Compositeness of baryonic resonances: Application to the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances

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We present a formulation of the compositeness for baryonic resonances to discuss the meson-baryon molecular structure inside the resonances. For this purpose, we derive a relation between the residue of the scattering amplitude at the resonance pole position and the two-body wave function of the resonance in a sophisticated way, and we define the compositeness as the norm of the two-body wave functions. As applications, we investigate the compositeness of the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances from precise πN scattering amplitudes in a unitarized chiral framework with the interaction up to the next-to-leading order in chiral perturbation theory. The πN compositeness for the $\Delta(1232)$ resonance is evaluated in the πN single-channel scattering, and we find that the πN component inside $\Delta(1232)$ in the present framework is non-negligible, which supports the previous work. On the other hand, the compositeness for the $N(1535)$ and $N(1650)$ resonances is evaluated in a coupled-channel approach, resulting that the πN, ηN, $K\Lambda$, and $K\Sigma$ components are negligible for these resonances.

I. INTRODUCTION

Investigating the internal structure of hadrons is one of the most important topics in hadron physics [1], highly motivated by our expectation that there can exist exotic hadrons, which are not composed of a three-quark (qqq) system for baryons or of a quark-antiquark ($q\bar{q}$) one for mesons. Namely, while traditional quark models have succeeded in describing baryons and mesons with qqq and $q\bar{q}$, respectively, we may consider some exotic configurations for hadron structures, e.g., tetraquarks and pentaquarks, as long as they are color singlet states. Indeed, there are several candidates of exotic hadrons, which cannot be classified into the states predicted by the quark models. For instance, $\Lambda(1405)$ has been considered as an exotic hadron rather than a compact uds state because of its anomalously light mass; because the $\bar{K}N(I=0)$ interaction is strongly attractive, $\Lambda(1405)$ may be a $\bar{K}N$ molecular state [2]. Until now, great efforts have been continuously made in both experimental and theoretical sides to clarify the structure of exotic hadron candidates and to discover genuine exotic hadrons. In this context, it is encouraging that there have been experimental signals of exotic hadrons in the heavy quark sector: charged charmoniumlike states discovered by Belle [3] and charmonium pentaquarks by LHCb [4]. Moreover, it is interesting that an evidence of $\Lambda(1405)$ as a $\bar{K}N$ molecular state has come from lattice QCD simulations [5].

Among exotic configurations of hadrons, hadronic molecular states are of special interest, because they are composed of two or more asymptotic states of QCD, i.e., color singlet states, and hence one can define the structure of these hadrons in hadronic degrees of freedom without complicated treatment of QCD. Actually, because hadrons are color singlet states, their masses and interactions between them do not depend on the renormalization scheme of QCD, in contrast to the quark-gluon dynamics. This viewpoint of the study on composites of asymptotic states originates in the old work on the field renormalization constant intensively discussed in the 1960s [6–9]. One of the most prominent results in this approach is that the deuteron is dominated by the loosely bound proton-neutron component [10]. Then the structure of hadrons from the field renormalization constant and from the so-called compositeness was studied in, e.g., Refs. [11–25]. In particular, the compositeness is explicitly defined as contributions from two-body wave functions to the normalization of the total wave function for the resonance [13,22,26,27] and can be extracted from the pole properties of the scattering amplitude for two asymptotic states. Because the total wave function is normalized to be unity, we can discuss the composite fraction of hadrons by comparing the compositeness with unity. Here we should note that, in general, the compositeness as well as the wave function is not observable and hence a model-dependent quantity. However, for states lying near the two-body threshold, we can express the compositeness with observables such as the scattering length and effective range, as studied in Refs. [10,11,15,19,20,25]. Besides, in a certain model the compositeness has been utilized to study the internal structure of hadronic resonances from experimental observables as well, such as the $\bar{K}N$ component inside $\Lambda(1405)$ [28] and the $K\bar{K}$ components inside the scalar mesons $f_0(980)$ and $a_0(980)$ [29].

Because the compositeness can be extracted from the scattering amplitude, it is a good subject to apply the compositeness to the nucleon resonances, which we abbreviate as N^*, and to discuss the meson-baryon compositeness for the N^* resonances. This is because, at present, precise πN scattering amplitudes are available from many research groups, e.g., ANL-Osaka [30], Jülich [31], and Dubna-Mainz-Taipei [32] in

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the so-called dynamical approaches and Bonn-Gatchina [33] and GWU [34] in the on-shell K-matrix approaches. In principle, we can extract the πN and other meson-baryon compositeness from the precise πN scattering amplitudes via properties of the resonance poles.

In this paper we focus on the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances because there are several implications that these hadrons may have certain fractions of the meson-baryon components. For $\Delta(1232)$, there are several suggestions that the effect of the meson cloud seems to be large, for instance, in the $M1$ transition form factor for $\gamma^* N \to \Delta(1232)$ at $Q^2 = 0$ [35]. The πN compositeness for $\Delta(1232)$ has been already studied in a simple phenomenological model [17], implying large contribution of the πN component to the internal structure of $\Delta(1232)$. For $N(1535)$ and $N(1650)$, there are several studies that they can be dynamically generated from meson-baryon degrees of freedom without introducing explicit resonance poles in the so-called chiral unitary approach [36–43]. In this approach, the πN and its coupled-channel amplitude is obtained based on the combination of chiral perturbation theory and the unitarization of the scattering amplitude [36–45]. The results in the chiral unitary approach might suggest that $N(1535)$ and $N(1650)$ are meson-baryon molecular states. Of special interest is the relation among $N(1535)$, $N(1650)$, and other dynamically generated resonances in the chiral unitary approach such as $\Lambda(1405)$ and $\Xi(1690)$. Namely, it is suggested in Ref. [46] that, in a flavor SU(3) symmetric world in the chiral unitary approach, $N(1535)$ and $N(1650)$ degenerate, together with the one of the two-$\Lambda(1405)$ pole, $\Xi(1690)$, and so on, into two degenerated octets as dynamically generated states. Because both $\Lambda(1405)$ and $\Xi(1690)$ in the chiral unitary approach in the physical world are respectively found to be indeed the $\bar{K} N$ [22] and $\bar{K} \Sigma$ [47] molecular states in terms of the compositeness, the degeneracy in the flavor SU(3) symmetric world implies that both $N(1535)$ and $N(1650)$ may be meson-baryon molecular states as well. However, the compositeness for $N(1535)$ was studied in the chiral unitary approach with the simplest interaction, i.e., the Weinberg-Tomozawa term, in Ref. [22], and the result indicated the large component originating from contributions other than the pseudoscalar meson-baryon dynamics considered for $N(1535)$.

Motivated by these observations, in the present study, we evaluate the compositeness for the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances and $N(940)$ from the precise πN scattering amplitudes in the chiral unitary approach, taking into account the interaction up to the next-to-leading order from chiral perturbation theory and including an explicit Δ term for $\Delta(1232)$. For $\Delta(1232)$, we discuss its πN compositeness in the πN single-channel scattering, while we treat $N(1535)$ and $N(1650)$ in a πN-ηN-$K \Lambda$-$K \Sigma$ coupled-channel problem without introducing explicit bare states. The loop function in our approach is evaluated with the dimensional regularization. We fit the model parameters to the solution of the partial-wave analysis for the πN scattering amplitude and calculate the meson-baryon compositeness for $\Delta(1232)$, $N(1535)$, and $N(1650)$ from the πN scattering amplitude. A part of the study on $\Delta(1232)$ was already reported in Ref. [48]. In this paper, we show the details of the formulation, results, and discussions.

This paper is organized as follows. In Sec. II we formulate the compositeness for baryonic resonances to discuss the meson-baryon molecular structure inside the resonances. In the formulation we consider the case of a relativistic scattering of arbitrary spin particles. Next, in Sec. III we show our numerical calculations on the compositeness for the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances in the chiral unitary approach with the interaction up to the next-to-leading order in chiral perturbation theory. Section IV is devoted to the summary of this study and outlook.

II. COMPOSITENESS

First of all, we formulate the compositeness, which has been recently developed in the hadron physics so as to discuss the hadronic molecular components inside hadrons. The compositeness is defined as contributions from two-body wave functions to the normalization of the total wave function $|\Psi\rangle$ for the resonance state and corresponds to unity minus the field renormalization constant intensively discussed in the 1960s [6–9]. Although the compositeness is not observable and hence a model-dependent quantity, it will be an important piece of information on the internal structure of the resonance state.

In this section we first show how to extract the compositeness from the residue of the two-body to two-body scattering amplitude at the resonance pole in the nonrelativistic framework in Sec. II A. Next we extend our discussions to the relativistic case in Sec. II B. In both Sec. II A and Sec. II B we do not specify the form of the interaction so as to give the general formulas of the compositeness in terms of the residue of the scattering amplitude at the resonance pole position, and in Sec. II C we consider the formulation with the separable interaction, which is employed in our numerical calculations in Sec. III. Then we give several comments on the interpretation of the compositeness for resonance states in Sec. II D. In the following we take the rest frame of the center-of-mass motion, namely two scattering particles have equal and opposite momentum, and hence the resonance state is at rest with zero momentum.

A. Scattering amplitude and wave function in a nonrelativistic case

We consider a two-body to two-body coupled-channel scattering in a nonrelativistic condition governed by the interaction operator $\hat{V}$ for the two-body systems, with which we have only the two-body states in the practical model space. For simplicity, we assume that the interaction is a central force and neglect the spin of the scattering particles. Moreover, for the later applications we allow the interaction to depend intrinsically on the energy of the system E, which corresponds to the eigenenergy of the full Hamiltonian.[1]

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[1]If there are missing (or implicit) channels which are implemented in the interaction, such missing channels, regardless of one-body bare states or more than one-body scattering states of higher thresholds, can be origin of the intrinsic energy dependence of the interaction. See Ref. [22] for the details.

The scattering amplitude can be formally obtained with the Lippmann-Schwinger equation in an operator form,

$$\hat{T}(E) = \hat{V}(E) + \hat{V}(E)\frac{1}{E - \hat{H}_0}\hat{T}(E)$$

$$= \hat{V}(E) + \hat{V}(E)\frac{1}{E - \hat{H}}\hat{V}(E), \qquad (1)$$

with the T-matrix operator $\hat{T}$, the free Hamiltonian $\hat{H}_0$, and the full Hamiltonian $\hat{H} \equiv \hat{H}_0 + \hat{V}(E)$.

First, to evaluate the scattering amplitude from the Lippmann-Schwinger equation (1), we have to introduce the scattering states with which we calculate the matrix element of the T-matrix operator. We represent the jth-channel two-body scattering state with relative momentum $\boldsymbol{q}$ as $|\boldsymbol{q}_j\rangle$, which is an eigenstate of the free Hamiltonian $\hat{H}_0$,

$$\hat{H}_0|\boldsymbol{q}_j\rangle = E_j(q)|\boldsymbol{q}_j\rangle, \quad \langle\boldsymbol{q}_j|\hat{H}_0 = E_j(q)\langle\boldsymbol{q}_j|, \qquad (2)$$

where $q \equiv |\boldsymbol{q}|$ is the magnitude of the momentum $\boldsymbol{q}$ and the eigenenergy $E_j(q)$ contains the threshold energy,

$$E_j(q) \equiv m_j + M_j + \frac{q^2}{2\mu_j}, \quad \mu_j \equiv \frac{m_j M_j}{m_j + M_j}, \qquad (3)$$

with the masses of the jth-channel particles m_j and M_j. We fix the normalization of the scattering states as

$$\langle\boldsymbol{q}'_k|\boldsymbol{q}_j\rangle = (2\pi)^3\delta_{jk}\delta(\boldsymbol{q}' - \boldsymbol{q}). \qquad (4)$$

Now we can express scattering amplitude of the $k(\boldsymbol{q}) \to j(\boldsymbol{q}')$ scattering, where $\boldsymbol{q}^{(\prime)}$ is the relative momenta in the initial (final) state, as

$$\langle\boldsymbol{q}'_j|\hat{T}(E)|\boldsymbol{q}_k\rangle \equiv T_{jk}(E; \boldsymbol{q}', \boldsymbol{q}), \qquad (5)$$

which is obtained from the interaction

$$\langle\boldsymbol{q}'_j|\hat{V}(E)|\boldsymbol{q}_k\rangle \equiv V_{jk}(E; \boldsymbol{q}', \boldsymbol{q}). \qquad (6)$$

In this study we assume the time-reversal invariance of the scattering process. This constrains the interaction and amplitude, with an appropriate choice of phases of the states, as

$$V_{jk}(E; \boldsymbol{q}', \boldsymbol{q}) = V_{kj}(E; \boldsymbol{q}, \boldsymbol{q}'), \qquad (7)$$

$$T_{jk}(E; \boldsymbol{q}', \boldsymbol{q}) = T_{kj}(E; \boldsymbol{q}, \boldsymbol{q}'). \qquad (8)$$

The scattering amplitude $T_{jk}(E; \boldsymbol{q}', \boldsymbol{q})$ is a solution of the Lippmann-Schwinger equation in the following form:

$$T_{jk}(E; \boldsymbol{q}', \boldsymbol{q})$$

$$= V_{jk}(E; \boldsymbol{q}', \boldsymbol{q}) + \sum_l \int \frac{d^3k}{(2\pi)^3} \frac{V_{jl}(E; \boldsymbol{q}', \boldsymbol{k})T_{lk}(E; \boldsymbol{k}, \boldsymbol{q})}{E - E_l(k)}. \qquad (9)$$

In the actual scattering, the system in the initial and final states should be on mass shell and the energy should be determined as $E = E_j(q') = E_k(q)$. We call this scattering amplitude the on-shell amplitude. However, in the intermediate state the energy $E_l(k)$ takes different values from E. Moreover, we can mathematically perform the analytic continuation of the scattering amplitude by taking the value of the energy E different from $E_j(q') = E_k(q)$ as an off-shell amplitude.

This will be essential to extract the wave function from the scattering amplitude at the resonance pole position in the complex energy plane.

Next, it is useful to decompose the scattering amplitude into partial wave amplitudes,

$$T_{jk}(E; \boldsymbol{q}', \boldsymbol{q}) = \sum_{L=0}^{\infty}(2L + 1)T_{L,jk}(E; q', q)P_L(\hat{q}' \cdot \hat{q}), \quad (10)$$

and in a similar manner for the interaction V, where P_L is the Legendre polynomials and $\hat{q}^{(\prime)}$ is the unit vector for the direction of $\boldsymbol{q}^{(\prime)}$: $\hat{q}^{(\prime)} \equiv \boldsymbol{q}^{(\prime)}/q^{(\prime)}$. Each partial-wave amplitude can be extracted as

$$T_{L,jk}(E; q', q) = \frac{1}{2}\int_{-1}^{1}d(\hat{q}' \cdot \hat{q})P_L(\hat{q}' \cdot \hat{q})T_{jk}(E; \boldsymbol{q}', \boldsymbol{q}). \qquad (11)$$

Because the Legendre polynomials satisfy the relation

$$\int d\Omega_{\boldsymbol{k}} P_L(\hat{q}' \cdot \hat{k})P_{L'}(\hat{k} \cdot \hat{q}) = \frac{4\pi}{2L + 1}\delta_{LL'}P_L(\hat{q}' \cdot \hat{q}) \qquad (12)$$

for the integral with respect to the solid angle of a vector $\boldsymbol{k}$, $\Omega_{\boldsymbol{k}}$, we can rewrite the Lippmann-Schwinger equation (9) as

$$T_{L,jk}(E; q', q)$$

$$= V_{L,jk}(E; q', q) + \sum_l \int_0^{\infty}\frac{dk}{2\pi^2}k^2$$

$$\times \frac{V_{L,jl}(E; q', k)T_{L,lk}(E; k, q)}{E - E_l(k)}. \qquad (13)$$

We note that in our formulation the on-shell scattering amplitude in each partial wave satisfies the optical theorem from the unitarity of the S matrix in the normalization

$$\text{Im } T_{L,jj}^{\text{on-shell}}(E) = -\sum_k \frac{\mu_k q_k}{2\pi}\left|T_{L,jk}^{\text{on-shell}}(E)\right|^2, \qquad (14)$$

where $q_k \equiv \sqrt{2\mu_k(E - m_k - M_k)}$ is the on-shell relative momentum in the kth channel and the sum runs over the open channels.

Let us now suppose that there is a resonance state $|\psi_{LM}\rangle$ in the partial wave L with its azimuthal component M. Here, to ensure a finite normalization of the resonance wave function $|\psi_{LM}\rangle$, we employ the Gamow vector, which was first introduced to describe unstable nuclei [49–52]. The resonance state $|\psi_{LM}\rangle$ as the Gamow vector is a solution of the Schrödinger equation,

$$\hat{H}|\psi_{LM}\rangle = [\hat{H}_0 + \hat{V}(E_{\text{pole}})]|\psi_{LM}\rangle = E_{\text{pole}}|\psi_{LM}\rangle, \qquad (15)$$

with the eigenenergy E_{pole}. We note that the resonance eigenenergy is, in general, complex, $E_{\text{pole}}^* \neq E_{\text{pole}}$; $\text{Re } E_{\text{pole}}$ and $-2\,\text{Im } E_{\text{pole}}$ are the mass and width of the resonance state, respectively. Then, to establish the normalization of the resonance state as the Gamow vector, we take $\langle\psi_{LM}^*|$ instead of $\langle\psi_{LM}|$ for the bra vector of the resonance. In this notation we can normalize the resonance wave function in the following manner:

$$\langle\psi_{LM'}^*|\psi_{LM}\rangle = \delta_{M'M}. \qquad (16)$$

The Schrödinger equation for the resonance bra state is expressed with the same eigenenergy as

$$\langle \psi_{LM}^* | \hat{H} = \langle \psi_{LM}^* | [\hat{H}_0 + \hat{V}(E_{\text{pole}})] = \langle \psi_{LM}^* | E_{\text{pole}}. \quad (17)$$

Here we summarize the two-body component of the resonance wave function in momentum space. Namely, because the interaction is assumed to be a central force, for the L-wave resonance, the wave function in momentum space can be written as a product of the radial part $R_j(q)$ in jth channel and the spherical harmonics $Y_{LM}(\hat{q})$ as

$$\langle \boldsymbol{q}_j | \psi_{LM} \rangle = R_j(q) Y_{LM}(\hat{q}). \quad (18)$$

We fix the normalization of the spherical harmonics $Y_{LM}(\hat{q})$ as

$$\int d\Omega_q Y_{LM}(\hat{q}) Y_{L'M'}^*(\hat{q}) = 4\pi \delta_{LL'} \delta_{MM'}. \quad (19)$$

From the bra state $\langle \psi_{LM}^* |$, the two-body wave function can be evaluated as

$$\langle \psi_{LM}^* | \boldsymbol{q}_j \rangle = R_j(q) Y_{LM}^*(\hat{q}). \quad (20)$$

Here we emphasize that, while we take the complex conjugate for the spherical harmonics, we do not take it for the radial part. This is because, while the spherical part can be calculated and normalized in a usual sense, the radial part should be treated so as to remove the divergence of the wave function at $q \to \infty$ [49–52] when we calculate the norm. From the above wave function, we can calculate the norm with respect to the jth-channel two-body wave function, X_j, in the following manner:

$$X_j \equiv \int \frac{d^3 q}{(2\pi)^3} \langle \psi_{LM}^* | \boldsymbol{q}_j \rangle \langle \boldsymbol{q}_j | \psi_{LM} \rangle = \int_0^\infty \frac{dq}{2\pi^2} q^2 [R_j(q)]^2. \quad (21)$$

This quantity is referred to as the compositeness. In this construction, the compositeness X_j is given by the complex number squared of the radial part $R_j(q)$ rather than by the absolute value squared, which is essential to normalize the resonance wave function. Therefore, in general, the compositeness becomes complex for resonance states. We also note that the sum of the norm X_j should be unity if there is no missing channels, which would be an eigenstate of the free Hamiltonian, to describe the resonance state. However, in actual calculations we may have contributions from missing channels, which can be implemented as the energy dependence of the interaction. As we discuss when we introduce the scattering amplitude and its residue at the resonance pole position, we do not make the sum of the compositeness X_j coincide with unity by hand. Instead, the value of the norm is automatically fixed when we calculate the residue of the scattering amplitude. Here we representatively denote the missing channels as $|\psi_0\rangle$, which represents not only one-body bare states but also more than one-body scattering states. In the present notation we can decompose unity in terms of the eigenstates of the free Hamiltonian:

$$\mathbb{1} = |\psi_0\rangle\langle\psi_0| + \sum_j \int \frac{d^3 q}{(2\pi)^3} |\boldsymbol{q}_j\rangle\langle\boldsymbol{q}_j|. \quad (22)$$

Therefore, the normalization of the resonance wave function $|\psi_{LM}\rangle$ is expressed as

$$\langle \psi_{LM}^* | \psi_{LM} \rangle = Z + \sum_j X_j = 1, \quad (23)$$

where we have introduced the missing-channel contributions Z defined as

$$Z \equiv \langle \psi_{LM}^* | \psi_0 \rangle \langle \psi_0 | \psi_{LM} \rangle. \quad (24)$$

Note that the quantity Z, which has been referred to as the elementariness,[2] becomes complex for resonance states as well. The explicit form of the elementariness Z will be given in Sec. II C in our model.

We now establish the way to extract the compositeness from the off-shell scattering amplitude obtained by the analytic continuation for the energy. The key is the fact that the resonance wave function appears as the residue at the resonance pole of the scattering amplitude. Namely, near the resonance pole, the off-shell scattering amplitude is dominated by the resonance pole term in the expansion by the eigenstates of the full Hamiltonian, and hence we have [see the last expression in Eq. (1)]

$$\hat{T}(E) \approx \sum_{M=-L}^{L} \hat{V}(E_{\text{pole}}) | \psi_{LM} \rangle \frac{1}{E - E_{\text{pole}}} \langle \psi_{LM}^* | \hat{V}(E_{\text{pole}}), \quad (25)$$

where we have summed up the possible azimuthal component M. Calculating the matrix element of this T-matrix operator, we obtain

$$T_{jk}(E; \boldsymbol{q}', \boldsymbol{q})$$
$$\approx \sum_{M=-L}^{L} \frac{\langle \boldsymbol{q}'_j | \hat{V}(E_{\text{pole}}) | \psi_{LM} \rangle \langle \psi_{LM}^* | \hat{V}(E_{\text{pole}}) | \boldsymbol{q}_k \rangle}{E - E_{\text{pole}}}. \quad (26)$$

Then we need to evaluate the matrix elements in the numerator, $\langle \boldsymbol{q}'_j | \hat{V}(E_{\text{pole}}) | \psi_{LM} \rangle$ and $\langle \psi_{LM}^* | \hat{V}(E_{\text{pole}}) | \boldsymbol{q}_k \rangle$. We can evaluate the former one by using the Schrödinger equation as

$$\langle \boldsymbol{q}_j | \hat{V}(E_{\text{pole}}) | \psi_{LM} \rangle = \langle \boldsymbol{q}_j | (\hat{H} - \hat{H}_0) | \psi_{LM} \rangle$$
$$= [E_{\text{pole}} - E_j(q)] \langle \boldsymbol{q}_j | \psi_{LM} \rangle, \quad (27)$$

and from Eq. (18) we obtain

$$\langle \boldsymbol{q}_j | \hat{V}(E_{\text{pole}}) | \psi_{LM} \rangle = \gamma_j(q) Y_{LM}(\hat{q}), \quad (28)$$

with

$$\gamma_j(q) \equiv [E_{\text{pole}} - E_j(q)] R_j(q). \quad (29)$$

In a similar manner we can calculate the latter matrix element as

$$\langle \psi_{LM}^* | \hat{V}(E_{\text{pole}}) | \boldsymbol{q}_j \rangle = \gamma_j(q) Y_{LM}^*(\hat{q}). \quad (30)$$

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[2]Although Z is called elementariness, it contains contributions not only from elementary one-body states but also from more than one-body scattering states.

By using the above matrix elements, we can rewrite the scattering amplitude near the resonance pole as

$$T_{jk}(E; \boldsymbol{q}', \boldsymbol{q}) \approx \frac{\gamma_j(q')\gamma_k(q)}{E - E_{\text{pole}}} \sum_{M=-L}^{L} Y_{LM}(\hat{q}')Y_{LM}^*(\hat{q})$$

$$= (2L + 1)\frac{\gamma_j(q')\gamma_k(q)}{E - E_{\text{pole}}} P_L(\hat{q}' \cdot \hat{q}), \quad (31)$$

where we have used the formula for the spherical harmonics and Legendre polynomials:

$$\sum_{M=-L}^{L} Y_{LM}(\hat{q}')Y_{LM}^*(\hat{q}) = (2L + 1)P_L(\hat{q}' \cdot \hat{q}). \quad (32)$$

The expression in Eq. (31) indicates that the partial wave amplitude in L wave contains the resonance pole, as we expected:

$$T_{L,jk}(E; q', q) = \frac{\gamma_j(q')\gamma_k(q)}{E - E_{\text{pole}}} + (\text{regular at } E = E_{\text{pole}}). \quad (33)$$

Furthermore, the residue of the partial-wave amplitude contains information on the resonance wave function via the expression in Eq. (29). Actually, we can calculate the jth-channel compositeness, X_j, by using the residue $\gamma_j(q)$ as

$$X_j = \int_0^\infty \frac{dq}{2\pi^2} q^2 [R_j(q)]^2$$

$$= \int_0^\infty \frac{dq}{2\pi^2} q^2 \left[\frac{\gamma_j(q)}{E_{\text{pole}} - E_j(q)}\right]^2. \quad (34)$$

This is the formula to evaluate the jth-channel compositeness X_j from the residue of the partial-wave amplitude T_L at the resonance pole. An important point is that the residue $\gamma_j(q)$ is obtained from the Lippmann-Schwinger equation without introducing any extra factor to scale the value of the compositeness X_j. In this sense, the value of the norm in Eq. (21) is automatically fixed when we calculate the residue of the scattering amplitude. Indeed, it was proved in Ref. [52] that the wave function from the residue of the scattering amplitude is correctly normalized to be unity for a general energy-independent interaction.[3]

Here we note that the compositeness X_j is not observable and, hence, in general, a model-dependent quantity. This can be understood with the property of the residue $\gamma_j(q)$. Namely, while the on-shell scattering amplitude for open channels is observable, the off-shell amplitude with the energy analytically continued to the resonance pole position is not observable. Therefore, to calculate the residue $\gamma_j(q)$, in general, one needs some model or assumptions for the analytic continuation. In other words, we have to fix the functional form when we evaluate the off-shell scattering amplitude. This is reflected as the model dependence of the residue γ_j and hence the compositeness X_j. However, in certain cases we can express

the compositeness only with the observable quantities. A special case is that the pole exists very close to the on-shell energies, in which we can directly relate the compositeness with threshold parameters such as the scattering length and effective range [10,11,15,19,20,25].

Finally, we comment on the semirelativistic case, in which the eigenenergy of the free Hamiltonian (3) is replaced with

$$E_j(q) \equiv \sqrt{q^2 + m_j^2} + \sqrt{q^2 + M_j^2}. \quad (35)$$

Even in this case we can follow the same discussion, and we obtain the same formula for the compositeness (34) but with the semirelativistic eigenenergy $E_j(q)$ in Eq. (35).

B. Scattering amplitude and wave function in a relativistic case

We extend our discussions to the relativistic case of the two-body to two-body scattering $k(p^\mu, q^\mu) \to j(p'^\mu, q'^\mu)$, where j and k are channel indices and p^μ, q^μ, p'^μ, and q'^μ are the momenta of particles whose masses are M_k, m_k, M_j, and m_j, respectively. We first consider a scattering of two spinless particles, and then we treat a two-body scattering of arbitrary spins by using the partial-wave amplitude. We take the center-of-mass frame, where the total energy-momentum of the system becomes $P^\mu = p^\mu + q^\mu = p'^\mu + q'^\mu = (w, \mathbf{0})$, with the center-of-mass energy w. The conventions used in this study are summarized in Appendix A.

In general, the scattering amplitude of two spinless particles is expressed as a function of the Mandelstam variable $s \equiv w^2$ and momenta q^μ and q'^μ. The scattering amplitude is a solution of the Lippmann-Schwinger equation in a relativistic form:

$$T_{jk}(s; q'^\mu, q^\mu)$$

$$= V_{jk}(s; q'^\mu, q^\mu) + i \sum_l \int \frac{d^4k}{(2\pi)^4}$$

$$\times \frac{V_{jl}(s; q'^\mu, k^\mu)T_{lk}(s; k^\mu, q^\mu)}{(k^\mu k_\mu - m_j^2)[(P-k)^\mu(P-k)_\mu - M_j^2]}. \quad (36)$$

Here we allow that the interaction kernel $V_{jk}(s; q'^\mu, q^\mu)$ may contain, in addition to the tree-level parts, contributions from t- and u-channel loops in a usual manner of the quantum field theory. For the on-shell amplitude s is related to the momenta q^μ and q'^μ, while the off-shell amplitude can be obtained with the analytic continuation to the complex values of s.

From the Lippmann-Schwinger equation (36), we construct an analog to the scattering equation in the nonrelativistic case, which will be essential to relate the scattering amplitude with the wave function clearly. To this end, we assume an on-shell condition to the energy $q^{(\prime)0}$ inside $V_{jk}(s; q'^\mu, q^\mu)$ by making it a function of the Mandelstam variable s as

$$q^0 \to \omega_k(s) \equiv \frac{s + m_k^2 - M_k^2}{2\sqrt{s}},$$

$$q'^0 \to \omega'_j(s) \equiv \frac{s + m_j^2 - M_j^2}{2\sqrt{s}}. \quad (37)$$

---

[3] An analysis on the relation between the residue of the scattering amplitude and the wave function will be presented in detail elsewhere [53].

In this assumption, we can treat the interaction kernel as in the nonrelativistic form

$$V_{jk}(s; q'^\mu, q^\mu) \rightarrow V_{jk}(s; \boldsymbol{q}', \boldsymbol{q}), \qquad (38)$$

and, hence, after performing the k^0 integral, the Lippmann-Schwinger equation (36) becomes

$$T_{jk}(s, \boldsymbol{q}', \boldsymbol{q}) = V_{jk}(s, \boldsymbol{q}', \boldsymbol{q}) + \sum_l \int \frac{d^3 k}{(2\pi)^3} \frac{\sqrt{s_l(k)}}{2\omega_l(k)\Omega_l(k)}$$

$$\times \frac{V_{jl}(s, \boldsymbol{q}', \boldsymbol{k}) T_{lk}(s, \boldsymbol{k}, \boldsymbol{q})}{s - s_l(k)}, \qquad (39)$$

with

$$\omega_j(q) \equiv \sqrt{q^2 + m_j^2}, \quad \Omega_j(q) \equiv \sqrt{q^2 + M_j^2}, \qquad (40)$$

$$s_j(q) \equiv [\omega_j(q) + \Omega_j(q)]^2. \qquad (41)$$

Now we can perform the partial-wave decomposition in a similar manner as in the nonrelativistic case. In particular, in the present formulation the partial-wave amplitude $T_{L, jk}$, extracted in the same way as in Eq. (11), is the solution of the Lippmann-Schwinger equation

$$T_{L, jk}(s; q', q) = V_{L, jk}(s; q', q) + \sum_l \int_0^\infty \frac{dk}{2\pi^2} \frac{k^2 \sqrt{s_l(k)}}{2\omega_l(k)\Omega_l(k)}$$

$$\times \frac{V_{L, jl}(s, q', k) T_{L, lk}(s, k, q)}{s - s_l(k)} \qquad (42)$$

and satisfies the optical theorem

$$\text{Im} \, T_{L, jj}^{\text{on-shell}}(s) = -\sum_k \frac{q_k(s)}{8\pi\sqrt{s}} \left| T_{L,jk}^{\text{on-shell}}(s) \right|^2, \qquad (43)$$

where $q_k(s)$ is the kth-channel center-of-mass momentum in the relativistic form

$$q_k(s) \equiv \frac{\lambda^{1/2}\left(s, m_k^2, M_k^2\right)}{2\sqrt{s}}, \qquad (44)$$

with the Källen function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$, and the sum runs over the open channels.

Of special interest is the expression of the Lippmann-Schwinger equation in Eq. (39), with which we can apply the relativistic formulation of the wave function developed in Refs. [22,54,55]. In this formulation, the two-body equation for the resonance state $|\Psi_{LM}\rangle$, in the partial wave L with its azimuthal component M, is expressed in an extended form of the Schrödinger equation as [22]

$$[\hat{\mathcal{K}} + \hat{\mathcal{V}}(s_{\text{pole}})]|\Psi_{LM}\rangle = s_{\text{pole}}|\Psi_{LM}\rangle,$$
$$\langle\Psi_{LM}^*|[\hat{\mathcal{K}} + \hat{\mathcal{V}}(s_{\text{pole}})] = s_{\text{pole}}\langle\Psi_{LM}^*|, \qquad (45)$$

where $\hat{\mathcal{K}}$ and $\hat{\mathcal{V}}$ are the kinetic energy and interaction operators, respectively, and s_{pole} is the resonance pole position with respect to the Mandelstam variable s. The kinetic operator $\hat{\mathcal{K}}$ corresponds to the free Hamiltonian in the nonrelativistic framework and has eigenstates of the jth-channel two-body covariant scattering state with the relative momentum $\boldsymbol{q}$, $|\boldsymbol{q}_j\rangle_{\text{co}}$, with which eigenvalues of the kinetic operator are

$$\hat{\mathcal{K}}|\boldsymbol{q}_j\rangle_{\text{co}} = s_j(q)|\boldsymbol{q}_j\rangle_{\text{co}}, \quad {}_{\text{co}}\langle\boldsymbol{q}_j|\hat{\mathcal{K}} = s_j(q) \, {}_{\text{co}}\langle\boldsymbol{q}_j|. \qquad (46)$$

In this study we take the same normalization of the covariant scattering state as in Ref. [22]:

$$_{\text{co}}\langle\boldsymbol{q}_k'|\boldsymbol{q}_j\rangle_{\text{co}} = \frac{2\omega_j(q)\Omega_j(q)}{\sqrt{s_j(q)}}(2\pi)^3\delta_{jk}\delta(\boldsymbol{q}' - \boldsymbol{q}). \qquad (47)$$

The factor $2\omega_j(q)\Omega_j(q)/\sqrt{s_j(q)}$ guarantees that the measure of the integral in the expression of the compositeness is Lorentz invariant, as we see later. The wave function in momentum space is expressed as

$$_{\text{co}}\langle\boldsymbol{q}_j|\Psi_{LM}\rangle = R_j(q)Y_{LM}(\hat{q}),$$
$$\langle\Psi_{LM}^*|\boldsymbol{q}_j\rangle_{\text{co}} = R_j(q)Y_{LM}^*(\hat{q}). \qquad (48)$$

With this scattering state, we can calculate the interaction kernel V_{jk} in the manner

$$_{\text{co}}\langle\boldsymbol{q}_j'|\hat{\mathcal{V}}(s)|\boldsymbol{q}_k\rangle_{\text{co}} = V_{jk}(s; \boldsymbol{q}', \boldsymbol{q}), \qquad (49)$$

and, similarly, the scattering amplitude is calculated as

$$_{\text{co}}\langle\boldsymbol{q}_j'|\hat{T}(s)|\boldsymbol{q}_k\rangle_{\text{co}} = T_{jk}(s; \boldsymbol{q}', \boldsymbol{q}). \qquad (50)$$

Now the scattering equation (39) is expressed as an equation of operators, as in Eq. (1) in the nonrelativistic case:

$$\hat{T}(s) = \hat{\mathcal{V}}(s) + \hat{\mathcal{V}}(s)\frac{1}{s - \hat{\mathcal{K}}}\hat{T}(s)$$

$$= \hat{\mathcal{V}}(s) + \hat{\mathcal{V}}(s)\frac{1}{s - \hat{\mathcal{K}} - \hat{\mathcal{V}}(s)}\hat{\mathcal{V}}(s). \qquad (51)$$

Actually, we can easily see that this operator equation becomes the Lippmann-Schwinger equation (39) by using the normalization (47). In this sense, thanks to the on-shell condition of the energy (37), Eqs. (51) and (45) become analogs to the Lippmann-Schwinger equation and the Schrödinger equation in the nonrelativistic case, respectively.

In this formulation, we can take the same strategy to calculate the relativistic wave function as in the nonrelativistic case. Near the resonance pole, the scattering amplitude is dominated by the resonance pole term in the expansion by the eigenstates of $\hat{\mathcal{K}} + \hat{\mathcal{V}}$ as

$$\hat{T}(s) \approx \sum_{M=-L}^{L} \hat{\mathcal{V}}(s_{\text{pole}})|\Psi_{LM}\rangle \frac{1}{s - s_{\text{pole}}}\langle\Psi_{LM}^*|\hat{\mathcal{V}}(s_{\text{pole}}), \quad (52)$$

and hence the partial-wave amplitude near the resonance pole position is expressed as

$$T_{L,jk}(s; q', q) = \frac{\gamma_j(q')\gamma_k(q)}{s - s_{\text{pole}}} + (\text{regular at } s = s_{\text{pole}}), \quad (53)$$

where we define the residue as

$$_{\text{co}}\langle\boldsymbol{q}_j|\hat{\mathcal{V}}(s_{\text{pole}})|\Psi_{LM}\rangle = \gamma_j(q)Y_{LM}(\hat{q}),$$
$$\langle\Psi_{LM}^*|\hat{\mathcal{V}}(s_{\text{pole}})|\boldsymbol{q}_j\rangle_{\text{co}} = \gamma_j(q)Y_{LM}^*(\hat{q}), \qquad (54)$$

with

$$\gamma_j(q) \equiv [s_{\text{pole}} - s_j(q)]R_j(q). \qquad (55)$$

Then we can calculate the norm of the jth-channel two-body state, as the compositeness X_j, from the residue of the

scattering amplitude at the resonance pole:

$$
X_j = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_j(q)}}{2\omega_j(q)\Omega_j(q)} \langle \Psi_{LM}^* | \boldsymbol{q}_j \rangle_{\text{co}} \, {}_{\text{co}} \langle \boldsymbol{q}_j | \Psi_{LM} \rangle
$$

$$
= \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_j(q)}}{2\omega_j(q)\Omega_j(q)} \left[\frac{\gamma_j(q)}{s_{\text{pole}} - s_j(q)} \right]^2. \quad (56)
$$

This is the formula to evaluate the jth-channel compositeness X_j from the residue of the partial-wave amplitude T_L at the resonance pole for a relativistic scattering of two spinless particles.

Finally, let us consider a two-body relativistic scattering of arbitrary spins. The partial-wave amplitude in this condition can be specified by the orbital angular momentum L and a certain index α, which represents the quantum number of the scattering, such as isospin. The optical theorem for the partial-wave amplitude is chosen to be the same as that in Eq. (43),

$$
\text{Im} T_{\alpha L, jj}^{\text{on-shell}}(w) = -\sum_k \frac{q_k(s)}{8\pi w} |T_{\alpha L, jk}^{\text{on-shell}}(w)|^2, \quad (57)
$$

where w is the center-of-mass energy, $s \equiv w^2$, and the sum runs over the open channels. In general, the off-shell amplitude $T_{\alpha L, jk}$ is a function of the center-of-mass energy w and momenta q^μ and q'^μ, but to relate the scattering amplitude with the wave function clearly, we assume the on-shell condition for the energy $q^{(\prime)0}$ so that it is a function of the center-of-mass energy w as in Eq. (37). Then, the partial-wave amplitude can be expressed near the resonance pole position as

$$
T_{\alpha L, jk}(w; q', q) = \frac{\gamma_j'(q')\gamma_k'(q)}{w - w_{\text{pole}}} + (\text{regular at } w = w_{\text{pole}}),
$$
$$(58)$$

where $w_{\text{pole}} \equiv \sqrt{s_{\text{pole}}}$ is the pole position in terms of the center-of-mass energy w. Now we extend the expression of the compositeness X_j in the last line in Eq. (56) to the scattering of arbitrary spin particles. Namely, in this study we define the compositeness for a two-body system with arbitrary spin particles by using the residue of the partial-wave amplitude as

$$
X_j \equiv 2w_{\text{pole}} \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_j(q)}}{2\omega_j(q)\Omega_j(q)} \left[\frac{\gamma_j'(q)}{s_{\text{pole}} - s_j(q)} \right]^2. \quad (59)
$$

This is the formula to evaluate the jth-channel compositeness X_j from the resonance pole of the partial-wave amplitude $T_{\alpha L}$ for a relativistic scattering of arbitrary spin particles. We note that this formula of the compositeness is valid even for baryonic resonances described with explicit Dirac γ matrices. In the following we use this expression to evaluate the compositeness of the N^* resonances.

C. Compositeness with separable interaction

Up to now we have considered the nonrelativistic and relativistic systems without specifying any explicit models for the interaction. In the following we consider the interaction of the separable type, i.e., the interaction $V(w; q', q)$ which can be factorized into the q-dependent part and q'-dependent one. The separable interaction is employed in the description of the N^* resonances in the chiral unitary approach in

Sec. III. We here concentrate on the scattering of the πN and other coupled channels in a relativistic framework, and hence the partial-wave amplitude is specified by isospin I, orbital angular momentum L, and total angular momentum $J = L \pm 1/2$, as $T_{IL}^\pm$. To fix the interaction, we first note that the radial wave function $R_j(q)$ in L wave behaves as $\sim q^L$ for the small q region:

$$
R_j(q) = \mathcal{O}(q^L) \quad \text{for small } q. \quad (60)
$$

Therefore, without loss of generality we can express the residue of the partial-wave amplitude $\gamma_j'(q)$ as

$$
\gamma_j'(q) = g_j q^L f_j(q), \quad (61)
$$

where a constant g_j is the coupling constant of the resonance to the jth-channel two-body state and a function $f_j(q)$ satisfies $f_j(0) = 1$ and $f_j(q) \to 0$ for $q \to \infty$ so as to tame the ultraviolet divergence of the integrals. Then it is interesting that we can obtain the residue in Eq. (61) exactly with the separable interaction of the form

$$
V_{IL, jk}^\pm(w; q', q) = V_{IL, jk}'^\pm(w) q'^L q^L f_j(q') f_k(q), \quad (62)
$$

where $V_{IL, jk}'^\pm$ depends only on the center-of-mass energy w. This form of the interaction was proposed in Refs. [13,17] so as to evaluate the compositeness for higher partial-wave states in a proper way. With this interaction, the full amplitude in L wave can be obtained as

$$
T_{IL, jk}^\pm(w; q', q) = T_{IL, jk}'^\pm(w) q'^L q^L f_j(q') f_k(q), \quad (63)
$$

where $T_{IL, jk}'^\pm(w)$ is a solution of the Lippmann-Schwinger equation in an algebraic form:

$$
T_{IL, jk}'^\pm(w) = V_{IL, jk}'^\pm(w) + \sum_l V_{IL, jl}'^\pm(w) G_{L, l}(w) T_{IL, lk}'^\pm(w).
$$
$$(64)$$

In this expression, $G_{L, j}$ is the loop function of the two-body state in jth channel, and in this study we take the expression

$$
G_{L, j}(w) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_j(q)}}{2\omega_j(q)\Omega_j(q)} \frac{q^{2L}[f_j(q)]^2}{s - s_j(q)}, \quad (65)
$$

in accordance with the discussion in Sec. II B. We note that, in this construction, we have correct behavior of the on-shell amplitude near the threshold:

$$
T_{IL, jk}^\pm(w)^{\text{on-shell}} \propto q'^L q^L. \quad (66)
$$

An important point in this approach is that the scattering amplitude $T_{IL, jk}'^\pm$, as well as the interaction $V_{IL, jk}'^\pm$, depends only on the energy w. Owing to this fact, the residue of $T_{IL, jk}'^\pm$ at the resonance pole position does not depend on the relative momentum q and hence a constant:

$$
T_{IL, jk}'^\pm(w) = \frac{g_j g_k}{w - w_{\text{pole}}} + (\text{regular at } w = w_{\text{pole}}). \quad (67)
$$

One can easily confirm that the constant of the residue g_j coincides with the prefactor in Eq. (61). Now, with the full amplitude $T_{IL, jk}^\pm$ in Eq. (63), we can straightforwardly

calculate the compositeness (59) in the present approach as

$$X_j = 2 w_{\text{pole}} g_j^2 \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{s_j(q)}}{2\omega_j(q)\Omega_j(q)} \left[\frac{q^L f_j(q)}{s_{\text{pole}} - s_j(q)} \right]^2$$

$$= -g_j^2 \left[\frac{dG_{L,j}}{dw} \right]_{w=w_{\text{pole}}}, \qquad (68)$$

where we have replaced the integral part in the middle of the equation with the derivative of the loop function in the last. Furthermore, in the present approach we can express the elementariness Z as

$$Z = -\sum_{j,k} g_k g_j \left[G_{L,j} \frac{dV'^{\pm}_{IL,jk}}{dw} G_{L,k} \right]_{w=w_{\text{pole}}}. \qquad (69)$$

Actually, with this expression we can show the normalization of the total wave function for the resonance state,

$$\langle \Psi^*_{LM} | \Psi_{LM} \rangle$$

$$= \sum_j X_j + Z$$

$$= -\sum_{j,k} g_k g_j \left[\delta_{jk} \frac{dG_{L,j}}{dw} + G_{L,j} \frac{dV'^{\pm}_{IL,jk}}{dw} G_{L,k} \right]_{w=w_{\text{pole}}}$$

$$= 1, \qquad (70)$$

where the condition of the correct normalization as unity is guaranteed by a generalized Ward identity proven in Ref. [56]. The elementariness Z measures the contributions from missing channels which are effectively taken into account in the two-body interaction in the practical model space, including both one-body bare states and more than one-body scattering states, on the assumption that the energy dependence of the interaction originates from channels which do not appear as explicit degrees of freedom.

Finally, we note that in our numerical calculations we employ the dimensional regularization to calculate the integral of the loop function, which is achieved by setting $f_j(q) = 1$ and modifying the integration variable as $d^4 k \to \mu_{\text{reg}}^{4-d} d^d k$, with the regularization scale μ_{reg}. The problems concerned with the dimensional regularization are discussed in the next section.

D. Interpretation of compositeness for resonances

Before going to the numerical results of the compositeness for N^* resonances, we here give how to interpret and treat complex values of the compositeness for resonance states, especially in the relation to the probabilistic interpretation. In this section we consider a single-channel problem for simplicity. The extension to the general coupled-channel case is given at the end.

As we have mentioned, the compositeness X and elementariness Z are, in general, complex for resonance states, which is inevitable when the correct normalization of the resonance wave function is required. This fact indicates that we cannot interpret them as probabilities because their values are not real and not bounded. In this line, several ways to make the compositeness real values have been proposed. For

instance, it has been suggested to use $1 - |Z|$ [57], $|X|$ [13,24], Re(X) [17], and $(1 - |Z| + |X|)/2$ [25] as the "probability" of the compositeness. All of them return to the same non-negative value $1 - Z = X$ for a stable bound state. However, for a resonance state these values, except for the last one, are not bounded in the range [0, 1], so the values $1 - |Z|, |X|$, and Re(X) cannot be treated as probability in a strict sense.

In contrast to these real values, to interpret the compositeness and elementariness, we propose to use simple but reasonable values defined as

$$\tilde{X} \equiv \frac{|X|}{1 + U}, \quad \tilde{Z} \equiv \frac{|Z|}{1 + U}, \qquad (71)$$

with

$$U \equiv |X| + |Z| - 1. \qquad (72)$$

Obviously, both $\tilde{X}$ and $\tilde{Z}$ are real, bounded in the range [0, 1], and automatically satisfy the sum rule:

$$\tilde{X} + \tilde{Z} = 1. \qquad (73)$$

We then require that we can interpret $\tilde{X}$ and $\tilde{Z}$ from the complex compositeness and elementariness as the "probability" if and only if U is much smaller than unity, $U \ll 1$. This is essentially an expression of the condition pointed out in Ref. [22], in which they proposed that reasonable interpretation can be obtained if $|\text{Im}(Z)|, |\text{Im}(X)| \ll 1$ and $0 \lesssim \text{Re}(Z), \text{Re}(X) \lesssim 1$, where X and Z have similarity with those of the stable bound states. In other words, we can have a resonance wave function which is similar to the wave function of the stable bound state. Here we note that, when $U \ll 1$ is satisfied, $\tilde{X}$ and $\tilde{Z}$ in Eq. (71) take very similar values to the quantities proposed in Ref. [25]:

$$\tilde{X}_{\text{KH}} \equiv \frac{1 - |Z| + |X|}{2}, \quad \tilde{Z}_{\text{KH}} \equiv \frac{1 - |X| + |Z|}{2}. \qquad (74)$$

Actually, a straightforward calculation provides

$$\tilde{X} - \tilde{X}_{\text{KH}} = \frac{|X|}{1 + U} - \frac{1 - |Z| + |X|}{2} = \frac{(|Z| - |X|)U}{2(1 + U)}, \qquad (75)$$

which should be much smaller than unity for $U \ll 1$. A similar result for $\tilde{Z} - \tilde{Z}_{\text{KH}}$ is obtained by exchanging $|X|$ with $|Z|$. We also mention that, with the condition $U \ll 1$, the quantities $1 - |Z|, |X|$, and Re(X) will take values similar to $\tilde{X}$.

Finally, an important property is that we can straightforwardly extend $\tilde{X}$ and $\tilde{Z}$ to the general coupled-channel case. This can be done as

$$\tilde{X}_j \equiv \frac{|X_j|}{1 + U}, \quad \tilde{Z} \equiv \frac{|Z|}{1 + U}, \qquad (76)$$

with

$$U \equiv \sum_j |X_j| + |Z| - 1. \qquad (77)$$

Again $\tilde{X}_j$ and $\tilde{Z}$ are real, bounded in the range [0, 1], and automatically satisfy the sum rule:

$$\sum_j \tilde{X}_j + \tilde{Z} = 1. \qquad (78)$$

In the following we use these real values as well as the original compositeness X_j and elementariness Z when we discuss the internal structure of N^* resonances.

III. NUMERICAL RESULTS

Let us now consider the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances and evaluate their meson-baryon compositeness from the residue of the scattering amplitude at the resonance pole position by using the formula developed in Sec. II C. We employ the chiral unitary approach to calculate the scattering amplitude. The chiral unitary approach is most successful in description of the $\Lambda(1405)$ resonance [58–64] and is applied to the πN scattering and several N^* resonances as well [36–45]. In this study, the interaction kernel is taken from chiral perturbation theory up to the next-to-leading order, and we construct separable interactions to evaluate the πN scattering amplitude. The loop function is evaluated with the dimensional regularization. The model parameters are fitted so that the partial-wave amplitudes reproduce the solution of the partial-wave analysis for the πN scattering amplitude obtained in Ref. [34], to which we refer as WI 08. Throughout the numerical calculations we use isospin symmetric masses for mesons and baryons.

A. The $\Delta(1232)$ resonance

First we consider the $\Delta(1232)$ resonance and calculate its πN compositeness. In this study we construct the πN single-channel scattering amplitude in s and p waves by using the unitarization of the chiral interaction up to the next-to-leading order plus the s- and u-channel $\Delta(1232)$ exchanges. In the analysis we also consider the πN compositeness for the ground-state nucleon $N(940)$.

Because both $N(940)$ and $\Delta(1232)$ exist in p-wave πN state with the orbital angular momentum $L = 1$, we have to employ the loop function with $L = 1$. As we will see, there is ambiguity in calculating the πN compositeness with the $L = 1$ loop function evaluated with the dimensional regularization. We discuss this ambiguity as well.

1. Scattering amplitude

Let us consider the πN scattering amplitude in isospin I, orbital angular momentum L, and total angular momentum $J = L \pm 1/2$, which is denoted by $T_{IL}^{\pm}(w; \mathbf{q}', \mathbf{q})$ as a function of the center-of-mass energy w and relative momentum in the initial (final) state $\mathbf{q}^{(\prime)}$. An important property of the scattering amplitude $T_{IL}^{\pm}$ is that, for each isospin and angular momentum, it should satisfy the optical theorem from the unitarity of the scattering matrix. Namely, below the inelastic threshold, the on-shell amplitude should satisfy

$$\text{Im}\, T_{IL}^{\pm}(w)^{\text{on-shell}}$$
$$= -\frac{\rho_{\pi N}(s)}{2} |T_{IL}^{\pm}(w)^{\text{on-shell}}|^2 \theta(w - m_\pi - M_N), \quad (79)$$

where $s \equiv w^2$, $\theta(x)$ is the Heaviside step function, m_π and M_N are the pion and nucleon masses, respectively, and $\rho_{\pi N}(s)$ is

the phase space defined as

$$\rho_{\pi N}(s) \equiv \frac{q_{\pi N}(s)}{4\pi\sqrt{s}}, \quad q_{\pi N}(s) \equiv \frac{\lambda^{1/2}(s, m_\pi^2, M_N^2)}{2\sqrt{s}}, \quad (80)$$

with the Källen function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The optical theorem (79) can be equivalently rewritten in the following form:

$$\text{Im}[T_{IL}^{\pm}(w)^{\text{on-shell}}]^{-1} = \frac{\rho_{\pi N}(s)}{2}\theta(w - m_\pi - M_N). \quad (81)$$

The chiral unitary approach is a model to construct the scattering amplitude which satisfies the optical theorem (79) with the interaction taken from chiral perturbation theory. To formulate the chiral unitary approach, we first fix the interaction kernel for the scattering equation. In this study we employ chiral perturbation theory up to $\mathcal{O}(p^2)$ for the πN interaction kernel $V_{IL}^{\pm}$. The interaction kernel consists of the Weinberg-Tomozawa term V_{WT}, s- and u-channel $N(940)$ exchanges V_{s+u}, next-to-leading order contact term V_2, and s- and u-channel $\Delta(1232)$ exchanges V_Δ (see Fig. 1). They are projected to the partial-wave components as

$$V_{IL}^{\pm}(w; |\mathbf{q}'|, |\mathbf{q}|) = [V_{\text{WT}} + V_{s+u} + V_2 + V_\Delta]_{IL}^{\pm}. \quad (82)$$

The explicit expression of each term is given in Appendix B. The interaction kernel has six model parameters altogether: the low-energy constants c_1, c_2, c_3, and c_4, the bare Δ mass M_Δ, and the $\pi N \Delta$ bare coupling constant $g_{\pi N \Delta}$. Then, according to the discussion in Sec. II C, we factorize the relative momenta of the order of the orbital angular momentum $|\mathbf{q}'|^L |\mathbf{q}|^L$, which is essential to evaluate the compositeness for higher partial-wave states [13,17], as

$$V_{IL}^{\pm}(w; |\mathbf{q}'|, |\mathbf{q}|) = |\mathbf{q}'|^L |\mathbf{q}|^L V_{IL}'^{\pm}(w), \quad (83)$$

where we have applied the on-shell condition to $V_{IL}'^{\pm}$ so that it depends only on the center-of-mass energy w, by replacing the
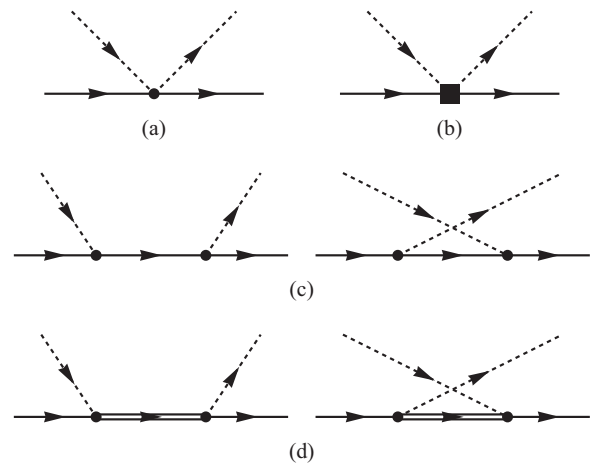

FIG. 1. Feynman diagrams for the interaction kernel: (a) Weinberg-Tomozawa term, (b) next-to-leading order term, (c) s- and u-channel $N(940)$ exchange terms, and (d) s- and u-channel $\Delta(1232)$ exchange terms. The solid, dashed, and double lines represent baryons, mesons, and $\Delta(1232)$, respectively. The dots and square represent the $\mathcal{O}(p^1)$ and $\mathcal{O}(p^2)$ vertices from chiral perturbation theory, respectively.

pion momentum q^μ in $V'^{\pm}_{IL}$ with the corresponding on-shell values:

$$q^0 \to \omega(w) \equiv \frac{s + m_\pi^2 - M_N^2}{2w}, \quad |\boldsymbol{q}| \to q_{\pi N}(s). \quad (84)$$

With this interaction kernel, the full scattering amplitude can be obtained as

$$T^{\pm}_{IL}(w; |\boldsymbol{q}'|, |\boldsymbol{q}|) = |\boldsymbol{q}'|^L |\boldsymbol{q}|^L T'^{\pm}_{IL}(w), \quad (85)$$

where $T'^{\pm}_{IL}(w)$ is a solution of the Lippmann-Schwinger equation in an algebraic form but without the factor of the external momenta $|\boldsymbol{q}'|^L |\boldsymbol{q}|^L$:

$$T'^{\pm}_{IL}(w) = V'^{\pm}_{IL} + V'^{\pm}_{IL} G_L T'^{\pm}_{IL} = \frac{1}{1/V'^{\pm}_{IL}(w) - G_L(w)}. \quad (86)$$

Here G_L is the loop function containing the contribution from the internal momentum $|\boldsymbol{q}|^{2L}$, and in this study we take the expression

$$G_L(w) = i \int \frac{d^4 q}{(2\pi)^4} \frac{|\boldsymbol{q}|^{2L}}{(q^2 - m_\pi^2)[(P-q)^2 - M_N^2]}, \quad (87)$$

with $P^\mu = (w, \boldsymbol{0})$.

Next let us focus on the loop function G_L. In this study we evaluate the loop function with the subtraction scheme, rather than a cutoff, and the dimensional regularization. Because the loop function contains the internal momentum $|\boldsymbol{q}|^{2L}$, the integral in Eq. (87) diverges logarithmically for $L = 0$ and it becomes worse for $L > 0$. Therefore, to make the integral finite, we need to subtract the divergences $L + 1$ times in the subtraction scheme (see Appendix C for the details). For instance, the $L = 0$ loop function is evaluated with a subtraction constant a as

$$G_{L=0}(w; a)$$
$$\equiv \frac{1}{16\pi^2} \left\{ a + \frac{s + m_\pi^2 - M_N^2}{2s} \ln\left(\frac{m_\pi^2}{M_N^2}\right) \right.$$
$$\left. - \frac{\lambda^{1/2}(s, m_\pi^2, M_N^2)}{s} \text{artanh}\left[\frac{\lambda^{1/2}(s, m_\pi^2, M_N^2)}{m_\pi^2 + M_N^2 - s}\right] \right\}, \quad (88)$$

where the regularization scale is fixed as $\mu_{\text{reg}} = M_N$, as in Appendix C. When we calculate the $L = 1$ loop function for $\Delta(1232)$ in the P_{33} amplitude and for $N(940)$ in P_{11}, we need two subtraction constants. We now eliminate one of the two subtraction constants by requiring that the nucleon pole does not shift in the unitarization of the πN scattering amplitude in P_{11}, which constrains the $L = 1$ loop function as

$$G_{L=1}(M_N) = 0. \quad (89)$$

Physically, this means that we do not perform the renormalization of the nucleon mass, but the wave function renormalization of the nucleon is allowed to take place because $dG_{L=1}/dw(M_N)$ may not be zero. As derived in Appendix C, the condition (89) brings the loop function in the following

expression:

$$G_{L=1}(w) = G_{\pi N, L=1}(w; \tilde{A}), \quad (90)$$

$$G_{\pi N, L=1}(w; \tilde{A}) = \frac{s - M_N^2}{4}\tilde{A} + \frac{s G_{\pi N}(w)}{4}$$
$$- \frac{m_\pi^2 + M_N^2}{2} G_{\pi N}(w) + \frac{(m_\pi^2 - M_N^2)^2}{4}$$
$$\times \left[\frac{G_{\pi N}(w) - G_{\pi N}(0)}{s} + \frac{G_{\pi N}(0)}{M_N^2}\right]. \quad (91)$$

Here $\tilde{A}$ is the remaining subtraction constant, which becomes a model parameter, and $G_{\pi N}(w)$ is the $L = 0$ loop function with the condition $G_{\pi N}(M_N) = 0$:

$$G_{\pi N}(w) = G_{L=0}(w; 0) - G_{L=0}(M_N; 0). \quad (92)$$

One can easily check that the loop function $G_{\pi N, L=1}$ satisfies $G_{\pi N, L=1}(M_N) = 0$. Besides, we note that the S_{11} and S_{31} amplitudes are not important in the study on $\Delta(1232)$. Therefore, to calculate the S_{11} and S_{31} amplitudes, we also require the $L = 0$ loop function to be zero at $w = M_N$, for simplicity:

$$G_{L=0}(w) = G_{\pi N}(w). \quad (93)$$

This condition is achieved also by the natural renormalization scheme [65], which can exclude explicit pole contributions from the loop functions.

Now we have the formulation to calculate the scattering amplitude for $\Delta(1232)$ in the chiral unitary approach. In the present formulation, we have seven model parameters. They are fixed so as to reproduce the solution of the πN partial-wave analysis WI 08 [34]. In the fitting procedure, we introduce a normalized scattering amplitude,

$$L_{2I\,2J}(w) = -\frac{\rho_{\pi N}(s) q_{\pi N}(s)^{2L}}{2} T'^{\pm}_{IL}(w)^{\text{on-shell}}, \quad (94)$$

which satisfies the optical theorem

$$\text{Im} L_{2I\,2J}(w) = |L_{2I\,2J}(w)|^2 \theta(w - m_\pi - M_N), \quad (95)$$

below the inelastic threshold for the πN state. We fit six πN amplitudes—$S_{11}(w)$, $S_{31}(w)$, $P_{11}(w)$, $P_{31}(w)$, $P_{13}(w)$, and $P_{33}(w)$—to the WI 08 solution up to 1.35 GeV in intervals of 4 MeV, in which only $\Delta(1232)$ appears as the N^* resonance.[4] We note that the WI 08 solution does not provide errors for the scattering amplitude. For the calculation of the χ^2 value, in this study we introduce a common error 0.01 for both the real and the imaginary parts of the scattering amplitude in every quantum number. From the best fit to the WI 08 solution, we obtain the model parameters listed in the second column of Table I, to which we refer as the "Naive"

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[4]The present energy range is even below the first excitation in P_{11}, i.e., the Roper resonance. Nevertheless, we can, in principle, calculate the compositeness for the Roper resonance by introducing scattering states of higher thresholds relevant to the Roper resonance and by fitting higher energy regions as well.

	Naive	Constrained
c_1 (GeV^{-1})	-0.111	-0.047
c_2 (GeV^{-1})	0.725	0.810
c_3 (GeV^{-1})	-1.797	-1.784
c_4 (GeV^{-1})	0.089	0.512
$g_{\pi N\Delta}$	1.808	1.507
M_Δ (MeV)	1296.0	1320.6
$\tilde{A}$	-3.61×10^{-3}	-4.82×10^{-3}
$\chi^2/N_{\text{d.o.f.}}$	$486.3/809$	$1239.9/809$

parameters. We mention that the low-energy constants found in this fitting are, in general, not identical to the ones from tree-level chiral perturbation theory, because we have fit them to the scattering amplitude including the $\Delta(1232)$ resonance region rather than fit them to the masses of baryons nor to low-energy phenomena around the πN threshold. We also show the P_{33} amplitude in the theoretical calculation and the WI 08 solution in Fig. 2, which shows a good reproduction of the P_{33} amplitude by the parameter set Naive.

2. Compositeness

Now that we have determined the scattering amplitude, let us evaluate the πN compositeness for $\Delta(1232)$ and $N(940)$ from their pole positions and residues. In the present formulation, the scattering amplitude has the resonance pole in the expression

$$T'^{\pm}_{IL}(w) = \frac{g^2}{w - w_{\text{pole}}} + (\text{regular at } w = w_{\text{pole}}), \quad (96)$$

where g is the coupling constant of the resonance to the πN state and w_{pole} is the pole position in the complex w
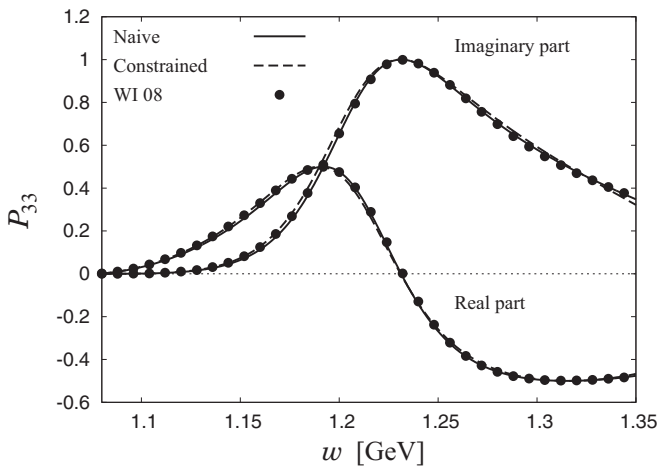

FIG. 2. Scattering amplitude P_{33} with parameter sets Naive and Constrained fitted to the WI 08 solution [34]. Two theoretical curves are very similar. The number of the plotted data points is 1/2 of the total in the fits for a better visualization.

plane. We note that g and w_{pole} contains information on the structure of the resonance, and this is formulated in terms of the compositeness as developed in Sec. II C,

$$X_{\pi N} = -g^2 \frac{dG_L}{dw}(w = w_{\text{pole}}), \quad (97)$$

which measures the amount of the two-body composite fraction inside the resonance. In addition, we can calculate the elementariness as well:

$$Z = -g^2 \left[G_L^2 \frac{dV'^{\pm}_{IL}}{dw} \right]_{w=w_{\text{pole}}}. \quad (98)$$

The elementariness Z measures the contributions from missing channels which are effectively taken into account in the πN interaction in the practical model space, on the assumption that the energy dependence of the interaction originates from channels which do not appear as explicit degrees of freedom. It is important that we have the normalization of the total wave function as

$$X_{\pi N} + Z = 1. \quad (99)$$

However, in general, both the compositeness $X_{\pi N}$ and the elementariness Z are complex for the resonance states, which are difficult to interpret. Therefore, we introduce quantities which are real, bounded in the range [0, 1], and automatically satisfy the sum rule,

$$\tilde{X}_{\pi N} \equiv \frac{|X_{\pi N}|}{1 + U}, \quad \tilde{Z} \equiv \frac{|Z|}{1 + U}, \quad (100)$$

with

$$U \equiv |X_{\pi N}| + |Z| - 1. \quad (101)$$

Obviously, we have the sum rule for $\tilde{X}_{\pi N}$ and $\tilde{Z}$:

$$\tilde{X}_{\pi N} + \tilde{Z} = 1. \quad (102)$$

We can interpret $\tilde{X}_{\pi N}$ and $\tilde{Z}$ from the complex compositeness and elementariness as the "probability" if and only if U is much smaller than unity, $U \ll 1$.

Now we calculate the pole positions, coupling constants, compositeness, and elementariness in the parameter set Naive and list them in the second and third columns of Table II. First, the $\Delta(1232)$ pole position in the parameter set Naive is very similar to that reported by Particle Data Group: $w_{\text{pole}} = (1210 \pm 1) - (50 \pm 1)i$ MeV [1]. The πN compositeness is

TABLE II. Properties of $\Delta(1232)$ and $N(940)$. We do not calculate U, $\tilde{X}_{\pi N}$, and $\tilde{Z}$ for $N(940)$ because it is a stable state.

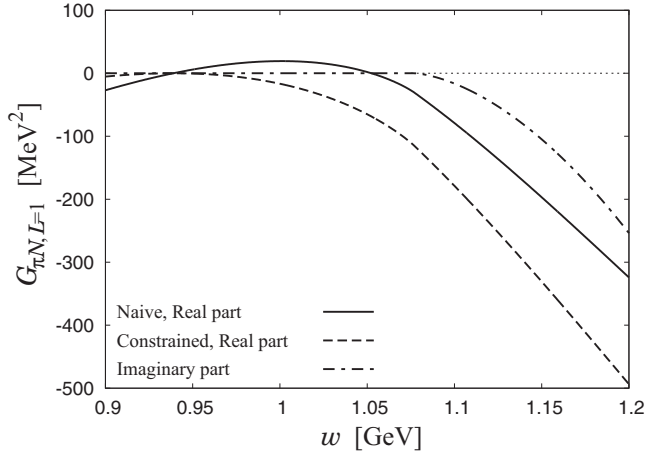|	Naive		Constrained	
	$\Delta(1232)$	$N(940)$	$\Delta(1232)$	$N(940)$
w_{pole} (MeV)	$1209.8 - 47.6i$	938.9	$1206.9 - 49.6i$	938.9
g (MeV$^{-1/2}$)	$0.383 - 0.053i$	0.560	$0.395 - 0.061i$	0.516
$X_{\pi N}$	$0.69 + 0.39i$	-0.18	$0.87 + 0.35i$	0.00
Z	$0.31 - 0.39i$	1.18	$0.13 - 0.35i$	1.00
U	0.30	$-$	0.31	$-$
$\tilde{X}_{\pi N}$	0.61	$-$	0.71	$-$
$\tilde{Z}$	0.39	$-$	0.29	$-$

FIG. 3. Loop function $G_{\pi N, L=1}$ around the energy from the nucleon mass M_N to the πN threshold. The imaginary part of the loop function takes the same value in both of the parameter sets Naive and Constrained.

evaluated as $X_{\pi N} = 0.69 + 0.39i$, which implies that the $\Delta(1232)$ resonance contains a significant πN component. Thus, our result in the refined model reconfirms the calculation in Ref. [17]. We note that the imaginary part of the compositeness is non-negligible as well, but the value of $U = 0.30$ is less than one-third. This implies that we may interpret $\tilde{X}_{\pi N}$ and $\tilde{Z}$ as the "probability" to find the πN composite and missing-channel contributions, respectively. From the values of $\tilde{X}_{\pi N}$ and $\tilde{Z}$, we may conclude that the $\Delta(1232)$ resonance in the present refined model contains a significant πN component.

On the other hand, for $N(940)$, the wave function renormalization takes place owing to $dG_{L=1}/dw(M_N) \neq 0$, and its πN compositeness becomes finite in the parameter set Naive. However, its value is real but negative. This result is unphysical, because we cannot interpret it as a probability although $N(940)$ is a stable state. The origin of the negative compositeness is the fact that the derivative of the $L=1$ loop function, $dG_{L=1}/dw$, becomes positive at the nucleon pole $w = M_N$, which can be seen in Fig. 3 (solid line). As a result, the compositeness becomes negative even when the coupling constant is real: $g = 0.560$ MeV$^{-1/2}$.

Physically, the derivative of the loop function, dG_L/dw, should not be positive below the πN threshold. We can see this by looking into the expression of the loop function G_L (see Appendix C),

$$G_L(w) \equiv -\int_{s_{th}}^{\infty} \frac{ds'}{2\pi} \frac{\rho_{\pi N}(s') q_{\pi N}(s')^{2L}}{s' - s - i0}, \quad (103)$$

with $s_{th} \equiv (m_\pi + M_N)^2$. When differentiating the loop function with respect to w, the integrand is positive definite regardless of the value of $s'(>s_{th})$. Therefore, the condition that the derivative of the loop function becomes negative (positive) at the nucleon mass is (un-)physical. However, in actual calculations, the positive value for the derivative of the loop function can happen to appear according to the value of the subtraction constant $\tilde{A}$.

Based on this discussion, to resolve the problem that the derivative of the loop function becomes positive, in addition to $G_{L=1}(M_N) = 0$ we further constrain the subtraction constant so that the derivative of the $L = 1$ loop function should be nonpositive at the nucleon pole: $dG_{L=1}/dw(M_N) \leqslant 0$. With this additional constraint, we obtain the best fit of the scattering amplitude to the WI 08 solution as the parameter set "Constrained," whose values are listed in the third column of Table I. This gives a slightly worse χ^2 value but we cannot see any clear discrepancy from curves in Fig. 2 (dashed line). Properties of $\Delta(1232)$ and $N(940)$ in the parameter set Constrained are listed in the fourth and fifth columns of Table II. The values of the coupling constants and compositeness are very similar to the parameter set Naive, except for the πN compositeness for $N(940)$, which now becomes non-negative. The result in the present model indicates that the $N(940)$ state is, as expected, not described in the πN molecular picture.

Finally, we note that there is ambiguity in calculating the πN compositeness $X_{\pi N}$ for p-wave resonances from the loop function with the subtraction scheme and the dimensional regularization (91). Namely, as discussed in Ref. [65], we can consider a shift of the subtraction constant $\tilde{A}$, which can be compensated by the corresponding shift of the interaction V so as not to change the full amplitude T. However, this shift of the subtraction constant can change the value of $dG_{L=1}/dw$ and hence that of $X_{\pi N}$, because the subtraction constant survives when we differentiate $G_{\pi N, L=1}(w)$ [see the structure in Eq. (91)]. Nevertheless, if we have a constraint $dG_{L=1}/dw(M_N) \leqslant 0$, such a shift of the subtraction constant is also constrained and $dG_{L=1}/dw$ cannot be close to zero around the $\Delta(1232)$ energy region. This can be seen from lines in Fig. 3. Namely, if the subtraction constant $\tilde{A}$ could increase arbitrarily only with the constraint $G_{L=1}(M_N) = 0$, the real part of the loop function in $w \geqslant M_N$ could shift upward in Fig. 3 and eventually become flat around $w = 1.2$ GeV, with which the πN compositeness for $\Delta(1232)$ would be negligible owing to the negligible value of $dG_{L=1}/dw(w_{pole})$. However, in such a case the derivative at $w = M_N$, $dG_{L=1}/dw(M_N)$, should be largely positive and hence it should be excluded. Therefore, in the present formulation, we cannot arbitrarily shift the value of the πN compositeness for $\Delta(1232)$ without changing the scattering amplitude. In particular, in the present calculation $\tilde{A}$ takes its maximal value under the constraint $dG_{L=1}/dw(w = M_N) \leqslant 0$, as seen from $dG_{L=1}/dw(w = M_N) = 0$ in Fig. 3. As a consequence, the present calculation would give a minimal value of $|X_{\pi N}|$ for $\Delta(1232)$ in our approach from the viewpoint of the shift of the subtraction constant. In the same manner, $N(940)$ could have a certain positive value of the πN compositeness by the shift of the subtraction constant. We note that such ambiguity will not take place when we use a usual cutoff scheme for the loop function rather than the dimensional regularization. In this condition, the derivative of the loop function at the nucleon mass will be definitely negative and nonzero, and hence $X_{\pi N}$ for the nucleon will be positive and nonzero, say, 0.1.

In summary, from the precise πN scattering amplitude we have found that, in the real part, the πN compositeness is larger than the elementariness for the $\Delta(1232)$ resonance. Its imaginary part is non-negligible, but the value of U is less than

one-third. Therefore, we may conclude that the πN component in the $\Delta(1232)$ resonance is large. The large real part of the πN compositeness and its non-negligible imaginary part might be the origin of the large meson cloud effect observed in, e.g., the $M1$ transition form factor of the $\gamma^* N \to \Delta(1232)$ process in the small momentum transfer region [35]. We mention that we have had two problems on the πN compositeness for the p-wave states; one is the negative πN compositeness for $N(940)$ in the naive fitting, and the other is ambiguity owing to the shift of the subtraction constant. Both originate from the value of the subtraction constant used in the analysis, and we have discussed the problems from the viewpoint of the shift of the subtraction constant and constraint on it at the energy of the nucleon mass. As a result, we have shown that in our approach the value of $|X_{\pi N}|$ for $\Delta(1232)$ cannot be small.

B. The $N(1535)$ and $N(1650)$ resonances

Next we consider the $N(1535)$ and $N(1650)$ resonances, both of which are S_{11} states in the πN scattering, and calculate their meson-baryon compositeness. In this study we describe these resonances in an s-wave πN-ηN-$K\Lambda$-$K\Sigma$ coupled-channel scattering in the chiral unitary approach. Here we do not introduce explicit pole terms for $N(1535)$ and $N(1650)$, in contrast to the case of $\Delta(1232)$. Actually, it is a good starting point to examine the picture of dominant meson-baryon components for them, because they can be discussed with meson-baryon dynamics in s wave. We regard that missing contributions are implemented into the energy dependence of the interaction, not as explicit channels coupling to meson-baryon states. Nevertheless, the essential part of the discussion about the elementariness is not changed, as we have done in the previous section.

1. Scattering amplitude

First we construct the interaction kernel in the chiral unitary approach. In this study we take into account the Weinberg-Tomozawa term [Fig. 1(a)] and the next-to-leading order term [Fig. 1(b)] for the interaction kernel, as done in Ref. [41]. From the Lagrangian of chiral perturbation theory, we obtain the interaction before the s-wave projection,

$$\mathcal{V}_{jk} = A_{\mathrm{WT}}\not{R} + A_{\mathrm{M}} + A_{14}(q \cdot q') + A_{57}[\not{q}, \not{q}'] \\ + A_{811}[\not{q}'(q \cdot P) + \not{q}(q' \cdot P)], \quad (104)$$

where $j, k = \pi N$, ηN, $K\Lambda$, and $K\Sigma$ are the channel indices, q^μ and q'^μ are the meson momenta in the initial and final states, respectively, $\not{q} \equiv \gamma^\mu q_\mu$ with the Dirac γ matrices γ^μ, $R^\mu \equiv q^\mu + q'^\mu$, and A_{WT}, A_{M}, A_{14}, A_{57}, and A_{811} are the coefficients of the meson-baryon couplings determined by flavor SU(3) symmetry together with the low-energy constants, meson decay constants, and meson masses. The expression of the coefficients A_{WT}, A_{M}, A_{14}, A_{57}, and A_{811} as well as the pertinent Lagrangian of chiral perturbation theory can be found in Ref. [41]. We have 14 low-energy constants in the coefficients, b_1 to b_{11}, b_0, b_D, and b_F, and we treat them as the model parameters. The meson decay constants are chosen at their physical values: $f_\pi = 92.4$ MeV, $f_K = 1.2 f_\pi$, and

$f_\eta = 1.3 f_\pi$. The interaction $\mathcal{V}$ is projected to the s wave as

$$V^+_{I=1/2\,L=0,\,jk}(w) = [\bar{u}_j \mathcal{V}_{jk} u_k]_{s\text{-wave}}, \quad (105)$$

where w is the center-of-mass energy, u_j is the Dirac spinor for the jth-channel baryon, whose normalization is summarized in Appendix A, and $\bar{u}_j \equiv u_j^\dagger \gamma^0$. The s-wave projection of each term can be performed as[5]

$$[\bar{u}_j \not{R} u_k]_{s\text{-wave}} = \mathcal{N}_j \mathcal{N}_k (2w - M_j - M_k), \quad (106)$$

$$[\bar{u}_j u_k]_{s\text{-wave}} = \mathcal{N}_j \mathcal{N}_k, \quad (107)$$

$$[\bar{u}_j (q \cdot q') u_k]_{s\text{-wave}} = \mathcal{N}_j \mathcal{N}_k \omega_j(w) \omega_k(w), \quad (108)$$

$$[\bar{u}_j [\not{q}, \not{q}'] u_k]_{s\text{-wave}} \\ = \mathcal{N}_j \mathcal{N}_k [2\omega_j(w)\omega_k(w) - 2(w - M_j)(w - M_k)], \quad (109)$$

$$[\bar{u}_j [\not{q}'(q \cdot P) + \not{q}(q' \cdot P)] u_k]_{s\text{-wave}} \\ = \mathcal{N}_j \mathcal{N}_k w[(w - M_j)\omega_k(w) + (w - M_k)\omega_j(w)], \quad (110)$$

where $\mathcal{N}_j$ is the normalization factor for the Dirac spinor,

$$\mathcal{N}_j \equiv \sqrt{E_j(w) + M_j}, \quad E_j(w) \equiv \frac{s + M_j^2 - m_j^2}{2w}, \quad (111)$$

and $\omega_j(w)$ is the energy of the jth-channel meson,

$$\omega_j(w) \equiv \frac{s + m_j^2 - M_j^2}{2w}. \quad (112)$$

Here $s \equiv w^2$ and m_j and M_j are the masses of meson and baryon in jth channel, respectively. We note that the constructed interaction kernel $V^+_{I=1/2\,L=0,\,jk}$, which is abbreviated as V_{jk} in the following, is a function only of the center-of-mass energy w; because $L = 0$, we do not need to factorize the relative momenta $|\boldsymbol{q}'|^L|\boldsymbol{q}|^L$ in contrast to the p-wave scattering in the previous section.

By using this interaction kernel, the full scattering amplitude T_{jk} is a solution of the Lippmann-Schwinger equation in an algebraic form

$$T_{jk}(w) = V_{jk}(w) + \sum_l V_{jl}(w) G_l(w) T_{lk}(w), \quad (113)$$

with the jth-channel loop function G_j. Here we take a covariant expression for the loop function

$$G_j(w) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m_j^2)[(P - q)^2 - M_j^2]}, \quad (114)$$

with $P^\mu = (w, \boldsymbol{0})$, and calculate the integral with the dimensional regularization. Its expression is shown in Eq. (C6) in Appendix C. In this study, to fix the subtraction constant in the loop function, we require the natural renormalization scheme [65]. According to the discussion in Ref. [65], we introduce a matching energy scale, at which the full scattering amplitude T coincides with the chiral interaction V for the consistency of the low-energy theorem with respect to the spontaneous breaking of chiral symmetry. We fix the matching

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[5]We note that the term $\bar{u}(q \cdot q')u$ has a higher-order s-wave part proportional to $|\boldsymbol{q}|^2|\boldsymbol{q}'|^2$, which is neglected in this study.

TABLE III. Fitted parameters for the πN amplitude S_{11}. The χ^2 value divided by the number of degrees of freedom is $\chi^2/N_{\text{d.o.f.}} = 94.6/167$.

b_1 (GeV^{-1})	0.469	b_8 (GeV^{-1})	0.523
b_2 (GeV^{-1})	-0.048	b_9 (GeV^{-1})	-1.246
b_3 (GeV^{-1})	1.244	b_{10} (GeV^{-1})	0.574
b_4 (GeV^{-1})	-1.507	b_{11} (GeV^{-1})	-0.845
b_5 (GeV^{-1})	-1.091	b_0 (GeV^{-1})	-5.513
b_6 (GeV^{-1})	-0.722	b_D (GeV^{-1})	1.708
b_7 (GeV^{-1})	3.009	b_F (GeV^{-1})	2.516

energy as the lowest mass of the "target" baryons in the scattering [65], i.e., the nucleon mass M_N,

$$G_j(w = M_N) = 0, \qquad (115)$$

for every j. With this condition, the loop function G_j becomes physical, i.e., negative in the region $M_N < w \leqslant m_j + M_j$, which can exclude explicit pole contributions from the loop functions [65] and fix the loop function without any model parameter.

Now we can evaluate the scattering amplitude in chiral unitary approach with 14 model parameters in the interaction kernel. In the present calculation, the model parameters are fixed so as to reproduce the S_{11} solution of the πN partial-wave analysis WI 08 [34] up to $w \leqslant 1.8$ GeV. As in the previous section, we introduce a normalized scattering amplitude,

$$S_{11}(w) = -\frac{\rho_{\pi N}(s)}{2} T_{\pi N \to \pi N}(w)^{\text{on-shell}}, \qquad (116)$$

which satisfies the optical theorem (95) below the inelastic threshold for the πN state. As we have already mentioned, the WI 08 solution does not provide errors for the scattering amplitude. For the calculation of the χ^2 value, we introduce errors 0.01 for $w \leqslant 1.35$ GeV and 0.02 for $w > 1.35$ GeV to the data, which are motivated by the expectation of the three-body effects above the $\pi\pi N$ threshold [37]. From the best fit, we obtain the model parameters listed in Table III with a good value of χ^2. We also show the S_{11} amplitude in the theoretical calculation and the WI 08 solution in Fig. 4, which shows a good reproduction of the S_{11} amplitude up to $w = 1.8$ GeV.

2. Compositeness

From the S_{11} scattering amplitude, we can search for the N^* poles in the complex energy plane. As a result, we find two poles located at $w_{\text{pole}} = 1496.4 - 58.7i$ MeV between the ηN and the $K\Lambda$ thresholds and $w_{\text{pole}} = 1660.7 - 70.0i$ MeV between the $K\Lambda$ and the $K\Sigma$ thresholds for the $N(1535)$ and $N(1650)$ resonances, respectively. These pole positions are consistent with the values of Particle Data Group: $w_{\text{pole}} = (1510 \pm 20) - (85 \pm 40)i$ MeV for $N(1535)$ and $w_{\text{pole}} = (1655 \pm 15) - (67.5 \pm 17.5)i$ MeV for $N(1650)$ [1]. In the following we evaluate the compositeness of the $N(1535)$ and $N(1650)$ resonances by using this scattering amplitude. Here we note that, because both resonances are in the s-wave πN-ηN-$K\Lambda$-$K\Sigma$ coupled-channel scattering, we do not have the problems concerned with the subtraction constant in
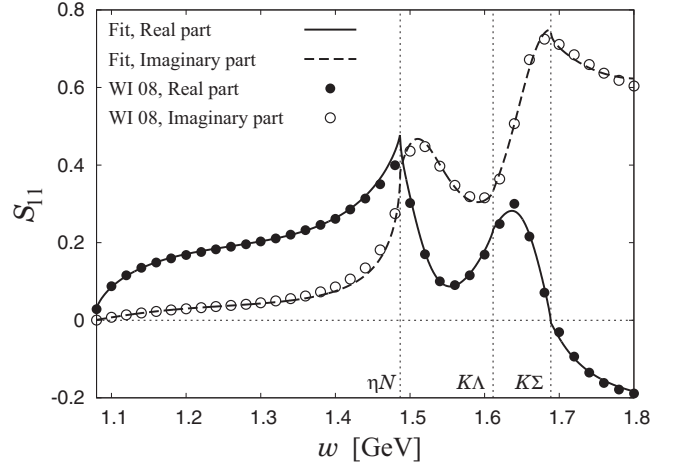

FIG. 4. Scattering amplitude S_{11} with parameters fitted to the WI 08 solution [34]. The number of the plotted data points is 1/5 of the total in the fits for a better visualization. The vertical dotted lines indicate the ηN, $K\Lambda$, and $K\Sigma$ thresholds, respectively.

the loop function, in contrast to the case of $\Delta(1232)$ and $N(940)$ in the previous section. Namely, even if we shift the subtraction constant of the $L = 0$ loop function, it does not change the value of $dG_{L=0}/dw$ and that of the compositeness, because the subtraction constant is eliminated when we perform the derivative of the $L = 0$ loop function. In other words, the integral in $dG_{L=0}/dw$ converges.

According to the scheme developed in Sec. II C, we extract the compositeness from the scattering amplitude. From the residue $g_j g_k$ of the scattering amplitude at the resonance pole position w_{pole},

$$T_{jk}(w) = \frac{g_j g_k}{w - w_{\text{pole}}} + (\text{regular at } w = w_{\text{pole}}), \qquad (117)$$

we evaluate the compositeness as the norm of the two-body wave function,

$$X_j = -g_j^2 \frac{dG_j}{dw}(w = w_{\text{pole}}), \qquad (118)$$

and the elementariness as well,

$$Z = -\sum_{j,k} g_k g_j \left[G_j \frac{dV_{jk}}{dw} G_k \right]_{w=w_{\text{pole}}}, \qquad (119)$$

which measures the contributions from missing channels on the assumption that the energy dependence of the interaction originates from missing channels. The normalization of the total wave function is achieved as

$$\sum_j X_j + Z = 1. \qquad (120)$$

From the compositeness and elementariness, both of which are complex for resonances in general, we calculate quantities which are real, bounded in the range [0, 1], and automatically satisfy the sum rule,

$$\tilde{X}_j \equiv \frac{|X_j|}{1+U}, \quad \tilde{Z} \equiv \frac{|Z|}{1+U}, \qquad (121)$$

TABLE IV. Properties of $N(1535)$ and $N(1650)$.

	$N(1535)$	$N(1650)$
w_{pole} (MeV)	$1496.4 - 58.7i$	$1660.7 - 70.0i$
$g_{\pi N}$ (MeV$^{1/2}$)	$47.1 - 7.3i$	$49.8 - 23.1i$
$g_{\eta N}$ (MeV$^{1/2}$)	$68.9 - 42.4i$	$-19.0 + 11.1i$
$g_{K\Lambda}$ (MeV$^{1/2}$)	$85.0 + 14.4i$	$-29.9 + 37.1i$
$g_{K\Sigma}$ (MeV$^{1/2}$)	$-31.4 + 17.5i$	$-73.8 + 6.0i$
$X_{\pi N}$	$-0.02 + 0.03i$	$0.00 + 0.04i$
$X_{\eta N}$	$0.04 + 0.37i$	$0.00 + 0.01i$
$X_{K\Lambda}$	$0.14 + 0.00i$	$0.08 + 0.05i$
$X_{K\Sigma}$	$0.01 - 0.02i$	$0.09 - 0.12i$
Z	$0.84 - 0.38i$	$0.84 + 0.01i$
U	0.48	0.13
$\tilde{X}_{\pi N}$	0.03	0.04
$\tilde{X}_{\eta N}$	0.25	0.01
$\tilde{X}_{K\Lambda}$	0.09	0.08
$\tilde{X}_{K\Sigma}$	0.01	0.13
$\tilde{Z}$	0.62	0.74

with

$$U \equiv \sum_j |X_j| + |Z| - 1. \quad (122)$$

Obviously, we have the sum rule for $\tilde{X}_j$ and $\tilde{Z}$:

$$\sum_j \tilde{X}_j + \tilde{Z} = 1. \quad (123)$$

We can interpret $\tilde{X}_j$ and $\tilde{Z}$ as the "probability" if and only if U is much smaller than unity, $U \ll 1$.

The numerical results of the coupling constants, compositeness, and elementariness are listed in Table IV both for the $N(1535)$ and $N(1650)$ resonances.

For the $N(1535)$ resonance, its coupling constants show an ordering similar to that in Ref. [41]; in particular, $|g_{K\Lambda}|$ is the largest and $|g_{\eta N}|$ comes next, which is consistent with the result in Ref. [41]. However, the values of the compositeness in the $K\Lambda$ and ηN channels are not comparable to unity, and the elementariness Z dominates the sum rule (120). Therefore, our result implies that $N(1535)$ has a large component originating from contributions other than the pseudoscalar meson-baryon dynamics considered. This conclusion was already drawn in Ref. [22] with the simplest interaction, i.e., the Weinberg-Tomozawa term, and we confirm this with our refined model for the precise S_{11} amplitude. The result of the compositeness means that the missing-channel contribution Z dominates the sum rule even if we do not take into account a bare-state contribution explicitly. The missing channel can contribute to the appearance of the resonance through the energy dependence of the interaction and the low-energy constants. In other words, in the present framework, information on the $N(1535)$ resonance is encoded in the energy dependence of the chiral interaction and the low-energy constants in it. However, in the present model space, we cannot conclude what the missing channel is; we expect that this will be genuine one-body state, but other channels such as vector meson-baryon and meson-meson-baryon systems could be

the origin.[6] We also note that the value of U is not small compared to unity, owing to the non-negligible imaginary part of $\tilde{X}_{\eta N}$ and $\tilde{Z}$. Therefore, modified quantities $\tilde{X}_j$ and $\tilde{Z}$ cannot be interpreted as probabilities to find the composite and missing fractions, respectively. In particular, although $\tilde{X}_{\eta N}$ is one-fourth, we cannot conclude a non-negligible ηN component for $N(1535)$.

Next, for the $N(1650)$ resonance, $|g_{K\Sigma}|$ is the largest among the absolute values of the coupling constants, as in Refs. [41]. However, the ordering of the coupling constants is not consistent. We expect that this is mainly because the accuracy of the fitting. Actually, our fitting can be more accurate, as seen in the better reproduction of the $N(1650)$ pole position reported by Particle Data Group. As for the component of $N(1650)$, we can see that the elementariness Z dominates the sum rule (120). In addition, the value of U for $N(1650)$ is much smaller than unity. Therefore, we can safely interpret the modified quantities $\tilde{X}_j$ and $\tilde{Z}$ as probabilities. The result listed in Table IV indicates that $\tilde{Z}$ is dominant and hence the $N(1650)$ resonance is indeed dominated by contributions other than the pseudoscalar meson-baryon dynamics considered.

Finally, it is interesting to compare the structure of $N(1535)$ and $N(1650)$ with that of $\Lambda(1405)$ and $\Xi(1690)$, all of which are considered to be dynamically generated in the chiral unitary approach. The compositeness of $\Lambda(1405)$ was evaluated in the chiral unitary approach in Ref. [22] with the leading plus next-to-leading order chiral interaction [66,67], concluding that the higher pole of $\Lambda(1405)$ is indeed dominated by the $\bar{K}N$ composite state. In contrast to $\Lambda(1405)$, the compositeness of $N(1535)$ and $N(1650)$ is negligible or not large, although we describe $N(1535)$ and $N(1650)$ in the meson-baryon degrees of freedom, as in the $\Lambda(1405)$ case. This difference of the structure is expected to originate from the different thresholds and model parameters (low-energy constants and subtraction constants), both of which should degenerate in the SU(3) symmetric world. In particular, when we shift the system from the SU(3) symmetric world to the physical one, the situation in the $S = 0$ sector would change most drastically; the πN threshold becomes the lowest one and the other channels such as ρN, $\pi \Delta$, and genuine qqq states would contribute to the πN scattering. In this study these are reflected to the low-energy constants in the next-to-leading order as the missing channels. Actually, while the chiral unitary approach can reproduce the phenomena around the $\bar{K}N$ threshold for $\Lambda(1405)$ even with the simplest interaction, i.e., the Weinberg-Tomozawa interaction [66,67], the πN scattering amplitude cannot be reproduced well in the chiral unitary approach around the N^* region only with the Weinberg-Tomozawa interaction [38]. Significant contributions in the next-to-leading order can introduce missing channels through the low-energy constants, and hence the compositeness (elementariness) is small (large) for $N(1535)$ and $N(1650)$. Besides, we also mention the fate of the dynamically generated resonances in $S = 0$ and

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[6]If we can reproduce well the S_{11} amplitude with the two bare pole terms corresponding to $N(1535)$ and $N(1650)$ and energy-independent meson-baryon interaction, we can conclude that $N(1535)$ and $N(1650)$ originate from one-body states, respectively.

$S = -2$ channels in the chiral unitary approach. Interestingly, the Clebsch-Gordan coefficients for the Weinberg-Tomozawa interaction term are the same for $S = 0$ (πN, ηN, $K\Lambda$, $K\Sigma$) and $S = -2$ ($\pi\Xi$, $\eta\Xi$, $\bar{K}\Lambda$, $\bar{K}\Sigma$) channels. On the one hand, in the $S = -2$ channel, it was suggested in Ref. [47] that $\Xi(1690)$ can be dynamically generated near the $\bar{K}\Sigma$ threshold with a dominant $\bar{K}\Sigma$ compositeness in s wave in the chiral unitary approach, consistently with the experimental observations. On the other hand, such a dynamically generated N^* state would exist if the Weinberg-Tomozawa interaction were dominant in the $S = 0$ chiral unitary approach, but, in fact, no hadronic molecular N^* state appears in $S = 0$ owing to the significant contributions from the next-to-leading-order terms. Nevertheless, we think that, although the internal structure of $N(1535)$, $\Lambda(1405)$, and $\Xi(1690)$ is different, our result does not mean that they are not members of the same flavor SU(3) multiplet. Namely, when we consider the SU(3) multiplets, we measure the masses of hadrons from the vacuum, that is, from zero. On the other hand, when we consider the internal structure in terms of the compositeness, we measure the masses from the thresholds. As a consequence, the breaking effect of flavor SU(3) symmetry is non-negligible in terms of the compositeness, because the meson-baryon thresholds are different for each state.

In summary, we have evaluated the compositeness of the $N(1535)$ and $N(1650)$ resonances from the precise S_{11} scattering amplitude. The results indicate that both of them are dominated by components other than the pseudoscalar meson-baryon dynamics considered. An important finding is that the missing-channel contribution Z dominates the sum rule even if we do not take into account a bare-state contribution explicitly. The missing channel can contribute to the appearance of the resonance through the energy dependence of the interaction. Finally, we note that we do not have problems concerned with the subtraction constant in the loop function, in contrast to the case of $\Delta(1232)$ and $N(940)$ in the previous subsection, because $N(1535)$ and $N(1650)$ are in the s-wave πN-ηN-$K\Lambda$-$K\Sigma$ coupled-channel scattering and, hence, $dG_{L=0}/dw$ converges.

IV. SUMMARY AND OUTLOOK

In this study we have presented a formulation of the compositeness for baryonic resonances to discuss the meson-baryon molecular structure inside the resonances. For this purpose, we have shown that the residue of the scattering amplitude at the resonance pole position contains the wave function of the resonance with respect to the two-body channel, in both the nonrelativistic and the relativistic formulations. Then we have defined the compositeness for the resonance state as a norm of the two-body wave function extracted from the residue of the scattering amplitude. An important point to be noted is that the value of compositeness, i.e., the norm of the two-body wave function, is automatically fixed when we calculate the residue of the scattering amplitude, without normalizing the wave function by hand. We have also defined the missing-channel contribution, which we call elementariness, as unity minus the sum of the compositeness, which measures the contributions from missing channels on the assumption

that the energy dependence of the interaction originates from missing channels. In addition, from the compositeness and elementariness, we have introduced quantities which are real, bounded in the range [0, 1], and automatically satisfy the sum rule. These quantities can be interpreted as probabilities in a certain class of resonances.

The formulated compositeness and elementariness were applied to the $\Delta(1232)$, $N(1535)$, and $N(1650)$ resonances and $N(940)$ in the chiral unitary approach, because there are several implications that these resonances may have certain fractions of the meson-baryon components. In the present model, we have determined the separable interaction of the pseudoscalar meson-octet baryon from chiral perturbation theory up to the next-to-leading order. The $\Delta(1232)$ resonance and $N(940)$ were described in the πN single-channel scattering, while the $N(1535)$ and $N(1650)$ resonances were in the s-wave πN-ηN-$K\Lambda$-$K\Sigma$ coupled-channel scattering. In both cases, the loop function was evaluated with the subtraction scheme and the dimensional regularization. In particular, we have to introduce two subtraction constants for $\Delta(1232)$ and $N(940)$ to calculate their compositeness in a proper way. The model parameters were fixed so that the πN scattering amplitude precisely reproduces the solution of the partial-wave analysis.

As a result for $\Delta(1232)$, we have found that the real part of the πN compositeness is larger than the elementariness. The imaginary part of the πN compositeness for $\Delta(1232)$ is non-negligible, but the sum of the absolute values of the compositeness and elementariness is close to unity. Therefore, we may conclude that the πN component in $\Delta(1232)$ is significant. We have also had two problems on the πN compositeness for the p-wave states; one is the negative πN compositeness for $N(940)$ in the naive fitting, and the other is ambiguity owing to the shift of the subtraction constant. Both originate from the value of the subtraction constant used in the analysis, and we have discussed the problems from the viewpoint of the shift of the subtraction constant and constraint on it at the energy of the nucleon mass. As a consequence, we have shown that in our approach the absolute value of the πN compositeness for $\Delta(1232)$ cannot be small.

For $N(1535)$ and $N(1650)$, on the other hand, we have found that both of them are dominated by components other than the pseudoscalar meson-baryon dynamics considered. An important finding was that, even if we do not take into account a bare pole term for the resonance explicitly, a missing channel can contribute to the appearance of the resonance through the energy dependence of the interaction and the low-energy constants. Because both resonances are in the s-wave πN-ηN-$K\Lambda$-$K\Sigma$ coupled-channel scattering, we do not have problems concerned with the subtraction constant in the loop function, in contrast to the case of $\Delta(1232)$ and $N(940)$.

Finally, we mention that the large absolute value of the πN compositeness for the $\Delta(1232)$ resonance would lead to the large meson cloud effect observed in, e.g., the $M1$ transition form factor of the $\gamma^* N \to \Delta(1232)$ process in the small momentum transfer region. However, our result relies on the separable interaction in the form of Eq. (83), which might be an oversimplified interaction in describing the πN scattering amplitude, especially in the $\Delta(1232)$ resonance region. In this

sense, it would be better to evaluate the compositeness for $\Delta(1232)$ in solving the integral equation for the scattering amplitude, such as in the dynamical approaches, so as to conclude the πN structure of $\Delta(1232)$ more clearly.

APPENDIX A: CONVENTIONS

In this Appendix we summarize our conventions of meson-baryon scatterings used in this paper.

Throughout this paper we employ the metric in four-dimensional Minkowski space defined as $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and the Einstein summation convention is used for the Lorentz index μ. The Dirac matrices γ^μ ($\mu = 0, 1, 2, 3$) satisfies the anticommutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (A1)$$

The Dirac spinor $u(\boldsymbol{p}, s)$, where $\boldsymbol{p}$ is three-momentum of the field and s represents its spin, is introduced as the positive energy solution of the Dirac equation for baryons. In this paper the Dirac spinor is normalized as

$$\bar{u}(\boldsymbol{p}, s')u(\boldsymbol{p}, s) = 2M\delta_{ss'}, \quad (A2)$$

where $\bar{u} \equiv u^\dagger \gamma^0$ and M is the mass of the Dirac field.

To describe the meson-baryon scatterings, we introduce meson and baryon one-particle states. The meson states $|\boldsymbol{k}, j\rangle$ are normalized as

$$\langle \boldsymbol{k}', j'|\boldsymbol{k}, j\rangle = 2\omega_j(k)(2\pi)^3 \delta^3(\boldsymbol{k}' - \boldsymbol{k})\delta_{jj'}, \quad (A3)$$

where $\boldsymbol{k}$ is the three-momentum of the meson, j indicates the channel, and $\omega_j(k) \equiv \sqrt{k^2 + m_j^2}$ is the meson energy with $k \equiv |\boldsymbol{k}|$. The baryon states $|\boldsymbol{p}, s, j\rangle$, on the other hand, are normalized as

$$\langle \boldsymbol{p}', s', j'|\boldsymbol{p}, s, j\rangle = 2E_j(p)(2\pi)^3 \delta^3(\boldsymbol{p}' - \boldsymbol{p})\delta_{ss'}\delta_{jj'}, \quad (A4)$$

where $\boldsymbol{p}$ is the three-momentum of the baryon, s is its spin, and $E_j(p) \equiv \sqrt{p^2 + M_j^2}$. By using these states, we can compose jth-channel meson-baryon two-body state in the center-of-mass frame with the relative momentum $\boldsymbol{q}$, which we simply write as $|j\rangle$:

$$|j\rangle \equiv |\boldsymbol{q}, j\rangle \otimes |-\boldsymbol{q}, s, j\rangle. \quad (A5)$$

Then we can evaluate the scattering amplitude $\mathcal{T}_{jk}$ as the matrix element of the T matrix with respect to the scattering states $|j\rangle$,

$$\langle j|\hat{T}|k\rangle = \bar{u}_j \mathcal{T}_{jk} u_k, \quad (A6)$$

where u_j is the Dirac spinor for the jth-channel baryon. Note that we do not sum the channel components j and k. In this Appendix, characters in the script style denote four-times-four matrices that are sandwiched by the Dirac spinors, except for Lagrangians.

The scattering amplitude $\bar{u}_j \mathcal{T}_{jk} u_k$ can be decomposed into the partial-wave amplitudes. For this purpose we write the scattering amplitude in the center-of-mass frame as

$$\langle j|\mathcal{T}|k\rangle = \bar{u}_j \mathcal{T}_{jk} u_k$$
$$= \chi_j^\dagger [g_{jk}(w, x) - ih_{jk}(w, x)(\hat{q}_j \times \hat{q}_k) \cdot \boldsymbol{\sigma}]\chi_k, \quad (A7)$$

where χ_j is the Pauli spinor for the jth-channel baryon, $w \equiv \omega_j + E_j = \omega_k + E_k$ is the center-of-mass energy, $x = \cos\theta$ with the center-of-mass scattering angle θ, and $\hat{q}_j$ is the unit vector in the direction of the relative three-momentum in the jth channel. In our notation we can calculate the differential cross section of the meson-baryon scatterings in the center-of-mass frame as

$$\frac{d\sigma_{k \to j}}{d\Omega} = \frac{1}{64\pi^2 s}\frac{q_j}{q_k}[|g_{jk}(w, x)|^2 + |h_{jk}(w, x)|^2 \sin^2\theta], \quad (A8)$$

where $s \equiv w^2$ and q_j is the j-channel center-of-mass momentum. Then g_{jk} and h_{jk} are expressed in terms of the partial–wave amplitudes $T_L^\pm(w)$ as

$$g_{jk}(w, x) = \sum_{L=0}^\infty [(L+1)T_L^+(w) + LT_L^-(w)]_{jk} P_L(x), \quad (A9)$$

$$h_{jk}(w, x) = \sum_{L=1}^\infty [T_L^+(w) - T_L^-(w)]_{jk} P_L'(x), \quad (A10)$$

with the Legendre polynomials $P_L(x)$ and $P_L'(x) \equiv dP_L/dx$. In terms of $T_L^\pm(w)$ the optical theorem can be expressed as

$$\text{Im}[T_L^\pm(w)]_{jj} = -\frac{1}{8\pi w}\sum_k q_k |[T_L^\pm(w)]_{jk}|^2, \quad (A11)$$

where the sum runs over the open channels.

The partial-wave amplitude $T_L^\pm(w)$ can be extracted from the scattering amplitudes g_{jk} and h_{jk} by using the orthonormal relation

$$\int_{-1}^1 dx\, P_L(x)P_M(x) = \frac{2}{2L+1}\delta_{LM}, \quad (A12)$$

and a relation

$$\int_{-1}^1 dx\, P_L'(x)[P_{M\pm1}(x) - xP_M(x)] = \left(\frac{1}{2L+1} \mp 1\right)\delta_{LM}. \quad (A13)$$

Actually, from these relations we can extract $T_L^\pm$ as

$$T_L^\pm(w)_{jk} = \frac{1}{2}\int_{-1}^1 dx\{g_{jk}P_L(x) - h_{jk}[P_{L\pm1}(x) - xP_L(x)]\}. \quad (A14)$$

It is useful to express the meson-baryon scattering amplitude $\mathcal{T}$ as

$$\mathcal{T} = A(s, t) + \frac{\not{R}}{2}B(s, t), \quad (A15)$$

where s and t are Mandelstam variables, $\not{p} \equiv \gamma^\mu p_\mu$, and $R^\mu \equiv q^\mu + q'^\mu$ with $q^{(\prime)\mu}$ being the meson momentum in the initial (final) state. In particular, in the single-channel problem, from $A(s,t)$ and $B(s,t)$ we can calculate the scattering amplitudes $g(w,x)$ and $h(w,x)$ as

$$
g = \frac{1}{2(E+M)}\big\{[4M(E+M) - t]A
$$
$$
+ [(w+M)t + 4(E+M)(wE - M^2)]B\big\}, \quad \text{(A16)}
$$

$$
h = \frac{A - (w+M)B}{E+M}|\boldsymbol{q}|^2, \quad \text{(A17)}
$$

where M and E are mass and energy of the baryon, respectively.

Finally, we mention that the πN scattering amplitude used in Sec. III, which we denote as $L_{2I\,2J}$ with isospin I and total angular momentum $J = L \pm 1/2$, is expressed with the scattering amplitude $T_L^\pm$ in our convention as

$$
L_{2I\,2J}(w) = -\frac{q_{\pi N}}{8\pi w}T_L^\pm(w). \quad \text{(A18)}
$$

Therefore, below the inelastic threshold the πN scattering amplitude $L_{2I\,2J}$ satisfies the optical theorem

$$
\mathrm{Im}\, L_{2I\,2J} = |L_{2I\,2J}|^2 \quad \text{(A19)}
$$

for each partial wave.

APPENDIX B: πN INTERACTION KERNEL FROM CHIRAL PERTURBATION THEORY

In this Appendix we show the expressions of the πN interaction kernel used in Sec. III A. In general, the πN scattering has two isospin components, namely, $I = 3/2$ and $I = 1/2$, respectively. The $\pi N(I = 3/2)$ amplitude corresponds to the $\pi^+ p \to \pi^+ p$ one:

$$
\mathcal{T}_{\pi N(I=3/2)} = \mathcal{T}_{\pi^+ p \to \pi^+ p}. \quad \text{(B1)}
$$

The $\pi N(I = 1/2)$ amplitude can be written in terms of the $\pi^- p \to \pi^- p$ and $\pi^+ p \to \pi^+ p$ amplitudes:

$$
\mathcal{T}_{\pi N(I=1/2)} = \tfrac{3}{2}\mathcal{T}_{\pi^- p \to \pi^- p} - \tfrac{1}{2}\mathcal{T}_{\pi^+ p \to \pi^+ p}. \quad \text{(B2)}
$$

By using crossing symmetry, we can convert the $\pi^+ p \to \pi^+ p$ amplitude into $\pi^- p \to \pi^- p$ one in terms of A and B in Eq. (A15):

$$
A_{\pi^- p \to \pi^- p}(s,t) = A_{\pi^+ p \to \pi^+ p}(u,t), \quad \text{(B3)}
$$

$$
B_{\pi^- p \to \pi^- p}(s,t) = -B_{\pi^+ p \to \pi^+ p}(u,t). \quad \text{(B4)}
$$

Therefore, in the following we consider only the $\pi^+ p \to \pi^+ p$ amplitude.

In this study we employ chiral perturbation theory up to $\mathcal{O}(p^2)$ for the $\pi^+ p \to \pi^+ p$ interaction kernel $\mathcal{V}$. The interaction kernel consists of the Weinberg-Tomozawa term $\mathcal{V}_{\mathrm{WT}}$, u-channel $N(940)$ exchange $\mathcal{V}_{s+u}$, next-to-leading order contact term $\mathcal{V}_2$, and s- and u-channel $\Delta(1232)$ exchanges $\mathcal{V}_\Delta$:

$$
\mathcal{V} = \mathcal{V}_{\mathrm{WT}} + \mathcal{V}_{s+u} + \mathcal{V}_2 + \mathcal{V}_\Delta. \quad \text{(B5)}
$$

The interaction kernel $\mathcal{V}$ in the isospin basis is projected to the partial-wave amplitude $V_{IL}^\pm(w)$ in the same way as $\mathcal{T} \to T_{IL}^\pm(w)$ shown in Appendix A. This partial-wave amplitude

corresponds to the interaction kernel in Eq. (82) and then is unitarized as in Sec. III A.

The leading-order [$\mathcal{O}(p^1)$] pion-nucleon Lagrangian can be expressed as

$$
\mathcal{L}_{\pi N}^{(1)} = \bar{N}(i\not{D} - M)N + \frac{g}{2}\bar{N}\not{u}\gamma_5 N. \quad \text{(B6)}
$$

In the expression, $N = (p,n)^t$ is the nucleon fields, M and g are the nucleon mass and the nucleon axial charge in the chiral limit, respectively, and $D_\mu \equiv \partial_\mu + \Gamma_\mu$ is the covariant derivative with $\Gamma_\mu = [u^\dagger, \partial_\mu u]/2$, where u is the square root of U in the nonlinear representation,

$$
u(x) \equiv \sqrt{U(x)} = \exp\left[i\frac{\vec{\pi}(x)\cdot\vec{\tau}}{2f}\right]
$$
$$
= 1 + i\frac{\vec{\pi}(x)\cdot\vec{\tau}}{2f} - \frac{\vec{\pi}(x)^2}{8f^2} + \mathcal{O}[(\vec{\pi}\cdot\vec{\tau})^3], \quad \text{(B7)}
$$

with the pion decay constant f in the chiral limit and $\vec{\tau}$ being the Pauli matrices acting in isospin space. The pion fields $\vec{\pi}$ are expressed as

$$
\vec{\pi}(x)\cdot\vec{\tau} = \begin{pmatrix} \pi^0(x) & \sqrt{2}\pi^+(x) \\ \sqrt{2}\pi^-(x) & -\pi^0(x) \end{pmatrix}, \quad \text{(B8)}
$$

and we further define u_μ as

$$
u_\mu \equiv iu^\dagger \partial_\mu U u^\dagger = -\frac{\partial_\mu\vec{\pi}(x)\cdot\vec{\tau}}{f} + \mathcal{O}[(\vec{\pi}\cdot\vec{\tau})^3]. \quad \text{(B9)}
$$

By using them we obtain the leading-order πN interaction as

$$
\mathcal{V}_{\mathrm{WT}} = \frac{\not{R}}{2}\frac{1}{2f_\pi^2}, \quad \text{(B10)}
$$

$$
\mathcal{V}_{s+u} = -\frac{g_A^2}{f_\pi^2}M_N + \frac{\not{R}}{2}\left[-\frac{g_A^2}{2f_\pi^2}\frac{u + 3M_N^2}{u - M_N^2}\right], \quad \text{(B11)}
$$

where we have replaced f, M, and g with their physical values f_π, M_N, and g_A, respectively. In this study we fix $f_\pi = 92.4$ MeV, $M_N = 938.92$ MeV, and $g_A = 1.267$. Therefore, the leading-order term does not have model parameters.

The next-to-leading-order [$\mathcal{O}(p^2)$] pion-nucleon Lagrangian can be expressed, after neglecting irrelevant terms for the πN scattering, as

$$
\mathcal{L}_{\pi N}^{(2)} = c_1\langle\chi_+\rangle\bar{N}N - \frac{c_2}{4M^2}\langle u_\mu u_\nu\rangle[\bar{N}D^\mu D^\nu N + (\mathrm{H.c.})]
$$
$$
+ \frac{c_3}{2}\langle u_\mu u^\mu\rangle\bar{N}N - \frac{c_4}{4}\bar{N}\gamma^\mu\gamma^\nu[u_\mu, u_\nu]N + \cdots, \quad \text{(B12)}
$$

where $\langle A\rangle$ denotes the trace of the 2×2 matrix A in the flavor space, and

$$
\chi_+ \equiv u^\dagger\chi u^\dagger + u\chi^\dagger u, \quad \text{(B13)}
$$

with $\chi = 2B_0\hat{m}$ ($\hat{m} = m_u = m_d$ in the isospin symmetric limit), which can be replaced with the squared pion mass: $\chi \approx m_\pi^2$. By using them we obtain the next-to-leading order

πN interaction as

$$\mathcal{V}_2 = \frac{4c_1 m_\pi^2}{f_\pi^2} - \frac{c_2}{8 f_\pi^2 M_N^2}[(s-u)^2 - t^2] - \frac{c_3(2m_\pi^2 - t)}{f_\pi^2}$$
$$- \frac{c_4}{2 f_\pi^2}(s-u) + \frac{\not{R}}{2}\frac{2c_4 M_N}{f_\pi^2}. \tag{B14}$$

The mass of pion is fixed as $m_\pi = 138.04$ MeV. The low-energy constants c_1, c_2, c_3, and c_4 are model parameters to be fixed so that we reproduce the scattering amplitude obtained in the partial-wave analysis.

The s- and u-channel $\Delta(1232)$ exchange term, $\mathcal{V}_\Delta$, is calculated with the Lagrangian [68],

$$\mathcal{L}_{\pi N \Delta} = \frac{g_{\pi N\Delta}}{m_\pi} \bar{\Delta}^\mu \vec{T}^\dagger (g_{\mu\nu} - \gamma_\mu \gamma_\nu) N \partial^\nu \vec{\pi} + (\text{H.c.}), \tag{B15}$$

with the bare $\pi N\Delta$ coupling constant $g_{\pi N\Delta}$ and the $1/2 \rightarrow 3/2$ isospin transition operator $\vec{T}$, which satisfies

$$T_b T_a^\dagger = \delta_{ba} - \tfrac{1}{3}\tau_b \tau_a. \tag{B16}$$

The spin-$\frac{3}{2}$ propagator with the momentum P^μ is

$$-i\frac{\not{P} + M_\Delta}{P^2 - M_\Delta^2}\left(g_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu - \frac{2P_\mu P_\nu}{3M_\Delta^2} + \frac{P_\mu\gamma_\nu - P_\nu\gamma_\mu}{3M_\Delta}\right), \tag{B17}$$

with the bare Δ mass M_Δ. Then the interaction $\mathcal{V}_\Delta$ is expressed as

$$\mathcal{V}_\Delta = A_\Delta(s,t) + \frac{1}{3}A_\Delta(u,t) + \frac{\not{R}}{2}\left[B_\Delta(s,t) - \frac{1}{3}B_\Delta(u,t)\right], \tag{B18}$$

with

$$A_\Delta(s,t) = -\frac{g_{\pi N\Delta}^2}{6M_\Delta^2 m_\pi^2 (s - M_\Delta^2)}\Big\{3M_\Delta^2(M_\Delta + M_N)\big(2M_N^2 - 2s - t + 2m_\pi^2\big) + 2M_\Delta\big[M_N^2\big(m_\pi^2 - 2s\big) + 2s^2 - m_\pi^2 s - m_\pi^4\big]$$
$$- M_N\big[M_N^4 - 2M_N^2\big(m_\pi^2 - s\big) - 3s^2 + 2m_\pi^2 s + m_\pi^4\big]\Big\}, \tag{B19}$$

$$B_\Delta(s,t) = -\frac{g_{\pi N\Delta}^2}{6M_\Delta^2 m_\pi^2 (s - M_\Delta^2)}\Big\{12M_\Delta^3 M_N + 3M_\Delta^2\big(4M_N^2 - t\big) + 2M_\Delta M_N\big(M_N^2 - m_\pi^2 - 5s\big)$$
$$- M_N^4 + 2M_N^2\big(m_\pi^2 - 3s\big) - \big(m_\pi^2 - s\big)^2\Big\}. \tag{B20}$$

In this interaction kernel, both the bare $\pi N\Delta$ coupling constant $g_{\pi N\Delta}$ and Δ mass M_Δ are model parameters.

As one can see, we have six model parameters altogether in the interaction kernel $\mathcal{V}$: the low-energy constants c_1, c_2, c_3, and c_4, the Δ bare mass M_Δ, and the $\pi N\Delta$ bare coupling constant $g_{\pi N\Delta}$. They, together with the subtraction constant of the loop function, are fixed so as to reproduce the scattering amplitude obtained in the partial-wave analysis, as explained in Sec. III A.

APPENDIX C: THE LOOP FUNCTION

In this Appendix we summarize formulas of the loop function with the angular momentum L, $G_L(w)$,

$$G_L(w) = i \int \frac{d^4 q}{(2\pi)^4}\frac{|\boldsymbol{q}|^{2L}}{(q^2 - m^2)[(P-q)^2 - M^2]}, \tag{C1}$$

where $P^\mu = (w, \mathbf{0})$ and m and M are the meson and baryon masses, respectively. By calculating the integrals with respect to q^0 and the solid angle of $\boldsymbol{q}$ and changing the integral variable from $|\boldsymbol{q}|$ to $s' = [\sqrt{\boldsymbol{q}^2 + m^2} + \sqrt{\boldsymbol{q}^2 + M^2}]^2$, we can rewrite the loop function as

$$G_L(w) = -\int_{s_{\text{th}}}^\infty \frac{ds'}{2\pi}\rho(s')\frac{q(s')^{2L}}{s' - s}, \tag{C2}$$

where $s \equiv w^2$, s_{th} is the threshold of the system, $s_{\text{th}} \equiv (m + M)^2$, and

$$\rho(s) \equiv \frac{q(s)}{4\pi\sqrt{s}}, \quad q(s) \equiv \frac{\lambda^{1/2}(s, m^2, M^2)}{2\sqrt{s}}, \tag{C3}$$

with the Källen function $\lambda(x, y, z)$. The integral in Eq. (C2) diverges logarithmically for $L = 0$ and worse for $L > 0$. However, we can "subtract" these divergences without changing analytic properties of the loop function. Namely, using an identity

$$\frac{1}{s' - s} = \frac{s - s_0}{(s' - s)(s' - s_0)} + \frac{1}{s' - s_0}, \tag{C4}$$

we can reduce the order of the divergence; the first term in the right-hand side of the identity (C4) reduces the order of s' of the integrand in Eq. (C2) by one, and the second term becomes a constant with respect to s after the integral. Iterating the substitution of this identity $L + 1$ times, we can derive the following expression of the loop function:

$$G_L(w) = -\sum_{m=0}^L \tilde{a}_m(s_0)(s - s_0)^m - (s - s_0)^{L+1}$$
$$\times \int_{s_{\text{th}}}^\infty \frac{ds'}{2\pi}\rho(s')\frac{q(s')^{2L}}{(s' - s)(s' - s_0)^{L+1}}. \tag{C5}$$

Now this integral converges. In the expression, $\tilde{a}_m$ ($m = 0, 1, \ldots, L$) is the subtraction constant at a certain scale s_0. An important point is that the loop function in Eq. (C5) has the

same analytic properties, i.e., the same dependence on w as that in Eq. (C2).

Next we consider two special cases, $L = 0$ and 1, which are relevant to our study in Sec. III. First the $L = 0$ case is well studied in various references. Utilizing the dimensional regularization for the loop function in Eq. (C1) with $L = 0$, we obtain the same structure as Eq. (C5) up to a constant

$$
G_0(w) = \frac{1}{16\pi^2} \left\{ a(\mu_{\rm reg}) + \ln\left(\frac{M^2}{\mu_{\rm reg}^2}\right) \right.
$$
$$
+ \frac{s + m^2 - M^2}{2s} \ln\left(\frac{m^2}{M^2}\right) - \frac{\lambda^{1/2}(s, m^2, M^2)}{s}
$$
$$
\left. \times \operatorname{artanh}\left[\frac{\lambda^{1/2}(s, m^2, M^2)}{m^2 + M^2 - s}\right] \right\}, \quad \text{(C6)}
$$

where $a(\mu_{\rm reg})$ is the subtraction constant at the regularization scale $\mu_{\rm reg}$. Because the loop function and scattering amplitude should finally scale independent, any change of the scale $\mu_{\rm reg}$ is absorbed by a change of the subtraction constant $a(\mu_{\rm reg})$ such that $a(\mu_{\rm reg}') - a(\mu_{\rm reg}) = \log(\mu_{\rm reg}'^2/\mu_{\rm reg}^2)$. For later convenience, here and in the following we fix the regularization scale as $\mu_{\rm reg} = M$ and write the loop function with $\mu_{\rm reg} = M$ as $G(w; a)$,

$$
G(w; a)
$$
$$
\equiv \frac{1}{16\pi^2} \left\{ a + \frac{s + m^2 - M^2}{2s} \ln\left(\frac{m^2}{M^2}\right) \right.
$$
$$
\left. - \frac{\lambda^{1/2}(s, m^2, M^2)}{s} \operatorname{artanh}\left[\frac{\lambda^{1/2}(s, m^2, M^2)}{m^2 + M^2 - s}\right] \right\}, \quad \text{(C7)}
$$

where a is the subtraction constant at $\mu_{\rm reg} = M$. Second, for the $L = 1$ case, we have to take into account another $q(s')^2$ factor in the integral. To this end we express $q(s)^2$ as

$$
q(s)^2 = \frac{\lambda(s, m^2, M^2)}{4s} = \frac{s}{4} - \frac{m^2 + M^2}{2} + \frac{(m^2 - M^2)^2}{4s}. \quad \text{(C8)}
$$

From this expression we can decompose $G_{L=1}(w)$ as

$$
G_1(w) = \sum_{n=1}^{3} H_n(w), \quad \text{(C9)}
$$

$$
H_1(w) = -\frac{1}{4} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s') \frac{s'}{s' - s}, \quad \text{(C10)}
$$

$$
H_2(w) = \frac{m^2 + M^2}{2} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s') \frac{1}{s' - s}, \quad \text{(C11)}
$$

$$
H_3(w) = -\frac{(m^2 - M^2)^2}{4} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s') \frac{1}{s'(s' - s)}. \quad \text{(C12)}
$$

To make them converge, we need two, one, and no subtractions for H_1, H_2, and H_3, respectively. The first term H_1 is calculated, with two subtraction constants $\tilde{b}_1$ and a_1, as

$$
H_1(w) = -\frac{1}{4} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s') \left(1 + \frac{s}{s' - s}\right)
$$
$$
= -\tilde{b}_1 + \frac{s}{4} G(w; a_1), \quad \text{(C13)}
$$

where a constant $\tilde{b}_1$ has been introduced so as to "renormalize" an infinite constant

$$
\tilde{b}_1 \equiv \frac{1}{4} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s'), \quad \text{(C14)}
$$

and the other integral corresponds to the $L = 0$ loop function $G(w; a_1)$, defined in Eq. (C7), with a subtraction constant a_1. Next, the second term H_2 is calculated in a similar manner to the $L = 0$ loop function as

$$
H_2(w) = -\frac{m^2 + M^2}{2} G(w; a_2), \quad \text{(C15)}
$$

where a subtraction constant a_2 has been introduced. Finally, the third term H_3 is calculated as

$$
H_3(w) = -\frac{(m^2 - M^2)^2}{4} \int_{s_{\rm th}}^{\infty} \frac{ds'}{2\pi} \rho(s') \frac{1}{s}\left(\frac{1}{s' - s} - \frac{1}{s'}\right)
$$
$$
= \frac{(m^2 - M^2)^2}{4s} [G(w; 0) - G(0; 0)], \quad \text{(C16)}
$$

where we have not introduced any subtraction constant because the integral in H_3 does not diverge. As a consequence, we obtain the $L = 1$ loop function $G_1(w)$ as

$$
G_1(w) = -\tilde{b}_1 + \frac{s}{4} G(w; a_1) - \frac{m^2 + M^2}{2} G(w; a_2)
$$
$$
+ \frac{(m^2 - M^2)^2}{4s} [G(w; 0) - G(0; 0)]. \quad \text{(C17)}
$$

In this expression, we have three subtraction constants $\tilde{b}_1$, a_1, and a_2, but one can easily see that $\tilde{b}_1$ and a_2 are not independent. Hence, we have two independent subtraction constants for $G_1(w)$.

In Sec. III A we require that the P_{11} unitarized partial wave of the πN scattering keeps the nucleon pole at the same position as in the interaction kernel, i.e., the physical nucleon mass M_N. This can be achieved with the condition that the loop function with the partial wave $L = 1$ vanishes at $w = M_N$. Thus, we require

$$
G_{\pi N, L=1}(M_N) = 0. \quad \text{(C18)}
$$

Then, with the condition (C18), we can eliminate one of the two independent subtraction constants in Eq. (C17). As a result, without loss of generality we can express the loop function $G_{\pi N, L=1}$ as

$$
G_{\pi N, L=1}(w; \tilde{A})
$$
$$
= \frac{s - M_N^2}{4} \tilde{A} + \frac{s G_{\pi N}(w)}{4} - \frac{m_\pi^2 + M_N^2}{2} G_{\pi N}(w)
$$
$$
+ \frac{(m_\pi^2 - M_N^2)^2}{4} \left[\frac{G_{\pi N}(w) - G_{\pi N}(0)}{s} + \frac{G_{\pi N}(0)}{M_N^2}\right], \quad \text{(C19)}
$$

where m_π is the pion mass and we have introduced the loop function $G_{\pi N}(w)$:

$$G_{\pi N}(w) = G(w; 0) - G(M_N; 0)$$

$$= \frac{1}{16\pi^2}\left\{\left(\frac{s + m_\pi^2 - M_N^2}{2s} - \frac{m_\pi^2}{2M_N^2}\right)\ln\left(\frac{m_\pi^2}{M_N^2}\right) - \frac{\lambda^{1/2}\left(s, m_\pi^2, M_N^2\right)}{s}\mathrm{artanh}\left[\frac{\lambda^{1/2}\left(s, m_\pi^2, M_N^2\right)}{m_\pi^2 + M_N^2 - s}\right]\right.$$

$$\left. + \frac{\lambda^{1/2}\left(M_N^2, m_\pi^2, M_N^2\right)}{s}\mathrm{artanh}\left[\frac{\lambda^{1/2}\left(M_N^2, m_\pi^2, M_N^2\right)}{m_\pi^2}\right]\right\}. \tag{C20}$$

which satisfies $G_{\pi N}(M_N) = 0$. We note that $G_{\pi N, L=1}$ contains one parameter $\tilde{A}$ as the remaining subtraction constant.

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