

Finite valence nucleon number and rotation-vibration interactions

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Background: A characteristic observable of nuclear collective motion is the relative $B(E2)$ values from the γ band to the ground band in even-even deformed nuclei. The Alaga rules provide an idealized set of benchmarks for these observables. However, deviations from the Alaga rules are universally observed and have been traditionally and successfully interpreted in terms of parameterized γ -band-ground-band bandmixing. An alternate approach, partial dynamical symmetries, has no bandmixing whatsoever and is parameter free, yet mimics closely the effects of bandmixing, due solely to the effects of finite valence nucleon number.

Purpose: To investigate the relation between these two seemingly contradictory approaches to understand how they can produce such similar results.

Method: To derive approximate relations between the two formalisms.

Results: A consistent relationship is found linking bandmixing to finite valence nucleon number effects on interband γ to ground-band $B(E2)$ values.

Conclusions: Two disparate approaches to one of the iconic characteristics of deformed nuclei are shown to be intimately related. Moreover, a systematic difference in their predictions also emerges naturally from the derivation. The qualitative linkage of valence nucleon number and the separation of vibrational and rotational degrees of freedom has long been assumed but never before explicitly demonstrated through complementary models.

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Despite the complexity of nucleonic motion in atomic nuclei, these many-body strongly interacting systems display remarkable regularities. Many of these can be understood in terms of idealized models, and many more can be interpreted in terms of simple perturbations to these models. It is the purpose of this Rapid Communication to investigate the relation between two seemingly dissimilar models, based on different premises, that yet produce very similar predictions for some of the most iconic observables in collective nuclei.

Some of the most characteristic observables in deformed even-even nuclei, used as benchmarks of structure for over 60 years, are the $B(E2)$ values from the $K = 2 \gamma$ vibrational band to the ground band. This intrinsic mode is usually the lowest lying collective one and hence, if we do not understand its properties, we can hardly claim to have in hand an adequate understanding of deformed nuclei.

Relative interband γ to ground-band $B(E2)$ values can be very simply modeled and are widely known experimentally. In the limit of zero rotation-vibration interaction, that is, if the wave functions can be written as a product of an intrinsic wave function (independent of spin), and a rotational D_{JMK} factor specifying the rotational character of the state, these $B(E2)$ values follow the so-called Alaga rules [1]. The Alaga rules do

not depend on the intrinsic structure of the states involved but only depend on the spins involved in the transitions and on the K values of the intrinsic excitations. They are calculated simply from squares of appropriate Clebsch-Gordan coefficients. They are given in the third column of Table I. For example, the Alaga rules for the relative $B(E2)$ values from the 2^+ state of the γ band to the 0^+ , 2^+ , and 4^+ levels of the ground band are in the relation 70, 100, and 5, respectively.

It is one of the triumphs of the collective model of deformed nuclei that these utterly simple, parameter-free predictions work as well as they do, as seen in the comparisons of columns 2 (experimental results) and 3 (Alaga rules) in Table I. Nevertheless, hardly unexpectedly, there are also clear empirical deviations from the Alaga rules. It turns out that these deviations behave in characteristic and systematic ways in all deformed nuclei. In particular, the deviations grow with spin, transitions that decrease the spin (e.g., $2^+ \rightarrow 0^+$) are always lower than the Alaga rules, and transitions that increase the spin (e.g., $2^+ \rightarrow 4^+$) are always larger than the Alaga rules.

For over half a century, these deviations from the Alaga rules have been interpreted in terms of rotation-vibration interactions, that is, in terms of bandmixing [2–6], between the states of the γ band and the ground band. In most cases,

TABLE I. Relative $B(E2)$ values for the decay of the γ band to the ground band in ^{168}Er . For each initial level, one transition is normalized to 100 and its error is included in those for the other transitions. These data are compared to the Alaga rules, to a bandmixing calculation [2–6] with mixing parameter $Z_\gamma = 0.035$, and to the parameter-free PDS predictions. The latter are taken from Ref. [7–9]. The quantity Z_γ is related to the spin-independent part of the mixing amplitude [2–6].

$J_i^\pi \rightarrow J_f^\pi$	^{168}Er	ALAGA	$Z_\gamma = 0.035$	PDS
$2_\gamma^+ \rightarrow 0^+$	56.2(11)	70	56.9	64.3
$2_\gamma^+ \rightarrow 2^+$	100	100	100	100
$2_\gamma^+ \rightarrow 4^+$	7.3(4)	5	7.6	6.3
$3_\gamma^+ \rightarrow 2^+$	100	100	100	100
$3_\gamma^+ \rightarrow 4^+$	62.6(14)	40	62.9	49.3
$4_\gamma^+ \rightarrow 2^+$	19.3(4)	34	20.2	28.1
$4_\gamma^+ \rightarrow 4^+$	100	100	100	100
$4_\gamma^+ \rightarrow 6^+$	13.1(12)	8.64	16.0	12.5
$5_\gamma^+ \rightarrow 4^+$	100	100	100	100
$5_\gamma^+ \rightarrow 6^+$	123(14)	57.1	117	79.6
$6_\gamma^+ \rightarrow 4^+$	11.2(10)	26.9	11.0	20.3
$6_\gamma^+ \rightarrow 6^+$	100	100	100	100
$6_\gamma^+ \rightarrow 8^+$	37.6(72)	10.6	23.6	18.0

the data for a given nucleus can be reproduced in terms of a single mixing parameter which varies smoothly with N and Z .

Recently, another interpretation, in terms of an approach called partial dynamical symmetry (PDS) [7], which has no mixing whatsoever between these bands, has also been shown to account [7–10] for a sizable fraction (but, notably, not all) of the deviations from the Alaga rules in a parameter-free way. The PDS predictions of deviations of these $B(E2)$ ratios from the Alaga rules arise solely from the effects of finite valence nucleon number. Given this situation, it would seem imperative to investigate how such dissimilar models can produce such similar predictions and to elucidate the relation between bandmixing and valence nucleon number.

It is the purpose of this Rapid Communication to do this by deriving a simple quantitative relation between these two models. These same derivations will also account for the observed characteristic differences between the two models as well.

In the bandmixing formalism [2–6] deviations from the Alaga rules for $E2$ transitions from the $K = 2$ γ band to the ground band ($K = 0$) are calculated using K -projection raising and lowering operators, leading to simple expressions for these corrections, in terms of a parameter, Z_γ , directly related to the mixing amplitude. Note that, like the Alaga rules, the correction factors are completely independent of the intrinsic structure of the ground and excited modes, depending only on the K values involved and on J , where J is the spin of the states that mix. These expressions are shown in Table II, and show the same characteristic features as seen in the data, namely, that the corrections to the Alaga rule increase with spin, and that (for positive Z_γ), transitions that decrease the spin are reduced by the mixing and that transitions that increase the spin are increased.

TABLE II. Expressions used to calculate bandmixing correction factors for the γ to ground-band $E2$ matrix elements. The corrections to the $B(E2)$ values are the squares of these expressions. Table based on Refs. [2–6].

J_i	J_f	Correction factor $\gamma \rightarrow g$
$J_f - 2$	J_f	$1 + (2J_f + 1)Z_\gamma$
$J_f - 1$	J_f	$1 + (J_f + 2)Z_\gamma$
J_f	J_f	$1 + 2Z_\gamma$
$J_f + 1$	J_f	$1 - (J_f - 1)Z_\gamma$
$J_f + 2$	J_f	$1 - (2J_f + 1)Z_\gamma$

Table I includes the results of a bandmixing calculation using a value of $Z_\gamma = 0.035$, chosen to fit the data. The bandmixing calculation reproduces the data extraordinarily well, except for the 6^+ to 8^+ transition but, in that case, the experimental error is rather large and the deviation of the bandmixing calculation from the data is only about 2σ .

Despite the evident success of this approach, there is, however, another model, mentioned above, called partial dynamical symmetry [7], that produces similar results for deformed nuclei with a seemingly different physical mechanism. The concept of a PDS is related to that of dynamical symmetries (DS) of the interacting boson model (IBM) [11]. In a PDS, some of the states retain the pure symmetry of the parent DS while others are mixed with other representations. There are several PDSs, depending on the parent DS of the IBM. The relevant case for deformed nuclei is a PDS based on the SU(3) limit, applicable to an axially symmetric deformed rotor. In this PDS, the ground and $K = 2$ γ bands retain absolutely pure SU(3) symmetry while most other states have broken symmetry. This allows one to remove the degeneracy of the first excited $K = 0$ band and the $K = 2$ (γ) band, which is seldom if ever seen experimentally, while yet retaining SU(3) wave functions for the ground and γ bands. This PDS has the remarkable property that γ -band to ground-band relative $B(E2)$ values are parameter free, that is, fixed by the model. The predictions therefore form a stringent test of the model.

Recently, the first extensive test of this SU(3)-based PDS was carried out in a study of 47 deformed and transitional rare earth nuclei [9]. A subsequent study looked at the actinides and $A \sim 100$ region with similar conclusions [10]. A consistent pattern was found for deformed nuclei, namely that, even though the ground and γ bands were pure SU(3), the PDS also gives deviations from the Alaga rules, that these deviations are in the direction of the data, and that they therefore mimic the characteristic trends in the dependence on spin and spin change as the bandmixing formalism. Interestingly, though, the PDS results systematically account only for about half of the experimental deviations from the Alaga rules. The PDS results for ^{168}Er are included in Table I where these features are seen. Similar results are found for other well-deformed nuclei where sufficient data are available. The success of the parameter-free PDS predictions is quite impressive and raises the question of how such disparate theoretical approaches as the PDS and bandmixing can obtain such similar results.

As noted above, the PDS predictions deviate from the Alaga rules solely due to the explicit inclusion of finite valence nucleon number effects, being larger for smaller numbers of valence nucleons. This dependence is intriguing as it reflects a well-known observed property of actual nuclei: The separation of rotational and vibrational degrees of freedom is sensitive to the number of shell-model configurations that can be mixed by residual interactions in the shell model and that number increases combinatorially with the number of valence nucleons. Therefore, one expects the interband $B(E2)$ values to approach the Alaga rules near midshell and deviate from them more as the number of valence nucleons decreases. The fact that the PDS automatically embodies such a dependence suggests that there could, in fact, be not only a relation between the PDS and bandmixing results for a given nucleus, but that the PDS inherently reflects the empirical variation of the strength of the rotation-vibration interaction across an entire region.

In the rest of this Rapid Communication we will show an approximate derivation of the relation between these models. The derivation is aided by the physics of the correction

factors. In the bandmixing approach the admixed transition amplitudes correspond to large intraband matrix elements and therefore the mixing amplitudes (Z_γ values) themselves are quite small. In the PDS approach, for boson numbers typical of deformed nuclei (typically ≥ 11), the finite boson number effects are also small. Thus, in both cases, we can make simplifying expansions of the expressions for the correction factors and drop small terms. Though the derivation is therefore utterly simple it has not been recognized analytically before.

Above, we gave the expressions for the $B(E2)$ values from the γ band to the ground band in the bandmixing formalism. Now we display the corresponding expressions for the PDS, highlighting the dependence on valence nucleon number. Since the ground and γ bands have pure SU(3) character in the PDS, we can use the SU(3) formulas derived by Van Isacker [12] using the $E2$ operator $T^{(E2)} = \alpha Q_{SU(3)} + \theta[(s^\dagger d) + (d^\dagger s)]^2$ for the γ -to-ground-band $B(E2)$ values. The term in α vanishes for SU(3) wave functions and so the $T^{(E2)}$ values are given by the term in θ . These are given below for the examples of J to J and $(J+2)$ transitions where J is the spin of the initial γ band level:

$$B(E2 : \gamma, J \rightarrow g, J) = \theta^2 \frac{2}{3} N \frac{3(J-1)(J+2)}{2(2J-1)(2J+3)} \frac{2(N-1)(2N-J-2)(2N-J)(2N+J-1)(2N+J+1)}{N(2N-3)(2N-1)[8(N-1)^2 - J(J+1)]} \quad (1)$$

$$B(E2 : \gamma, J \rightarrow g, J+2) = \theta^2 \frac{2}{3} N \frac{(J-1)J}{4(2J+1)(2J+3)} \frac{2(N-1)(2N-J-2)(2N+J-1)(2N+J+1)(2N+J+3)}{N(2N-3)(2N-1)[8(N-1)^2 - J(J+1)]}, \quad (2)$$

where N is the boson number equal to half the number of valence protons and neutrons each counted to the nearest closed shell. Strictly speaking, these formulas give the $B(E2)$ values in the Vergados basis while the predictions of the PDS require the Elliott basis. The Elliott-Vergados transformation adds unneeded complexity for our purposes here since the differences are very small and are well within the approximations used in the derivations below. The exact Elliott results were used in all the numerical values shown in this paper. Equations (1) and (2) look complicated but actually have a simple structure. Aside from an overall normalization constant, $\theta^2 \frac{2}{3} N$ (which cancels out in $B(E2)$ ratios), there are two factors. The first, depending only on the spins of the levels, is simply the Alaga rule itself, while the extensive second factor contains the dependence on boson number, N . The question is to see why such seemingly different expressions as those in Table II for the bandmixing formalism and those in Eqs. (1) and (2) give such similar trends and why the PDS results only account for about half the deviations of the data from the Alaga rules as the bandmixing approach with fitted choice of Z_γ .

To this end, we derive approximate correction factors, CF, for ratios of γ band to ground band $B(E2)$ values in both the bandmixing (BM) and PDS formalisms. To be specific let us consider, as an example, the ratio $B(E2 : 2_\gamma^+ \rightarrow 4_{gr}^+)/B(E2 : 2_\gamma^+ \rightarrow 2_{gr}^+)$. For the case of bandmixing, we get from Table II:

$$CF_{BM}(2_\gamma^+ \rightarrow 4_{gr}^+/2_\gamma^+ \rightarrow 2_{gr}^+) = \left(\frac{1+9Z_\gamma}{1+2Z_\gamma} \right)^2. \quad (3)$$

Since Z_γ is small (~ 0.04), we can drop quadratic terms in Z_γ in squaring the correction factor, giving

$$CF_{BM} \sim (1+18Z_\gamma)/(1+4Z_\gamma) \sim (1+18Z_\gamma)(1-4Z_\gamma),$$

where, in the second step, we have expanded the denominator and kept only the first term. Again multiplying out and dropping quadratic terms in Z_γ gives

$$CF_{BM} \sim 1+14Z_\gamma. \quad (4)$$

Now consider the PDS expressions. As noted, the normalization factors cancel out in the $B(E2)$ ratios and the first factors after the normalization just give the ratio of Alaga rules. The correction factors are therefore given solely by the last factors that depend on the boson number, N . First note that the denominators in these factors are identical for transitions from even spin members of the γ and hence cancel out in the ratio. (Likewise, they are equal for transitions starting from odd spin states.) Also, many of the factors in the numerators are identical. Taking the same ratio as above then gives

$$CF_{PDS} = \frac{2N+5}{2N-2}. \quad (5)$$

Dividing numerator and denominator by $2N$, and expanding the denominator and dropping terms in $(1/2N)^2$, gives

$$CF_{PDS} = \frac{1+5/2N}{1-2/2N} \sim 1 + \frac{7}{2N}. \quad (6)$$

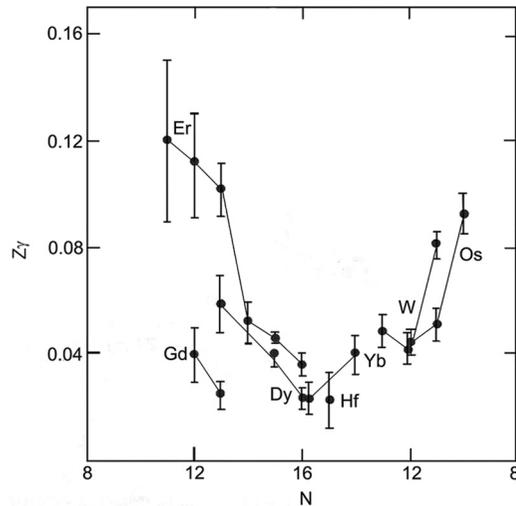


FIG. 1. Z_γ values as a function of half the number of valence nucleons extracted for rare-earth transitional and deformed nuclei. Figure taken from Ref. [13].

This certainly resembles the form of Eq. (4). In fact, we note, interestingly, that the coefficient of Z_γ is twice that of $1/2N$.

Empirically, the strength of the bandmixing varies across the deformed region, minimizing near midshell where the number of valence nucleons is largest, giving the largest configuration spaces, the most collectivity, and the best separation of intrinsic and rotational degrees of freedom. This is seen in Fig. 1, taken from Ref. [13], which also discusses the relation to valence nucleon number.

In addition to noting the generally parabolic behavior of Z_γ across the region, we see in fact that the approximation $Z_\gamma \sim 1/2N$ describes the trend for well-deformed nuclei quite well. So we can make the ansatz that $Z_\gamma \sim 1/2N$. Making that substitution in Eq. (6) gives

$$CF_{\text{PDS}} \sim 1 + 7Z_\gamma. \quad (7)$$

This is identical to Eq. (4) except for the factor of two difference in the correction term. Exactly similar results occur for other ratios, such as $4^+ \rightarrow 2^+ / 4^+ \rightarrow 4^+$ and $6^+ \rightarrow 4^+ / 6^+ \rightarrow 6^+$, with the same factor of two relation in the coefficients of the correction terms. These results are summarized in Table III. These results can be written in the general form given in Table IV, which, again, shows the similarity of the two models and the factor of 2 difference in the correction terms.

Equations (4) and (7), and analogous ones for other ratios, to within the accuracies of the expansions involved, show that indeed the model of γ -band-ground-band mixing and the PDS model produce corrections to the relative $B(E2)$ values of the Alaga rules that are very similar. They are identical in form and have coefficients of the correction terms differing by a factor of two, when the empirical relation $Z = 1/2N$ is made. Moreover, the PDS predictions of relative $B(E2)$ values across a deformed region reflect the empirical behavior of the deviations from the Alaga rules with proton and neutron number. This accounts for the similar behavior of these two

TABLE III. Comparison of bandmixing and PDS corrections to the Alaga rules for ratios of γ to ground-band $B(E2)$ values (see text).

$B(E2)$ ratio ($J_i - J_f$)/($J_i - J_f'$)	Bandmixing	PDS
$2^+ \rightarrow 0^+ / 2^+ \rightarrow 2^+$	$1 - 6Z_\gamma$	$1 - 3Z_\gamma$
$2^+ \rightarrow 4^+ / 2^+ \rightarrow 2^+$	$1 + 14Z_\gamma$	$1 + 7Z_\gamma$
$3^+ \rightarrow 4^+ / 3^+ \rightarrow 2^+$	$1 + 14Z_\gamma$	$1 + 7Z_\gamma$
$4^+ \rightarrow 2^+ / 4^+ \rightarrow 4^+$	$1 - 14Z_\gamma$	$1 - 7Z_\gamma$
$4^+ \rightarrow 6^+ / 4^+ \rightarrow 4^+$	$1 + 22Z_\gamma$	$1 + 11Z_\gamma$
$5^+ \rightarrow 6^+ / 5^+ \rightarrow 4^+$	$1 + 22Z_\gamma$	$1 + 11Z_\gamma$
$6^+ \rightarrow 4^+ / 6^+ \rightarrow 6^+$	$1 - 22Z_\gamma$	$1 - 11Z_\gamma$
$6^+ \rightarrow 8^+ / 6^+ \rightarrow 6^+$	$1 + 30Z_\gamma$	$1 + 15Z_\gamma$

models seen for ^{168}Er in Table I, for the other nuclei discussed in Refs. [9] and [10], and for the fact that the PDS only goes about halfway from the Alaga rules to the data.

In addition, since N is a simple property of the shell model and the number of nucleons whereas Z_γ has to be extracted from the measured transition intensities for each nucleus, the relation $Z_\gamma \sim 1/2N$ suggests that one can estimate the amount of bandmixing in other nuclei even without branching ratio data. It will be interesting to see if this idea works on other mass regions. To the extent that it does work, it renders the bandmixing approach approximately parameter free.

It is useful to understand the accuracy of the approximate results we have obtained since the expansions made depend on J , on Z_γ , and on $1/2N$. Looking at the spin dependence of the factors multiplying Z_γ in Table II and the factors involving boson number in Eqs. (1)–(2) (as well as the numerical coefficients in Table III), it is clear that the approximations are better for low spin and higher boson numbers. The quality of the approximations is seen in Fig. 2, which compares the PDS formulas with exact calculations. For the first few levels of the

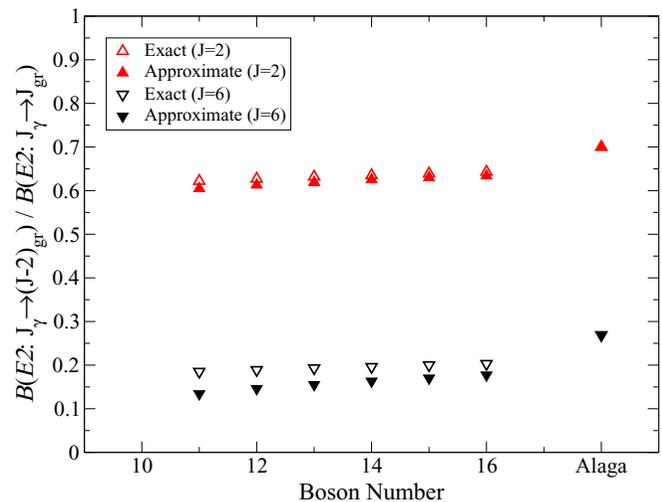


FIG. 2. PDS calculations for $B(E2)$ ratios from the 2^+ and 6^+ states of the γ band to the ground band calculated with the exact PDS formulas and with the approximations derived in this paper.

TABLE IV. Generalized relations for the PDS corrections to the γ - to ground-band $B(E2)$ ratios corresponding to the specific ratios in Table III.

$B(E2)$ ratio	Bandmixing	PDS
J -even ($J \rightarrow J + 2$)/($J \rightarrow J$)	$1 + 2(2J + 3)Z_\gamma$	$1 + (2J + 3)Z_\gamma$
J -even ($J \rightarrow J - 2$)/($J \rightarrow J$)	$1 - 2(2J - 1)Z_\gamma$	$1 - (2J - 1)Z_\gamma$
J -odd ($J \rightarrow J + 1$)/($J \rightarrow J - 1$)	$1 + 2(2J + 1)Z_\gamma$	$1 + (2J + 1)Z_\gamma$

γ band and for boson numbers appropriate for well-deformed nuclei near midshell ($N > 12$) they are quite good. For higher spin levels (e.g., $J = 6$) and lower boson numbers they start to break down.

To summarize, we have shown a quantitative relation linking the behavior of two seemingly very dissimilar models of collective behavior in deformed nuclei—a rotational model with rotation-vibration interactions and a model of pure rotational bands whose predictions depend only on the number of valence nucleons. We have seen that both models predict deviations of iconic observables, namely γ to ground band relative $B(E2)$ values, from the Alaga rules and also why the predictions of these models also differ from each other in a characteristic way. The results provide a quantitative link (long assumed in a qualitative way) between valence nucleon number (size of shell model configuration space) and the

separation of rotational and intrinsic degrees of freedom. They thus also offer a way to estimate bandmixing in nuclei where branching ratios from the γ to ground band are not known.

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