Classic calculations of static properties of nucleons reexamined

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Classic calculations of the magnetic moments μ_p and μ_n of the nucleons using the traditional exponential kernel show instability with respect to variations of the Borel mass as well as arbitrariness with respect to the choice of the onset of perturbative QCD. The use of a polynomial kernel, the coefficients of which are determined by the masses of the nucleon resonances stabilizes the calculation and provides much better damping of the unknown contribution of the nucleon continuum. The method is also applied to the evaluation of the coupling g_A of proton to the axial current and to the strong part of the neutron-proton mass difference δM_{np}. All these quantities depend sensitively on the value of the 4-quark condensate $\langle 0 \mid \bar{q}q\bar{q}q \mid 0\rangle$, and the value $\langle 0 \mid \bar{q}q\bar{q}q \mid 0\rangle \simeq 1.6\langle 0 \mid \bar{q}q \mid 0\rangle^2$ reproduces the experimental results.

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I. INTRODUCTION

The QCD sum rule method introduced 35 years ago by Shifman, Vainshtein, and Zakharov [1] has been a powerful analytic approach to the problem of extracting low-energy physical quantities from QCD expressions valid in the space-like asymptotic domain. The method starts from a dispersion integral

$$\text{Residue} = \frac{1}{\pi}\int_{\text{th}}^{\infty} dt\, e^{-t/M^2}\text{Im}P(t). \tag{1}$$

The residue contains the physical quantity of interest and the integral runs from some physical threshold to infinity. The integral is then split into two parts

$$\int_{\text{th}}^{\infty} dt\, e^{-t/M^2}\text{Im}P(t) = \int_{\text{th}}^{s_0} dt\, e^{-t/M^2}\text{Im}P(t) + \int_{s_0}^{\infty} dt\, e^{-t/M^2}\text{Im}P(t), \tag{2}$$

where the divider s_0 signals the onset of perturbative QCD. In the first integral on the right-hand side of the equation above, $\text{Im}P(t)$ describes the unknown contribution of the resonances. The second integral takes into account the contribution of the QCD part of the amplitude when $P(t)$ is replaced by its QCD expression. M^2, the square of the Borel mass, is a parameter introduced in order to suppress the unknowns of the problem. If M^2 is small, the damping of the first unknown integral on the right-hand side of Eq. (2) is good but the contribution of the unknown higher order nonperturbative condensates increases rapidly. If M^2 increases, the contribution of the unknown condensates decreases but the damping in the resonances region worsens. An intermediate value of M^2 has to be chosen. Because M^2 is a nonphysical parameter the results should be independent of it in a relatively broad window; this is not the case in the problems at hand. The choice of the parameter s_0 which signals the onset of perturbative QCD is another source of uncertainty.

In this work I shall re-examine the classic calculations of the magnetic moments of the nucleons [2,3] and the coupling of protons to the axial current [4,5], and I shall use polynomial kernels in dispersion integrals in order to eliminate the contribution of the unknown integrands. The coefficients of these polynomials are determined by the masses of the nucleon resonances themselves and involve none of the instability and arbitrariness inherent in the use of exponential kernels. The same kernels have been used to evaluate the neutron-proton mass difference [6] and the nucleon mass [7].

The method will be first applied to the calculation of the magnetic moments of the nucleons μ_p and μ_n. As a second application I shall consider the coupling g_A of the proton to the axial-vector current [4,5], and I shall finally review briefly a previous calculation of the strong part of the neutron-proton mass difference [6].

All the quantities turn out to depend sensitively on the four-quark condensate

$$\langle 0 \mid \bar{q}q\bar{q}q \mid 0\rangle = \kappa\langle 0 \mid \bar{q}q \mid 0\rangle^2. \tag{3}$$

The value of κ has been the subject of much investigation [8] and the result is estimated to vary between 1 (vacuum dominance) and 4. An interesting result of the present investigation is that the single value $\kappa \simeq 1.6$ reproduces the experimental values of all four quantities μ_p, μ_n, g_A, and δM_{np}.

II. NUCLEON MAGNETIC MOMENTS

I shall concentrate on the work of Balitsky and Yung [2] because it offers better convergence of the operator product expansion (OPE) than that of Ioffe and Smilga [3]. Starting from the 3-point function

$$W_{\mu\nu}(p) = \frac{i}{2}\iint dx\, dy\, e^{ip.x} y_\nu \langle 0|T j_\mu(y) \times \eta\left(\frac{x}{2}\right)\eta\left(\frac{-x}{2}\right)|0\rangle - \mu \leftrightarrow \nu, \tag{4}$$

where j_μ is the electromagnetic current and $\eta(x) = \epsilon_{abc}[u^a(x)C\gamma_\mu u^b(x)]\gamma_5\gamma_\mu d^c(x)$ is the proton current of Ioffe and Smilga [3]. The double-nucleon pole contribution to the tensor $W_{\mu\nu}(p)$ has the form

$$W_{\mu\nu}(p) = \frac{-\lambda^2}{\left(p^2 - m_N^2\right)^2}\left[\frac{i}{2}\{\bar{p},\sigma_{\mu\nu}\}F_m^N + i\sigma_{\mu\nu}\left(F_m^N + F_e^N\right) + \frac{i}{2}\left(F_m^N - F_e^N\right)\bar{p}\sigma_{\mu\nu}\bar{p}\right], \quad (5)$$

where $\bar{p} = p_\mu\gamma^\mu$, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$, $F_m^N = 2\frac{m_N}{e}\mu_N$ and $F_e^N = \frac{e_N}{e}$ are the values of the magnetic and electric form factors of the nucleons at zero momentum transfer, and λ denotes the coupling of the nucleon current to the nucleon

$$\langle 0|\eta|N\rangle = \lambda U_N. \quad (6)$$

In general

$$W_{\mu\nu}(p) = -i\{\bar{p},\sigma_{\mu\nu}\}W_1(p^2) - i\sigma_{\mu\nu}W_2(p^2) - ip\sigma_{\mu\nu}\bar{p}W_3(p^2). \quad (7)$$

$W_1(p^2)$ is selected, which I call $W(p^2)$ for simplicity. $W(p^2)$ has first to be evaluated in the deep Euclidean region. This is carefully done by Balitsky and Yung [2] who exploit both local and bilocal representations of the OPE and who use vector meson dominance to estimate the bilocal contributions. The result is

$$W^{\rm QCD}(t) = W_{\rm pert}(t) + \frac{c_1}{t} + \frac{c_2}{t^2} + \frac{c_3}{t^3} + \cdots, \quad (8)$$

with

$$c_1 = -\frac{4}{3}\frac{e_u}{m_V^2}\langle 0|\bar{q}q\bar{q}q|0\rangle,$$

$$c_2 = -\left[\frac{8}{27}\frac{e_u}{m_V^2}\langle 0|\bar{q}q|0\rangle\langle 0|\bar{u}\sigma Gu|0\rangle + \frac{1}{3}\left(e_d + \frac{2}{3}e_u\right)\langle 0|\bar{q}q\bar{q}q|0\rangle\right]$$
$$\simeq -\frac{1}{3}\left[\frac{8}{9}e_u\frac{m_0^2}{m_V^2} + \left(e_d + \frac{2}{3}e_u\right)\right]\langle 0|\bar{q}q\bar{q}q|0\rangle, \quad (9)$$

where $m_V = m_\rho \simeq m_\omega$ and $m_0^2 = \frac{\langle 0|\bar{u}\sigma G|0\rangle}{\langle 0|\bar{q}q|0\rangle} \backsimeq 0.8\,\text{GeV}^2$ and c_3 is an unknown term which will be used to estimate the error.

For small and moderate momentum transfers, $W(t = p^2)$ has double and single poles

$$W(t) = \frac{\lambda^2 F_m^p(t)}{\left(t - m_N^2\right)^2} + \frac{b_1}{\left(t - m_N^2\right)} + \cdots. \quad (10)$$

The single pole arises from the unknown nucleon-continuum transitions and the remainder from the continuum-continuum intermediate states. As a function of t, $W(t)$ is analytic in the complex t plane with poles shown in Eq. (10) and a cut along the positive t axis starting at $t_{\rm th} = (m_N + m_\pi)^2$.

Consider the contour C consisting of two straight lines just above and below the cut and running from threshold to a large value R and a circle of radius R and consider the integral $\int_C dt(m_N^2 - t)f(t)W(t)$. The factor $(m_N^2 - t)$ has been introduced in order to eliminate the unknown single-pole contribution and $f(t)$ is an entire function. On the circle, $W(t)$ can be replaced by $W^{\rm QCD}(t)$ to a good approximation except possibly near the real axis. Repeated application of Cauchy's theorem leads to

$$-\frac{1}{2}\lambda^2 F_m^N f\left(m_N^2\right) = \frac{1}{\pi}\int_{\rm th}^R dt\left(m_N^2 - t\right)f(t)\,{\rm Im}\,W(t) - \frac{1}{2\pi i}\int_0^R dt\left(m_N^2 - t\right)f(t)\,{\rm Disc}\,W_{\rm pert}(t) + m_N^2 c_1 - \left(1 + a_1 m_N^2\right)c_2, \quad (11)$$

where

$$\frac{1}{2\pi i}{\rm Disc}\,W_p(t) = \frac{e_u}{16\pi^4}t \quad (12)$$

to lowest order in a_s.

The second term on the right-hand side of Eq. (11) equals the contribution of the integral on the circle of $W_{\rm pert}^{\rm QCD}(t)$. The last two terms represent the contribution of the integral on the circle of the first two terms of the nonperturbative expansion of $W^{\rm QCD}(t)$ [for the choice I shall adopt $f(t) = 1 - a_1 t - a_2 t^2$] and the small contribution of the unknown next two nonperturbative terms has been neglected (note that the use of the exponential kernel would introduce an infinite number of such unknowns in the game).

The first term on the right-hand side of Eq. (11), which represents the contribution of the physical continuum, constitutes the main uncertainty of the calculation. The choice of the so-far arbitrary function $f(t)$ aims at reducing this term as much as possible in order to allow its neglect. The commonly used choice is $f(t) = e^{-t/M^2}$ where M^2 is the Borel mass parameter and it is hoped that the result is not too sensitive to it. This is not the case in the problem at hand. I will choose instead a simple polynomial

$$f(t) = p_0(t) = 1 - a_1 t - a_2 t^2. \quad (13)$$

The coefficients $a_{1,2}$ are chosen in order to minimize $f(t)$ on the interval $I : 2 \preccurlyeq t \preccurlyeq 3\,\text{GeV}^2$ where the nucleon $\frac{1}{2}^+$ and $\frac{1}{2}^-$ resonances lie. Minimizing $\int_I dt f(t)^2$ yields, for example,

$$a_1 = 0.807\ \text{GeV}^{-2}, \quad a_2 = -0.160\ \text{GeV}^{-4}. \quad (14)$$

With this choice the relative damping $p_0(t)/p_0(m_N^2)$ does not exceed 6% on the interval I. Then

$$\frac{1}{2}\lambda^2 F_m^p p_0\left(m_N^2\right) = \frac{e_u}{16\pi^4}\int_0^R dt\,t\left(m_N^2 - t\right)p_0(t) - m_N^2 c_1 + \left(1 + a_1 m_N^2\right)c_2, \quad (15)$$

where the contribution of the integral over the resonance region [the first term on the right-hand side of Eq. (11)] has been neglected because of the damping polynomial. The choice of R is determined by stability considerations, it should not be too small as this would invalidate the OPE on the circle nor should it be too large because $p_0(t)$ would start enhancing the contribution of the continuum instead of suppressing it. It turns out indeed that the integral on the right-hand side of Eq. (15) is stable for $2 \lesssim R \lesssim 3\,\text{GeV}^2$ and that it contributes little compared to the nonperturbative terms shown in Eq. (9). The result is essentially proportional to the 4-quark condensate.

Numerically then

$$\tfrac{1}{2}\lambda^2 F_m^p f\left(m_N^2\right) \simeq 0.80\langle 0|\overline{q}q\overline{q}q|0\rangle. \tag{16}$$

The coupling λ has been obtained by a similar method [6,7]

$$(2\pi)^4\lambda^2 m_N p_0\left(m_N^2\right) = -B_3 I_1(R) - B_7 + a_1 B_9 \tag{17}$$

with

$$\begin{aligned} B_3 &= 4\pi^2\left(1+\frac{3}{2}a_s\right)\langle 0|\overline{q}q|0\rangle, \\ B_7 &= -\frac{4}{3}\pi^4\langle 0|\overline{q}q|0\rangle\langle 0|a_s\bar{G}G|0\rangle, \\ B_9 &= -(2\pi)^6\frac{136}{81}a_s\langle 0|(\overline{q}q)^3|0\rangle, \\ I_1(R) &= \int_0^R dt\, tp_0(t). \end{aligned} \tag{18}$$

Numerically

$$\lambda^2 p_0\left(m_N^2\right) = \frac{1.21\ \mathrm{GeV}^6}{32\pi^4}. \tag{19}$$

So that with

$$\langle 0|\overline{q}q\overline{q}q|0\rangle = \kappa\langle 0|\overline{q}q|0\rangle^2 \tag{20}$$

I finally get

$$F_m^p = 1.50\kappa. \tag{21}$$

At this point it is interesting to compare my approach to the use of the exponential kernel. One would have instead of Eq. (15)

$$\begin{aligned} \lambda^2 F_m^p e^{-\frac{m_N^2}{M^2}} &= \frac{e_u}{16\pi^4}\int_0^R dt\, t\left(m_N^2 - t\right)e^{-\frac{t}{M^2}} - m_N^2 c_1 \\ &\quad + \left(1-\frac{m_N^2}{M^2}\right)c_2 \end{aligned} \tag{22}$$

and

$$(2\pi)^4\lambda^2 e^{-\frac{m_N^2}{M^2}} = -B_3 M^4 E_1(R/M^2) - B_7 + \frac{1}{M^2}B_9, \tag{23}$$

where

$$E_1 = \int_0^{R/M^2} dx\, x\, e^{-x}.$$

The disadvantage of the standard approach is that the result is very sensitive to the value of the Borel mass parameter M^2. It actually varies by a factor of 2.8 when M^2 varies between 0.8 and 1.4 GeV^2. The use of the polynomial kernel has stabilized the calculation. The magnetic moment of the neutron is obtained by the exchange $u \leftrightarrow d$ with the result

$$F_m^n = -1.26\kappa. \tag{24}$$

The method can likewise be applied to the calculation of the proton to the axial-vector current to which I turn in the next section.

III. COUPLING OF THE PROTON TO THE AXIAL-VECTOR CURRENT

Following Belyaev and Kogan [4], I start with the polarization amplitude in an external axial-vector field

$$\Pi^A(q^2) = i\int dx\, e^{iqx}\langle 0|T\eta(x)\eta(0)|0\rangle_A. \tag{25}$$

$\Pi^A(q^2)$ also has double- and single-nucleon poles

$$\Pi^A(t) = -\frac{\bar{\lambda}^2 g_A}{\left(t-m_N^2\right)^2} + \frac{b\bar{\lambda}^2}{\left(t-m_N^2\right)} + \cdots, \tag{26}$$

and in the deep Euclidean region

$$\Pi^A_{\mathrm{QCD}}(t) = t\ln(-t) + \frac{c_1}{t} - \frac{20}{9}a_{qq}^2\frac{1}{t^2} + \frac{c_3}{t^3} + \cdots, \tag{27}$$

where

$$\begin{aligned} c_1 &= \tfrac{1}{4}\langle 0|g_s\bar{G}G|0\rangle + \tfrac{16}{9}\pi^2 m_1^2 f_\pi^2, \\ c_3 &= -\tfrac{7}{6}m_0^2 a_{qq}^2, \end{aligned} \tag{28}$$

with $m_1^2 = 1.5\,\mathrm{GeV}^2$, $\bar{\lambda}^2 = 2(2\pi^4)\lambda^2$, and $a_{qq}^2 = 4\pi^2\langle\bar{q}q\rangle^2$.

The contribution of the single pole has been estimated to be quite small [3,4]. I shall neglect this contribution in order not to eliminate it by multiplying by $(t - m_N^2)$ as before because this would introduce higher order unknown condensates whose contribution could be large because the convergence of the asymptotic series is not good enough.

The method used in the preceding section is repeated and the same damping polynomial $p_0(t)$ is used which gives

$$-\bar{\lambda}^2 p_0'\left(m_N^2\right)g_A = \int_0^R dt\, tp_0(t) + c_1 + a_1\frac{20}{9}a_{qq}^2 - a_2 c_3. \tag{29}$$

The final numerical result is

$$g_A = 0.39 + 0.61\kappa. \tag{30}$$

IV. RESULTS AND CONCLUSIONS

The results obtained for the magnetic moments and the axial-vector coupling of the nucleon

$$\begin{aligned} F_m^p &= 1.50\kappa, \\ F_p^n &= -1.26.\kappa, \quad g_A = 0.39 + 0.61\kappa, \end{aligned} \tag{31}$$

are very sensitive to the value of κ on which no consensus exists.

Another physical quantity which is also very sensitive to this parameter is the strong part of the proton-neutron mass difference

$$\delta M_{np} = (m_d - m_u)U = 2.60 \pm 0.50\ \mathrm{MeV}. \tag{32}$$

This quantity was evaluated in [6] with the same approach used here with the result

$$U = 1.03\kappa - 0.57, \tag{33}$$

and the values taken for the quark masses are $m_u = 2.9 \pm 0.2$ and $m_d = 5.3 \pm 0.4$ MeV [9]. The expression obtained in [6] differs slightly from Eq. (32) because the condensate B_9 appearing in Eq. (18) was taken with the wrong sign.

Of course our results are affected by the uncertainties of the calculation. It is interesting, however, that a single value of κ reproduces the experimental values of the nucleon magnetic moments, the coupling to the axial-vector current, and the neutron-proton mass difference. Indeed, taking $\kappa = 1.6$ in Eqs. (20), (23), (29), and (32), then

$$F_m^p = 2.40, \quad F_m^n = -2.01,$$
$$g_A = 1.36, \quad \delta M_{np} = 2.6 \text{ MeV}, \tag{34}$$

which compare well with the experimental numbers

$$F_m^p = 2.71, \quad F_m^n = -1.91,$$
$$g_A = 1.27, \quad \delta M_{np} = (2.6 \pm .5) \text{ MeV}. \tag{35}$$

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).
[2] I. I. Balitsky and A. V. Yung, Phys. Lett. B **129**, 328 (1983).
[3] B. L. Ioffe and A. V. Smilga, Nucl. Phys. B **232**, 109 (1984).
[4] V. M. Belyaev and Y. I. Kogan, Phys. Lett. B **136**, 273 (1984).
[5] D. Espriu and A. Palanques-Mestre, Z. Phys. C **23**, 97 (1984).
[6] N. F. Nasrallah, Phys. Rev. C **87**, 055203 (2013).
[7] N. F. Nasrallah and K. Schilcher, Phys. Rev. C **89**, 045202 (2014).
[8] O. Catà, Phys. Rev. D **81**, 054011 (2010), and references therein.
[9] C. A. Dominguez, N. F. Nasrallah, R. H. Rontsch, and K. Schilcher, Phys. Rev. D **79**, 014009 (2009).