Cumulants of multiplicity distributions in most-central heavy-ion collisions

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I investigate the volume corrections on cumulants of total charge distributions and net proton distributions. The required volume information is generated by an optical Glauber model. I find that the corrected statistical expectations of multiplicity distributions mimic the negative binomial distributions at noncentral collisions, and they tend to approach the Poisson ones at most-central collisions due to the "boundary effects," which suppress the volume corrections. However, net proton distributions and reference multiplicity distributions are sensitive to the external volume fluctuations at most-central collisions, which imply that one has to consider the details of volume distributions in event-by-event multiplicity fluctuation studies.

I. INTRODUCTION

In search of critical end points (CEPs) in a quantum chromodynamics (QCD) phase diagram [1–6], event-by-event multiplicity fluctuations [3–6] have been regarded as a very useful tool in relativistic heavy-ion experiments [7–11]. Besides the fluctuation data and theoretical studies on critical fluctuations [12–15], it is clear that a sufficient understanding of noncritical statistical fluctuations is also important [16–26]. As one of the noncritical corrections, the volume corrections on the cumulants of multiplicity distributions have been discussed in previous studies [27–31]. Especially in Ref. [29], Skokov and his collaborators have derived a general formalism for the corrected cumulants. However, quantitative estimations of the volume corrections on data are obscure in previous statistical studies.

The effect of volume corrections on multiplicity distributions are caused by two things: (i) the distributions of volume in heavy-ion collisions (HICs) and (ii) the multiplicity fluctuations of reference particles used for centrality definitions. To quantitatively estimate this effect in theoretical studies, therefore, the multiplicity fluctuations of fluctuation measures and reference particles, as well as the volume distributions, need to be investigated.

In this work, I continue my studies from Ref. [32]. In Ref. [32], I derived a general formalism for the multiplicity distributions measured in experiments. With the corrected expression, I calculated the volume corrections under the Poisson approximations. I found that the statistical expectations of multiplicity distributions mimic the negative binomial distributions (NBDs) in noncentral collisions. I also offered some reasonable explanations for the experimental data of (net) charge distributions. This work indicated the importance of volume corrections in heavy-ion experiments.

In this work, I extend the study to the most-central collisions, i.e., the top few centrality percentages. Different from the platformlike distributions in noncentral collisions, the volume distributions in most-central collisions need more phenomenological considerations. This is due to the nontrivial features of volume distributions in most-central collisions, which are reflected in the rapid decreasing of the probability distributions of the reference multiplicity when the corresponding reference multiplicity is above an appropriate value. Such tail distributions are caused by the finite size and energy of the colliding nuclei, the fluctuating positions of the nucleons in the colliding nuclei, etc. In this work, I use an optical Glauber model to give more realistic descriptions of this information in relativistic heavy-ion collisions.

Meanwhile, the multiplicity distributions measured in most-central heavy-ion experiments also reveal some nontrivial features. Different from the large deviations in noncentral collisions, the scale variances of charged hadron distributions are close to the trivial Poisson expectations in most-central collisions [8,33,34]. Moreover, the cumulant products of net proton distributions showed obvious nonmonotonic behavior at 0%–5% centrality percentage [9], which was regarded as one of the most striking observables in searching for the CEP. The 0%–5% centrality percentage is the transition range of the volume distributions, and I show in this work that the volume distributions in this centrality range are very sensitive to the parameters of the phenomenological model. To study the corresponding volume corrections, therefore, more realistic investigations of the volume distributions are required. This, as far as I know, has not been addressed in previous statistical fluctuation studies.

The paper is organized as follows. In Sec. II, I give the main formalism used in this work. The analytical properties of charge fluctuations have also been discussed with a toy volume distribution. In Sec. III I show some numerical results on the cumulants of total charge fluctuations and discuss the effect of Gaussian-type external volume fluctuations on net proton distributions. I give a summary in the final section.

II. THE MODEL

In this section, I first review the main formalism of volume corrections given in Ref. [32] by using another expression, and then I employ an optical Glauber model to describe the required volume distributions in cumulant calculations. Finally, some analytical solutions are discussed by using a toy volume distribution.

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A. Volume corrections

To bridge the gaps between experimental measurements and theoretical calculations, I have derived a general expression in a statistical model [32] for recent data on multiplicity distributions at the BNL Relativistic Heavy Ion Collider (RHIC) [7–11,34,35]. The conditional probability distribution for the distribution of multiplicity q in a given reference multiplicity bin k reads

$$\mathscr{P}_{B|A}(q|k) = \frac{\int d\Omega F(\Omega) P_B(q; \Omega) P_A(k; \Omega)}{\mathscr{P}_A(k)}, \quad (1)$$

with $\mathscr{P}_A(k)$ being the distribution of reference multiplicity,

$$\mathscr{P}_A(k) = \int d\Omega F(\Omega) P_A(k; \Omega). \quad (2)$$

Here q represents the multiplicity of moment-analysis particles in subevent B, and k represents the multiplicity of particles for the centrality definition in subevent A. The latter k is also called the reference multiplicity. $P_A(k; \Omega)$ and $P_B(q; \Omega)$ stand for the corresponding multiplicity distributions in a specific statistical ensemble with a set of principal thermodynamic variables Ω. I have assumed that subevent A and subevent B are independent of each other in each event (thermal system).

If one only considers the distribution of the system volume V, Eqs. (1) and (2) can be written as

$$\mathscr{P}_{B|A}(q|k) = \frac{1}{\mathscr{P}_A(k)} \int dV F(V) P_A(k; V) P_B(q; V),$$

$$\mathscr{P}_A(k) = \int dV F(V) P_A(k; V). \quad (3)$$

In experiments, total charged hadrons are usually chosen as reference particles for centrality definitions [7–11,34,35]. As shown in Ref. [32], the multiplicity fluctuations of total charges can be described by the Poisson distributions in a fixed volume. Therefore $P_A(k; V)$ can be described by a Poisson distribution, and then the volume V [volume distribution $F(V)$] can be substituted by the Poisson parameters λ [$f(\lambda)$]. Consequently,

$$\mathscr{P}_{B|A}(q|k) = \mathscr{N}(k) \int d\lambda f(\lambda) \frac{\lambda^k e^{-\lambda}}{k!} P_B(q; \lambda),$$

$$\mathscr{P}_A(k) = \int d\lambda f(\lambda) \frac{\lambda^k e^{-\lambda}}{k!}, \quad (4)$$

where $\mathscr{N}(k) = 1/\mathscr{P}_A(k)$ is normalization factor.

Similar to Ref. [29], the first four cumulants of $\mathscr{P}_{B|A}(q|k)$ can be written as

$$\frac{c_1}{\langle \lambda \rangle} = \kappa_1,$$

$$\frac{c_2}{\langle \lambda \rangle} = \kappa_2 + \kappa_1^2 v_2,$$

$$\frac{c_3}{\langle \lambda \rangle} = \kappa_3 + 3\kappa_2 \kappa_1 v_2 + \kappa_1^3 v_3, \quad (5)$$

$$\frac{c_4}{\langle \lambda \rangle} = \kappa_4 + \left(4\kappa_3 \kappa_1 + 3\kappa_2^2\right) v_2 + 6\kappa_2 \kappa_1^2 v_3 + \kappa_1^4 v_4.$$

Here κ_1, κ_2, κ_3, and κ_4 are the first four reduced cumulants of $P_B(q; \lambda)$:

$$\kappa_1 = \frac{\sum q P_B(q; \lambda)}{\lambda} \equiv \frac{\bar{q}}{\lambda},$$

$$\kappa_2 = \frac{\sum (\Delta q)^2 P_B(q; \lambda)}{\lambda},$$

$$\kappa_3 = \frac{\sum (\Delta q)^3 P_B(q; \lambda)}{\lambda}, \quad (6)$$

$$\kappa_4 = \frac{\sum (\Delta q)^4 P_B(q; \lambda) - 3(\sum (\Delta q)^2 P_B(q; \lambda))^2}{\lambda},$$

with $\Delta q \equiv q - \bar{q}$, and

$$v_2 = \frac{\langle (\Delta \lambda)^2 \rangle}{\langle \lambda \rangle}, \quad v_3 = \frac{\langle (\Delta \lambda)^3 \rangle}{\langle \lambda \rangle},$$

$$v_4 = \frac{\langle (\Delta \lambda)^4 \rangle - 3\langle (\Delta \lambda)^2 \rangle^2}{\langle \lambda \rangle}, \quad (7)$$

with $\Delta \lambda = \lambda - \langle \lambda \rangle$, $\langle (...) \rangle \equiv \mathscr{N}(k) \int d\lambda f(\lambda) \frac{\lambda^k e^{-\lambda}}{k!} (...)$, and

$$\langle \lambda^m \rangle = \mathscr{N}(k) \int d\lambda f(\lambda) \frac{\lambda^k e^{-\lambda}}{k!} \lambda^m = \frac{(k+m)!}{k!} \frac{\mathscr{P}_A(k+m)}{\mathscr{P}_A(k)}. \quad (8)$$

For the charge fluctuations in this work, I assume $P_B(q; \lambda)$ is a Poisson distribution with parameter $\mu = \bar{q} = b\lambda$. The reduced cumulants read

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = b. \quad (9)$$

For the net conserved charge fluctuations, I assume $P_B(q; \lambda)$ is a Skellam distribution with parameters $\mu_+ = b_+ \lambda$ for the positive conserved charges and $\mu_- = b_- \lambda$ for the negative conserved charges. The reduced cumulants read

$$\kappa_1 = \kappa_3 = b_+ - b_- \equiv \Delta b; \quad \kappa_2 = \kappa_4 = b_+ + b_- \equiv b_{\pm}. \quad (10)$$

The values of b, b_+, and b_- are related to the multiplicity ratios of different particle species with different acceptance cuts, which can be described by the statistical model if the kinematic cuts on particles of interest can be well simulated. For example, in the net proton case, there is a simple relation between b_- and b_+ in the classical statistical model with a grand canonical ensemble, $b_-/b_+ = \exp{(-2\mu_B/T)}$, where μ_B and T are the baryon chemical potential and the temperature.

In noncentral heavy-ion collisions, one has $\mathscr{P}_A(k + m)/\mathscr{P}_A(k) \simeq 1$ for small m and

$$\langle \lambda \rangle = k+1, \quad v_2 = 1, \quad v_3 = 2, \quad v_4 = 6. \quad (11)$$

Then the first four cumulants of the total charge distributions can be written as

$$\frac{c_1}{k+1} = b \equiv M/(k+1), \quad \frac{c_2}{k+1} = b + b^2,$$

$$\frac{c_3}{k+1} = b + 3b^2 + 2b^3, \quad (12)$$

$$\frac{c_4}{k+1} = b + 7b^2 + 12b^3 + 6b^4.$$

And the first four cumulants of the net conserved charge distributions can be written as

$$
\begin{aligned}
\frac{c_1^N}{k+1} &= \Delta b \equiv \frac{M_+}{k+1} - \frac{M_-}{k+1}, \\
\frac{c_2^N}{k+1} &= b_\pm + \Delta b^2, \\
\frac{c_3^N}{k+1} &= \Delta b + 3 b_\pm \Delta b + 2 \Delta b^3, \\
\frac{c_4^N}{k+1} &= b_\pm + 4 \Delta b^2 + 3 b_\pm^2 + 12 b_\pm \Delta b^2 + 6 \Delta b^4.
\end{aligned}
\tag{13}
$$

Equations (12) and (13) are the appropriate solutions obtained in Ref. [32] in noncentral collisions. To obtain the exact values of cumulants in a given reference multiplicity bin k_0, one only needs the information of the mean multiplicity of moment analysis particles, similar to the Poisson and Skellam expectations. For the (net) charge distributions reported by the STAR Collaboration, the volume corrections are important because the reduced cumulants b (b_+, b_-) are of the order of $O(1)$. Therefore the volume corrections play a crucial role in describing the negative binomial multiplicity distributions of (net) charges [32].

Some remarks are in order here. In the above discussions, I have assumed that b, b_+, and b_- are independent of centrality (volume). This is because, in typical statistical models, the Poisson parameters μ (μ_+, μ_-) and λ are both proportional to volume. However, this assumption can be contaminated by various effects, such as (i) the unplatform distribution of multiplicity as a function of pseudorapidity and (ii) the uncertainties from experiment inefficiencies, etc. To reduce the uncertainties from centrality-dependent b (b_+, b_-), therefore, I suggest the following centrality definition approach. First, each event is divided into two subevents with different (pseudo)rapidity cuts: $y_{\min} < y < y_{\max}$ and $-y_{\max} < y < -y_{\min}$. Here a rapidity gap, $2y_{\min}$, is employed to reduce the autocorrelation effect[1] of these two subevents in each event. These kinematic cuts will significantly reduce the uncertainties from source (i) in symmetric nuclear collisions. Second, one of the subevents is randomly chosen for centrality definition, and the other one is used for moment analysis in each event. This approach will largely suppress the contributions from source (ii). Finally, with the role reversal of these two subevents, all the events can be used to calculate the corresponding cumulants twice. In this sense this approach redoubles the statistics, and it is extremely useful due to the statistic-hungry properties of the cumulant calculations. Note that this approach cannot reduce the nonlinear contributions from fluctuations of intensive variables, e.g., μ_B and T.

B. Volume distributions

As mentioned above, the volume distributions have been substituted by $f(\lambda)$, the distributions of Poisson parameters λ. The reason that $\mathscr{P}_A(k+m)/\mathscr{P}_A(k) \simeq 1$ is used in Ref. [32] in noncentral collisions is that the volume distributions are platformlike distributions in the corresponding centrality range. To give the cumulant calculations in most-central collisions, however, the details of $f(\lambda)$ need to be investigated.

In this work an optical Glauber model [36–40] is used to simulate the distribution of $f(\lambda)$ (see the Appendix for the details):

$$
f(\lambda) = \mathscr{R} \int P(n;\zeta)[1 - P_0(\zeta)] 2\pi \zeta \, d\zeta,
\tag{14}
$$

where $\mathscr{R}$ is a normalization factor and ζ is the impact parameter. The correlation function can be written as [37]

$$
P(\lambda;\zeta) = \frac{1}{\sqrt{2\pi a\lambda}} \exp\left[-\frac{[\lambda - hn(\zeta)]^2}{2a\lambda} \right],
\tag{15}
$$

which stands for Gaussian-type volume fluctuation.[2] The parameter a is the strength of fluctuation. In the absence of fluctuation $a = 0$, the Gaussian distribution becomes a δ function. Here

$$
n(\zeta) = \left[\frac{(1-x)}{2} n_{\mathrm{part}}(\zeta) + x n_{\mathrm{coll}}(\zeta) \right] \Big/ n(0),
\tag{16}
$$

with $n_{\mathrm{part}}(\zeta)$ being the number of participant nucleons and $n_{\mathrm{coll}}(\zeta)$ being the number of binary nucleon-nucleon collisions. The fitting parameter h is determined by the size and the energy of the colliding nuclei. The fraction parameter is chosen as $x = 0.12$ in this work. $P_0(\zeta)$ is the probability of no interaction among the nuclei [37]:

$$
P_0(\zeta) = \left[1 - \frac{n_{\mathrm{coll}}(\zeta)}{N_A N_B} \right]^{N_A N_B},
\tag{17}
$$

with the mass numbers of the collisional nuclei N_A and N_B.

C. Analytical solutions

Before presenting the numerical results of cumulant calculations in the next section, I now discuss some analytical properties of the multiplicity distributions by using a toy volume distribution.

Assume that the distribution $f(\lambda)$ is a platform distribution in the range $\lambda \in [0,h]$, i.e.,

$$
f(\lambda) = \begin{cases} 1/h, & \lambda \leqslant h, \\ 0, & \lambda > h, \end{cases}
\tag{18}
$$

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[1] In this work, the autocorrelation means the correlation of subevents A and B in one event. It can be generated by collective flow, hadronization, jet, resonance decays, etc. This is different from the one discussed in the present work, i.e., the correlation from event-by-event analysis.

[2] The external volume fluctuation is very different from the so-called "volume fluctuation" in Ref. [29]. The so-called volume fluctuation is actually the volume correction discussed in the previous subsection. While the external volume fluctuation is the fluctuating term of the Glauber volume distribution. In hydrodynamics, it can be generated by fluctuating initial conditions, fluctuating freeze-out hypersurfaces, etc. The volume corrections still exist in the absence of external volume fluctuations.

where h corresponds to the upper boundary of the system volume.

From the above assumptions and the Poisson approximation of $P_B(q;\mu)$ with parameter $\mu = b\lambda$, Eq. (4) can be rewritten as

$$\mathscr{P}_{B|A}(q|k) = \frac{b^q}{q!} \frac{\int_0^h d\lambda \lambda^{k+q} e^{-(1+b)\lambda}}{\int_0^h d\lambda \lambda^k e^{-\lambda}}$$

$$= \frac{1}{q!} \left(\frac{b}{1+b}\right)^q \left(1 - \frac{b}{1+b}\right)^{k+1}$$

$$\times \frac{\gamma[k+q+1, h(1+b)]}{\gamma(k+1, h)}, \qquad (19)$$

where $\gamma(s,x)$ is the lower incomplete γ function.[3]

At noncentral collisions, $k + q \ll h$,

$$\mathscr{P}_{B|A}(q|k) \simeq \text{NBD}(q;r,p) \equiv \frac{(r+q-1)!}{(r-1)!q!} p^q (1-p)^r, \qquad (20)$$

which mimics a standard negative binomial distribution with parameters $r = k + 1$ and $p = b/(1+b)$. These results are consistent with the approximate solutions given in Ref. [32] [see also Eq. (12) in this work].

At most-central collisions, $k \gg h$,

$$\mathscr{P}_{B|A}(q|k) \simeq \frac{e^{-bh}(bh)^q}{q!} \frac{k+1}{k+q+1}, \qquad (21)$$

where I have used

$$\gamma(s,x) = x^s (s-1)! e^{-x} \sum_{m=0}^{\infty} \frac{x^m}{(s+m)!} \qquad (22)$$

and the leading-order approximation

$$\sum_{m=0}^{\infty} \frac{x^m}{(s+m)!} \simeq \frac{1}{s!} \qquad (23)$$

when $x/s \ll 1$. For the distribution with $q \ll k$,

$$\mathscr{P}_{B|A}(q|k) \sim \frac{e^{-bh}(bh)^q}{q!}, \qquad (24)$$

which mimics a Poisson distribution with parameter $\tilde{\mu} = bh$.

Therefore, as the reference multiplicity k increases, the distribution $M(k)$ will be saturated at high reference-multiplicity range because the parameter $\tilde{\mu}$ is independent of k. Moreover, the variances will change the value from the NBD expectations to the Poisson predictions. These features have been also observed in experiments [8,33,34], which indicate that the volume corrections are weak in most-central heavy-ion collisions. The suppression of volume corrections in most-central collisions has been found in previous work [29] but with so-called symmetric volume fluctuations. The different contributions of volume corrections on cumulants of multiplicity distributions in noncentral and most-central collisions are due to the tail distributions of the volume at its upper boundary and I call this the "boundary effect" in this work.

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[3]The author thanks T. S. Biro for a discussion of this point.

III. RESULTS AND DISCUSSIONS

In this section I present the results of the corrected cumulants of total charge distributions and net proton distributions.[4]

The free parameters in the model are the multiplicity ratios b (b_+ and b_-), the Glauber parameters h and a, and some other parameters in the optical Glauber model. To determine these values at a given collision energy, the information of reference multiplicity distribution $\mathscr{P}_A(k)$ and the mean multiplicity distributions of the fluctuation measures $\mathscr{M}(k)$ are required. Unfortunately, both of them are not available in current experiments. Instead of realistic baseline predictions, the main purpose of this work is to investigate the sensitivity of higher-order cumulants of multiplicity fluctuations on the collision geometry in most-central collisions. Therefore, as the first attempt, the fitting parameter of the optical Glauber model in this work is set to $h = 400$ and the strength parameter of external Gaussian fluctuation is set to $a = 0.01$ as the default choice unless stated explicitly. For the net proton distribution, the Poisson parameter for proton distribution is set to $\mu_+ = b_+\lambda = 0.04\lambda$, and the one for antiproton distribution is set to $\mu_- = b_-\lambda = 0.01\lambda$.

Figure 1(a) shows the distribution of the Poisson parameter $f(\lambda)$ and the corresponding distribution of the reference multiplicity $\mathscr{P}_A(k)$ [see also the black-solid curve in Fig. 2(a)]. The contributions of event-by-event Poisson fluctuations on $\mathscr{P}_A(k)$ are obvious in a high reference multiplicity range, i.e., most-central collisions.

A. Total charges

In the total charges case, one has $b = 1$ if the centralities are also defined by the multiplicity of total charges with the recommended centrality definition approach. The numerical results for cumulants of total charge distributions are shown in Figs. 1(b) and 1(c). The corresponding bands represent the NBD expectations carried out by Eq. (20).

The numerical results can be well described by the NBD expectations in a wide centrality range, which is consistent with my previous conclusions [32]. At most-central collisions, due to the boundary effects, the mean values of total charge distributions tend to saturate in the high reference multiplicity range. Meanwhile, the variances of total charge distributions tend to approach their mean value, which make the distributions mimic the Poisson distributions at the second-order moment level.

To further investigate this transition, Fig. 1(d) shows the scale variance of total charge fluctuations $\omega = \sigma^2/M$. It is quite clear that, from peripheral to central collisions, the scale variances of total charge fluctuations vary from being NBD expectations ($\omega = 1 + M/(k+1) \simeq 2$) to Poisson expectations ($\omega = 1$). The deviations of scale variances from the Poisson expectations in a high reference multiplicity range (ultracentral collisions) are due to the external Gaussian fluctuation on collision geometry [see Eq. (14)].

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[4]The relations between the net proton distributions and their corresponding net conserved charge distributions, i.e., net baryon distributions, are discussed in Refs. [41,42].
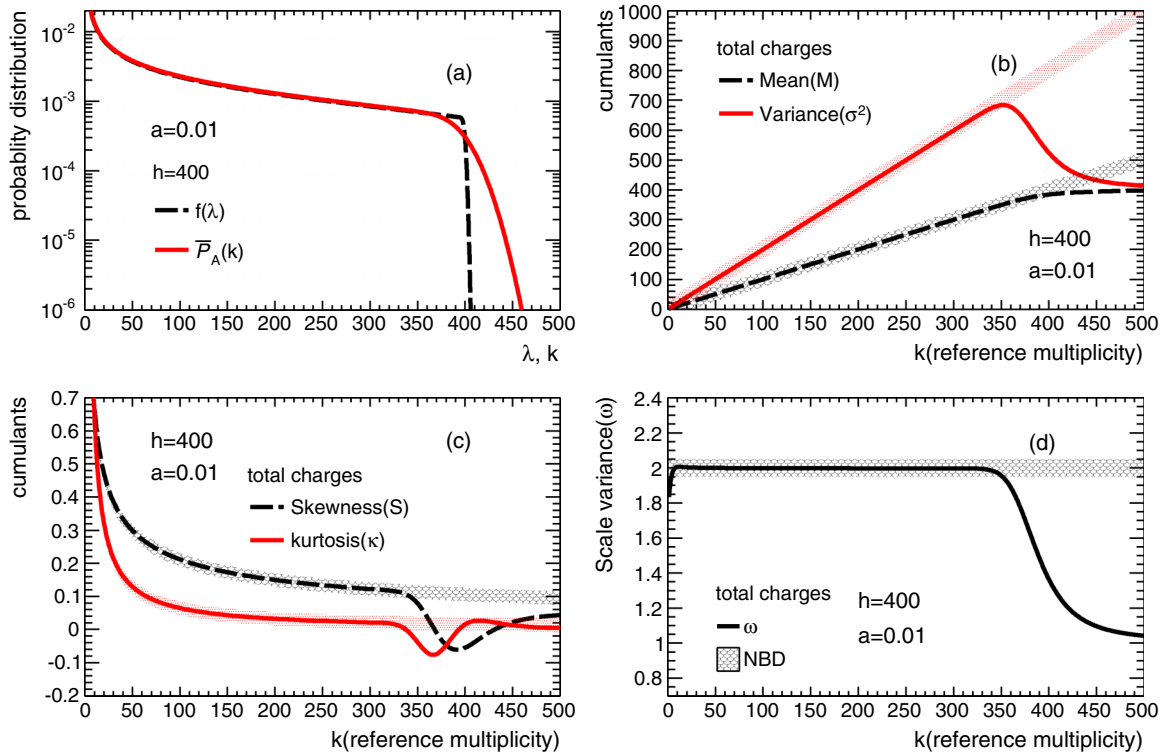
FIG. 1. (a) Probability distribution of the Poisson parameter λ (black dashed line) and the reference multiplicity (red solid line). (b) Mean value (M) and variance (σ^2) of total charge distributions. (c) Skewness (S) and kurtosis (κ) of total charge fluctuations. (d) The scale variance ($\omega = \sigma^2/M$) of total charge distributions. The notation $\bar{P}_A(k)$ in panel (a) stands for $\mathscr{P}_A(k)$ in the text. The bands in panels (b), (c), and (d) represent the corresponding NBD expectations from Eq. (20).
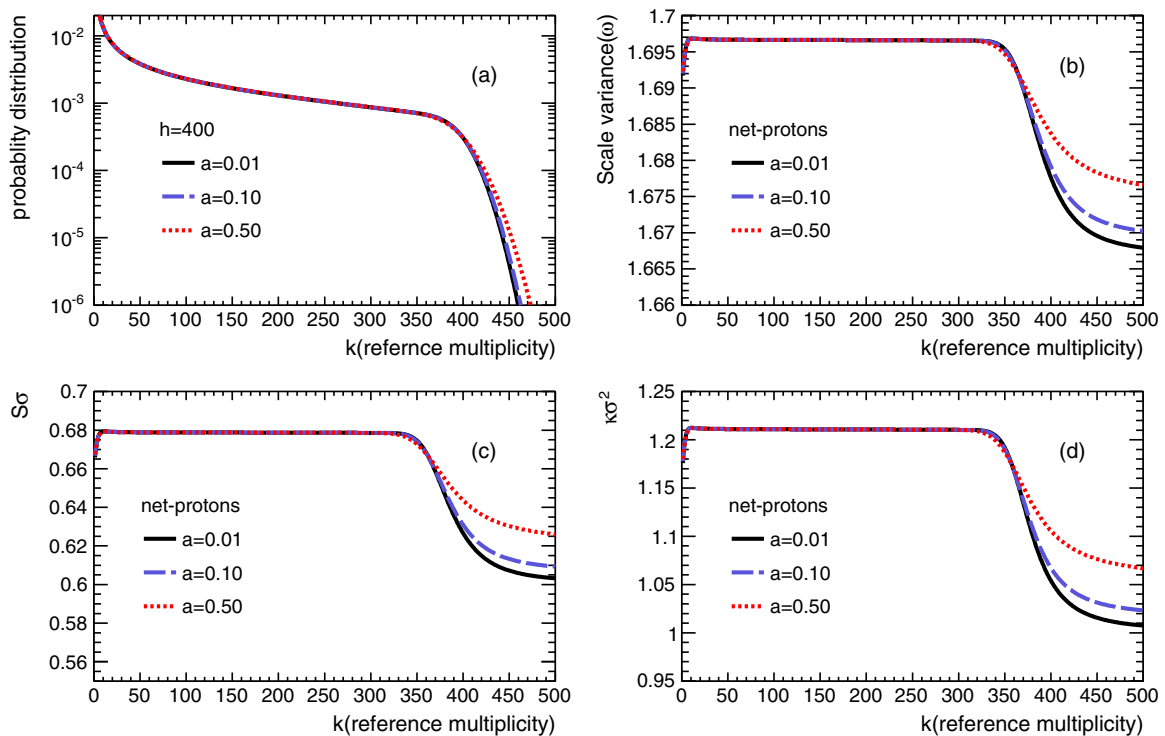

FIG. 2. (a) Distributions of reference multiplicity and cumulant products, (b) $\omega = c_2/c_1$, (c) $S\sigma = c_3/c_2$, and (d) $\kappa\sigma^2 = c_4/c_2$ of net proton distributions with various strengths of Gaussian-type external volume fluctuations: $a = 0.01$ (black solid lines), $a = 0.10$ (blue dashed lines), and $a = 0.50$ (red dotted lines).

It is interesting to find that the centrality dependence of skewness and kurtosis have nonmonotonic behavior in a high reference multiplicity range, and the minimum values of skewness and kurtosis can be negative. Although the volume correction effects are suppressed by the boundary effects, the nontrivial behavior of high-order cumulants of total charge distributions imply that realistic descriptions of the volume distributions (e.g., collision geometry, etc.) are important in studying the transition of multiplicity distribution from noncentral to most-central heavy-ion collisions.

B. Net protons

Now I focus on fluctuations of net protons. I calculate the cumulant products $\omega \equiv c_2/c_1$, $S\sigma \equiv c_3/c_2$, and $\kappa\sigma^2 \equiv c_4/c_2$ of net proton distribution as shown in Figs. 2(b), 2(c), and 2(d) (black solid lines).

Except for the most-central collision range, the numerical results are consistent with the approximate solutions given in Ref [32], i.e.,

$$\omega = \beta(1-\alpha) + \frac{1+\alpha}{1-\alpha} = 1.697, \tag{25}$$

$$S\sigma = 2\beta(1-\alpha) + \frac{\beta(1-\alpha^2)+1-\alpha}{\beta(1-\alpha)^2+1+\alpha} = 0.679, \tag{26}$$

$$\kappa\sigma^2 = 6\beta\left(\gamma - \frac{2\alpha}{\gamma}\right) + 1 = 1.211, \tag{27}$$

where $\alpha = M_-/M_+ = b_-/b_+$, $\beta = M_+/(k+1) = b_+$, and $\gamma = \beta(1-\alpha)^2+1+\alpha$. The approximate solutions of these cumulant products are independent of the centralities. In general, the cumulant products can been used to extract the chemical freeze-out parameters from the hadron resonance gas model [6,21]. However, I note that the effects of volume corrections need to be taken into account for these extractions.

Although I cannot determine the sources and magnitudes of the Gaussian-type external volume fluctuations, it is still interesting to study the effect of such external fluctuations on the measured cumulants. I therefore calculate the cumulant products of net proton distributions with various strength parameters: $a = 0.01$ (default), $a = 0.10$, and $a = 0.50$. Note that for some specific calculations at given collision energies, these parameters in the Glauber model could be constrained by the data of $\mathscr{P}_A(k)$ if it becomes available.

Figure 2(a) shows that the distribution of reference multiplicity has a wider tail for a larger strength parameter a. The effect of different external volume fluctuations can be distinguished only in the most-central collision range [see Figs. 2(b)–2(d)]. This is because for the cumulants of net proton distributions calculated from Eq. (5), the contribution from the distribution of volume was related to the ratio of $\mathscr{P}_A(k+m)/\mathscr{P}_A(k)$ [see Eq. (8)]. The differences of this ratio in different scenarios are obvious only in most-central collisions [see Fig. 2(a)]. It is worth noting that the magnitude of the external Gaussian-type fluctuation's strength can be reflected not only in the second-order cumulant (ω) of net proton distribution but also in the more sensitive higher-order ones, i.e, $S\sigma$ and $\kappa\sigma^2$. The results imply that, besides the non-Gaussian critical fluctuations suggested in Ref. [12], the

nonmonotonic behavior of higher-order cumulants beyond variances in a high reference multiplicity range could be also generated by volume corrections with different Gaussian-type volume fluctuations.

In the future, I will consider several other effects in multiplicity fluctuation studies, e.g, quantum effects, resonance decays, experimental acceptance, the correlation between different moment-analysis particles, the correlation between moment-analysis particles and reference particles, etc. As I have discussed in Ref. [32], these corrections are especially important in the study of net proton fluctuations. Therefore, instead of an exploratory study given in this work, more elaborate studies are needed to pin down the exact statistical predictions of measured cumulants in heavy-ion collisions.

IV. CONCLUSIONS

I have extended my previous work of corrected cumulants of (net conserved) charge distributions to most-central heavy-ion collisions. The required volume distributions are simulated by an optical Glauber model. Under the Poisson approximations, I calculated the corrected cumulants of total charge (net proton) distributions from a general formalism according to the data. To reduce the uncertainties between experimental measurements and theoretical calculations, as well as to redouble the statistics, I also suggested a special approach for centrality definition.

I found that the statistical expectations of multiplicity distribution mimic the NBD in noncentral collisions, but tend to approach the Poisson distribution in most-central collisions. This transition is because of the boundary effects that were caused by the upper boundary of the system volume in HICs, which significantly suppress the volume corrections. I have further investigated the effect of external Gaussian-type volume fluctuations on the high-order cumulants of net proton distributions.

The results indicate that the noncritical volume corrections on high-order cumulants of multiplicity distributions become weak in most-central collisions. However, the sensitivity of the net proton distribution and reference multiplicity distributions to the external volume fluctuations implies that the details of volume distribution in relativistic heavy-ion collisions need to be considered carefully in event-by-event multiplicity fluctuation studies.

APPENDIX: OPTICAL GLAUBER MODEL

The density distribution of the colliding nuclei in the Glauber model [36–40] is given by Woods-Saxon profiles,

$$\rho_N(r) = \frac{\rho_0}{\exp\left[(r-R_N)/\xi\right]+1}, \tag{A1}$$

with the nuclear radius $R_N = (1.12 N^{1/3} - 0.86 N^{-1/3})$ fm, the normal nuclear density $\rho_0 = 0.1699$ fm^{-3}, and the surface diffuseness $\xi = 0.54$ fm. Here N is the mass number of the nuclei. The nuclear thickness function is obtained from the optical path-length through the nucleus along the beam direction:

$$T_N = \int_{-\infty}^{+\infty} dz \rho_N(x,y,z). \qquad (A2)$$

The number density of binary collisions with impact parameter ζ reads

$$n_{bc}(x,y;\zeta) = \sigma_{in} T_{N_A}(x + \zeta/2, y) T_{N_B}(x - \zeta/2, y), \qquad (A3)$$

where σ_{in} is the total inelastic cross section. Then the total number of binary collisions is

$$n_{coll}(\zeta) = \int dx dy n_{bc}(x,y;\zeta). \qquad (A4)$$

The number density of wounded collisions (participants) with impact parameter ζ is

$$n_{wn}(x,y;\zeta) = \left[1 - \left(1 - \frac{\sigma_{in} T_{N_B}\left(x - \frac{\zeta}{2}\right)}{N_B} \right)^{N_B} \right]$$

$$\times T_A\left(x + \frac{\zeta}{2}, y\right) + (N_A \leftrightarrow N_B), \qquad (A5)$$

where $N_A = N_B = 197$ for Au + Au collisions. Then the total number of wounded collisions (participants) is

$$n_{part}(\zeta) = \int dx dy n_{wn}(x,y;\zeta). \qquad (A6)$$

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