Odd-even staggering of binding energy for nuclei in the *sd* shell

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In this paper we study odd-even staggering phenomena of binding energy in the framework of the nuclear shell model for nuclei in the *sd* shell. We decompose the USDB effective interaction into the monopole interaction and multipole (residual) interactions. We extract the empirical proton-neutron interaction, the Wigner energy, and the one-neutron separation energy using calculated binding energies. The monopole interaction, which represents the spherical mean field, provides contributions to the empirical proton-neutron interaction, the symmetry energy, and the Wigner energy. It does not induce odd-even staggering of the empirical proton-neutron interaction or the one-neutron separation energy. Isovector monopole and quadrupole pairing interactions and isoscalar spin-1 pairing interactions play a key role in reproducing an additional binding energy in both even-even and odd-odd nuclei. The Wigner energy coefficients are sensitive to residual two-body interactions. The nuclear shell structure has a strong influence on the evolution of the one-neutron separation energy, but not on empirical proton-neutron interactions. The so-called three-point formula is a good probe of the shell structure.

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I. INTRODUCTION

Proton-neutron interactions are very important in atomic nuclei. Proton-neutron interactions provide a dominant contribution to the attractive mean-field potential in nuclei, and residual proton-neutron interactions play a key role in nuclear deformations, collective motions, and phase transitions [1–3]. Microscopically, proton-neutron correlations have been studied using the SO(5) and SO(8) models of isoscalar and isovector pairing [4–9], mean-field approaches [10–12], the nuclear shell model with the "pair counting" operators [4,13,14], the interacting boson model [15,16], and the nucleon-pair approximation of the shell model [17]. Empirically, proton-neutron interactions have been studied mainly using two approaches: the $N_\mathrm{p}N_\mathrm{n}$ scheme (product of the valence proton number and the valence neutron number) [18] and empirical proton-neutron interactions [19–33].

The empirical proton-neutron interaction between the last i proton(s) and the last j neutron(s) in a nucleus is defined by

$$\delta V_{i\text{p-}j\text{n}}(Z,N) \equiv B(Z,N-j) + B(Z-i,N) - B(Z,N) - B(Z-i,N-j), \qquad (1)$$

where B is the binding energy. Here B is positive, and $\delta V_{i\text{p-}j\text{n}}$ is negative. The empirical proton-neutron interaction exhibits good systematics and regularities. In our previous work [33] we studied two of them. One is that the $\delta V_{1\text{p-}1\text{n}}$ of even-A nuclei is systematically stronger than that of the neighboring odd-A nuclei (we call it odd-even staggering of $\delta V_{1\text{p-}1\text{n}}$), and the other is that the $\delta V_{i\text{p-}j\text{n}}$ of $N = Z$ nuclei is much stronger than that of neighboring $N \neq Z$ nuclei, which is known as the Wigner effect (or the Wigner energy, alternatively). For nuclei in the *sd* shell, these two patterns are well reproduced by the nuclear shell model with the USDB effective interaction [34].

This work goes one step further along the same lines as Ref. [33] in the shell model, but with the so-called monopole interaction and multipole interactions. In Ref. [35] Zuker showed that a shell-model Hamiltonian can be decomposed into a monopole part and multipole parts. The monopole interaction represents the spherical mean field, and the multipole interaction represents residual two-body correlations. In this paper we extract the monopole and multipole interactions from the USDB interaction and make use of them to calculate the binding energies and $\delta V_{i\text{p-}j\text{n}}$ for nuclei in the *sd* shell. We show that isovector monopole pairing, isovector quadrupole pairing, and isoscalar spin-1 pairing correlations play a key role in the odd-even staggering of $\delta V_{1\text{p-}1\text{n}}$ and in the Wigner energy. On the other hand, the odd-even staggering phenomenon of one-neutron separation energy has been well known, which is explained by the neutron-neutron pairing correlation. It is interesting to revisit the odd-even staggering of the one-neutron separation energy using the shell model with the monopole and multipole interactions.

This paper is organized as follows. In Sec. II we introduce the shell-model-interaction decomposition method suggested by Zuker. In Sec. III we study the odd-even staggering of $\delta V_{1\text{p-}1\text{n}}$, the Wigner energy, and the odd-even staggering of the one-neutron separation energy. In Sec. IV we summarize our results.

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II. MULTIPOLE DECOMPOSITION OF SHELL-MODEL INTERACTIONS

The shell-model-interaction decomposition method is presented in Refs. [35,36]. Here we briefly review the essential ideas and formulas.

The shell-model effective interaction is defined by

$$H_{\text{eff}} = H_0 + V. \tag{2}$$

Here the first term, H_0, is the single-particle energy term,

$$H_0 = \sum_j \epsilon_j \sum_{m\tau} a^{\dagger}_{jm\tau} a_{jm\tau},$$

where ϵ_j is the single-particle energy of orbit j. The second term in Eq. (2), V, is the two-body interaction term,

$$V = \sum_{JT} \sum_{j_1\leqslant j_2} \sum_{j_3\leqslant j_4} \frac{V_{JT}(j_1j_2j_3j_4)}{\sqrt{(1+\delta_{j_1j_2})(1+\delta_{j_3j_4})}} \times \sum_{m\tau} A^{JT}_{m\tau}(j_1j_2)^{\dagger} A^{JT}_{m\tau}(j_3j_4), \tag{3}$$

where $V_{JT}(j_1j_2j_3j_4)$ are two-body matrix elements. $V_{JT}(j_1j_2j_3j_4)$ is isoscalar if $T=0$ and isovector if $T=1$. $A^{JT}_{m\tau}(j_1j_2)^{\dagger}$ is the creation operator of a nucleon pair with spin J and isospin T, i.e.,

$$A^{JT}_{m\tau}(j_1j_2)^{\dagger} = \left(a^{\dagger}_{j_1} \times a^{\dagger}_{j_2}\right)^{JT}_{m\tau}.$$

The form of V in Eq. (3) is called the pairing-type form.

Using Eq. (3) and angular momentum algebra, V can be rewritten in the so-called multipole-type form, i.e.,

$$V = \sum_{J'T'} (-)^{J'+T'} \sum_{j_1\leqslant j_2} \sum_{j_3\leqslant j_4} \frac{\omega_{J'T'}(j_1j_3j_2j_4)}{\sqrt{(1+\delta_{j_1j_2})(1+\delta_{j_3j_4})}} \times Q^{J'T'}(j_1j_3)\cdot Q^{J'T'}(j_2j_4) + \sum_{j_1\leqslant j_2\leqslant j_4} \frac{\sqrt{2(2j_2+1)}\,\omega_{00}(j_1j_4j_2j_2)}{\sqrt{(1+\delta_{j_1j_2})(1+\delta_{j_2j_4})}} Q^{00}(j_1j_4), \tag{4}$$

where

$$\omega_{J'T'}(j_1j_3j_2j_4) = \sum_{JT} (-)^{j_2+j_3+J+J'+T+T'+1} \times \sqrt{(2J+1)(2T+1)}\,V_{JT}(j_1j_2j_3j_4) \times \begin{Bmatrix} j_1 & j_3 & J' \\ j_4 & j_2 & J \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T' \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}. \tag{5}$$

$\begin{Bmatrix} j_1 & j_3 & J' \\ j_4 & j_2 & J \end{Bmatrix}$ is a six-j symbol, and $Q^{J'T'}(j_1j_3)$ is a multipole operator with spin J' and isospin T', i.e.,

$$Q^{J'T'}(j_1j_3) \equiv \left(a^{\dagger}_{j_1} \times \tilde{a}_{j_3}\right)^{J'T'},$$

$$\tilde{a}_{jm\tau} \equiv (-)^{j-m}(-)^{\frac{1}{2}-\tau} a_{j-m-\tau}.$$

In Ref. [35] Zuker defines the monopole interaction. In one major shell, the monopole interaction is written

$$H_m = H_0 + V_m,$$

where

$$V_m = \sum_{JT} \sum_{j_1\leqslant j_2} \frac{V_T(j_1j_2)\sum_{m\tau} A^{JT}_{m\tau}(j_1j_2)^{\dagger} A^{JT}_{m\tau}(j_1j_2)}{\sqrt{(1+\delta_{j_1j_2})(1+\delta_{j_3j_4})}}, \tag{6}$$

$$V_T(j_1j_2) = \frac{\sum_J V_{JT}(j_1j_2j_1j_2)(2J+1)\left[1-(-)^{J+T}\delta_{j_1j_2}\right]}{(2j_1+1)\left[(2j_2+1)+(-)^T\delta_{j_1j_2}\right]}. \tag{7}$$

V_m in Eqs. (6) and (7) is also written [35,36]

$$V_m = \sum_{j_1j_2} \left[\left(a_{j_1j_2} - b_{j_1j_2} \frac{3\delta_{j_1j_2}}{4(1+\delta_{j_1j_2})} \right) \frac{n_{j_1}(n_{j_2}-\delta_{j_1j_2})}{1+\delta_{j_1j_2}} + b_{j_1j_2} \frac{T_1\cdot T_2}{1+\delta_{j_1j_2}} \right],$$

where

$$a_{j_1j_2} = \tfrac{1}{4}[3V_1(j_1j_2) + V_0(j_1j_2)],$$
$$b_{j_1j_2} = V_1(j_1j_2) - V_0(j_1j_2).$$

The monopole interaction provides the average energy of configurations at a fixed number of particles and isospin in each orbit, which is the spherical mean-feld energy [35,36]. It exhausts the contributions of the monopole (namely, $J'=0$) terms in Eq. (4), and the other terms (the multipole $J>0$ terms) generate residual correlations beyond the mean field.

In Ref. [35] it is shown that the multipole interactions, V_Q, V_S, V_D, V_P, and V_M, play important roles in shell-model descriptions of low-lying states of atomic nuclei. V_Q is the quadrupole-quadrupole interaction [namely, the $(J',T')=(2,0)$ term in Eq. (4)], which generates the SU(3) rotational motion and the quadrupole deformation in a nucleus [37]. V_S and V_D are the isovector monopole and quadrupole pairing interaction [namely, the $(J,T)=(0,1)$ and $(2,1)$ terms in Eq. (3)], respectively; they are the most important pairing interactions in the isovector channel. V_P is the isoscalar spin-1 pairing interaction [namely, the $(J,T)=(1,0)$ term in Eq. (3)], which may generate an isoscalar proton-neutron pairing correlation. V_M consists of three multipole-multipole interactions: the quadrupole-quadrupole interaction [the $(J',T')=(2,0)$ term], the hexadecapole-hexadecapole interaction [the $(J',T')=(4,0)$ term], and the $\sigma\tau\cdot\sigma\tau$ interaction [the $(J',T')=(1,1)$ term].

In this work we study nuclei in the *sd* shell using the shell model with the USDB effective interaction [34] (denoted H_{USDB}) and schematic Hamiltonians consisting of the monopole interaction and the multipole interactions, such as

$$H_{mQ} = H_m + V_Q,$$
$$H_{mQS} = H_m + V_Q + V_S,$$
$$H_{mQSDP} = H_m + V_Q + V_S + V_D + V_P,$$
$$H_{mM} = H_m + V_M,$$
$$H_{mMS} = H_m + V_M + V_S,$$
$$H_{mMSDP} = H_m + V_M + V_S + V_D + V_P.$$

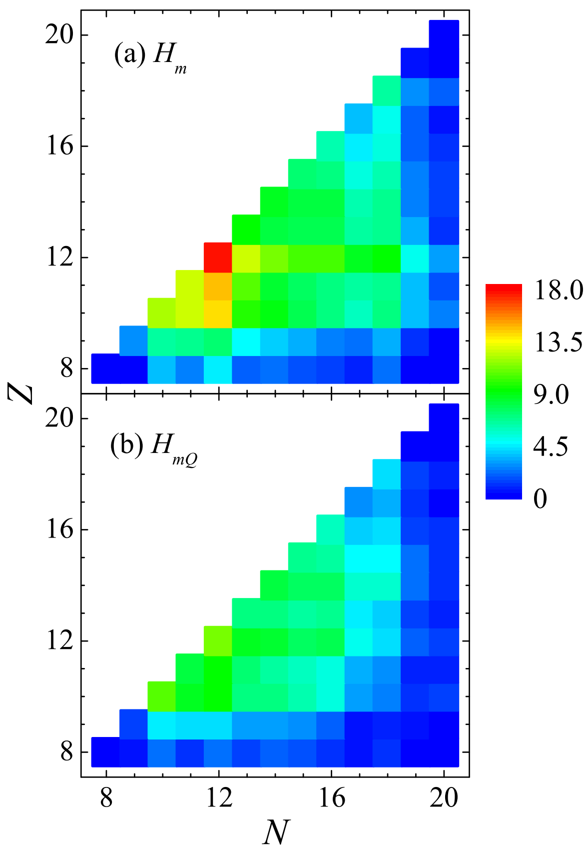

FIG. 1. Differences (in MeV) between the binding energies calculated by the shell model with the USDB interaction and those calculated with H_m and H_{mQ}. One sees that the binding energy calculated with H_m is close to that calculated with the USDB interaction for spherical nuclei but not for deformed nuclei.

In the schematic Hamiltonians, H_m, V_Q, V_M, V_S, V_D, and V_P are extracted from $H_{\rm USDB}$ one by one. We exemplify this using the case of H_{mMS}. First, the monopole interaction, H_m, is obtained from Eqs. (6) and (7), in which the two-body matrix elements, $V_{JT}(j_1j_2j_3j_4)$, are directly taken from $H_{\rm USDB}$. Next, the multipole-multipole interaction, V_M, is obtained by Eqs. (4) and (5), in which $V_{JT}(j_1j_2j_3j_4)$ are taken from $(H_{\rm USDB} - H_m)$. Finally, the pairing interaction, V_S, is obtained by Eq. (3), in which $V_{JT}(j_1j_2j_3j_4)$ are taken from $(H_{\rm USDB} - H_m - H_M)$.

Using the shell model with the USDB interaction and the schematic Hamiltonians mentioned above, we calculate the ground-state energies and the empirical proton-neutron interactions for 65 nuclei in the *sd* shell ($^{18-25}$F, $^{20-28}$Ne, $^{22-29}$Na, $^{24-30}$Mg, $^{26-31}$Al, $^{28-34}$Si, $^{30-35}$P, $^{32-36}$S, $^{34-37}$Cl, $^{36-38}$Ar, and $^{38-39}$K). In Fig. 1 one sees that for spherical nuclei the binding energies calculated with the monopole interaction, H_m, are close to those calculated with the USDB interaction. For describing deformed nuclei (especially ^{24}Mg), H_m (the spherical mean field) is no longer enough; the quadrupole-quadrupole interaction and other residual interactions are necessary.

III. RESULTS AND DISCUSSION

By summing $\delta V_{1p\text{-}1n}$ over all valence protons and valence neutrons, the integrated proton-neutron interaction between

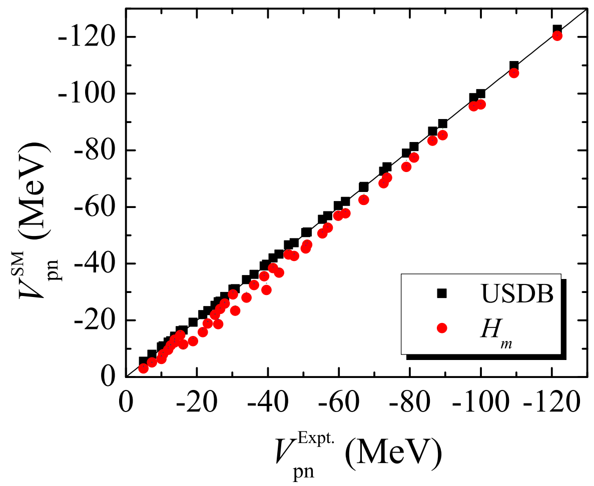

FIG. 2. Integrated proton-neutron interactions, $V_{\rm pn}$, extracted from experimental data on binding energies and those obtained by the shell model with the USDB interaction and H_m. The straight line is plotted according to $V_{\rm pn}^{\rm SM} = V_{\rm pn}^{\rm Expt.}$. H_m provides an important contribution to $V_{\rm pn}$.

valence nucleons in a nucleus is given by

$$V_{\rm pn}(Z,N) \equiv \sum_{Z_x=Z_0+1}^{Z} \sum_{N_x=N_0+1}^{N} \delta V_{1p\text{-}1n}(Z_x,N_x)$$
$$= B(Z,N_0) + B(Z_0,N) - B(Z,N) - B(Z_0,N_0),$$

where Z_0 and N_0 are the nearest magic numbers below the nucleus. For nuclei in the *sd* shell, we have $Z_0 = N_0 = 8$ and

$$V_{\rm pn}(Z,N) = B(Z,8) + B(8,N) - B(Z,N) - B(8,8).$$

As shown in Fig. 2, the integrated proton-neutron interactions, $V_{\rm pn}$, obtained by the shell model with the USDB interaction are in good agreement with those extracted from experimental data on binding energies. In Ref. [33] we demonstrate that the isoscalar proton-neutron interactions are consistent with the empirical proton-neutron interactions. In the present work we show that the monopole interaction (namely, the spherical mean field) plays a key role in the integrated proton-neutron interaction, which is similar to the conclusions in Refs. [30,32].

In the Weizsäcker nuclear binding energy formula, the average energy of empirical proton-neutron interactions is explained by the symmetry energy [31]. In Ref. [38] Jiang *et al.* extracted symmetry energy coefficients for atomic nuclei using local mass relations. In this work we extract the volume symmetry energy coefficient, $c_{\rm sym}^{(V)}$, using the same method as in Ref. [38]. Binding energies are either taken from experimental data or calculated by the shell model. Results for nuclei in the *sd* shell are presented in Table I. One sees all the symmetry energy coefficients are close to each other, including that calculated with H_m. This indicates that correlations beyond the spherical mean field do not have much influence on the symmetry energy in atomic nuclei. We note that the $c_{\rm sym}^{(V)}$ calculated with H_m is slightly larger than that obtained with the USDB interaction.

TABLE I. The symmetry energy coefficient extracted using local mass relations in Ref. [38] for nuclei in the sd shell. Expt.: the result extracted from experimental data on binding energies. All other entries under "Result" represent results obtained by the nuclear shell model with different interactions. In Ref. [38] the symmetry energy coefficient $c_{\rm sym}^{(V)}$ is equal to 32.10 MeV.

Result	$c_{\rm sym}^{(V)}$ (MeV)
Expt.	32.49
USDB	33.17
H_m	34.74
H_{mQ}	34.60
H_{mQS}	34.49
H_{mQSD}	35.09
H_{mQSP}	34.38
H_{mQSDP}	34.86
H_{mM}	34.42
H_{mMS}	34.55
H_{mMSD}	34.33
H_{mMSP}	34.24
H_{mMSDP}	34.04

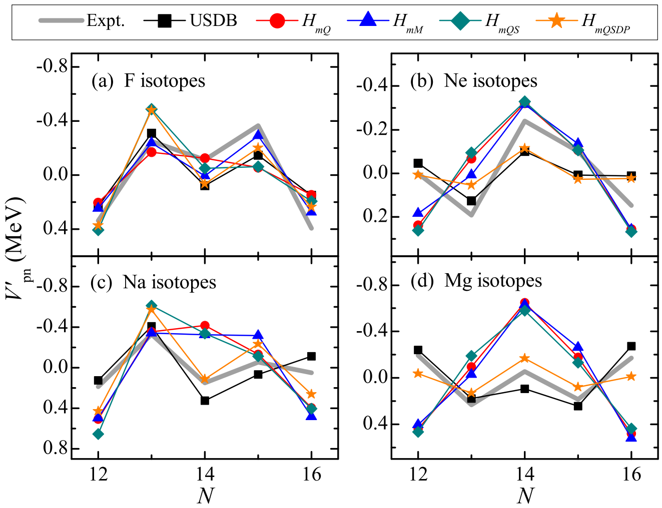

FIG. 4. $V'_{\rm pn}$ versus neutron number N for F, Ne, Na, and Mg isotopes.

A. Odd-even staggering of $\delta V_{\rm 1p\text{-}1n}$

In Fig. 3 one sees that $\delta V_{\rm 1p\text{-}1n}$ values extracted from experimental data on binding energies (thick gray lines) exhibit odd-even staggering behavior, i.e., the $\delta V_{\rm 1p\text{-}1n}$ values of even-A nuclei are systematically stronger than those of odd-A nuclei. It was suggested that proton-neutron pairing interactions induce an additional binding energy in odd-odd nuclei, which leads to the odd-even staggering of $\delta V_{\rm 1p\text{-}1n}$ [39,40]. Similarly, in Ref. [41] the odd-even staggering of $\delta V_{\rm 1p\text{-}1n}$ was explained by additional binding energies in odd-odd nuclei due to configuration mixing originating from residual proton-neutron interactions. In our previous work [33] we found that there exists an additional binding energy for both even-even and odd-odd nuclei, which leads to the odd-even staggering of $\delta V_{\rm 1p\text{-}1n}$.

As shown in Fig. 3, $\delta V_{\rm 1p\text{-}1n}$ values obtained by the shell model with H_m (red circles) do not exhibit odd-even staggering behavior, which means that the spherical mean field does not induce odd-even staggering. With consideration of quadrupole deformation, $\delta V_{\rm 1p\text{-}1n}$ values obtained with H_{mQ} (blue triangles) exhibit very weak odd-even staggering behavior. By considering more residual interactions, the odd-staggering phenomenon is well reproduced (we exemplify $\delta V_{\rm 1p\text{-}1n}$ with H_{mM}, H_{mQS}, and H_{mMS} in Fig. 3).

Now let us focus on the additional binding energy for even-even and odd-odd nuclei. We study the integrated proton-neutron interaction, $V_{\rm pn}$, which can be separated into two parts: a smooth part (denoted $V_{\rm smooth}$) and the residual part (denoted $V'_{\rm pn}$). The smooth part of $V_{\rm pn}$ is described by a linear function, $V_{\rm smooth}(N;Z) \equiv a_Z N + b_Z$. Here a_Z and b_Z are parameters to be fixed. The residual part of $V_{\rm pn}$ is then defined by $V'_{\rm pn} \equiv V_{\rm pn} - V_{\rm smooth} = V_{\rm pn}(Z,N) - a_Z N - b_Z$, which is expected to be sensitive to odd-even staggering phenomena in the empirical proton-neutron interactions. In this work we calculate $V'_{\rm pn}$ using the shell model with the schematic Hamiltonians. The results for F, Ne, Na, and Mg isotopes are presented in Fig. 4.

In Fig. 4 one sees that the $V'_{\rm pn}$ values obtained by the shell model with H_{mQSDP} are close to those obtained by the shell model with the USDB interaction or extracted from experimental data on binding energies. For the F and Na isotopes, the $V'_{\rm pn}$ values for odd-odd nuclei are smaller than those for neighboring odd-A nuclei; for the Ne and Mg isotopes, the $V'_{\rm pn}$ values for ven-even nuclei are smaller than

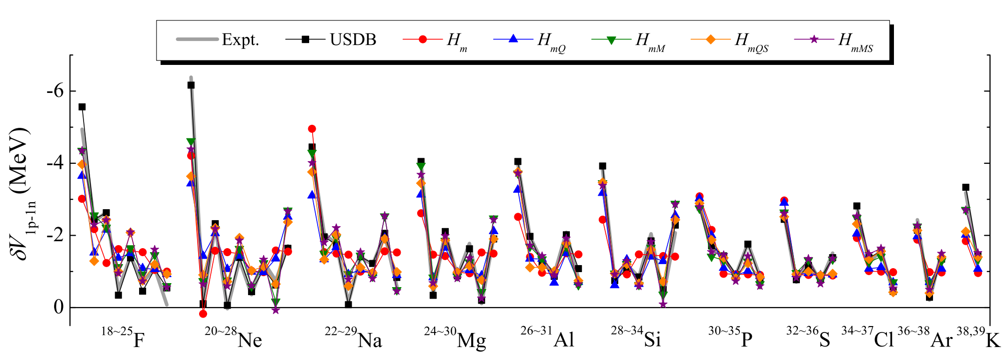

FIG. 3. $\delta V_{\rm 1p\text{-}1n}$ for nuclei in the sd shell.

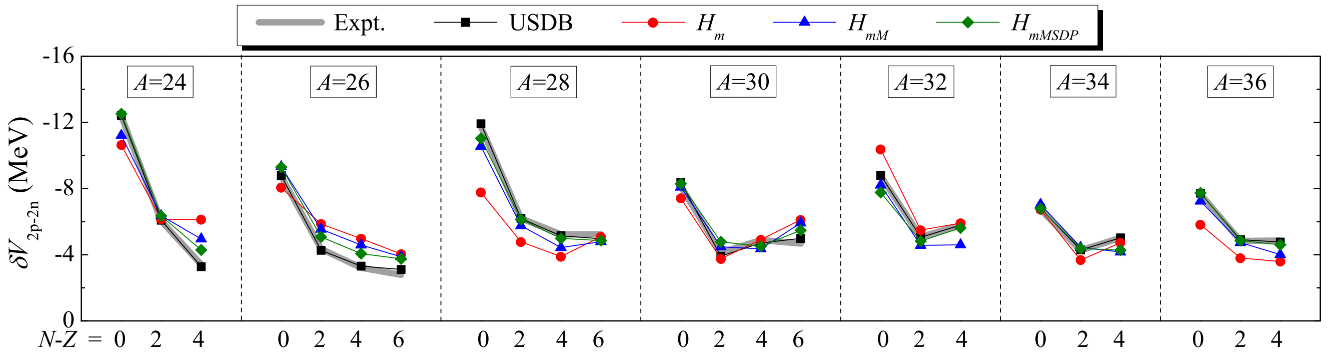

FIG. 5. $\delta V_{2p\text{-}2n}$ for nuclei in the sd shell.

those for neighboring odd-A nuclei. This result indicates an additional binding for both even-even and odd-odd nuclei.

The results obtained by the shell model with H_{mQ}, H_{mM}, and H_{mQS} show different scenarios. As shown in Fig. 4, V'_{pn} values obtained with H_{mQ} do not show odd-even staggering but exhibit a parabolic behavior with the neutron number. This means that H_{mQ} provides an additional binding energy for neither even-even nor odd-odd nuclei. For the F and Na isotopes, V'_{pn} values obtained with H_{mM} and H_{mQS} for odd-odd nuclei are smaller than those for neighboring odd-A nuclei (not obvious for the Na isotopes); for the Ne and Mg isotopes, V'_{pn} values obtained with H_{mM} and H_{mQS} exhibit a parabolic behavior with the neutron number. This result means that H_{mM} and H_{mQS} provide an additional binding energy for odd-odd nuclei but not for even-even nuclei. The isovector quadrupole pairing interaction and the isoscalar spin-1 proton-neutron pairing interaction are important in reproducing the additional binding energy in even-even nuclei.

B. Wigner energy

In nuclear binding energy formulas, the symmetry energy depends on $(N-Z)^2$ (or the squared isospin, T^2), and the Wigner energy depends on $|N-Z|$ (or T). In a few microscopic models these two energies are unified by the so-called total symmetry energy, which depends on $T(T+X)$, where the value of X represents the ratio of the Wigner energy to the symmetry energy. In Wigner's SU(4) spin-isospin symmetry theory [42] the total symmetry energy is proportional to $T(T+4)$. In the seniority scheme in a single-j shell or the SU(2) symmetry theory [43] it is proportional to $T(T+1)$, which is an eigenvalue of the operator, $\boldsymbol{T}^2$. Fitting the experimental data on binding energies, one usually obtains an empirical value of X ranging between 1 and 4 [44].

The Wigner energy has constantly attracted attention [22,33,45–54] among nuclear physicists. In the recent literature a more sophisticated definition of the Wigner energy is as follows:

$$B_{\mathrm{W}}(Z,N) = -W(A)|N-Z| - d(A)\delta_{Z,N}\pi_{\mathrm{np}}. \qquad (8)$$

Here $\delta_{Z,N}$ is equal to 1 if $N=Z$ and equal to 0 if $N \neq Z$; π_{np} is equal to 1 for odd-odd nuclei and vanishes for even-even and odd-A nuclei. The first term in Eq. (8) is usually called the "W term," and the second term is usually called the "d term" and represents an underbinding in the ground state of $N=Z$ odd-odd nuclei. In binding energy formulas, X is calculated as

$$X = \frac{B_{\mathrm{W}}}{E_{\mathrm{sym}}} \approx \frac{AW(A)}{c_{\mathrm{sym}}^{(V)}}, \qquad (9)$$

where E_{sym} is the symmetry energy.

In Figs. 3 and 5 one sees that the $\delta V_{1p\text{-}1n}$ and $\delta V_{2p\text{-}2n}$ for $N=Z$ nuclei are much stronger than those for neighboring $N \neq Z$ nuclei. This phenomenon originates from the Wigner energy [22,45]. In Refs. [22,46] it is reported that the Wigner energy is strongly suppressed if isoscalar interactions are removed. In our previous work [33] it is shown that the values of $\delta V_{2p\text{-}2n}$ decrease dramatically when isoscalar interactions are removed, but $\delta V_{2p\text{-}2n}$ values for $N=Z$ nuclei continue to be much stronger than those for their $N \neq Z$ neighbors. This indicates that removing isoscalar interactions strongly suppresses the total symmetry energy, but the value of X remains nearly unchanged.

In Fig. 5 one sees that $\delta V_{2p\text{-}2n}$ values obtained by the shell model with H_m for $N=Z$ nuclei are much stronger than those for their $N \neq Z$ neighbors. The spherical mean field provides contributions to the Wigner energy. $\delta V_{2p\text{-}2n}$ values obtained by the shell model with H_{mMSDP} are close to those extracted from experimental data on binding energies.

In Ref. [46] Satula *et al.* extracted the Wigner energy coefficients, W and d, using local mass relations in the zeroth-order approximation, and in Ref. [54] Cheng *et al.* extracted them using local mass relations in the first-order approximation. In this work we extract W and d using the same method as in Ref. [54] and calculate X using Eq. (9). The values of $c_{\mathrm{sym}}^{(V)}$ are taken from Table I.

As shown in Fig. 6, the values of W, d, and X obtained by the shell model with H_m, H_{mQ}, and H_{mQSDP} are much smaller than those extracted from experimental data on binding energies; those obtained with H_{mM}, H_{mMS}, H_{mMSD}, and H_{mMSP} are reasonably good; and those obtained with H_{mMSDP} are close to the empirical ones. We note that all the values of W, d, and X obtained with eight schematic Hamiltonians, H_m, H_{mQ}, H_{mQSDP}, H_{mM}, H_{mMS}, H_{mMSD}, H_{mMSP}, and H_{mMSDP}, are smaller than those obtained with the USDB interaction. Residual two-body interactions beyond V_M, V_S, V_D, and V_P are necessary for precise description of the Wigner energy coefficients.

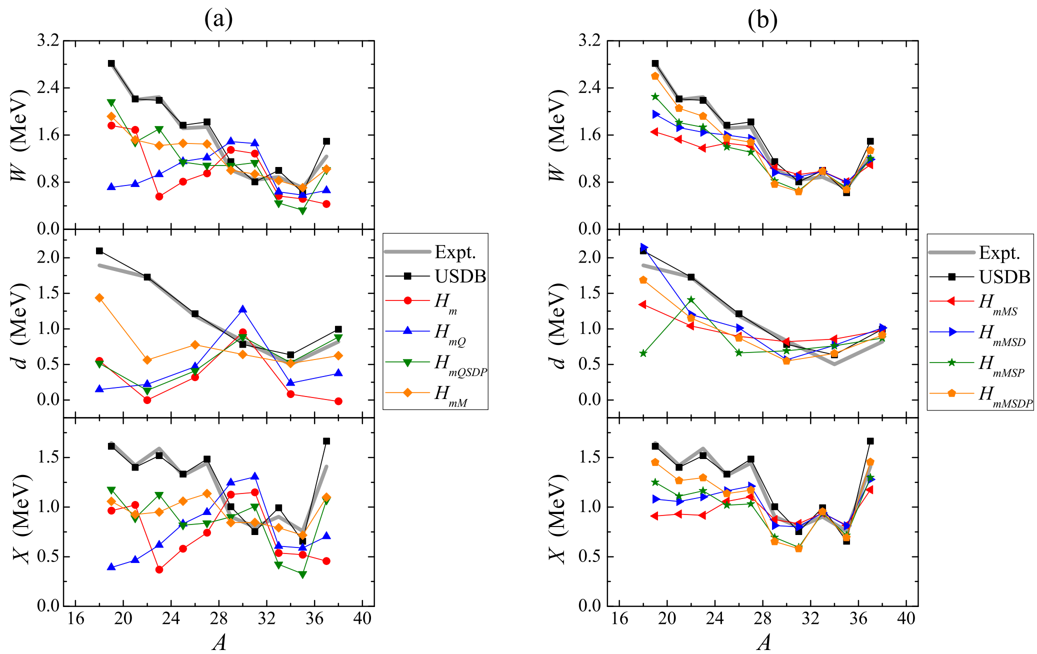

FIG. 6. Wigner energy coefficients, W, d, and X, for nuclei in the sd shell. (a) and (b) are the same except that the interactions are different.

The value of X extracted from experimental data on binding energies ranges between 0.8 and 1.6 for nuclei in the sd shell. In a single-j shell calculation with the monopole interaction H_m, one obtains precisely $X = 1$ [32,43]. In our sd-shell calculation, the X obtained by the shell model with H_m is not equal to 1 but evolves rapidly as the mass number A changes. This is the consequence of the shell effect. Interestingly, we find that X obtained with H_{mM}, H_{mMS}, and H_{mMSD} evolves smoothly with values $\sim$1.

C. Odd-even staggering of the one-neutron separation energy

In the final part of this section, we arrive at the odd-even staggering of the one-neutron separation energy, $S_{\rm n}$, for O isotopes in the sd shell. In Figure 7(a) one sees that $S_{\rm n}$ values taken from experimental data (or those calculated by the shell model with the USDB interaction) for even-N nuclei are 3–4 MeV larger than those for neighboring odd-N nuclei. This odd-even staggering phenomenon is regarded as a consequence of the neutron-neutron monopole pairing. In Refs. [55,56] the odd-even staggering is described by the so-called three-point formula:

$$\Delta_{\rm n}^{(3)}(Z,N) \equiv \frac{(-)^{N+1}}{2}[S_{\rm n}(Z,N+1) - S_{\rm n}(Z,N)] = \frac{(-)^{N+1}}{2}[B(Z,N+1) + B(Z,N-1) - 2B(Z,N)]. \qquad (10)$$

The results of $\Delta_{\rm n}^{(3)}$ for O isotopes in the sd shell are shown in Fig. 7(b).

Let us focus on $^{17-22}$O and $^{25-27}$O. As shown in Fig. 7, $S_{\rm n}$ obtained by the shell model with H_m exhibits a linear dependence on the neutron number, and the values of $\Delta_{\rm n}^{(3)}$ with H_m are close to 0. The reason is simple: the spherical mean field does not contribute to the odd-even staggering of the one-neutron separation energy. $S_{\rm n}$ values with H_{mQ} exhibit odd-even staggering with a magnitude of about 1.5 MeV, and $\Delta_{\rm n}^{(3)}$ with H_{mQ} are equal to about 0.75 MeV. This result indicates that the quadrupole-quadrupole interaction,

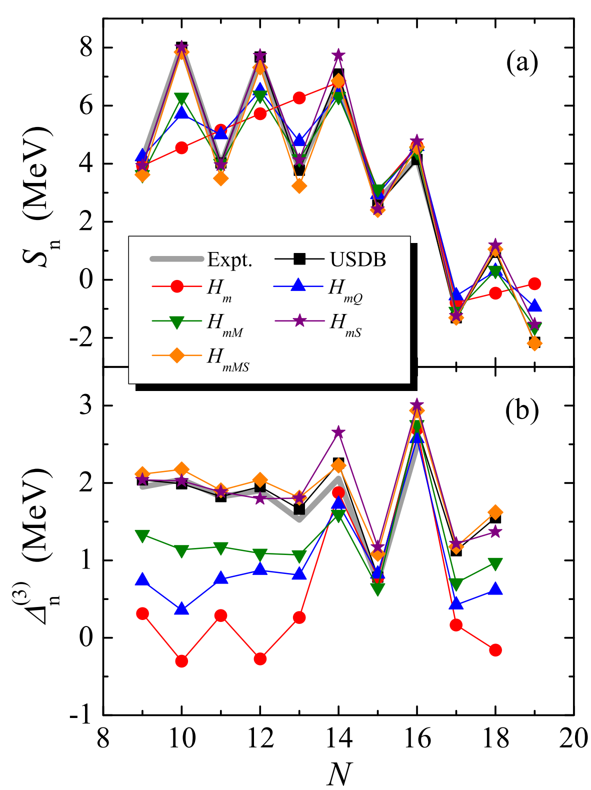

FIG. 7. One-neutron separation energy, $S_{\rm n}$, and the three-point formula [Eq. (10)], $\Delta_{\rm n}^{(3)}$, for O isotopes in the sd shell.

which may generate the quadrupole deformation in a nucleus, provides a small contribution to the odd-even staggering of S_n and to $\Delta_n^{(3)}$. It should be noted that the result of the deformed mean-field calculation is slightly different from ours: the Kramers degenerate single-particle levels provide very strong odd-even staggering for $\Delta_n^{(3)}$ (see Fig. 1(b) in Ref. [55]). One possible reason is that the shell-model calculation conserves the spin symmetry, while the deformed mean-field calculation does not. The S_n and $\Delta_n^{(3)}$ values with H_{mS} and H_{mMS} are close to those taken from the experimental data. The neutron-neutron monopole pairing interaction plays a key role in the odd-even staggering of S_n.

For ^{22}O, ^{23}O, and ^{24}O we find that the S_n and $\Delta_n^{(3)}$ obtained by the shell model with H_m (namely, the spherical mean field) evolve rapidly as the neutron number changes. This anomaly originates from the nuclear shell structure: for $^{17-22}$O the valence neutrons occupy the $0d_{5/2}$ orbit, for $^{23-24}$O the valence neutrons begin to occupy the $1s_{1/2}$ orbit, and for $^{25-28}$O the valence neutrons begin to occupy the $0d_{3/2}$ orbit. The sudden drop in S_n for $N = 14$–17 is due to the effective single-particle energy differences of the three orbits. $\Delta_n^{(3)}$ [Eq. (10)] is a good probe of the nuclear shell structure. In Fig. 7, one sees that the S_n and $\Delta_n^{(3)}$ obtained by the shell model with the USDB interaction and those with H_m have nearly the same values for ^{22}O, ^{23}O, and ^{24}O, which indicates that the contribution from the neutron-neutron monopole pairing interaction to the odd-even staggering of S_n is strongly suppressed.

IV. SUMMARY

In this paper we study odd-even staggering phenomena of the nuclear binding energy in terms of the nuclear shell model for nuclei in the *sd* shell. We decompose the USDB effective interaction [34] into the monopole interaction (H_m, which represents the spherical mean field) and residual interactions using Zuker's method [35]. We use schematic Hamiltonians as well as the USDB interaction to study the odd-even staggering of $\delta V_{1p\text{-}1n}$, the Wigner energy, and the odd-even staggering of the one-neutron separation energy, S_n.

The monopole interaction (namely, the spherical mean field) is responsible for bulk properties of the empirical proton-neutron interactions, the symmetry energy, and the one-nucleon separation energy. However, the monopole interaction plus the quadrupole-quadrupole interaction (namely, H_{mQ}) cannot explain the odd-even staggering of $\delta V_{1p\text{-}1n}$. In our previous work [33] we found that there is an additional binding energy in both even-even and odd-odd nuclei, which leads to the odd-even staggering of $\delta V_{1p\text{-}1n}$. In this work we show that although $\delta V_{1p\text{-}1n}$ values obtained by the shell model with H_{mM} and H_{mQS} exhibit odd-even staggering behavior, these two schematic Hamiltonians cannot explain the additional binding in even-even nuclei. Interestingly, the odd-even staggering of $\delta V_{1p\text{-}1n}$ values and the additional binding in both even-even and odd-odd nuclei are well reproduced with H_{mQSDP}. The isovector quadrupole pairing interaction and the isoscalar spin-1 pairing interaction play an important role.

We extract the Wigner energy coefficients, W, d, and X, using local mass relations, in which the binding energies are taken from experimental data or calculated by the shell model. It is shown that the monopole interaction provides contributions to the Wigner energy. The Wigner energy coefficients, W, d, and X, obtained by the shell model with H_{mMSDP} are in good accordance with those extracted from experimental data on binding energies. All the values of W, d, and X obtained with the schematic Hamiltonians, H_m, H_{mQ}, H_{mQSDP}, H_{mM}, H_{mMS}, H_{mMSD}, H_{mMSP}, and H_{mMSDP}, are smaller than the empirical values. Residual two-body interactions beyond V_M, V_S, V_D, and V_P are necessary for precise description of the Wigner energy coefficients.

The importance of the neutron-neutron pairing correlation in the odd-even staggering phenomenon of S_n has been well known. The spherical mean field makes no contribution to the odd-even staggering of S_n or to the three-point formula, $\Delta_n^{(3)}$. Our shell-model calculation with H_{mQ} shows that the quadrupole-quadrupole interaction, which may generate the quadrupole deformation in a nucleus, provides a small contribution. The result of the deformed mean-field calculation [55] is slightly different from ours; one possible reason is that the mean-field calculation breaks rotational invariance. The nuclear shell structure has a strong influence on the evolution of S_n but not on the empirical proton-neutron interaction. The three-point formula, Eq. (10), is a good probe of the nuclear shell structure.

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[1] A. de-Shalit and M. Goldhaber, Phys. Rev. **92**, 1211 (1953).
[2] I. Talmi, Rev. Mod. Phys. **34**, 704 (1962).
[3] P. Federman and S. Pittel, Phys. Rev. C **20**, 820 (1979).
[4] J. Engel, K. Langanke, and P. Vogel, Phys. Lett. B **389**, 211 (1996).
[5] J. Engel, S. Pittel, M. Stoitsov, P. Vogel, and J. Dukelsky, Phys. Rev. C **55**, 1781 (1997).
[6] J. Doběs and S. Pittel, Phys. Rev. C **57**, 688 (1998).
[7] O. Juillet and S. Josse, Eur. Phys. J. A **8**, 291 (2000).
[8] S. Lerma H., B. Errea, J. Dukelsky, and W. Satula, Phys. Rev. Lett. **99**, 032501 (2007).
[9] J. Dukelsky, B. Brrea, S. Lerma H., G. G. Dussel, C. Esebbag, and N. Sandulescu, Int. J. Mod. Phys. E **17**, 2155 (2008).
[10] A. L. Goodman, Adv. Nucl. Phys. **11**, 263 (1979); Phys. Rev. C **60**, 014311 (1999).
[11] W. Satula and R. Wyss, Phys. Lett. B **393**, 1 (1997).
[12] J. Terasaki, R. Wyss, and P.-H. Heenen, Phys. Lett. B **437**, 1 (1998).

[13] K. Langanke, D. J. Dean, P. B. Radha, Y. Alhassid, and S. E. Koonin, Phys. Rev. C **52**, 718 (1995).
[14] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson, R. J. Liotta, and R. Wyss, Phys. Rev. C **84**, 021301(R) (2011).
[15] S. Zerguine and P. Van Isacker, Phys. Rev. C **83**, 064314 (2011).
[16] P. van Isacker, Phys. Scripta T **150**, 014042 (2012); Int. J. Mod. Phys. E **22**, 1330028 (2013).
[17] G. J. Fu, J. J. Shen, Y. M. Zhao, and A. Arima, Phys. Rev. C **87**, 044312 (2013); G. J. Fu, Y. M. Zhao, and A. Arima, *ibid.* **90**, 054333 (2014); **91**, 054322 (2015).
[18] R. F. Casten, Phys. Lett. B **152**, 145 (1985); Phys. Rev. Lett. **54**, 1991 (1985); Nucl. Phys. A **443**, 1 (1985); R. F. Casten and N. V. Zamfir, J. Phys. G **22**, 1521 (1996).
[19] M. K. Basu and D. Banerjee, Phys. Rev. C **3**, 992 (1971); **4**, 652 (1971).
[20] J. Jänecke, Phys. Rev. C **6**, 467 (1972); J. Jänecke and H. Behrens, *ibid.* **9**, 1276 (1974).
[21] J. Y. Zhang, R. F. Casten, and D. S. Brenner, Phys. Lett. B **227**, 1 (1989).
[22] D. S. Brenner, C. Wesselborg, R. F. Casten, D. D. Warner, and J. Y. Zhang, Phys. Lett. B **243**, 1 (1990).
[23] R. B. Cakirli, D. S. Brenner, R. F. Casten, and E. A. Millman, Phys. Rev. Lett. **94**, 092501 (2005).
[24] R. B. Cakirli and R. F. Casten, Phys. Rev. Lett. **96**, 132501 (2006).
[25] D. S. Brenner, R. B. Cakirli, and R. F. Casten, Phys. Rev. C **73**, 034315 (2006).
[26] Y. Oktem, R. B. Cakirli, R. F. Casten, R. J. Casperson, and D. S. Brenner, Phys. Rev. C **74**, 027304 (2006).
[27] M. Stoitsov, R. B. Cakirli, R. F. Casten, W. Nazarewicz, and W. Satula, Phys. Rev. Lett. **98**, 132502 (2007).
[28] R. B. Cakirli, K. Blaum, and R. F. Casten, Phys. Rev. C **82**, 061304(R) (2010).
[29] G. J. Fu, H. Jiang, Y. M. Zhao, S. Pittel, and A. Arima, Phys. Rev. C **82**, 034304 (2010).
[30] M. Bender and P.-H. Heenen, Phys. Rev. C **83**, 064319 (2011).
[31] G. J. Fu, Y. Lei, H. Jiang, Y. M. Zhao, B. Sun, and A. Arima, Phys. Rev. C **84**, 034311 (2011).
[32] C. Qi, Phys. Lett. B **717**, 436 (2012).
[33] G. J. Fu, J. J. Shen, Y. M. Zhao, and A. Arima, Phys. Rev. C **87**, 044309 (2013).
[34] B. A. Brown and W. A. Richter, Phys. Rev. C **74**, 034315 (2006).
[35] A. P. Zuker, Nucl. Phys. A **576**, 65 (1994); M. Dufour and A. P. Zuker, Phys. Rev. C **54**, 1641 (1996).
[36] E. Caurier, G. Martinez-Pinedo, F. Nowacki, A. Poves, and A. P. Zuker, Rev. Mod. Phys. **77**, 427 (2005).
[37] J. P. Elliott, Proc. R. Soc. London, Ser. A **245**, 128 (1958); M. Harvey, Adv. Nucl. Phys. **1**, 67 (1968); K. Brahmam and R. Raju, At. Data Nucl. Data Tables **16**, 165 (1975).
[38] H. Jiang, G. J. Fu, Y. M. Zhao, and A. Arima, Phys. Rev. C **85**, 024301 (2012).
[39] A. de-Shalit, Phys. Rev. **105**, 1528 (1957).
[40] W. A. Friedman and G. F. Bertsch, Phys. Rev. C **76**, 057301 (2007).
[41] Z. C. Gao and Y. S. Chen, Phys. Rev. C **59**, 735 (1999).
[42] E. Wigner, Phys. Rev. **51**, 947 (1937); **51**, 106 (1937).
[43] I. Talmi, *Simple Models of Complex Nuclei* (Harwood Academic, Chur, Switzerland, 1993).
[44] A. O. Macchiavelli, P. Fallon, R. M. Clark, M. Cromaz, M. A. Deleplanque, R. M. Diamond, G. J. Lane, I. Y. Lee, F. S. Stephens, C. E. Svensson, K. Vetter, and D. Ward, Phys. Rev. C **61**, 041303(R) (2000).
[45] P. Van Isacker, D. D. Warner, and D. S. Brenner, Phys. Rev. Lett. **74**, 4607 (1995).
[46] W. Satula, D. J. Dean, J. Gary, S. Mizutori, and W. Nazarewicz, Phys. Lett. B **407**, 103 (1997).
[47] A. S. Jensen, P. G. Hansen, and B. Jonson, Nucl. Phys. A **431**, 393 (1984).
[48] N. Zeldes, Phys. Lett. B **429**, 20 (1998).
[49] M. W. Kirson, Phys. Lett. B **661**, 246 (2008).
[50] R. A. Senkov, G. F. Bertsch, B. A. Brown, Y. L. Luo, and V. G. Zelevinsky, Phys. Rev. C **78**, 044304 (2008).
[51] K. Neergård, Phys. Lett. B **537**, 287 (2002); **572**, 159 (2003); Phys. Rev. C **80**, 044313 (2009).
[52] I. Bentley and S. Frauendorf, Phys. Rev. C **88**, 014322 (2013).
[53] I. Bentley, K. Neergård, and S. Frauendorf, Phys. Rev. C **89**, 034302 (2014).
[54] Y. Y. Cheng, M. Bao, Y. M. Zhao, and A. Arima, Phys. Rev. C **91**, 024313 (2015).
[55] W. Satula, J. Dobaczewski, and W. Nazarewicz, Phys. Rev. Lett. **81**, 3599 (1998).
[56] S. A. Changizi, C. Qi, and R. Wyss, Nucl. Phys. A **940**, 210 (2015).