can be used to extract the unit cross section between the absolute DGT strength and the forward angle DCX cross sections. The isospin structure of DGT strength is also given by using the isospin coupling scheme of DGT states.

The sum rule for the single GT transitions is well known and proportional to the neutron excess,

$$S_- - S_+ = \sum_f |\langle f|\hat{O}_-(\text{GT})|i\rangle|^2 - \sum_f |\langle f|\hat{O}_+(\text{GT})|i\rangle|^2$$
$$= 3(N - Z), \quad (1)$$

where the GT transition operators read

$$\hat{O}_\pm(\text{GT}) = \sum_\alpha \sigma(\alpha)t_\pm(\alpha). \quad (2)$$

The GT sum rule is model independent and gives a good guidance to perform the single charge exchange reactions such as (p,n) and $(^3\text{He},t)$ reactions, to observe GTR strength in many nuclei (see, for example, a review article Ref. [8]).

The DGT transition operator $\hat{O}_\pm(\text{GT})^2$ can be projected to good multipole states to be

$$[\hat{O}_\pm(\text{GT}) \times \hat{O}_\pm(\text{GT})]^J_\mu, \quad J = 0,2. \quad (3)$$

The sum rule strength is defined as

$$D^J_\pm = \sum_{J_f M_f, \mu} \left| \langle J_f, M_f | [\hat{O}_\pm \times \hat{O}_\pm]^J_\mu | J_i M_i \rangle \right|^2$$
$$= \langle J_i M_i | \sum_\mu (-1)^\mu [\hat{O}_\mp \times \hat{O}_\mp]^J_\mu [\hat{O}_\pm \times \hat{O}_\pm]^J_{-\mu} | J_i M_i \rangle, \quad (4)$$

where J_i and J_f are the angular momenta of the initial and final states, respectively. The sum rule strength is also expressed by the reduced matrix element,

$$D^J_\pm = \frac{1}{2J_i + 1} \sum_{J_f} |\langle J_f||[\hat{O}_\pm \times \hat{O}_\pm]^J||J_i\rangle|^2, \quad (5)$$

where the double bar means the reduced matrix element for the angular momentum. Hereafter we denote the initial state $|J_i M_i\rangle$ by a simple notation $|i\rangle$. The sum rule value for $J = 0$ excitations is then evaluated as

$$D_-^{(J=0)} - D_+^{(J=0)} = \langle i|[[\hat{O}_+ \times \hat{O}_+]^{(J=0)}, [\hat{O}_- \times \hat{O}_-]^{(J=0)}]|i\rangle$$
$$= 2(N - Z)(N - Z + 1) + \tfrac{4}{3}[(N - Z)S_+$$
$$- \langle i|[i\hat{\Sigma} \cdot (\hat{O}_- \times \hat{O}_+) + \hat{\Sigma} \cdot \hat{\Sigma}]|i\rangle], \quad (6)$$

where

$$\hat{\Sigma} = \sum_\alpha \sigma(\alpha) \qquad (7)$$

and $(\hat{O}_- \times \hat{O}_+)$ is the vector product of two operators. In the last line of Eq. (6), we use an identity

$$(\hat{O}_+ \times \hat{O}_-) \cdot \hat{\Sigma} = -\hat{\Sigma} \cdot (\hat{O}_- \times \hat{O}_+) + 2i\,\hat{\Sigma} \cdot \hat{\Sigma} \qquad (8)$$

which makes the final formula of the sum rule (6) different from those in Refs. [9,11], but it is equivalent to the one in Ref. [10]. It is important for quantitative study to rewrite the formula by using the identity (8) since the terms including $\hat{O}_+$ on the right can be omitted in $N > Z$ nuclei in a good accuracy. The sum rule for $J^\pi = 2^+$ final states can be obtained in a similar way to that for $J = 0$ states:

$$
\begin{aligned}
&D_-^{(J=2)} - D_+^{(J=2)} \\
&= \langle i | \sum_\mu (-1)^\mu \big[[\hat{O}_+ \times \hat{O}_+]_\mu^{(J=2)}, [\hat{O}_- \times \hat{O}_-]_{-\mu}^{(J=2)}\big] | i \rangle \\
&= 10(N - Z)(N - Z - 2) \\
&\quad + \frac{10}{3}[2(N - Z)S_+ + \langle i | [i\,\hat{\Sigma} \cdot (\hat{O}_- \times \hat{O}_+) + \hat{\Sigma} \cdot \hat{\Sigma}] | i \rangle].
\end{aligned}
\qquad (9)
$$

It should be noticed that the sum rule for $J = 2$ final states is not given in Refs. [9,10], but given in Ref. [11] in a different form.

Next, we consider the DGT transition to the DIAS state. The DIAS state is defined as

$$|\text{DIAS}\rangle = \frac{T_- T_-}{N_f} |i\rangle, \qquad (10)$$

where $T_- = \sum_\alpha t_-(\alpha)$ is the isospin lowering operator and the normalization factor is $N_f = \sqrt{2(N-Z)(N-Z-1)}$. The

double Fermi transition to DIAS state is evaluated to be

$$|\langle \text{DIAS} | T_- T_- | i \rangle|^2 = 2(N-Z)(N-Z-1), \qquad (11)$$

in which the mother state is assumed to have a good isospin quantum number $T = T_z = (N-Z)/2$. The DGT transition to DIAS is also expressed as

$$
\begin{aligned}
D_-^{(J=0)}(\text{DIAS}) &= |\langle \text{DIAS} | [\hat{O}_- \times \hat{O}_-]^{(J=0)} | i \rangle|^2 \\
&= \left| \frac{1}{N_f} \langle \hat{0} | T_+ T_+ [\hat{O}_- \times \hat{O}_-]^{(J=0)} | i \rangle \right|^2 \\
&= \frac{1}{3N_f^2} [8\Sigma_{\text{sum}} - 2(S_- + S_+)]^2 \\
&= \frac{1}{6(N-Z)(N-Z-1)} \\
&\quad \times [8\Sigma_{\text{sum}} - 2(S_- + S_+)]^2,
\end{aligned}
\qquad (12)
$$

where Σ_{sum} is the sum rule of the isovector spin transition,

$$\Sigma_{\text{sum}} = \langle i | \sum_{\alpha,\mu} \sigma_\mu(\alpha) t_z(\alpha) \sum_{\beta,\mu'} \sigma_{\mu'}(\beta) t_z(\beta) | i \rangle. \qquad (13)$$

For a single-j configuration with the occupation probability v^2, the value Σ_{sum} is evaluated to be

$$
\begin{aligned}
\Sigma_{\text{sum}} &= \sum_{p,h,\mu} |\langle (ph)1\mu | \sigma_\mu t_z | \hat{0} \rangle|^2 \\
&= \frac{1}{4} \frac{v_h^2(2j_h + 1)(2j_h - 1)}{j_h} \quad \text{for } j_h = j_p + 1,
\end{aligned}
\qquad (14)
$$

where we use a filling approximation for the last occupied orbit.

Calculated sum rule values for DGT transitions to $J^\pi = 0^+$ and 2^+ in Eqs. (6) and (9), respectively, are listed in Table I for ten nuclei, together with those for DIAS states by Fermi and GT operators (11) and (12). In $N > Z$ nuclei with the

TABLE I. Various sum rule values for DGT transitions to $J^\pi = 0^+, 2^+$ and DIAS states. The double Fermi transitions to DIAS are also listed. For Eqs. (6) and (9), the spin sum rule $\hat{\Sigma} \cdot \hat{\Sigma}$ terms in the last line is obtained by shell model calculations: ^{6}He, ^{8}He, and ^{14}C in p-shell model space with CKII interaction, ^{18}O and ^{20}O in sd-shell model space with USDB interaction, and Ca isotopes in pf-shell model space with GXPF1A interaction. The single particle value of Σ_{sum} is used for ^{90}Zr. The single particle values from Eq. (13) are given in the bracket in the sixth column. In the second and third columns, the value in the bracket is the upper limit for $J = 0$, while that is the lower limit for $J = 2$. Notice that Vogel *et al.* in Ref. [9] and Zheng *et al.* in Ref. [10] calculated only the $J = 0$ channel and a factor 3 larger than the present one because of the exact angular momentum projection in the present results. The values Σ_{sum} for DIAS$^{(J=0)}$ (DGT) in Eq. (12) are calculated by using the same shell model calculations as for $\hat{\Sigma} \cdot \hat{\Sigma}$. The sum rule value DIAS$^{(J=0)}$ (DGT) with the single particle value Σ_{sum} (13) is also listed in the fifth column with bracket.

Initial state	$(J = 0)$	$(J = 2)$	DIAS	DIAS(DGT)	Σ_{sum}
^{6}He	12.0(12)	0.0 (0)	4	3.70(3.70)	0.211×10^{-3} (0.667)
^{8}He	39.7(40)	80.7 (80)	24	7.71(2.47)	0.052 (1.333)
^{14}C	8.98(12)	7.55 (0)	4	4.56 (0.15)	0.566 (1.333)
^{18}O	10.4(12)	3.96 (0)	4	7.72 (2.61)	0.297 (0.80)
^{20}O	35.5(40)	91.3 (80)	24	4.13 (1.74)	0.845 (1.60)
^{42}Ca	8.50(12)	8.75 (0)	4	3.80 (2.18)	0.66 (0.86)
^{44}Ca	32.6(40)	98.5(80)	24	2.31 (1.47)	1.38 (1.71)
^{46}Ca	72.3(84)	269.3(240)	60	1.89 (1.32)	2.20 (2.57)
^{48}Ca	135.5(144)	501.2(480)	112	3.70 (1.25)	1.59 (3.43)
^{90}Zr	196.3(220)	859.2(800)	180	(1.11)	(4.44)

good isospin quantum number, the vector product $(\hat{O}_- \times \hat{O}_+)$ and the sum S_+ in Eqs. (6) and (9) vanish because of the isospin raising operator in $\hat{O}_+$. The spin sum rule $\hat{\Sigma} \cdot \hat{\Sigma}$ term is obtained by shell model calculations: ^{6}He, ^{8}He, and ^{14}C with p-shell model space with CKII interaction, ^{18}O and ^{20}O with sd-shell model space with USDB interaction, and Ca isotopes with pf-shell model space with GXPF1A interaction. The single particle value is used for ^{90}Zr assuming the neutron excitation from $1g_{9/2}$ to $1g_{7/2}$. The value in the bracket is obtained without the spin sum rule term, i.e., the upper limit for $J = 0$, while the lower limit for $J = 2$. Notice that Vogel *et al.* in Ref. [9] and Zheng *et al.* in Ref. [10] calculated only the $J = 0$ channel and give a factor 3 larger than the present one because of no exact angular momentum projection in their results. The values Σ_{sum} for DIAS$^{(J=0)}$ (DGT) in Eq. (12) are calculated by using the same shell model wave functions as for $\hat{\Sigma} \cdot \hat{\Sigma}$ and shown in the sixth column in Table I. The single particle values of Eq. (13) are given in the bracket in the sixth column. The value DIAS$_-^{(J=0)}$ (DGT) with the single particle value Σ_{sum} is listed in the fifth column with brackets. In all the nuclei in the present study, the value of Σ_{sum} is one-fourth of $\hat{\Sigma} \cdot \hat{\Sigma}$ because of a semiclosed configuration (either proton or neutron configuration is a closed shell).

The values Σ_{sum}, equivalently $\hat{\Sigma} \cdot \hat{\Sigma}$, are almost negligible for p-shell nuclei ^{6}He and ^{8}He, because of good SU(4) spin-isospin symmetry in which the spin transitions are allowed only for spin and isospin flip transition between $(S, T) = (0, 1) \rightarrow (1, 0)$. This selection turns out to be the selection of transitions $(J^\pi, T) = (0^+, 1) \rightarrow (1^+, 0)$ in ^{6}He and $(J^\pi, T) = (0^+, 2) \rightarrow (1^+, 1)$ in ^{8}He, which correspond to the charge exchange GT transitions. The shell model values Σ_{sum} are smaller than the single-particle values in other nuclei by a factor $1.2 \sim 2.5$. The effect of the $\hat{\Sigma} \cdot \hat{\Sigma}$ term for $J = 0$ and $J = 2$ sum rules are negligible for ^{6}He and ^{8}He, while it amounts at most to 20% in other nuclei with $T > 1$ and it becomes much larger in $T = 1$ nuclei such as ^{14}C, ^{18}O, and ^{42}Ca.

In $N \gg Z$ nuclei, the sum rule values of DGT transitions are approximately proportional to the $(2J + 1)$ factor, i.e., the value for $J = 2$ is 5 times larger than that for $J = 0$ in the same nucleus. However, this proportionality is strongly modified in $N \sim Z$ nuclei. In the extreme, the sum rule for $J = 2$ transitions is smaller than that for $J = 0$ in the nuclei ^{6}He, ^{14}C, and ^{18}O with $N = Z + 2$, and the two values are almost equal in ^{42}Ca.

It is noticed that the DGT transition to DIAS state is much smaller than the sum rule values of DGT transitions for $J = 0$ and $J = 2$ states in nuclei with $N > Z + 2$. However, in $T = 1$ nuclei, the DGT transition to the DIAS state is comparable with the DGT sum rules (6) and (9). This characteristic turns out the possibility to extract the unit cross section for DGT transition in $T = 1$ nuclei, especially ^{14}C and ^{18}O, since the DIAS state might have a narrow width and may be distinguished clearly from other DGT states in double charge exchange reactions. This argument is entirely benefited from the fact that the spin-isospin response is much more favorable than the isospin response in the medium energy charge exchange reactions with $E_{\text{lab}} \sim 200$ MeV/u.

So far the isospin dependence of DGT sum rules in the grand-daughter nucleus is not considered. For the DGT transitions from a mother nucleus with the isospin T, five different isospin states $T'' = T + 2, T + 1, T, T - 1, T - 2$ are expected for the grand-daughter nucleus with $T_z = T - 2$. DIAS is a state with $T'' = T$. The isospin selection amplitude $A(T'')$ for different isospin T'' in the grand-daughter nucleus can be calculated by a combination of Clebsch-Gordan coefficients,

$$
\begin{aligned}
A(T'') &= \sum_{T'} \langle T T 1 - 1 | T' T - 1 \rangle \langle T' T - 1 1 - 1 | T'' T - 2 \rangle \\
&= \langle T T 2 - 2 | T'' T - 2 \rangle \\
&= \begin{cases}
\sqrt{\dfrac{2T - 3}{2T + 1}} & \text{for} \quad T'' = T - 2 \\[4pt]
\sqrt{\dfrac{2(2T - 1)}{(T + 1)(2T + 1)}} & \text{for} \quad T'' = T - 1 \\[4pt]
\sqrt{\dfrac{6}{(T + 1)(2T + 3)}} & \text{for} \quad T'' = T \\[4pt]
\sqrt{\dfrac{6}{(T + 1)(T + 2)(2T + 1)}} & \text{for} \quad T'' = T + 1 \\[4pt]
\sqrt{\dfrac{6}{(T + 1)(T + 2)(2T + 1)(2T + 3)}} & \text{for} \quad T'' = T + 2
\end{cases}
\end{aligned}
$$

(15)

The calculated values referring to mother nuclei with different isospins $T = 1 \sim 22$ are given in Table II. For the mother nucleus with $T = 1$, only three isospin states T'' are allowed by DGT excitations. For mother nuclei with large isospin $T > 2$, the isospin $T'' = T - 2$ give the major contributions. On the other hand, for mother nuclei with small isospin $T \leqslant 2$, the dominant configurations are $T'' = T$ for $T = 1$ and $T'' = T - 1$ for $T = 2$ mother nuclei.

In summary, we studied the sum rules of DGT excitations by the operator $(\sigma t_-)^2$. In nuclei $N > Z + 2$, the $J^\pi = 2^+$ excitations dominate the DGT strength, due to the multipole factor $(2J + 1)$, more than 0^+ excitations. However, in nuclei with $N \sim Z$, the 0^+ excitations become substantially stronger, even larger than 2^+ excitations in light nuclei with $T = 1$ such as ^{14}C an ^{18}O. The excitation to DIAS is also studied through $(\sigma t_-)^2$ and t_-^2 operators to investigate the possibility to extract the unit cross sections for DGT strength. We pointed out that the strength of DIAS excitations by the $(\sigma t_-)^2$ operator is competitive to DGT strength in the light $T = 1$ nuclei. This

TABLE II. Splitting of DGT strength to different isospin final state with $T'' = T \pm 2, T \pm 1, T$ for the initial state with $T = 1 \sim 22$. Probability $A(T'')^2$ of each isospin T'' for a given T is listed in %.

T	T''				
	$T - 2$	$T - 1$	T	$T + 1$	$T + 2$
1	–	–	60	33.3	6.7
2	20	40	28.6	10	1.4
3	42.9	35.7	16.7	4.3	0.5
4	55.6	31.1	10.9	2.2	0.2
5	63.6	27.3	7.7	1.3	0.1
8	76.5	19.6	3.5	0.4	0.0
22	91.1	8.3	0.6	0.0	0.0

characteristic feature may give a good opportunity to extract the unit cross section for DGT strength. The isospin structure of DGT is also studied in the grand-daughter nuclei. Among five possible isospin states $T'' = T \pm 2, T'' = T \pm 1, T'' = T$ in a grand-daughter nucleus referred to a mother state with the isospin T, the largest component appears for $T'' = T - 2$ states for $T > 2$ nuclei, while $T'' = T$ and $T'' = T - 1$ states are dominant in the final states in $T = 1$ and $T = 2$ nuclei, respectively.

[1] M. Takaki, T. Uesaka *et al.*, Proposal for experiment at RCNP, "Search for double Gamow Teller giant resonances in ^{48}Ti via the heavy-ion double charge exchange ^{48}Ca(^{12}C, ^{12}Be(0_2^+)) reaction" (2015), http://www.rcnp.osaka-u.ac.jp/Divisions/plan/b-pac/ex_appro/summary/pdf/E429.pdf.

[2] F. Cappuzzello, C. Agodi, M. Bondi, D. Carbone, M. Cavallaro, and A. Foti, J. Phys.: Conf. Series **630**, 012018 (2015).

[3] S. Mordechai and C. F. Moore, Nature **352**, 393 (1991).

[4] H. Ward *et al.*, Phys. Rev. Lett. **70**, 3209 (1993).

[5] Ph. Chomaz and N. Frascaria, Phys. Rep. **252**, 275 (1995).

[6] J. Blomgren *et al.*, Phys. Lett. B **362**, 34 (1995).

[7] T. Uesaka, M. Takaki *et al.*, Proposal for Nuclear Physics Experiment at RI Beam Factory "Search for Double Gamow-Teller Giant Resonances in $\beta\beta$-decay nuclei via the heavy-ion double charge exchange ^{48}Ca(^{12}C, ^{12}Be(0_2^+)) reaction" (2016) and private communications.

[8] M. Ichimura, H. Sakai, and T. Wakasa, Prog. Part. Nucl. Phys. **56**, 446 (2006).

[9] P. Vogel, M. Ericson, and J. D. Vergados, Phys. Lett. B **212**, 259 (1988).

[10] D. C. Zheng, L. Zamick, and N. Auerbach, Phys. Rev. C **40**, 936 (1989).

[11] K. Muto, Phys. Lett. B **277**, 13 (1992).