

Quantum canonical ensemble and correlation femtoscopy at fixed multiplicitiesS. V. Akkelin¹ and Yu. M. Sinyukov^{1,2,3}¹*Bogolyubov Institute for Theoretical Physics, Metrolohichna 14b, 03680 Kiev, Ukraine*²*ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für Schwerionenforschung, D-64291 Darmstadt, Germany*³*Tomsk State University, Department of Physics, Lenin Avenue 36, Tomsk, Russia*

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Identical particle correlations at fixed multiplicity are considered by means of quantum canonical ensemble of finite systems. We calculate one-particle momentum spectra and two-particle Bose-Einstein correlation functions in the ideal gas by using a recurrence relation for the partition function. Within such a model we investigate the validity of the thermal Wick's theorem and its applicability for decomposition of the two-particle distribution function. The dependence of the Bose-Einstein correlation parameters on the average momentum of the particle pair is also investigated. Specifically, we present the analytical formulas that allow one to estimate the effect of suppressing the correlation functions in a finite canonical system. The results can be used for the femtoscopy analysis of the $A + A$ and $p + p$ collisions with selected (fixed) multiplicity.

DOI: [10.1103/PhysRevC.94.014908](https://doi.org/10.1103/PhysRevC.94.014908)**I. INTRODUCTION**

The correlation femtoscopy method (for reviews see, e.g., Refs. [1–3]) uses momentum correlations of identical particles to extract information about the spatiotemporal structure of extremely small and short-lived systems created in nucleon and nuclear collisions. The method is grounded in the Bose-Einstein or Fermi-Dirac symmetric properties of the quantum states. Because in high-energy nucleus-nucleus or hadron-hadron collisions most of produced particles are pions, the Bose-Einstein correlations of two identical pions are usually analyzed to increase the statistics of the correlation femtoscopy measurements. Inasmuch as mean particle multiplicities increase with collision energy, one can divide a whole set of high-energy collision events into subsets with fixed charged-particle multiplicities. In recent papers [4] it was considered even a possibility of single-event correlation femtoscopy—at least, theoretically, because in reality not enough pairs remain for a statistically meaningful analysis in a single event. Typically, the ensemble of events with charged-particle numbers selected in some fixed multiplicity bins is analyzed. In heavy-ion collisions, the measurement of observables as a function of the multiplicity class has a long history and is regarded as a proxy for centrality dependence. Recently, due to the start of Large Hadron Collider (LHC) experiments where colliding energies per nucleon pair are in a TeV region, the fixed particle multiplicity technique has also been utilized for analysis of the Bose-Einstein correlations of identical particles in proton-proton collisions [5].

It is firmly established now that high-energy heavy-ion collisions are described basically by relativistic hydrodynamics (for recent reviews, see Ref. [6]) and that at the later dilute stage of matter evolution the hydrodynamics is followed by the highly dissipative hadronic gas expansion that is modeled by a hadronic cascade model like UrQMD [7] till the free streaming regime is reached. The so-called hadronization—transition from continuous medium (hydrodynamic) consideration of the system to its particle-based description—is associated typically with the lowest possible temperature when the system

is still close to local thermal and chemical equilibrium. It defines the isotherm that is often called the chemical freeze-out hypersurface. The situation is less clear in high-multiplicity proton-proton collisions, and there are also attempts to describe particle momentum spectra in such collisions using a hydrodynamic approach (see, e.g., Ref. [8]).

It is worth to noting that quantum statistics is not an inherent property of these, as well as many other, quasiclassical models. Typically, when hadrons are generated on the hadronization hypersurface, the single-particle weight is sampled according to the Bose-Einstein or Fermi-Dirac distributions in the grand canonical ensemble (with viscous corrections, if necessary) in the framework of the so-called Cooper-Frye prescription [9]. As for the two-identical-particle spectra, the current quasiclassical simulations utilize the factorized decomposition of the two-particle emission function into the single-particle ones with the additional multiplier proportional to the module squared of two-particle symmetrized or antisymmetrized amplitudes, so that the correlation function becomes (omitting all nonprincipal details) $C(p_1, p_2) \propto \langle 1 \pm \cos(p_1 - p_2)(x_1 - x_2) \rangle$, where angular brackets mean averaging over an emission function. Such a local “switching on” of the quantum statistic effects in two-particle cross sections is just like in the final-state-interaction method. However, the quantum statistics is not associated with local two-particle interaction but is the global effect, and complete symmetrization or antisymmetrization of the *total* system is required to find the correct results for one- and multi-particle momentum spectra. Such an analysis was done in Refs. [10,11]. It has been shown that for thermal identical particles the abovementioned procedure for one- and two-boson spectra evaluations in quasiclassical models is correct if one provides the symmetrization in an *ensemble* of initially independently emitted thermal Boltzmann particles with the Poisson distribution for the particle numbers in the ensemble. Otherwise, the above-described prescription for the simulation of the single- and double-particle spectra and the correlation function $C(p_1, p_2)$ is not correct. In particular, it is violated for the states with a fixed boson number. The

detailed grounding of such a conclusion, provided in Ref. [11] with corresponding numerical calculations, is based on the nonrelativistic Kopylov-Podgoretsky model [1] of initially Boltzmann independent factorized sources (with subsequent symmetrization).

In this article we use as a basis thermal canonical and grand-canonical bosonic ensembles for quantum relativistic ideal gases, which allows us to avoid the specific procedure of “switching on” the quantum statistics as well as the assumption of initially distinguishable sources. Our aim is to clarify the general reasons of violation of the standard prescriptions for $C(p_1, p_2)$ (see above) when one calculates the correlation function in events with fixed multiplicity or just on an event-by-event basis [4]. The answer is not trivial and depends on the applicability of the thermal Wick’s theorem [12]. It is beyond the scope of this study to give a comprehensive analysis and prescription for quasiclassical event generators dealing with $p + p$ and $A + A$ collisions. Rather, the purpose of this paper is to show how imposed particle number constraints affect the single-particle momentum density and the two-particle momentum correlation function in a canonical ensemble of a finite system, as well as to present the analytical estimates for these values in some tractable approximations. In particular, the conditions under which the standard decomposition of two-particle distribution can be applied are considered.

II. ONE- AND TWO- PARTICLE MOMENTUM SPECTRA IN GRAND-CANONICAL AND CANONICAL ENSEMBLES OF IDENTICAL BOSONS

We begin with a brief overview of the properties of the quantum grand-canonical ensemble of a noninteracting boson field with plane waves satisfying periodic boundary conditions on the walls of a cubic box (see, e.g., Ref. [13]).

A. Momentum spectra and Wick’s theorem in a grand-canonical ensemble

The basic object, a grand-canonical statistical operator, can be written as follows:

$$\rho = \exp[-\beta(\widehat{H} - \mu\widehat{N})], \quad (1)$$

where $\beta = 1/T$ is the inverse temperature, $\widehat{H} = \sum_p \epsilon_p a_p^+ a_p$ is the Hamiltonian, ϵ_p is the energy of the single-particle state, and $\widehat{N} = \sum_p a_p^+ a_p$ is the particle number operator. Creation, a_p^+ , and annihilation, a_p , operators satisfy the following canonical commutation relation in the discrete-mode representation:

$$[a_p, a_{p'}^+] = \delta_{pp'}, \quad (2)$$

where $\delta_{pp'}$ is the Kronecker delta function. For notational simplicity, here and below we write p instead of (p_x, p_y, p_z) . The expectation value of the operator \widehat{A} can be expressed as

$$\langle \widehat{A} \rangle = \frac{\text{Tr}[\rho \widehat{A}]}{Z}, \quad (3)$$

where Z is the grand-canonical partition function,

$$Z = \text{Tr}[\rho]. \quad (4)$$

In what follows we assume for simplicity that the chemical potential $\mu = 0$. For $\mu \neq 0$ the substitution $\epsilon_p \rightarrow \epsilon_p - \mu$ has to be utilized in corresponding expressions.

Using the eigenstates of the particle number operator,

$$|p_1, \dots, p_N\rangle = \frac{1}{\sqrt{N!}} a_{p_1}^+ \cdots a_{p_N}^+ |0\rangle, \quad (5)$$

and the identity

$$\sum_N \sum_{p_1, \dots, p_N} |p_1, \dots, p_N\rangle \langle p_1, \dots, p_N| = 1, \quad (6)$$

one can write the statistical operator (1) in the form

$$\rho = \sum_N \sum_{p_1, \dots, p_N} e^{-\beta \epsilon_{p_1} - \dots - \beta \epsilon_{p_N}} |p_1, \dots, p_N\rangle \langle p_1, \dots, p_N|. \quad (7)$$

Inasmuch as our aim here is to calculate the two-boson correlation function, we are interested in the expectation values of operators $a_{p_1}^+ a_{p_2}$ and $a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}$; other n -point operator functions can be calculated in a similar way, if necessary. For calculations we adapt the method proposed in Ref. [12] (see also Refs. [13,14]). The corresponding results are, of course, well known, but it allows us to show a simple example of calculations within the method of Ref. [12] and will help to reveal differences between calculations with and without a fixed particle number constraint.

Our starting point is the relationship

$$a_p \rho = \rho a_p e^{-\beta \epsilon_p}, \quad (8)$$

which can be proved by using an elementary operator algebra and Eq. (7). Then, using trace invariance under the cyclic permutation of an operator, we get

$$\begin{aligned} \text{Tr}[\rho a_{p_1}^+ a_{p_2}] &= e^{-\beta \epsilon_{p_2}} \text{Tr}[\rho a_{p_2} a_{p_1}^+] \\ &= e^{-\beta \epsilon_{p_2}} \text{Tr}[\rho a_{p_1}^+ a_{p_2}] + e^{-\beta \epsilon_{p_2}} \delta_{p_1 p_2} \text{Tr}[\rho]. \end{aligned} \quad (9)$$

From the above equation we have

$$\langle a_{p_1}^+ a_{p_2} \rangle = \frac{\delta_{p_1 p_2}}{e^{\beta \epsilon_{p_2}} - 1}, \quad (10)$$

which is a familiar Bose-Einstein distribution. Similarly, the trace $\text{Tr}[\rho a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}]$ can be expressed as

$$\begin{aligned} \text{Tr}[\rho a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}] &= e^{-\beta \epsilon_{p_2}} \delta_{p_1 p_2} \text{Tr}[\rho a_{p_2}^+ a_{p_1}] \\ &\quad + e^{-\beta \epsilon_{p_2}} \delta_{p_2 p_2} \text{Tr}[\rho a_{p_1}^+ a_{p_1}] \\ &\quad + e^{-\beta \epsilon_{p_2}} \text{Tr}[\rho a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}], \end{aligned} \quad (11)$$

and we have

$$\begin{aligned} \text{Tr}[\rho a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}] &= \frac{\delta_{p_1 p_2}}{e^{\beta \epsilon_{p_2}} - 1} \text{Tr}[\rho a_{p_2}^+ a_{p_1}] \\ &\quad + \frac{\delta_{p_2 p_2}}{e^{\beta \epsilon_{p_2}} - 1} \text{Tr}[\rho a_{p_1}^+ a_{p_1}]. \end{aligned} \quad (12)$$

Then, taking into account Eq. (10), $\langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle$ reads

$$\langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle = \langle a_{p_2}^+ a_{p_1} \rangle \langle a_{p_1}^+ a_{p_2} \rangle + \langle a_{p_1}^+ a_{p_1} \rangle \langle a_{p_2}^+ a_{p_2} \rangle, \quad (13)$$

which is nothing but the particular case of the thermal Wick's theorem [12] (see also Ref. [15] where the applicability of the thermal Wick's theorem for inhomogeneous locally equilibrated noninteracting systems is analyzed). Note that to derive Eqs. (10) and (13) we do not need an explicit expression for the grand-canonical partition function (4).

B. The momentum spectra in a canonical ensemble with a fixed particle number constraint (discrete-mode representation)

Now, let us apply the fixed particle number constraint to the grand-canonical statistical operator (1). For this aim, one can utilize the projection operator \mathcal{P}_N ,

$$\mathcal{P}_N = \sum_{p_1, \dots, p_N} |p_1, \dots, p_N\rangle \langle p_1, \dots, p_N|, \quad (14)$$

which automatically invokes the corresponding constraint. Then we assert that the canonical statistical operator with the constraint, ρ_N , is

$$\begin{aligned} \rho_N &= \mathcal{P}_N \rho \mathcal{P}_N \\ &= \sum_{p_1, \dots, p_N} e^{-\beta \epsilon_{p_1} - \dots - \beta \epsilon_{p_N}} |p_1, \dots, p_N\rangle \langle p_1, \dots, p_N|. \end{aligned} \quad (15)$$

The expectation value of the operator \hat{A} can be defined as

$$\langle \hat{A} \rangle_N = \frac{\text{Tr}[\rho_N \hat{A}]}{Z_N}, \quad (16)$$

where Z_N is the corresponding canonical partition function,

$$Z_N = \text{Tr}[\rho_N]. \quad (17)$$

To evaluate the expectation values of the operators $a_{p_1}^+ a_{p_2}$ and $a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2}$ with the canonical statistical operator ρ_N , see Eqs. (15) and (16), we first utilize elementary operator algebra to prove that

$$a_p \rho_N = \rho_{N-1} a_p e^{-\beta \epsilon_p}. \quad (18)$$

Then, to evaluate the trace $\text{Tr}[\rho_N a_{p_1}^+ a_{p_2}]$, we exploit its invariance under cyclic permutations and get

$$\begin{aligned} \text{Tr}[\rho_N a_{p_1}^+ a_{p_2}] &= e^{-\beta \epsilon_{p_2}} \text{Tr}[\rho_{N-1} a_{p_2} a_{p_1}^+] \\ &= e^{-\beta \epsilon_{p_2}} \text{Tr}[\rho_{N-1} a_{p_1}^+ a_{p_2}] + e^{-\beta \epsilon_{p_2}} \delta_{p_1 p_2} \text{Tr}[\rho_{N-1}]. \end{aligned} \quad (19)$$

From Eq. (19) we have the iteration relation

$$\langle a_{p_1}^+ a_{p_2} \rangle_N = e^{-\beta \epsilon_{p_2}} \delta_{p_1 p_2} \frac{Z_{N-1}}{Z_N} + e^{-\beta \epsilon_{p_2}} \frac{Z_{N-1}}{Z_N} \langle a_{p_1}^+ a_{p_2} \rangle_{N-1}. \quad (20)$$

By using Eq. (20) one can prove by induction (see also Ref. [16] and references therein) that

$$\langle a_{p_1}^+ a_{p_2} \rangle_N = \delta_{p_1 p_2} \sum_{i=1}^N e^{-i\beta \epsilon_{p_2}} \frac{Z_{N-i}}{Z_N}. \quad (21)$$

In the same way we obtain

$$\begin{aligned} \langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle_N &= e^{-\beta \epsilon_{p_2}} \frac{Z_{N-1}}{Z_N} \langle a_{p_2} a_{p_1}^+ a_{p_2}^+ a_{p_1} \rangle_{N-1} \\ &= e^{-\beta \epsilon_{p_2}} \frac{Z_{N-1}}{Z_N} (\langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle_{N-1} \\ &\quad + \delta_{p_1 p_2} \langle a_{p_2}^+ a_{p_1} \rangle_{N-1} + \delta_{p_2 p_2} \langle a_{p_1}^+ a_{p_1} \rangle_{N-1}), \end{aligned} \quad (22)$$

and one can prove by induction that

$$\begin{aligned} \langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle_N &= \delta_{p_1 p_2} \sum_{i=1}^N e^{-i\beta \epsilon_{p_2}} \frac{Z_{N-i}}{Z_N} \langle a_{p_2}^+ a_{p_1} \rangle_{N-i} \\ &\quad + \delta_{p_2 p_2} \sum_{i=1}^N e^{-i\beta \epsilon_{p_2}} \frac{Z_{N-i}}{Z_N} \langle a_{p_1}^+ a_{p_1} \rangle_{N-i}. \end{aligned} \quad (23)$$

Finally, using Eq. (21) and taking into account that $\langle a_{p_1}^+ a_{p_1} \rangle_0 = 0$, we see that Eq. (23) becomes

$$\begin{aligned} \langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle_N &= (\delta_{p_1 p_2} \delta_{p_2 p_1} + \delta_{p_2 p_2} \delta_{p_1 p_1}) \\ &\quad \times \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} e^{-i\beta \epsilon_{p_2}} e^{-j\beta \epsilon_{p_1}} \frac{Z_{N-i-j}}{Z_N}. \end{aligned} \quad (24)$$

It is immediately apparent from Eqs. (21), (23), and (24) that for the noninteracting canonical ensemble with the fixed particle number constraint the decomposition (13), which follows from the thermal Wick's theorem, is no more valid.¹ Also, notice that for practical utilizations of Eqs. (21) and (24) one needs first to calculate the canonical partition functions. The latter can be done by means of the recurrence relations as given in Ref. [18]. Below, for the reader's convenience, we present an elementary derivation of it. As the starting point we utilize the relation

$$\sum_p \langle a_p^+ a_p \rangle_N = N, \quad (25)$$

which follows from the definition of ρ_N [see Eqs. (15) and (16)]. Then, accounting for Eq. (21) we get

$$N Z_N = \sum_{i=1}^N \sum_p e^{-i\beta \epsilon_p} Z_{N-i}. \quad (26)$$

The above expression can be rewritten as

$$Z_N = \frac{1}{N} \sum_{i=0}^{N-1} Z_i \sum_p e^{-(N-i)\beta \epsilon_p}. \quad (27)$$

Taking into account that $Z_0 = \langle 0|0 \rangle = 1$, we get $Z_1 = \sum_p e^{-\beta \epsilon_p}$. Then the recurrence relation can be expressed in

¹Please note also Ref. [17], where expressions for these expectation values are derived in rather compact form at the price of an additional integration.

its final form as

$$Z_N = \frac{1}{N} \sum_{i=0}^{N-1} Z_i Z_1[(N-i)\beta], \quad (28)$$

where for notational convenience we have defined the quantities

$$Z_1(j\beta) = \sum_p e^{-j\beta\epsilon_p}. \quad (29)$$

Then, the use of Eqs. (28) and (29) allows one to determine the canonical partition functions and, therefore, to calculate Eqs. (21) and (24) in the canonical ensemble with periodical boundary conditions.

C. Canonical ensemble in the thermodynamic limit

It is worth noting that periodical boundary conditions are just a mathematical trick that allows convenient mathematical description of finite-volume systems. To eliminate these artificial assumptions, the transition to the thermodynamic limit is typically applied. Then, the number (or the mean number in a grand-canonical ensemble) of particles N and the volume V of the system go to ∞ while keeping the particle density $n = N/V$ constant. As a result, the discrete-mode representation tends to the continuous momentum representation of the canonical operators. In this limit, strictly speaking, normalization of the statistical operator fails because the partition function diverges, but the expectation values can still be defined. There are no problems with the application of this limit to the expectation values (10) and (13) calculated in the grand-canonical ensemble, but the computation of Eqs. (21) and (23) in the canonical ensemble is a more involved problem because Eqs. (21) and (23) explicitly depend on canonical partition functions. To overcome this problem, let us first utilize Eq. (26) to write Z_{N-j-1}/Z_{N-j} as

$$\frac{Z_{N-j-1}}{Z_{N-j}} = \frac{N-j}{N-j-1} \frac{\sum_{i=1}^{N-j-1} \sum_p e^{-i\beta\epsilon_p} Z_{N-j-i-1}}{\sum_{i=1}^{N-j} \sum_p e^{-i\beta\epsilon_p} Z_{N-j-i}}. \quad (30)$$

It is immediately apparent from Eq. (30) that for any fixed j in the thermodynamic limit

$$\lim_{N,V \rightarrow \infty} \frac{Z_{N-j-1}}{Z_{N-j}} = \lim_{N,V \rightarrow \infty} \frac{Z_{N-1}}{Z_N} \equiv \gamma, \quad (31)$$

and therefore,

$$\lim_{N,V \rightarrow \infty} \frac{Z_{N-j}}{Z_N} = \lim_{N,V \rightarrow \infty} \left(\frac{Z_{N-1}}{Z_N} \right)^j. \quad (32)$$

Thus we observe that Eq. (30) in the thermodynamic limit becomes the identity,

$$\frac{Z_{N-1}}{Z_N} = \frac{\sum_{i=1}^{\infty} \sum_p e^{-i\beta\epsilon_p} \left(\frac{Z_{N-1}}{Z_N} \right)^{i+1}}{\sum_{i=1}^{\infty} \sum_p e^{-i\beta\epsilon_p} \left(\frac{Z_{N-1}}{Z_N} \right)^i}. \quad (33)$$

By using Eq. (32), one can write the sum in Eqs. (21) and (24) for $\gamma e^{-\beta\epsilon_p} < 1$ as follows:

$$\langle a_{p_1}^+ a_{p_2} \rangle_{N,V \rightarrow \infty} = \frac{\delta_{p_1 p_2}}{e^{\beta\epsilon_{p_2}} \gamma^{-1} - 1} \quad (34)$$

and

$$\langle a_{p_1}^+ a_{p_2}^+ a_{p_1} a_{p_2} \rangle_{N,V \rightarrow \infty} = \frac{\delta_{p_1 p_2} \delta_{p_2 p_1} + \delta_{p_2 p_2} \delta_{p_1 p_1}}{(e^{\beta\epsilon_{p_1}} \gamma^{-1} - 1)(e^{\beta\epsilon_{p_2}} \gamma^{-1} - 1)}. \quad (35)$$

Note that the above expressions satisfy the principle of thermodynamic equivalence between the canonical ensemble and the grand-canonical ensemble with $e^{\beta\mu} = \frac{Z_{N-1}}{Z_N}$ (see also Ref. [19]).

Finally, redefining in the thermodynamic limit $\delta(\mathbf{k} - \mathbf{k}') = (2\pi)^{-3} V \delta_{\mathbf{k}\mathbf{k}'}$ and $a(\mathbf{k}) = [(2\pi)^{-3} V]^{1/2} [2E(k)]^{1/2} a_k$, $E(k) = \sqrt{k^2 + m^2}$, we get

$$\langle a^+(\mathbf{p}_1) a(\mathbf{p}_2) \rangle = \frac{2E(p_2) \delta(\mathbf{p}_1 - \mathbf{p}_2)}{e^{\beta E(p_2)} \gamma^{-1} - 1} \quad (36)$$

and

$$\begin{aligned} \langle a^+(\mathbf{p}_1) a^+(\mathbf{p}_2) a(\mathbf{p}_1) a(\mathbf{p}_2) \rangle \\ = \frac{4E(p_2) E(p_1) \{[\delta(\mathbf{p}_1 - \mathbf{p}_2)]^2 + [\delta(\mathbf{0})]^2\}}{(e^{\beta E(p_1)} \gamma^{-1} - 1)(e^{\beta E(p_2)} \gamma^{-1} - 1)}. \end{aligned} \quad (37)$$

It is easily seen from Eqs. (36) and (37) that $\langle a^+(\mathbf{p}_1) a^+(\mathbf{p}_2) a(\mathbf{p}_1) a(\mathbf{p}_2) \rangle$ can be written as

$$\begin{aligned} \langle a^+(\mathbf{p}_1) a^+(\mathbf{p}_2) a(\mathbf{p}_1) a(\mathbf{p}_2) \rangle &= \langle a^+(\mathbf{p}_2) a(\mathbf{p}_1) \rangle \langle a^+(\mathbf{p}_1) a(\mathbf{p}_2) \rangle \\ &+ \langle a^+(\mathbf{p}_1) a(\mathbf{p}_1) \rangle \langle a^+(\mathbf{p}_2) a(\mathbf{p}_2) \rangle. \end{aligned} \quad (38)$$

Equation (38) is the particular case of the thermal Wick's theorem [12]. Evidently, this is the consequence of general results about the equivalency of canonical and grand-canonical ensembles in the thermodynamic limit. However, when particle production occurs from extremely small systems with a typical size of 10^{-14} m created in relativistic nucleus and particle collisions, one should use an approximation of the thermodynamic limit with great caution especially in correlation femtoscopy analysis of the system's effective size.

III. TWO-BOSON CORRELATIONS AT FIXED MULTIPLICITIES AND FINITE VOLUMES

In the previous section we calculated the expectation values of creation and annihilation operators both in a finite volume with periodical boundary conditions and in the thermodynamic limit. We demonstrated that only for the latter case is the thermal Wick's theorem applied in the canonical ensemble of noninteracting particles with the fixed particle number constraint. Our aim in this section is to adjust these results for correlation femtoscopy analysis of systems created in high-energy particle and nucleus collisions. To do this, one needs to take into account, first, that experimentally measured observables are one- and two-particle momentum spectra in the continuous-mode momentum representation, and, second, that finite sizes measured by this method do not allow one

to treat these systems in the thermodynamic limit. Then, to be able to consider such systems in a simple approximation,² let us assume that the Compton wavelength and the thermal one are much less than the size of the system. This allows one to use the continuous-mode momentum representation as an approximation to the discrete momentum representation in a finite volume without invoking the thermodynamic limit. To evaluate quantities of interest, we use as the starting point Eqs. (21) and (24), where we substitute $(2\pi)^{-3}V\delta_{kk'} \rightarrow \delta_V(\mathbf{k} - \mathbf{k}')$ and $a_k \rightarrow a(\mathbf{k})[(2\pi)^{-3}V]^{-1/2}[2E(k)]^{-1/2}$. Note that the appearance of $\delta_V(\mathbf{k} - \mathbf{k}')$ instead of $(2\pi)^{-3}V\delta_{kk'}$ [or $\delta(\mathbf{k} - \mathbf{k}')$] does not mean modification of the commutation relation of the creation and annihilation operators but just some modification of the averaging of their products for the case when the spatial particle number density, which can be calculated from the one-particle Wigner function (see, e.g., Ref. [14]), comes to be not constant in full space but related to a finite system. Also, it is natural to assume that the corresponding density distribution is not sharp but has a smooth cutoff. The important example is that in a grand-canonical system with a Gaussian spatial particle number density distribution one should put

$$\delta_V(\mathbf{k} - \mathbf{k}') = \frac{R^3}{(2\pi)^{3/2}} e^{-(\mathbf{k}-\mathbf{k}')^2 R^2/2} \quad (39)$$

in Eqs. (10) and (13) to reproduce the corresponding density behavior and the Gaussian distribution of the particle momentum difference observed in Bose-Einstein two-particle correlation data.³ Let us assume that the same substitution (39) can be applied also in a canonical ensemble with fixed particle multiplicity, Eqs. (21) and (24). Then the parameter R is related to the volume V used in the discrete-mode “box” representation as follows:

$$R^3 = V(2\pi)^{-3/2}. \quad (40)$$

The computation of $\langle a^+(\mathbf{k})a^+(\mathbf{k}')a(\mathbf{k})a(\mathbf{k}') \rangle_N$ and $\langle a^+(\mathbf{k})a(\mathbf{k}) \rangle_N$ makes it possible to obtain the two-particle correlation function, which is defined as

$$C_N(\mathbf{p}, \mathbf{q}) = C_N \frac{\langle a^+(\mathbf{p}_1)a^+(\mathbf{p}_2)a(\mathbf{p}_1)a(\mathbf{p}_2) \rangle_N}{\langle a^+(\mathbf{p}_1)a(\mathbf{p}_1) \rangle_N \langle a^+(\mathbf{p}_2)a(\mathbf{p}_2) \rangle_N}, \quad (41)$$

where $\mathbf{p} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$, and C_N is the normalization constant; the latter is needed to normalize the theoretical correlation function in accordance with normalization that is applied by experimentalists: $C_N(\mathbf{p}, \mathbf{q}) \rightarrow 1$ for $|\mathbf{q}| \rightarrow \infty$.

²An exact approach should be based on a local equilibrium statistical operator for finite systems, similar to how it was realized in Ref. [15] for the particular case of a grand-canonical ensemble in a locally equilibrated longitudinally expanding system. However, for a canonical ensemble with a fixed particle number, this is an especially nontrivial problem.

³The corresponding results for a specific case that includes the symmetrization procedure for an initially nonsymmetrized amplitude of boson radiation from factorized independent (noncoherent) sources are presented in Ref. [11].

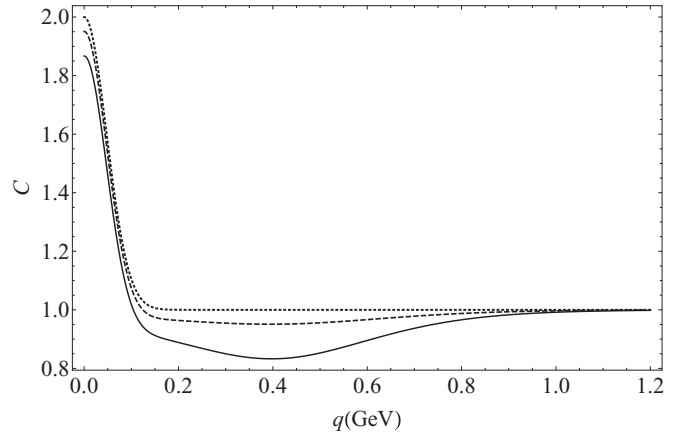


FIG. 1. The two-pion correlation functions $C(q_x = q, q_y = 0, q_z = 0; p_x = p, p_y = 0, p_z = 0)$, with $p = 0.2$ GeV/c, $R = 3$ fm, and $T = 0.06$ GeV in the case of the two-particle system: $N = 2$. The dotted line corresponds to the “standard” expression for the pure Bose-Einstein correlation function (CF) of the Gaussian source. The dashed line is related to the CF when one-boson spectra in the two-boson system is calculated from the two-particle spectra by integrating it over one of the momenta. The solid line corresponds to our approximation based on Eq. (42).

Substitution of Eqs. (21) and (24) into Eq. (41) yields

$$C_N(\mathbf{p}, \mathbf{q}) = C_N \frac{(1 + e^{-\mathbf{q}^2 R^2}) \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} e^{-i\beta E(p_2) - j\beta E(p_1)} \frac{Z_{N-i-j}}{Z_N}}{\sum_{i=1}^N \sum_{j=1}^N e^{-i\beta E(p_2)} e^{-j\beta E(p_1)} \frac{Z_{N-i}}{Z_N} \frac{Z_{N-j}}{Z_N}}, \quad (42)$$

where we performed the transition to the continuous-mode representation as is described above. Despite its complexity, this expression allows a trivial evaluation of the $|\mathbf{q}| \rightarrow \infty$ limit. Namely, because $E(p_1)$ and $E(p_2)$ tend to ∞ when $|\mathbf{q}| \rightarrow \infty$ at fixed \mathbf{p} , we see that $C_N(\mathbf{p}, \mathbf{q}) \rightarrow C_N \frac{Z_{N-2} Z_N}{Z_{N-1}^2}$ in this limit. If we now demand that $C_N(\mathbf{p}, \mathbf{q}) \rightarrow 1$ when $|\mathbf{q}| \rightarrow \infty$, we see that the normalization factor introduced in Eq. (41) is

$$C_N = \frac{Z_{N-1}^2}{Z_{N-2} Z_N}. \quad (43)$$

Note that systems with small particle numbers can be easily investigated by exploiting recurrence relations for the partition function and performing straightforward exact calculations. In Fig. 1 we illustrate and compare the results for two-pion correlation functions $C(q_x = q, q_y = 0, q_z = 0; p_x = p, p_y = 0, p_z = 0)$ (41), with $p = 0.2$ GeV/c, $R = 3$ fm, and $T = 0.06$ GeV in the case when the two-pion spectrum, see the numerator of Eq. (42), is calculated for the two-particle system: $N = 2$. Our aim here is, in particular, to compare the results of the calculations of the two-particle momentum correlations (41) by means of Eq. (42) with some other prescriptions. First, we present the “standard” correlation function $C(\mathbf{p}, \mathbf{q}) = 1 + \exp(-\mathbf{q}^2 R^2)$ that, in fact, corresponds to the correlation function (42) with $N = 2$ in the numerator and $N = 1$ in the denominator (the prefraction normalization constant C_2 is

modified correspondingly). Such a formal case was considered in Ref. [3], and it was shown that the above “standard” expression can be obtained only for a fairly large system when one can ignore the quantum uncertainty principle in particle radiation. It is worth noting that the same “standard” behavior of the two-particle correlation function takes place in the grand-canonical ensemble for bosons [10] when the (effective) system sizes are much larger than the thermal de Broglie wavelength [3,15].

The second result that we present for the two-particle correlation function corresponds to the case when the one-particle pion spectra [$\langle a^+(\mathbf{p}_1)a(\mathbf{p}_1) \rangle_2$ and $\langle a^+(\mathbf{p}_2)a(\mathbf{p}_2) \rangle_2$ in Eq. (41)] are calculated by integration of the two-particle momentum spectra [numerator of Eq. (42)] over one of the particle momenta.

The third result is based on our approximation (42) of the correlation function of a finite system without any modification. One can see that the last result qualitatively reproduces the previous one where the one-particle spectra are calculated from the two-particle ones in a direct way (that is, in fact, an exact self-consistent approach) and the two-particle spectra are taken in our finite-system approximation [numerator of Eq. (42)]. Namely, in both approaches the correlation functions are suppressed (their intercepts are reduced), they lose the Gaussian form even for the Gaussian source, and the correlation function $C(\mathbf{p}, \mathbf{q})$ becomes less than unity at intermediate q values and approaches the limiting value of 1 from below. Such a similarity of the results supports our approximation (42) of the two-particle momentum distribution function.⁴

To see the corresponding effects in the systems with finite but rather large N , as can happen in relativistic particle and nucleus collisions, let us first write the recurrence relation (28) as

$$Z_N = \frac{1}{N} \sum_{i=0}^{N-2} Z_i Z_1 [(N-i)\beta] + \frac{1}{N} Z_{N-1} Z_1(\beta). \quad (44)$$

Utilizing again Eq. (28), one can express Z_{N-1} as

$$Z_{N-1} = \frac{1}{N-1} \sum_{i=0}^{N-2} Z_i Z_1 [(N-i-1)\beta]. \quad (45)$$

Thus we observe that

$$\frac{Z_N}{Z_{N-1}} = \frac{N-1}{N} \frac{\sum_{i=0}^{N-2} Z_i Z_1 [(N-i)\beta]}{\sum_{i=0}^{N-2} Z_i Z_1 [(N-i-1)\beta]} + \frac{1}{N} Z_1(\beta). \quad (46)$$

Now, using Eq. (29), one can change the order of summations in Eq. (46) and get

$$\frac{Z_N}{Z_{N-1}} = \alpha_{N-2} \exp(-\beta m) \frac{N-1}{N} + \frac{Z_1(\beta)}{N}, \quad (47)$$

⁴It is worth noting that the same peculiarities of the two-particle momentum correlations for fixed- N systems were obtained in Ref. [11]. The ultimate reason for such behavior is the violation of the Wick's theorem for heat systems with fixed multiplicities.

where

$$\alpha_{N-2} = \frac{\sum_p f_{N-2}(\beta \epsilon_p) \exp(-\beta \epsilon_p + \beta m)}{\sum_p f_{N-2}(\beta \epsilon_p)}, \quad (48)$$

$$f_{N-2}(\beta \epsilon_p) = \sum_{i=0}^{N-2} Z_i(\beta) \exp[-(N-i-1)\beta \epsilon_p], \quad (49)$$

and m is a particle mass. It is apparent from Eq. (48) that $\alpha_{N-2} < 1$. Now, note that in the continuous-mode representation $\sum_p \rightarrow \frac{V}{(2\pi)^3} \int d^3 p$, $\epsilon_p \rightarrow E(p) = \sqrt{p^2 + m^2}$, and Eq. (47) takes the following form,

$$\frac{Z_N}{Z_{N-1}} = \frac{1}{(2\pi)^3} \frac{V}{N} I(\beta) + \alpha_{N-2} \exp(-\beta m), \quad (50)$$

where

$$I(\beta) \equiv \int d^3 p \exp[-\beta E(p)] = 4\pi \beta^{-1} m^2 K_2(\beta m) \quad (51)$$

and $V = (2\pi)^{3/2} R^3$ [see Eq. (40)]. Taking into account that $\alpha_{N-2} < 1$ and assuming the low particle number density approximation,

$$\frac{1}{(2\pi)^3} \frac{V}{N} I(\beta) \gg 1, \quad (52)$$

we then get⁵

$$\frac{Z_{N-1}}{Z_N} \simeq (2\pi)^3 \frac{N}{V} I^{-1}(\beta) \ll 1. \quad (53)$$

It is easily seen that $\frac{Z_{N-i-1}}{Z_N} = \frac{Z_{N-i-1}}{Z_{N-i}} \dots \frac{Z_{N-1}}{Z_N} \ll \frac{Z_{N-1}}{Z_N} \ll 1$, giving from Eqs. (42) and (43) the following approximate formula,

$$C(\mathbf{p}, \mathbf{q}) \simeq \frac{1 + (e^{-\beta E(p_1)} + e^{-\beta E(p_2)}) \frac{Z_{N-3}}{Z_{N-2}}}{1 + (e^{-\beta E(p_1)} + e^{-\beta E(p_2)}) \frac{Z_{N-2}}{Z_{N-1}}} (1 + e^{-\mathbf{q}^2 R^2}), \quad (54)$$

which can be further simplified as

$$C(\mathbf{p}, \mathbf{q}) \simeq \left[1 + (e^{-\beta E(p_1)} + e^{-\beta E(p_2)}) \left(\frac{Z_{N-3}}{Z_{N-2}} - \frac{Z_{N-2}}{Z_{N-1}} \right) \right] \times (1 + e^{-\mathbf{q}^2 R^2}). \quad (55)$$

Taking into account Eq. (53), the above expression reads

$$C(\mathbf{p}, \mathbf{q}) \simeq \left[1 - \frac{(2\pi)^3}{VI(\beta)} (e^{-\beta E(p_1)} + e^{-\beta E(p_2)}) \right] (1 + e^{-\mathbf{q}^2 R^2}). \quad (56)$$

To see qualitative peculiarities of the above expression explicitly, let us rewrite Eq. (56) for $\beta m \gg 1$.⁶ Then $I(\beta) \approx \lambda_T^{-3} e^{-\beta m}$, where $\lambda_T = (2\pi m T)^{-1/2}$ is the so-called

⁵An attentive reader will notice that Eq. (53) is associated with the Boltzmann approximation in the thermodynamic limit.

⁶Our conclusions are valid, in fact, for any βm values, because only the quantitative strength of the effect depends on the βm value.

thermal wavelength, $T = 1/\beta$, and Eq. (56) can be simplified as

$$C_N(\mathbf{p}, \mathbf{q}) \simeq \left[1 - (2\pi)^3 \frac{\lambda_T^3}{V} e^{\beta m} (e^{-\beta E(p_1)} + e^{-\beta E(p_2)}) \right] \times (1 + e^{-\mathbf{q}^2 R^2}). \quad (57)$$

It is instructive to compare the above expression with a typical experimental parametrization of the two-boson Bose-Einstein correlation function, which looks like

$$C_N(\mathbf{p}, \mathbf{q}) = 1 + \lambda(p) \exp(-R_G^2 \mathbf{q}^2). \quad (58)$$

Here $0 \leq \lambda(p) \leq 1$ describes the correlation strength, R_G is the Gaussian interferometry radius, and to allow comparison with our simple model we simplify the real experimental three-dimensional parametrization by assuming spherical symmetry in Eq. (58). To do this comparison, first note that $\lambda(p) = C_N(\mathbf{p}, \mathbf{0}) - 1$, and one can see from Eq. (57) that the intercept $C_N(\mathbf{p}, \mathbf{0})$ is

$$C_N(\mathbf{p}, \mathbf{0}) \simeq 2 \left[1 - 2(2\pi)^3 \frac{\lambda_T^3}{V} e^{-\beta E(p) + \beta m} \right]. \quad (59)$$

Therefore for the correlation strength parameter $\lambda(p)$ we get in the limit of small particle number densities (52)

$$\lambda(p) \simeq 1 - 4e^{-\beta E(p) + \beta m} (2\pi)^3 \frac{\lambda_T^3}{V}. \quad (60)$$

Note that the Bose-Einstein correlations are suppressed, $\lambda(p) < 1$, and that $\lambda(p) \rightarrow 1$ when $V \rightarrow \infty$; therefore Eq. (60) represents the finite-volume effect in a canonical ensemble with fixed particle multiplicity. One can see that the suppression becomes stronger when the system size tends to the thermal de Broglie wavelength. It has been shown in the model of independent incoherent emitters that $\lambda \rightarrow 0$ in a *finite* system at fixed multiplicity, if particle number tends to infinity [11,20]. It is interesting to note that λ is usually interpreted as a measure of coherence in a theoretical model: $\lambda = 0$ for a pure quantum state and $0 < \lambda \leq 1$ for a mixed quantum state (see, e.g., Refs. [2,3]). However, in the case of a thermal system with fixed multiplicity, the suppression of the correlation function is the result of violation of the Wick's theorem. Then the coherence effects need an additional specific treatment [3].

Using Eq. (60), we see that Eq. (57) can be written in the form

$$C(\mathbf{p}, \mathbf{q}) \simeq \frac{1}{2} [1 + \lambda(p) f(p, q)] (1 + e^{-R^2 \mathbf{q}^2}), \quad (61)$$

where the factor

$$f(p, q) = \frac{e^{-\beta E(p_1)} + e^{-\beta E(p_2)}}{2e^{-\beta E(p)}} \quad (62)$$

results in non-Gaussian behavior of the correlation function with respect to $|\mathbf{q}|$. Namely, for $|\mathbf{p}| \gg |\mathbf{q}|$ and $|\mathbf{p}| \gg m$,

$$\frac{e^{-\beta E(p_1)} + e^{-\beta E(p_2)}}{2e^{-\beta E(p)}} \approx \exp\left(-\frac{q^2}{8T\sqrt{p^2 + m^2}}\right) \times \cosh\left(\frac{pq}{2T\sqrt{p^2 + m^2}}\right). \quad (63)$$

Equations (58), (61), and (62) allow us to relate R and R_G for small $|\mathbf{q}|$, when $R^2 \mathbf{q}^2 \ll 1$ and $R_G^2 \mathbf{q}^2 \ll 1$. Then, expanding $C(\mathbf{p}, \mathbf{q})$ in \mathbf{q} and keeping the first term in the Taylor series, we get from Eq. (61)

$$C(\mathbf{p}, \mathbf{q}) \simeq \frac{1}{2} (1 + \lambda(p)) (2 - R^2 \mathbf{q}^2), \quad (64)$$

and we also get from Eq. (58)

$$C(\mathbf{p}, \mathbf{q}) \simeq 1 + \lambda(p) (1 - R_G^2 \mathbf{q}^2). \quad (65)$$

Here we assume that the system size is large enough in comparison with the thermal wavelength, etc., and replace $f(p, q)$ by 1. After equating Eqs. (64) and (65) we get eventually

$$R_G^2 = \frac{1 + \lambda(p)}{2\lambda(p)} R^2. \quad (66)$$

We remind the reader that this rough approximation is valid only when λ is close to unity. It is interesting to note that the Gaussian interferometry radius exhibits a decrease as the pair momentum increases that is similar to the well-known behavior of the homogeneity length that is associated with the Gaussian interferometry radius in a locally equilibrated expanding system [21].

IV. CONCLUSIONS

In this paper, we have analyzed the single- and two-particle momentum spectra for finite canonical systems of noninteracting particles with a fixed particle number constraint. We find that the corresponding expressions satisfy the thermal Wick's theorem in the thermodynamic limit and do not satisfy it in the general finite case. Our analysis implies that contrary to traditional beliefs (see, e.g., Ref. [4]), decomposition of two-particle distribution functions through one-particle ones is questionable for the class of events with fixed multiplicities originating from small thermal systems.

Furthermore, we evaluated the two-particle correlation function in a low particle number density approximation in a canonical ensemble model. An analysis of the two-particle correlation function indicates that even for Gaussian sources the correlation function is non-Gaussian and reaches values less than unity in some intermediate region of relative momentum of particles q ; the apparent source size (interferometry radius extracted from the typical experimental parametrization for small values of $|\mathbf{q}|$) decreases with the half-sum pair momentum $|\mathbf{p}|$ and at large $|\mathbf{p}|$ reaches the static Gaussian source value.

It was found that the finite-size effect in thermal systems with fixed multiplicity results in a reduction of the intercept of the correlation function. In approximation of small particle number density, the "coherence parameter" λ becomes smaller than unity and decreases when the system size approaches the thermal de Broglie wavelength. As $|\mathbf{p}|$ increases the parameter rises gradually and reaches the constant value of 1. It is interesting to note that measured with various methods of analysis by different LHC collaborations [5] values of λ in $p + p$ collisions are smaller than unity and exhibit a decrease as the average momentum of the pair increases. This is at variance with the behavior of λ in a canonical ensemble model with a

fixed particle number constraint and could be caused by a specific nonthermal mechanism of particle production in $p + p$ collisions, e.g., by pions originated from resonances, but analysis of the latter goes beyond the scope of the present article.

In the present work, to make the problem tractable, we used a simple static canonical ensemble model: while the particle-emitting sources produced in high-energy nucleus and particle collisions are expanding, interactions between the emitted particles are rather complicated and include the resonance decays, etc. Therefore, further investigations based on a more realistic model of an evolving source could be of great interest. However, we hope that the present analysis sheds light on the additional causes (besides the purity, long-lived resonances, and coherence) of the suppression of the correlation function intercept value expressed by the parameter λ . Namely, we found that this parameter is a measure of the degree of the Wick's theorem violation in a thermal

system with a fixed particle number constraint. The results of our analysis may be useful for interpretation of the results of the correlation femtoscopy of events with fixed multiplicities and also for event-generator modeling of the two-particle Bose-Einstein correlations arising in small thermal systems created in relativistic nucleus and particle collisions.

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