Particle-hole symmetry in generalized seniority, microscopic interacting boson (fermion) model, nucleon-pair approximation, and other models

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The particle-hole symmetry (equivalence) of the full shell-model Hilbert space is straightforward and routinely used in practical calculations. In this work I show that this symmetry is preserved in the subspace truncated up to a certain generalized seniority and give the explicit transformation between the states in the two types (particle and hole) of representations. Based on the results, I study particle-hole symmetry in popular theories that could be regarded as further truncations on top of the generalized seniority, including the microscopic interacting boson (fermion) model, the nucleon-pair approximation, and other models.

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I. INTRODUCTION

The particle-hole symmetry (equivalence) of the full shell-model Hilbert space is straightforward. A Slater determinant of $2N$ particles is the same state as a Slater determinant of $2(\Omega - N)$ holes within a model space of degeneracy $2\Omega = \sum_j (2j+1)$. Operators should be converted accordingly as in textbooks [1,2], and the final results are independent of whether choosing particles or holes as the degree of freedom. Practical shell-model calculations frequently encounter the dimension limitation, and various truncation schemes are necessary in reducing the dimension. Apparently it is desirable to preserve particle-hole symmetry when truncating; in this work I consider whether this is the case for some popular truncation schemes.

The seniority quantum number v was first introduced by Racah [3–5] as the number of unpaired nucleons in a single j level to incorporate pairing correlations. As a truncation scheme for the realistic multi-j shell model, the seniority v equals the total number of unpaired nucleons in all j levels. Obviously, a $2N$-particle Slater determinant of seniority v is the same state as a $2(\Omega - N)$-hole Slater determinant of seniority v. The particle-hole symmetry is preserved in the seniority truncated subspaces.

The generalized seniority quantum number S was also introduced [6–11] as the number of unpaired nucleons in a multi-j model, but the paired part wave function is uniquely written as the condensate of coherent pairs. The generalized seniority states are no longer Slater determinants, and the particle-hole symmetry is not obvious. Using commutator techniques in the J scheme, Talmi showed [12] that the $2N$-particle state of $S = 0$ is the same as the $2(\Omega - N)$-hole state of $S = 0$ with reciprocal coherent pair structures, and the $S = 2$ particle states span the same subspace as the $S = 2$ hole states. But for $S > 2$ the conclusion is absent. Along the same line, the particle-hole symmetry found by Johnson and Vincent [13] is restricted within the $\mathcal{S}$-$\mathcal{D}$ subspace, and their collective quadrupole pair operator $\mathcal{D}$ is defined with the seniority projection and thus is different from the usual one. Reference [13] was mainly written for the microscopic foundation of the interacting boson model. The particle-hole symmetry for arbitrary generalized seniority S was claimed in Ref. [10] but without a proof; in fact, their only results for $S = 2$ [Eqs. (2.44) and (2.95)] were misprinted.

In this work I show that particle-hole symmetry exists for arbitrary generalized seniority S. In addition, I give the explicit transformation of states between the particle and the hole representations, in both the M scheme and the J scheme. Based on the results, I consider particle-hole symmetry for popular theories that could be regarded as further truncations on top of the generalized seniority, including the microscopic interacting boson (fermion) model, the nucleon-pair approximation, and other models.

Section II discusses in the M scheme the particle-hole symmetry in generalized seniority. The M-scheme results are coupled into J-scheme expressions in Sec. III. I consider in Secs. IV–VI particle-hole symmetry in the microscopic interacting boson (fermion) model, the nucleon-pair approximation, and other popular truncation schemes. Section VII summarizes the work. The J-scheme expressions of the lowest generalized seniorities are given in the Appendix.

II. M-SCHEME GENERALIZED SENIORITY

In this section I show that particle-hole symmetry exists in generalized-seniority truncated subspaces. The time-reversal invariance is assumed but not necessarily the rotational symmetry; hence, the results are valid for deformed (Nilsson) single-particle levels. Briefly reviewing definitions of generalized seniority, the pair-creation operator

$$P_\alpha^\dagger = a_\alpha^\dagger a_{\tilde{\alpha}}^\dagger \tag{1}$$

creates a pair of particles on the single-particle level $|\alpha\rangle$ and its time-reversed partner $|\tilde{\alpha}\rangle$ ($|\tilde{\tilde{\alpha}}\rangle = -|\alpha\rangle$, $P_\alpha^\dagger = P_{\tilde{\alpha}}^\dagger$). The coherent pair-creation operator

$$P^\dagger = \sum_{\alpha \in \Lambda} v_\alpha P_\alpha^\dagger \tag{2}$$

creates a pair of particles coherently distributed with structure coefficients v_α over the entire single-particle space, where the summation index $\alpha \in \Lambda$ runs over the "pair index" space Λ

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that is half of the single-particle space (for example, only those orbits with a positive magnetic quantum number m). The unnormalized pair-condensate wave function of the $2N$-particle system

$$(P^\dagger)^N|0\rangle \tag{3}$$

builds in pairing correlations. Gradually breaking coherent pairs, the state with $S = 2s$ unpaired nucleons is

$$\underbrace{a^\dagger a^\dagger \cdots a^\dagger}_{S=2s}(P^\dagger)^{N-s}|0\rangle. \tag{4}$$

Loosely speaking, S is defined as the generalized-seniority quantum number [6–11]. More precisely, one distinguishes between the subspace of S unpaired nucleons and the subspace of generalized seniority S. Any state of $S' < S$ unpaired nucleons can be written as a linear combination of the states of S unpaired nucleons, after substituting several $P^\dagger$ by Eq. (2). Therefore, the subspace of S unpaired nucleons consists of the subspaces of generalized-seniority $S, S-2, \ldots, 2, 0$. Practical calculations usually truncate the full many-body space to the subspace of S unpaired nucleons (the subspace up to a certain generalized seniority S); the basis consists of all the states of the form (4). $S = 2N$ corresponds to the full space without truncation.

The full space has particle-hole symmetry. Assuming the single-particle space has degeneracy 2Ω, the Hilbert space consisting of Slater determinants of $2N$ particles is the same as that of $2\bar{N} \equiv 2(\Omega - N)$ holes. Now I consider whether the above two Hilbert spaces truncated to s broken pairs (up to generalized seniority $S = 2s$) are still the same $[0 \leqslant s \leqslant \min(N,\bar{N})]$. To simplify notations I define

$$\eta^s_{s'} = \frac{(N-s)!}{(\bar{N}-s')!}\prod_{\alpha\in\Lambda} v_\alpha.$$

Talmi has shown [12] that for $s = 0$ they are the same. The $2N$-particle pair condensate (3) is the same state as the $2\bar{N}$-hole pair condensate

$$(\bar{P})^{\bar{N}}|\bar{0}\rangle \tag{5}$$

with reciprocal pair structures

$$\bar{P} = \sum_{\alpha\in\Lambda}\frac{1}{v_\alpha}P_\alpha. \tag{6}$$

In Eq. (5),

$$|\bar{0}\rangle = \prod_{\alpha\in\Lambda} P^\dagger_\alpha|0\rangle \tag{7}$$

is the completely occupied state (closed shell). This result was rederived with the correct normalization in Eq. (2.90) of Ref. [10],

$$(P^\dagger)^N|0\rangle = \eta^0_0(\bar{P})^{\bar{N}}|\bar{0}\rangle. \tag{8}$$

For $s = 1$, Talmi proved [12] particle-hole symmetry through commutator techniques in the (coupled) J scheme. Here I prove it in the M scheme by using the identity (8) in Pauli-blocked spaces; this proof seems more clear and can be directly generalized to $s \geqslant 2$. I divide the $s = 1$ states into two types: $a^\dagger_\alpha a^\dagger_\beta(P^\dagger)^{N-1}|0\rangle$ (where α and β belong to different time-reversal pairs, $P_\alpha \neq P_\beta$) and $a^\dagger_\alpha a^\dagger_{\tilde{\alpha}}(P^\dagger)^{N-1}|0\rangle$. The first type is

$$\text{type I} = a^\dagger_\alpha a^\dagger_\beta(P^\dagger)^{N-1}|0\rangle = (P^\dagger_{[\alpha\beta]})^{N-1}a^\dagger_\alpha a^\dagger_\beta|0\rangle, \tag{9}$$

where $P^\dagger_{[\alpha\beta]} \equiv P^\dagger - v_\alpha P^\dagger_\alpha - v_\beta P^\dagger_\beta$ is the coherent pair-creation operator removing $P^\dagger_\alpha$ and $P^\dagger_\beta$ due to Pauli blocking. For convenience I introduce $|[\alpha\beta]\}$ to represent a subspace of the original single-particle space, by removing pairs of single-particle levels $|\alpha\rangle,|\tilde{\alpha}\rangle$ and $|\beta\rangle,|\tilde{\beta}\rangle$ from the latter. The vacuum state and the closed shell of $|[\alpha\beta]\}$ are represented by $|0\rangle_{[\alpha\beta]} = |0\rangle$ and $|\bar{0}\rangle_{[\alpha\beta]} = P_\alpha P_\beta|\bar{0}\rangle$. Within the subspace $|[\alpha\beta]\}$ the identity (8) still holds,

$$(P^\dagger_{[\alpha\beta]})^{N-1}|0\rangle_{[\alpha\beta]} = \frac{(N-1)!}{(\bar{N}-1)!}\left(\prod^{\gamma\neq\alpha,\beta}_{\gamma\in\Lambda} v_\gamma\right)(\bar{P}_{[\alpha\beta]})^{\bar{N}-1}|\bar{0}\rangle_{[\alpha\beta]}$$

$$= \frac{\eta^1_1}{v_\alpha v_\beta}(\bar{P}_{[\alpha\beta]})^{\bar{N}-1}|\bar{0}\rangle_{[\alpha\beta]}, \tag{10}$$

where $\bar{P}_{[\alpha\beta]} \equiv \bar{P} - P_\alpha/v_\alpha - P_\beta/v_\beta$, and the power of $\bar{P}_{[\alpha\beta]}$ corresponding to $(P^\dagger_{[\alpha\beta]})^{N-1}$ is computed as $(\Omega-2)-(N-1) = \bar{N}-1$. Equation (9) results from acting $a^\dagger_\alpha a^\dagger_\beta$ on Eq. (10). Using $a^\dagger_\alpha a^\dagger_\beta|\bar{0}\rangle_{[\alpha\beta]} = a_{\tilde{\alpha}}a_{\tilde{\beta}}|\bar{0}\rangle$, Eq. (9) becomes

$$\text{type I} = \eta^1_1\frac{a_{\tilde{\alpha}}a_{\tilde{\beta}}}{v_\alpha v_\beta}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle. \tag{11}$$

The result is an $s = 1$ state in the hole representation.

The second type is treated similarly.

$$\text{type II} = a^\dagger_\alpha a^\dagger_{\tilde{\alpha}}(P^\dagger)^{N-1}|0\rangle = (P^\dagger_{[\alpha]})^{N-1}P^\dagger_\alpha|0\rangle. \tag{12}$$

The identity (8) in the subspace $|[\alpha]\}$ gives $[(\Omega-1)-(N-1) = \bar{N}]$

$$(P^\dagger_{[\alpha]})^{N-1}|0\rangle_{[\alpha]} = \frac{(N-1)!}{\bar{N}!}\left(\prod^{\gamma\neq\alpha}_{\gamma\in\Lambda} v_\gamma\right)(\bar{P}_{[\alpha]})^{\bar{N}}|\bar{0}\rangle_{[\alpha]}$$

$$= \frac{\eta^1_0}{v_\alpha}(\bar{P}_{[\alpha]})^{\bar{N}}|\bar{0}\rangle_{[\alpha]}.$$

Therefore, Eq. (12) becomes

$$\text{type II} = \frac{\eta^1_0}{v_\alpha}(\bar{P}_{[\alpha]})^{\bar{N}}|\bar{0}\rangle. \tag{13}$$

The binomial expansion of $(\bar{P}_{[\alpha]})^{\bar{N}} = (\bar{P} - P_\alpha/v_\alpha)^{\bar{N}} = (\bar{P})^{\bar{N}} - \bar{N}(\bar{P})^{\bar{N}-1}P_\alpha/v_\alpha + \cdots$ has $\bar{N}+1$ terms, but terms with $(P_\alpha)^2$ or higher powers vanish when acting on $|\bar{0}\rangle$ due to Pauli's principle. Thus,

$$\text{type II} = \frac{\eta^1_0}{v_\alpha}(\bar{P})^{\bar{N}}|\bar{0}\rangle + \frac{\eta^1_1}{(v_\alpha)^2}a_{\tilde{\alpha}}a_{\tilde{\tilde{\alpha}}}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle, \tag{14}$$

where I have used $\eta^1_0\bar{N} = \eta^1_1$ and $P_\alpha = a_{\tilde{\alpha}}a_\alpha = -a_{\tilde{\alpha}}a_{\tilde{\tilde{\alpha}}}$. In the result the first component is an $s = 0$ hole state (linear combinations of the $s = 1$ hole states), and the second component is an $s = 1$ hole state. Combining Eqs. (11)

and (14), I write in summary

$$a_\alpha^\dagger a_\beta^\dagger (P^\dagger)^{N-1}|0\rangle = \eta_1^1 \frac{a_{\tilde{\alpha}} a_{\tilde{\beta}}}{v_\alpha v_\beta} (\bar{P})^{\bar{N}-1}|\bar{0}\rangle + \delta_{\beta\tilde{\alpha}} \frac{\eta_0^1}{v_\alpha} (\bar{P})^{\bar{N}}|\bar{0}\rangle. \tag{15}$$

Equation (15) tells that the $s = 1$ particle states can be expressed as the $s = 1$ hole states. The converse is also true. Thus, the particle space of $s = 1$ broken pairs and the hole space of $s = 1$ broken pairs are the same.

The $s \geqslant 2$ states could be treated similarly. In general, an unnormalized $s = h + k$ particle state is written as

$$\text{state} = \underbrace{a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger \cdots a_{\alpha_{2h}}^\dagger}_{2h} \underbrace{P_{\beta_1}^\dagger P_{\beta_2}^\dagger \cdots P_{\beta_k}^\dagger}_{k} (P^\dagger)^{N-s}|0\rangle, \tag{16}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_{2h}$ belong to different pairs of orbits ($P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_{2h}}$ are all different). The identity (8) in the Pauli-blocked subspace $|[\alpha_1 \cdots \alpha_{2h}, \beta_1 \cdots \beta_k]\}$ reads $[(\Omega - 2h - k) - (N - s) = \bar{N} - h]$

$$\begin{aligned}
&\left(P^\dagger_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]}\right)^{N-s}|0\rangle_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]} \\
&= \frac{(N-s)!}{(\bar{N}-h)!} \frac{\prod_{\gamma\in\Lambda} v_\gamma}{v_{\alpha_1}\cdots v_{\alpha_{2h}} v_{\beta_1}\cdots v_{\beta_k}} \left(\bar{P}_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]}\right)^{\bar{N}-h}|\bar{0}\rangle_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]} \\
&= \frac{\eta_h^s}{v_{\alpha_1}\cdots v_{\alpha_{2h}} v_{\beta_1}\cdots v_{\beta_k}} \left(\bar{P}_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]}\right)^{\bar{N}-h}|\bar{0}\rangle_{[\alpha_1\cdots\alpha_{2h},\beta_1\cdots\beta_k]}.
\end{aligned}$$

Therefore, Eq. (16) becomes

$$\text{state} = \eta_h^s \frac{a_{\tilde{\alpha}_1} a_{\tilde{\alpha}_2} \cdots a_{\tilde{\alpha}_{2h}}}{v_{\alpha_1}\cdots v_{\alpha_{2h}} v_{\beta_1}\cdots v_{\beta_k}} \left(\bar{P}_{[\beta_1\cdots\beta_k]}\right)^{\bar{N}-h}|\bar{0}\rangle. \tag{17}$$

Power expanding the right-hand side,

$$\begin{aligned}
\left(\bar{P}_{[\beta_1\cdots\beta_k]}\right)^{\bar{N}-h}|\bar{0}\rangle &= \left(\bar{P} - \frac{P_{\beta_1}}{v_{\beta_1}} - \frac{P_{\beta_2}}{v_{\beta_2}} - \cdots - \frac{P_{\beta_k}}{v_{\beta_k}}\right)^{\bar{N}-h}|\bar{0}\rangle \\
&= \sum_{0\leqslant n\leqslant k} \frac{(\bar{N}-h)!}{(\bar{N}-h-n)!} \sum_{\{\gamma_1\cdots\gamma_n\}\in\{\beta_1\cdots\beta_k\}} \frac{(-)^n P_{\gamma_1} P_{\gamma_2}\cdots P_{\gamma_n}}{v_{\gamma_1} v_{\gamma_2}\cdots v_{\gamma_n}} \bar{P}^{\bar{N}-h-n}|\bar{0}\rangle,
\end{aligned} \tag{18}$$

where the summation index $\{\gamma_1 \cdots \gamma_n\} \in \{\beta_1 \cdots \beta_k\}$ means taking n different elements $\{\gamma_1 \cdots \gamma_n\}$ from the set $\{\beta_1 \cdots \beta_k\}$, and summing over all possibilities. Consequently Eq. (17) becomes

$$\begin{aligned}
\text{state} &= \underbrace{a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger \cdots a_{\alpha_{2h}}^\dagger}_{2h} \underbrace{P_{\beta_1}^\dagger P_{\beta_2}^\dagger \cdots P_{\beta_k}^\dagger}_{k} (P^\dagger)^{N-s}|0\rangle \\
&= \frac{a_{\tilde{\alpha}_1} a_{\tilde{\alpha}_2} \cdots a_{\tilde{\alpha}_{2h}}}{v_{\alpha_1}\cdots v_{\alpha_{2h}} v_{\beta_1}\cdots v_{\beta_k}} \sum_{0\leqslant n\leqslant k} \eta_{h+n}^s \sum_{\{\gamma_1\cdots\gamma_n\}\in\{\beta_1\cdots\beta_k\}} \frac{(-)^n P_{\gamma_1} P_{\gamma_2}\cdots P_{\gamma_n}}{v_{\gamma_1} v_{\gamma_2}\cdots v_{\gamma_n}} \bar{P}^{\bar{N}-h-n}|\bar{0}\rangle,
\end{aligned} \tag{19}$$

where I have used $\eta_h^s(\bar{N}-h)!/(\bar{N}-h-n)! = \eta_{h+n}^s$. The result has components of the broken-pair number $s' = h + n = h, h+1, \ldots, s$. Hence, the particle states of s broken pairs can be expressed as the hole states of s broken pairs. The converse is also true. This proves particle-hole symmetry: the $2N$-particle space and the $2\bar{N}$-hole space truncated to arbitrary s broken pairs (up to generalized seniority $S = 2s$) are the same $[0 \leqslant s \leqslant \min(N,\bar{N})]$. This symmetry has been tested numerically by the fast algorithm that was developed [14] and applied [15,16] recently.

Odd-particle systems also have particle-hole symmetry in generalized seniority: the $(2N + 1)$-particle space and the $(2\bar{N} - 1)$-hole space truncated to arbitrary $S = 2s + 1$ unpaired particles and holes (up to generalized seniority $S = 2s + 1$) are the same $[0 \leqslant s \leqslant \min(N, \bar{N} - 1)]$. The actual transformation between the particle and the hole representations is $(s = h + k)$

$$\begin{aligned}
&\underbrace{a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger \cdots a_{\alpha_{2h+1}}^\dagger}_{2h+1} \underbrace{P_{\beta_1}^\dagger P_{\beta_2}^\dagger \cdots P_{\beta_k}^\dagger}_{k} (P^\dagger)^{N-s}|0\rangle \\
&= \frac{-a_{\tilde{\alpha}_1} a_{\tilde{\alpha}_2} \cdots a_{\tilde{\alpha}_{2h+1}}}{v_{\alpha_1}\cdots v_{\alpha_{2h+1}} v_{\beta_1}\cdots v_{\beta_k}} \sum_{0\leqslant n\leqslant k} \eta_{h+n+1}^s \sum_{\{\gamma_1\cdots\gamma_n\}\in\{\beta_1\cdots\beta_k\}} \frac{(-)^n P_{\gamma_1} P_{\gamma_2}\cdots P_{\gamma_n}}{v_{\gamma_1} v_{\gamma_2}\cdots v_{\gamma_n}} \bar{P}^{\bar{N}-1-h-n}|\bar{0}\rangle.
\end{aligned} \tag{20}$$

III. J-SCHEME GENERALIZED SENIORITY

In the previous section I show that particle-hole symmetry exists in the generalized-seniority truncated subspaces and find the transformation between the particle and the hole representations in the M scheme. In this section I assume the rotational symmetry and write the transformation in the (coupled) J scheme. The single-particle space is generally written as $\{j_1, j_2, \ldots, j_D\}$. The pair structure $v_{jm} = v_j$ is independent of the magnetic quantum number m. I choose the phase of the time-reversed orbit to be

$$\tilde{a}_{j,m} = (-)^{j-m} a_{j,-m}. \tag{21}$$

The tensor $\tilde{a}_j$ transforms in the same way as $a_j^\dagger$ under rotation.

In even systems the J-scheme transformation results from coupling the M-scheme transformation (19) with Clebsch-Gordan coefficients. For components with the maximal generalized seniority,

$$\begin{aligned}&\left(a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_{2s}}^\dagger\right)^{\tau,J} (P^\dagger)^{N-s}|0\rangle\\ &= \eta_s^s \frac{\left(\tilde{a}_{j_1}\tilde{a}_{j_2}\cdots\tilde{a}_{j_{2s}}\right)^{\tau,J}}{v_{j_1}v_{j_2}\cdots v_{j_{2s}}} (\bar{P})^{\bar{N}-s}|\bar{0}\rangle + O(s-1),\end{aligned} \tag{22}$$

where $O(s-1)$ represents terms of generalized seniority $2(s-1)$ and less, and τ collects all the intermediate angular momenta to specify the state in the selected coupling scheme. The result is neat: simply replacing $a_{j_i}^\dagger$ by $\tilde{a}_{j_i}$.

Odd-particle systems could be treated similarly. Coupling the M-scheme transformation (20) with Clebsch-Gordan coefficients, one has for components of the maximal generalized seniority

$$\begin{aligned}&\left(a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_{2s+1}}^\dagger\right)^{\tau,J} (P^\dagger)^{N-s}|0\rangle\\ &= -\eta_{s+1}^s \frac{\left(\tilde{a}_{j_1}\tilde{a}_{j_2}\cdots\tilde{a}_{j_{2s+1}}\right)^{\tau,J}}{v_{j_1}v_{j_2}\cdots v_{j_{2s+1}}} (\bar{P})^{\bar{N}-1-s}|\bar{0}\rangle + O(s-1).\end{aligned} \tag{23}$$

The Appendix provides the full expressions [explicit form of $O(s-1)$] for the simplest cases of $S = 2,3,4$ unpaired nucleons.

IV. MICROSCOPIC INTERACTING BOSON (FERMION) MODEL

In this section I show that the microscopic model space of the interacting boson (fermion) model (IBM and IBFM) [17–23], as a subspace of the full fermionic many-body space, has particle-hole symmetry. But the full mapping procedure preserves the symmetry only at the exact mid-shell. The usual mapping prescription adopts the particle (hole) representation in the lower (upper) shell [11]; there is no ambiguity at the switching mid-shell nucleus.

The IBM uses bosons of various multipolarities as building blocks of the model space. Microscopically, the bosons are identified [10,18,19,21] as collective nucleon pairs

$$B^\dagger = \sum_{j_1 j_2} \beta_{j_1 j_2} \left(a_{j_1}^\dagger a_{j_2}^\dagger\right)^\lambda \tag{24}$$

with the multipolarity λ and the pair structure $\beta_{j_1 j_2}$. Initially only $\mathcal{S}$ ($\lambda = 0$) and $\mathcal{D}$ ($\lambda = 2$) bosons are introduced, but later it is found that bosons with $\lambda > 2$ are frequently necessary [21]. [My $P^\dagger$ operator (2) is macroscopically the IBM $\mathcal{S}$ boson.] The mapping from the shell model determines the bosonic Hamiltonian.

Different mapping methods exist [19]. Here I refer to the Otsuka-Arima-Iachello mapping [18] summarized in a broader sense as seven steps. First, on the boson side:

(1) Select important bosons; for example, $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{G}$ ($\lambda = 4$) bosons.
(2) Construct the basis of the bosonic space Γ_b from operators $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{G}$.
(3) From all the allowed (by symmetry) terms consisting of $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{G}$ bosons, select the dominant terms entering into the bosonic Hamiltonian $\mathcal{H}$ and other observables such as the quadrupole moment $\mathcal{Q}$; each term has a strength parameter χ_i yet to be determined.
(4) Compute the matrices M_b of the bosonic operators $\mathcal{H}$ and $\mathcal{Q}$ within the bosonic space Γ_b; the matrix elements have parameters χ_i.

Next, on the fermion side:

(5) Choose the microscopic pair structures (24) for $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{G}$; identify microscopically each bosonic basis state with its fermionic state. These fermionic states span a subspace Γ_f of the full many-body space.
(6) Compute the matrices M_f of the fermionic operators H and Q within the fermionic space Γ_f.

Finally, for the mapping:

(7) Fix χ_i such that M_b from step 4 closely resembles, by certain criteria, M_f from step 6.

I consider mapping methods that fix in step 5 the microscopic pair structures (24) in each nucleus separately; hence, the pair structures vary along the isotopic (isobaric) chain for realistic Hamiltonians. The $\mathcal{S}$ pair is usually determined by minimizing the mean energy or from the BCS theory. The $\mathcal{D}$ and $\mathcal{G}$ pairs could be fixed by, for example, diagonalizing H in the generalized-seniority-2 sector. In a given nucleus, I consider whether the mapping results depend on choosing between the particle and the hole representations.

The subspace Γ_f of step 5 and the matrices M_f of step 6 are the same under the two representations. This has been proved by Talmi [12] and Johnson and Vincent [13] for $\mathcal{S}$ and $\mathcal{D}$ bosons; here I generalize their conclusion to cases with many kinds of bosons. Equation (22) immediately gives

$$\begin{aligned}&(B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J} (P^\dagger)^{N-s}|0\rangle\\ &= \eta_s^s (\bar{B}_1 \bar{B}_2 \cdots \bar{B}_s)^{\tau,J} (\bar{P})^{\bar{N}-s}|\bar{0}\rangle + O(s-1),\end{aligned} \tag{25}$$

where the s particle-pair operators $B_i^\dagger = \sum \beta_{j_1 j_2}^i (a_{j_1}^\dagger a_{j_2}^\dagger)^{\lambda_i}$ ($i = 1,2,\ldots,s$) could have different multipolarities λ_i and pair structures $\beta^i_{j_1 j_2}$. The corresponding hole-pair operators are

$$\bar{B}_i = \sum_{j_1 j_2} \frac{\beta^i_{j_1 j_2}}{v_{j_1} v_{j_2}} (\tilde{a}_{j_1} \tilde{a}_{j_2})^{\lambda_i}. \quad (26)$$

In the particle representation, step 5 microscopically identifies the bosonic basis state as

$$(\mathcal{B}_1^\dagger \mathcal{B}_2^\dagger \cdots \mathcal{B}_s^\dagger)^{\tau,J} (\mathcal{S}^\dagger)^{N-s}|0\rangle \Leftrightarrow \hat{\mathcal{P}}_{2s} (B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J} (P^\dagger)^{N-s}|0\rangle, \quad (27)$$

where $\hat{\mathcal{P}}_{2s}$ is the projection operator that keeps only components of the maximal generalized seniority $2s$ from the fermionic wave function $(B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J}(P^\dagger)^{N-s}|0\rangle$ [the left-hand side of Eq. (25)]. The projection is necessary [10,11,18] because the bosons $\mathcal{B}_i$ $(i = 1,2,\ldots,s)$ commute with the $\mathcal{S}$ boson, and the macroscopic bosonic state $(\mathcal{B}_1^\dagger \mathcal{B}_2^\dagger \cdots \mathcal{B}_s^\dagger)^{\tau,J}(\mathcal{S}^\dagger)^{N-s}|0\rangle$ is orthogonal to those of a different number of $\mathcal{S}^\dagger$. I clarify one point of the current mapping scheme. For example, in the $\mathcal{S}$-$\mathcal{D}$-$\mathcal{G}$ model space, although the fermionic state $(D^\dagger G^\dagger)^{J=6}(P^\dagger)^{N-2}|0\rangle$ is orthogonal to all the fermionic $s = 1$ states $D^\dagger(P^\dagger)^{N-1}|0\rangle$ and $G^\dagger(P^\dagger)^{N-1}|0\rangle$, the boson image is still defined with the projection $(\mathcal{D}^\dagger \mathcal{G}^\dagger)^{J=6}(\mathcal{S}^\dagger)^{N-2}|0\rangle \Leftrightarrow \hat{\mathcal{P}}_4(D^\dagger G^\dagger)^{J=6}(P^\dagger)^{N-2}|0\rangle$. Here the projected-out $s = 1$ components have the $\mathcal{I}$ boson (multipolarity $\lambda = 6$) not included in the $\mathcal{S}$-$\mathcal{D}$-$\mathcal{G}$ model space. Similarly in the hole representation the bosonic basis state is identified as

$$(\bar{\mathcal{B}}_1^\dagger \bar{\mathcal{B}}_2^\dagger \cdots \bar{\mathcal{B}}_s^\dagger)^{\tau,J}(\bar{\mathcal{S}}^\dagger)^{\bar{N}-s}|0\rangle \Leftrightarrow \hat{\mathcal{P}}_{2s}(\bar{B}_1 \bar{B}_2 \cdots \bar{B}_s)^{\tau,J}(\bar{P})^{\bar{N}-s}|\bar{0}\rangle. \quad (28)$$

Equations (25), (27), and (28) imply

$$\hat{\mathcal{P}}_{2s}(B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J}(P^\dagger)^{N-s}|0\rangle = \eta_s^s \hat{\mathcal{P}}_{2s}(\bar{B}_1 \bar{B}_2 \cdots \bar{B}_s)^{\tau,J}(\bar{P})^{\bar{N}-s}|\bar{0}\rangle. \quad (29)$$

Therefore, in step 5 the fermionic subspace Γ_f and its basis are the same under the two representations; the matrices M_f of step 6 are also the same. Orthogonalizing, for example, the fermionic particle states $\hat{\mathcal{P}}_4(B_1^\dagger B_1^\dagger)^J(P^\dagger)^{N-2}|0\rangle$ and $\hat{\mathcal{P}}_4(B_2^\dagger B_2^\dagger)^J(P^\dagger)^{N-2}|0\rangle$ (the two different bosons $\mathcal{B}_1$ and $\mathcal{B}_2$ commute in the IBM) does not affect the conclusion; the orthogonalization happens in the particle and the hole representations simultaneously.

However, in general the bosonic space Γ_b is different under the two representations: the particle Γ_b has N bosons and the hole Γ_b has $\bar{N}$ bosons. Diagonalizing the mapped bosonic Hamiltonian $\mathcal{H}$ in Γ_b also gives different results. Only at the exact mid-shell $N = \bar{N} = \Omega/2$ (Ω must be even) does the full mapping procedure preserve particle-hole symmetry. Examples include semimagic nuclei ${}^{116}_{50}\mathrm{Sn}_{66}$, ${}^{148}_{66}\mathrm{Dy}_{82}$, and ${}^{186}_{82}\mathrm{Pb}_{104}$, and the open-shell nucleus ${}^{170}_{66}\mathrm{Dy}_{104}$. The usual mapping prescription switches from the particle representation to the hole representation beyond the mid-shell [11]; at the switching nucleus of the particle number $2N = \Omega$ (Ω is even), both representations give the same results without ambiguity. The ambiguity reported in Ref. [24] is because they insist the pair structures (24) are *invariant* along the isotopic chain.

For odd systems with $2N + 1$ particles, the microscopic IBFM [22,23] uses the model space consisting of one (or more) unpaired fermion(s) and various bosons. From Eq. (23) we immediately have

$$(a_j^\dagger B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J}(P^\dagger)^{N-s}|0\rangle = -\frac{\eta_{s+1}^s}{v_j}(\tilde{a}_j \bar{B}_1 \bar{B}_2 \cdots \bar{B}_s)^{\tau,J}(\bar{P})^{\bar{N}-1-s}|\bar{0}\rangle + O(s-1), \quad (30)$$

where $B_i^\dagger$ and $\bar{B}_i$ are still defined by Eqs. (24) and (26). The normalization η_{s+1}^s/v_j is different when $a_j^\dagger$ is on different j levels. Projecting onto the maximal generalized seniority $S = 2s + 1$, the $O(s-1)$ terms drop out. The microscopic IBFM model space preserves particle-hole symmetry.

However, the full mapping procedure preserves particle-hole symmetry only if the spaces Γ_b of the two representations are the same. The particle Γ_b has N bosons (and one unpaired fermion) and the hole Γ_b has $\bar{N} - 1$ bosons. The condition of the same boson number $N = \bar{N} - 1$ implies that the nucleus has the particle number $2N + 1 = \Omega$ (Ω must be odd); this is at the exact mid-shell. Examples include semimagic nuclei ${}^{67}_{28}\mathrm{Ni}_{39}$ and ${}^{89}_{39}\mathrm{Y}_{50}$, and the open-shell nucleus ${}^{78}_{39}\mathrm{Y}_{39}$. The usual mapping prescription switches between the two representations in mid-shell nuclei and has no ambiguity.

If the microscopic IBM or IBFM is only used to truncate the shell-model space (to Γ_f of step 5), we diagonalize M_f of H from step 6 without doing further mapping. In this case the truncation scheme has particle-hole symmetry. Its validity, especially compared with the nucleon-pair approximation of Sec. V, deserves further study (both are inspired by IBM).

V. NUCLEON-PAIR APPROXIMATION

Inspired by the IBM, the nucleon-pair approximation (NPA) [25–29] further truncates the generalized-seniority subspace; the unpaired nucleons are coupled into collective pairs of certain multipolarities (quadrupole, octupole, hexadecapole, etc.). The NPA basis state is

$$(B_1^\dagger B_2^\dagger \cdots B_s^\dagger)^{\tau,J}(P^\dagger)^{N-s}|0\rangle \quad (31)$$

as appeared on the left-hand side of Eq. (25), where $B_i^\dagger = \sum \beta^i_{j_1 j_2}(a_{j_1}^\dagger a_{j_2}^\dagger)^{\lambda_i}$ is still defined by Eq. (24). In NPA we diagonalize the exact shell-model Hamiltonian inside the NPA subspace without mapping onto bosons.

Here I show that in general particle-hole symmetry is lost in the NPA subspace. As a counterexample, I consider the simplest version of NPA consisting of only $S^\dagger$ [my $P^\dagger$ (2)] and $D^\dagger$ pairs. From Eq. (19), the particle state of two $D^\dagger$ coupled to $J = 4$ is transformed as

$$(D^\dagger D^\dagger)^{J=4}(P^\dagger)^{N-2}|0\rangle = \eta_2^2(\bar{D}\bar{D})^{J=4}(\bar{P})^{\bar{N}-2}|\bar{0}\rangle + \eta_1^2(\bar{G})^{J=4}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle. \quad (32)$$

In the hole representation a new hexadecapole pair $\bar{G} = \sum \beta^G_{j_1 j_2}(\tilde{a}_{j_1}\tilde{a}_{j_2})^{\lambda=4}$ appears, and its structure $\beta^G_{j_1 j_2}$ is completely determined by the structure of $D^\dagger$. The particle-hole symmetry is broken.

Near the mid-shell, the NPA should be careful in choosing between the particles and the holes as the degree of freedom; the results are generally different.

VI. OTHER TRUNCATION SCHEMES

In this section I consider particle-hole symmetry in other popular truncation schemes on top of the generalized seniority. These schemes are frequently used to truncate the shell-model space; here they act in the same way on the *unpaired* nucleons of the generalized-seniority subspace (4).

In the multi-j model, I introduce n_j as the number of *unpaired* nucleons [particles (holes) in the particle (hole) representation] on the j level. The truncation $n_j \leqslant n_j^{\max}$, where $n_j^{\max}$ are preselected integers, preserves particle-hole symmetry. This is easily proved through Eq. (19): on the right-hand side the number of unpaired holes, n_j^{hole}, is less than (some β index is not selected into the γ indices) or equal to (all are selected) the number of unpaired particles, n_j^{particle}, of the left-hand side.

However, following the same argument, the truncation $n_j \geqslant n_j^{\min}$ ($n_j^{\min}$ are preselected integers) breaks particle-hole symmetry.

Another popular truncation scheme is cutting by mean energies of the basis states. For each basis state $|i\rangle$ in the form (4), we compute $E_i = \langle i|H|i\rangle$ and remove all the states with $E_i > E_{\max}$ ($E_{\max}$ is the energy cutoff). In general this scheme breaks particle-hole symmetry. As shown in Eq. (19), some hole states from the right-hand side possibly had higher mean energy than the particle state from the left-hand side.

The particle-hole symmetry in other truncation schemes could be analyzed through the transformations (19) and (20) for even and odd systems.

VII. CONCLUSIONS

In this work I show that particle-hole symmetry survives the truncation from the full shell-model space to the generalized-seniority subspace. The explicit transformations between the states in the particle and the hole representations are provided in both the M scheme and the J scheme.

Based on the results, I consider this symmetry in popular theories that could be regarded as further truncations on top of the generalized seniority. Specifically, the microscopic model space of the interacting boson (fermion) model, as a subspace of the full fermionic many-body space, has particle-hole symmetry. But the full mapping procedure preserves the symmetry only at the mid-shell. The usual mapping prescription adopts the particle (hole) representation in the lower (upper) shell; there is no ambiguity at the switching mid-shell nucleus. The nucleon-pair approximation breaks particle-hole symmetry. Other studied truncation schemes are restricting the unpaired nucleon number in each j level, and cutting by the mean energy of the basis states.

Practical calculations frequently truncate the shell-model space due to the dimension limit. Near the mid-shell, the results of this work guide the choice between the particle and the hole representations, for truncation schemes related to the generalized seniority. More care is due if the symmetry is broken.

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APPENDIX

I give the full expressions [$O(s-1)$ terms in Eqs. (22) and (23)] of the J-scheme transformations for the lowest generalized seniority $S = 2,3,4$. These results frequently find applications in other theories such as the boson mapping. In even systems and $S = 2s = 2$,

$$\left(a_{j_1}^\dagger a_{j_2}^\dagger\right)^J (P^\dagger)^{N-1}|0\rangle = C_{s=1} + \delta_{J0}\delta_{j_1 j_2}\frac{\eta_0^1 \hat{j}_1}{v_{j_1}}(\bar{P})^{\bar{N}}|\bar{0}\rangle,$$

where $\hat{j}_1 \equiv \sqrt{2j_1+1}$, and I have used $(a_j^\dagger a_j^\dagger)_0^0 = \sum_m a_{jm}^\dagger \tilde{a}_{jm}^\dagger/\sqrt{2j+1}$. $C_{s=1} = \eta_1^1 (\tilde{a}_{j_1}\tilde{a}_{j_2})^J (\bar{P})^{\bar{N}-1}|\bar{0}\rangle/(v_{j_1}v_{j_2})$ stands for the $s=1$ term as given in Eq. (22).

In even systems and $S = 2s = 4$, the unpaired part $(a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger a_{j_4}^\dagger)^{\tau,J}$ divides into several cases. If the four particles are on different j levels (j_1, j_2, j_3, and j_4 are all different), the $O(s-1)$ terms vanish and the full expression is given by Eq. (22). If only two of the four j's are the same (j, j_3, and j_4 are different),

$$\begin{aligned}&\left[(a_j^\dagger a_j^\dagger)^\lambda \left(a_{j_3}^\dagger a_{j_4}^\dagger\right)^{\lambda'}\right]^J (P^\dagger)^{N-2}|0\rangle\\&\quad = C_{s=2} + \delta_{\lambda 0}\delta_{\lambda' J}\frac{\eta_1^2 \hat{j}\left(\tilde{a}_{j_3}\tilde{a}_{j_4}\right)^J}{v_j v_{j_3} v_{j_4}}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle,\end{aligned}$$

where λ is even and $C_{s=2} = \eta_2^2[(\tilde{a}_j\tilde{a}_j)^\lambda(\tilde{a}_{j_3}\tilde{a}_{j_4})^{\lambda'}]^J(\bar{P})^{\bar{N}-2}|\bar{0}\rangle/(v_j^2 v_{j_3}v_{j_4})$ according to Eq. (22). If the four j's are pairwise equal ($j \neq j'$),

$$\begin{aligned}&[(a_j^\dagger a_j^\dagger)^\lambda (a_{j'}^\dagger a_{j'}^\dagger)^{\lambda'}]^J (P^\dagger)^{N-2}|0\rangle\\&\quad = C_{s=2} + \delta_{\lambda 0}\delta_{\lambda' J}(1-\delta_{\lambda' 0})\frac{\eta_1^2 \hat{j}(\tilde{a}_{j'}\tilde{a}_{j'})^J}{v_j v_{j'}^2}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle\\&\qquad + \delta_{\lambda' 0}\delta_{\lambda J}(1-\delta_{\lambda 0})\frac{\eta_1^2 \hat{j}'(\tilde{a}_j\tilde{a}_j)^J}{v_{j'} v_j^2}(\bar{P})^{\bar{N}-1}|\bar{0}\rangle\\&\qquad + \delta_{\lambda' 0}\delta_{\lambda 0}\delta_{J0}\frac{\eta_0^2 \hat{j}\hat{j}'}{v_j v_{j'}}(\bar{P})^{\bar{N}}|\bar{0}\rangle,\end{aligned}$$

where λ, λ' are even and $C_{s=2} = \eta_2^2[(\tilde{a}_j\tilde{a}_j)^\lambda(\tilde{a}_{j'}\tilde{a}_{j'})^{\lambda'}]^J(\bar{P})^{\bar{N}-2}|\bar{0}\rangle/(v_j^2 v_{j'}^2)$. If three or four particles are on the same j level, the result is complicated, involving various recoupling of the identical $a_j^\dagger$ operators; I skip it here.

In odd systems and $S = 2s+1 = 3$, I give the full expression for $(a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger)^{\tau,J}(P^\dagger)^{N-1}|0\rangle$. If the three particles are on different j levels (j_1, j_2, and j_3 are all different), the

$O(s-1)$ terms vanish and the full expression is given by Eq. (23). If only two of the three j's are the same ($j \neq j'$),

$$((a_j^\dagger a_j^\dagger)^\lambda a_{j'}^\dagger)^J (P^\dagger)^{N-1}|0\rangle = C_{s=1} - \delta_{\lambda 0}\delta_{Jj'} \frac{\eta_1^1 \hat{j} \tilde{a}_{j'}}{v_j v_{j'}} (\bar{P})^{\bar{N}-1}|\bar{0}\rangle,$$

where $C_{s=1} = -\eta_2^1((\tilde{a}_j\tilde{a}_j)^\lambda \tilde{a}_{j'})^J(\bar{P})^{\bar{N}-2}|\bar{0}\rangle/(v_j^2 v_{j'})$ according to Eq. (23). If the three j's are the same,

$$(a_j^\dagger a_j^\dagger)^0 a_{jm}^\dagger (P^\dagger)^{N-1}|0\rangle = C_{s=1} - \frac{(2j-1)\eta_1^1 \tilde{a}_{jm}}{\hat{j} v_j^2} (\bar{P})^{\bar{N}-1}|\bar{0}\rangle,$$

where $C_{s=1} = -\eta_2^1(\tilde{a}_j\tilde{a}_j)^0 \tilde{a}_{jm}(\bar{P})^{\bar{N}-2}|\bar{0}\rangle/v_j^3$. And for even $\lambda \neq 0$,

$$(a_j^\dagger a_j^\dagger)_0^\lambda a_{jm}^\dagger (P^\dagger)^{N-1}|0\rangle = C_{s=1} + 2(-)^{j-m} C_{jmj-m}^{\lambda 0} \frac{\eta_1^1 \tilde{a}_{jm}}{v_j^2} (\bar{P})^{\bar{N}-1}|\bar{0}\rangle,$$

where $C_{s=1} = -\eta_2^1(\tilde{a}_j\tilde{a}_j)_0^\lambda \tilde{a}_{jm}(\bar{P})^{\bar{N}-2}|\bar{0}\rangle/v_j^3$.

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