Entanglement in the states of the two-rotors model

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The eigenfunctions of the two-rotors model are superpositions of states corresponding to precessions of the rotors around two orthogonal axes. In the application of the model to a system of particles, such a structure becomes a coherent entanglement of many particles. In nuclear physics such an entanglement has not been directly confirmed. I show that it is possible to come to a definite conclusion about its existence by measuring the electromagnetic transition probabilities for the $J = 3$ member of the scissors-mode rotational band and for higher excited states with intrinsic energy twice that of the scissors mode. The present results are relevant to single-domain magnetic nanoparticles.

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I. INTRODUCTION

The two-rotors model (TRM) describes the dynamics of two rigid bodies rotating with respect to each other under an attractive force around their centers of mass fixed at one and the same point. It was devised as a model for deformed atomic nuclei, in which case the rigid bodies represent the proton and neutron systems [1]. The low-lying excited states predicted by this model were first observed [2] in the rare-earth nucleus ^{156}Gd and then in all deformed atomic nuclei [3] and were called scissors modes; see Fig. 1.

By analogy, similar collective excitations were predicted in several other systems [4] and as is well known they have been clearly observed in Bose-Einstein condensates [5]. Moreover, an application of the TRM to the evaluation of the magnetic susceptibility of single-domain magnetic nanoparticles stuck in rigid matrices has given results compatible with a vast body of experimental data with an agreement in some cases surprisingly good [6].

Figure 1, however, while very suggestive, does not give a complete representation of the TRM states, because the TRM Hamiltonian has a double-well potential and then at the classical level two states corresponding to the two minima. The present paper is devoted to the investigation of the consequences of this feature. To describe the problem it is necessary to define the model. I assume the two rotors to have axial symmetry [7]. The TRM Hamiltonian is then

$$H = \frac{1}{2\mathcal{I}_1}\vec{L}_1^2 + \frac{1}{2\mathcal{I}_2}\vec{L}_2^2 + V, \tag{1}$$

where $\vec{L}_1, \vec{L}_2$ are the angular momenta, $\mathcal{I}_1, \mathcal{I}_2$ are the moments of inertia of the two rotors with respect to the axes perpendicular to the symmetry axis, and V is the potential interaction between them. I assume the potential to be a function of the angle between the axes of the rotors. Denoting this angle by 2θ,

$$V = V[|\cos(2\theta)|]. \tag{2}$$

This potential is symmetric with respect to $\theta = \pi/4$ and has two degenerate minima at $\theta = 0, \pi/2$. So at the classical level the axes of the rotors will vibrate with respect to one another around these values of θ. At the quantum level the eigenfunctions will be superpositions of states describing these vibrations. To be definite, let us define an intrinsic frame of axes ξ, η, ζ:

$$\hat{\xi} = \frac{\hat{\zeta}_2 \times \hat{\zeta}_1}{2\sin\theta}, \quad \hat{\eta} = \frac{\hat{\zeta}_2 - \hat{\zeta}_1}{2\sin\theta}, \quad \hat{\zeta} = \frac{\hat{\zeta}_2 + \hat{\zeta}_1}{2\cos\theta}. \tag{3}$$

The eigenfunctions will be superpositions of states describing the precession of the proton and neutron axes around the ζ and the η axes [1]. Such superpositions are constrained by the condition that independent inversions of the orientation of the proton and neutron axes are not observable. In general, such constraint should be imposed on the absolute value of the wave functions [8]. In Ref. [1], however, they were enforced by requiring that the eigenfunctions, rather than their absolute value, should be invariant under these inversions, and I use here this restrictive requirement. As a consequence, they result in having the form schematically represented in Fig. 2.

As far as I know the actual occurrence of such an entanglement has never been directly confirmed in nuclear physics. I discuss this feature of the TRM in detail and I show how entanglement can be observed in atomic nuclei by studying the $J = 3$ member of the scissors-mode rotational band and states with higher intrinsic energy.

The determination of the eigenstates of the TRM requires the solution of the above-mentioned constraint that until now was worked out case by case. Here I present a rather general and practical procedure to do it. In this way I find that the solution for states with higher intrinsic energy used in a previous investigation [9] is not unique, as I incorrectly assumed.

In Sec. II I report the essentials of the TRM, in Sec. III I determine its eigenvalues and eigenstates, in Secs. IV and V I evaluate the electromagnetic (em) transition amplitudes in the scissors-mode rotational band and in the overtones respectively, in Sec. VI I discuss what we can learn about entanglement in atomic nuclei from new possible experiments, in Sec. VII I compare with other theoretical approaches, and finally in Sec. VIII I present my conclusions, including a conjecture concerning single domain magnetic nanoparticles [6]. In the Appendix I collect and derive some expressions of em operators. I set $\hbar = c = 1$.

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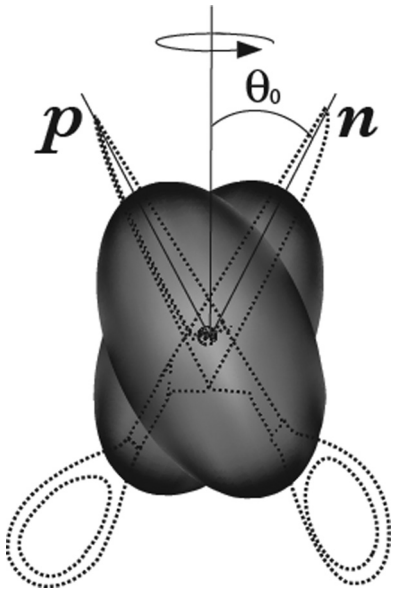

FIG. 1. Scissors modes in the two-rotors model: the proton (p) and neutron (n) rotors precess around the bisector of their axes.

II. THE TWO-ROTORS MODEL

The TRM Hamiltonian acts on the direction cosines of the rotor axes $\hat{\zeta}_1,\hat{\zeta}_2$. These variables can be replaced with the Euler angles α,β,γ that describe the orientation of the intrinsic frame plus the angle θ. The correspondence $\{\zeta_1,\zeta_2\} = \{\alpha,\beta,\gamma,\theta\}$ is one-to-one and regular for $0 < \theta < \pi/2$. *It is important to remember that this whole range of θ is necessary for the transformation to be one-to-one [10].*

Because of the axial symmetry the wave functions must satisfy the constraints

$$\hat{\zeta}_1 \cdot \vec{L}_1 = \hat{\zeta}_2 \cdot \vec{L}_2 = 0. \tag{4}$$

To get analytic results, these (weak) constraints on the wave functions were replaced with (strong) constraints on the operators and solved in terms of

$$\begin{aligned} \vec{L} &= \vec{L}_1 + \vec{L}_2 \\ \vec{S} &= \vec{L}_1 - \vec{L}_2 \end{aligned} \tag{5}$$

where

$$S_\xi = i\frac{\partial}{\partial\theta}, \quad S_\eta = -\cot\theta L_\zeta, \quad S_\zeta = -\tan\theta L_\eta. \tag{6}$$

Using the above change of variables, the TRM Hamiltonian becomes the sum of the rotational Hamiltonian of the two-rotors system as a whole plus an intrinsic Hamiltonian that in the reformulation of Ref. [10] reads

$$H = \frac{\vec{L}^2}{2\mathcal{I}} + H_{\rm intr}, \tag{7}$$

where $\mathcal{I} = \mathcal{I}_1\mathcal{I}_2/(\mathcal{I}_1+\mathcal{I}_2)$ and

$$\begin{aligned} H_{\rm intr} = \frac{1}{2\mathcal{I}}\left[\cot^2\theta L_\zeta^2 + \tan\theta^2 L_\eta^2 - \frac{\partial^2}{\partial\theta^2} - 2\cot(2\theta)\frac{\partial}{\partial\theta}\right] \\ + \frac{\mathcal{I}_1-\mathcal{I}_2}{4\mathcal{I}_1\mathcal{I}_2}\left[-\tan\theta L_\zeta L_\eta - \cot\theta L_\eta L_\zeta + iL_\zeta\frac{\partial}{\partial\theta}\right] + V. \end{aligned} \tag{8}$$

This Hamiltonian is invariant under separate inversions of the rotors axes. To define the action of such operators, I must write the unit length vectors $\hat{\zeta}_1,\hat{\zeta}_2$ in terms of the intrinsic and global variables:

$$\hat{\zeta}_1 = -\sin\theta\,\hat{\eta} + \cos\theta\,\hat{\xi}, \quad \hat{\zeta}_2 = \sin\theta\,\hat{\eta} + \cos\theta\,\hat{\xi}. \tag{9}$$

Then the inversion operators can be represented as

$$\mathcal{I}_{\zeta_1} = R_\zeta(\pi)R_\xi\left(\frac{\pi}{2}\right)R_\theta, \quad \mathcal{I}_{\zeta_2} = R_\eta(\pi)R_\xi\left(\frac{\pi}{2}\right)R_\theta, \tag{10}$$

where $R_\zeta(\pi), R_\eta(\pi), R_\xi(\frac{\pi}{2})$ are rotation operators around the intrinsic axes.

As I said, it was assumed that such inversions should leave the wave functions invariant. Invariance under separate inversions is equivalent to the conditions

$$\mathcal{I}_{\zeta_1}\mathcal{I}_{\zeta_2}\Psi = \Psi, \tag{11}$$

$$\mathcal{I}_{\zeta_1}\Psi = \Psi. \tag{12}$$

The range of θ can be separated into two regions,

$$s_{\rm I} = s(\theta)s\left(\frac{\pi}{4}-\theta\right), \quad s_{\rm II} = s\left(\frac{\pi}{2}-\theta\right)s\left(\theta-\frac{\pi}{4}\right), \tag{13}$$

where $s(x)$ is the step function: $s(x) = 1, x > 0$ and zero otherwise. The two regions are obtained from each other by the reflection of θ with respect to $\pi/4$. It is convenient to introduce the notation

$$R_\theta f(\theta) = f\left(\frac{\pi}{2}-\theta\right) = \overset{\circ}{f}(\theta), \tag{14}$$

so that $\overset{\circ}{s}_{\rm I} = s_{\rm II}$. With this notation, $\overset{\circ}{V} = V$.

The second term of $H_{\rm intr}$ is proportional to [10] $\theta_0\,|\mathcal{I}_1 - \mathcal{I}_2|/(4\mathcal{I}_1\mathcal{I}_2)$, where

$$\theta_0 = (\mathcal{I}C)^{-\frac{1}{4}} \tag{15}$$

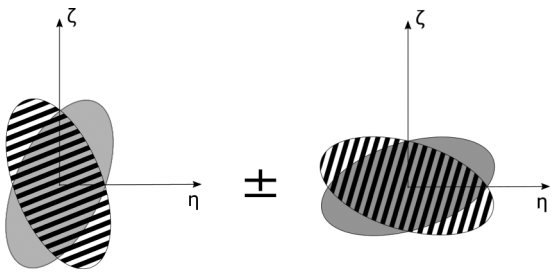

FIG. 2. The TRM Hamiltonian has a double-well potential, the two wells corresponding to the precession of the rotors axes around the ζ and η axes of the intrinsic frame. The eigenfunctions, therefore, are *necessarily a superposition of the states describing such precessions*. The requirement that they be invariant under inversion of the orientation of the neutron and proton axes, that is not observable, selects a definite superposition for each value of the total angular momentum.

is the zero-point oscillation parameter. It is therefore negligible for atomic nuclei (for which $|\mathcal{I}_1 - \mathcal{I}_2|/(4\mathcal{I}_1\mathcal{I}_2) \ll 1$ and $\theta_0 \sim 0.1$) but not for free nanoparticles (for which $|\mathcal{I}_1 - \mathcal{I}_2|/(4\mathcal{I}_1\mathcal{I}_2) \sim 1$ and $\theta_0 \leqslant 1$). I think, however, that the importance of the second term of $H_{\rm I}$ for different moments of inertia is attributable to the fact that the intrinsic frame I chose is not a principal frame, namely a frame in which the tensor of the moment of inertia of the two-rotors system is diagonal. I conjecture that in a principal system the second term will be small also for nanoparticles.

Neglecting the second term, the TRM Hamiltonian becomes then invariant also under the transformation

$$R = R_\xi\left(\frac{\pi}{2}\right)R_\theta. \tag{16}$$

Next I eliminate the linear derivative in the first term of $H_{\rm intr}$ by the transformation

$$(U\Phi)(\theta) = \frac{1}{\sqrt{2\sin(2\theta)}}\Phi'(\theta), \tag{17}$$

getting

$$H'_{\rm intr} = U H_{\rm intr} U^{-1} = \frac{1}{2\mathcal{I}}\left\{-\frac{d^2}{d\theta^2} - [2+\cot^2(2\theta)] + \cot^2\theta\, L_\zeta^2 + \tan^2\theta L_\eta^2\right\} + V(\theta). \tag{18}$$

At last I assume that the wave functions have such a fast falloff (which is completely justified in the case of nuclei) that I can perform the harmonic approximation for the potential and the circular functions,

$$\begin{aligned} &V \approx \frac{1}{2}C\,\theta^2, \quad S'_\xi = i\,\nabla_\theta, \quad S'_\eta = 0, \\ &S'_\zeta = -\frac{1}{\theta}L_\eta, \quad \text{in region I,} \end{aligned} \tag{19}$$

$$\begin{aligned} &V \approx \frac{1}{2}C\,\mathring{\theta}^2, \quad S'_\xi = -i\,\nabla_{\mathring{\theta}}, \quad S'_\eta = -\frac{1}{\mathring{\theta}}L_\zeta, \\ &S'_\zeta = 0, \quad \text{in region II,} \end{aligned} \tag{20}$$

where

$$\nabla_\theta = \frac{d}{d\theta} - \frac{1}{2\theta}. \tag{21}$$

I then write accordingly

$$H'_{\rm intr} \approx H_{\rm I}s_{\rm I} + H_{\rm II}s_{\rm II}, \tag{22}$$

where

$$\begin{aligned} H_{\rm I} &= \frac{1}{2}\omega\left[-\frac{d^2}{dx^2} + \frac{1}{x^2}\left(I_\zeta^2 - \frac{1}{4}\right) + x^2\right], \quad 0 \leqslant x \leqslant \frac{\pi}{4\theta_0}, \\ H_{\rm II} &= \frac{1}{2}\omega\left[-\frac{d^2}{d\,\mathring{x}^2} + \frac{1}{\mathring{x}^2}\left(I_\eta^2 - \frac{1}{4}\right) + \mathring{x}^2\right], \quad 0 \leqslant \mathring{x} \leqslant \frac{\pi}{4\theta_0}, \end{aligned} \tag{23}$$

with

$$x = \frac{\theta}{\theta_0}, \quad \mathring{x} = \frac{\mathring{\theta}}{\theta_0}, \tag{24}$$

$$\omega = \sqrt{\frac{C}{\mathcal{I}}}. \tag{25}$$

The harmonic approximation makes more evident that (18) is a double-well Hamiltonian, implying that in stationary states *the rotor axes oscillate simultaneously around the ζ and η axes*.

The eigenfunctions and eigenvalues of $H_{\rm I}$ are [10]

$$\varphi_{Kn}(x) = \sqrt{\frac{n!}{(n+K)!\,\theta_0}}\,x^{K+\frac{1}{2}}L_n^K(x^2)e^{-\frac{1}{2}x^2}, \tag{26}$$

$$\epsilon_{nK} = \omega(2n+K+1), \tag{27}$$

where L_n^K are Laguerre polynomials and the wave functions φ_{Kn} are normalized according to

$$\int_0^\infty dx[\varphi_{Kn}(x)]^2 = \frac{1}{2}. \tag{28}$$

Because in the harmonic approximation θ plays the role a a radius, I call n the radial quantum number.

In general, the eigenstates occur in doublets, whose energy splitting can be estimated with the WKB approximation

$$\delta E \approx E\exp\int_{-\theta(E)}^{\theta(E)}[-|p(\theta)|], \tag{29}$$

where $\theta(E)$ is the angle of inversion of the classical trajectory of energy E and $p(\theta)$ its conjugate momentum, $|p| = \sqrt{|2\mathcal{I}(E-V)|} \approx |\sin\theta|/\theta_0^2$. Because $\theta(E) \approx \theta_0$ for the states of interest,

$$\delta E \approx E\exp\left(-\frac{2}{\theta_0^2}\right). \tag{30}$$

For atomic nuclei in the rare-earth region $\theta_0^2 \sim 0.01$ and such energy splitting is to all effects negligible, but the situation is different for nanoparticles.

III. EIGENSTATES

I write the eigenfunctions in the form

$$\Psi_{IMmn} = \sum_{K\geqslant 0}\mathcal{F}^I_{MK}(\alpha,\beta,\gamma)\Phi^I_{mKn}(\theta), \tag{31}$$

where

$$\mathcal{F}^I_{MK} = \sqrt{\frac{2I+1}{16(1+\delta_{K0})\pi^2}}\left[\mathcal{D}^I_{MK} + (-1)^I\mathcal{D}^I_{M-K}\right]. \tag{32}$$

I, M, K are the nucleus angular momentum and its component on the z axis of the laboratory frame and the ζ axis of the intrinsic frame, and m is an additional quantum number to be specified in the sequel. Because all the states I consider have positive parity, I omit the parity quantum number. The combination of rotational matrices in the $\mathcal{F}$ is required by the condition (11). It remains to impose the condition (12).

The eigenstates are normalized according to

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\theta |\Psi_{IMmn}|^2 = 1. \quad (33)$$

The eigenstates of the Hamiltonian in region I are

$$\Psi^{(\mathrm{I})}_{L,M,K,n} = \mathcal{F}^I_{MK}(\alpha,\beta,\gamma)\varphi_{Kn}(\theta). \quad (34)$$

For each such eigenstate there is in region II the degenerate eigenstate

$$\Psi^{(\mathrm{II})}_{I,M,K,n} = \mathcal{G}^I_{M,K}\ \mathring{\varphi}_{K,n}, \quad (35)$$

where

$$I_\eta^2\, \mathcal{G}^I_{M,K} = K^2 \mathcal{G}^I_{M,K}. \quad (36)$$

The constraint (12) determines their amplitudes in the total eigenfunction.

When I express the $\mathcal{G}^I_{M,K}$ in terms of the $\mathcal{F}^I_{MK}$ the total eigenfunctions take the standard form (31). Notice that in region I the eigenstates have a unique component of I_ζ, while in region II they have all the components of I_ζ appearing in $\mathcal{G}^I_{M,K}$. The quantum number m is the component of the total angular momentum on the ζ axis in region I. Even if each of the rotors has axial symmetry, the two-rotors system does not have it, so that the component of angular momentum along any intrinsic axis is not conserved, resulting in a superposition of states with different K-quantum number.

To impose the constraint (12), I must determine the action of $R_\eta(\pi)$ and $R_\xi(\pi/2)$ on the $\mathcal{F}^I_{M,K}$ and the $\mathcal{G}^I_{M,K}$. For any component of $\hat{I}_k, k = \xi,\eta,\zeta$,

$$\exp(i\alpha\hat{I}_k) = i\frac{\hat{I}_k}{I_k}\sin(I_k\alpha) + \cos(I_k\alpha), \quad (37)$$

so that

$$\exp(i\pi\hat{I}_k)\psi_{I_k} = (-)^{I_k}\psi_{I_k},$$
$$\exp(i\pi/2\,\hat{I}_k)\psi_{I_k} = \left[i\frac{\hat{I}_k}{I_k}\sin(I_k\pi/2) + \cos(I_k\pi/2)\right]\psi_{I_k}. \quad (38)$$

Notice that the transformations in the last equation are simpler for I_k even.

To find the action of $R_\xi(\pi/2)$ on the $\mathcal{F}^I_{M,K}$ and the $\mathcal{G}^I_{M,K}$ I express these functions in terms of the eigenstates of $\hat{I}^2_\xi$

$$I_\xi^2\, \mathcal{K}^I_{M,K} = K^2 \mathcal{K}^I_{M,K}. \quad (39)$$

Because, as noted above, such an action is simpler for even values of K, it is convenient to express all the $\mathcal{F}^I_{M,K}$ and the $\mathcal{G}^I_{M,K}$ for K even and odd in terms of the $\mathcal{K}^I_{M,K}$ with even K.

IV. THE SCISSORS-MODE ROTATIONAL BAND

For the discussion of entanglement it is necessary to separate the contributions coming from regions I and II. To this end I introduce the parameters $r_{\mathrm{I}}, r_{\mathrm{II}}$ that in the TRM take the values

$$r_{\mathrm{I}} = r_{\mathrm{II}} = 1, \quad \text{in the TRM.} \quad (40)$$

A general feature is that the intraband magnetic transition amplitudes vanish, because they are proportional to

$$\langle\varphi_{1,0}|\nabla_\theta|\varphi_{1,0}\rangle = \langle\varphi_{1,0}|\frac{1}{\theta}|\varphi_{1,0}\rangle = 0. \quad (41)$$

A. The bandhead

The bandhead, the scissors mode, is a pure $K = 1$ state. Its wave function and transition amplitude are well known [1] but are reported for the sake of completeness,

$$\Psi_{1M1,0} = \mathcal{F}^1_{M1}\Phi^1_{1,1,0}, \quad (42)$$

where

$$\Phi^1_{1,1,0} = \varphi_{1,0} - \mathring{\varphi}_{1,0}. \quad (43)$$

The transition amplitude to the ground state is

$$\langle\Psi_{1M1,0}|\mathcal{M}(M1;\mu)|\Psi_{0,0,0,0}\rangle = \frac{i}{2\sqrt{3}}\frac{1}{\theta_0}\mathcal{M}(M1)C^{1M}_{001\mu}(r_{\mathrm{I}} + r_{\mathrm{II}}), \quad (44)$$

where the expression of

$$\mathcal{M}(M1) = \sqrt{\frac{3}{4\pi}}\frac{e}{2m} \quad (45)$$

is rederived in the Appendix.

B. The $J = 2$ member of the band

The $J = 2$ member of the band is also a pure $K = 1$ state, and its wave function and transition amplitude are also well known [1] but are reported for the sake of completeness

$$\Psi_{2M1,0} = \mathcal{F}^2_{M1}\Phi^2_{1,1,0}, \quad (46)$$

where

$$\Phi^2_{1,1,0} = \varphi_{1,0} + \mathring{\varphi}_{1,0}. \quad (47)$$

Its transition amplitude to the ground state is

$$\langle\Psi_{2M,1,0}|\mathcal{M}(E2;\mu)|\Psi_{0,0,0,0}\rangle = -ie\,Q_{20}\frac{1}{4}\theta_0\ C^{2M}_{002\mu}\,(r_{\mathrm{I}} + r_{\mathrm{II}}), \quad (48)$$

where Q_{20} is the quadrupole moment in the intrinsic frame.

C. The $J = 3$ member

The wave function of the $J = 3$ member is determined in the present paper. It can be written

$$\Psi_{3M1,0} = c\,\mathcal{F}^3_{M1}\,\varphi_{1,0} + s\,\mathcal{G}^3_{M1}\,\mathring{\varphi}_{1,0}, \quad c^2 + s^2 = 1, \quad (49)$$

where

$$\mathcal{G}^3_{M1} = \frac{1}{4}\left(\mathcal{F}^3_{M1} + \sqrt{15}\,\mathcal{F}^3_{M3}\right). \quad (50)$$

The eigenfunctions of $\hat{I}^2_\xi$ with eigenvalues 0,4, respectively, are

$$\mathcal{K}^3_{M0} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\,\mathcal{F}^3_{M1} - \sqrt{5}\,\mathcal{F}^3_{M3}\right),$$
$$\mathcal{K}^3_{M2} = \frac{1}{2\sqrt{2}}\left(\sqrt{5}\,\mathcal{F}^3_{M1} + \sqrt{3}\,\mathcal{F}^3_{M3}\right). \quad (51)$$

Expressing $\mathcal{F}^3_{M1}$ and $\mathcal{G}^3_{M1}$ in terms of $\mathcal{K}^3_{M0}$ and $\mathcal{K}^3_{M2}$ and imposing the constraint (12), I get

$$\Phi^3_{1,1,0} = \varphi_{1,0} + \frac{1}{4}\,\mathring{\varphi}_{1,0},$$
$$\Phi^3_{1,3,0} = \frac{\sqrt{15}}{4}\,\mathring{\varphi}_{1,0}. \tag{52}$$

Written in the standard form (31)

$$\Psi_{3M1,0} = \mathcal{F}^3_1\Big(\varphi_{10} + \frac{1}{4}\,\mathring{\varphi}_{10}\Big) + \frac{\sqrt{15}}{4}\mathcal{F}^3_3\,\mathring{\varphi}_{10}\,. \tag{53}$$

One can see how the intrinsic structure of the two-rotors system changes in the band with the angular momentum, with a strong departure from a rigid rotor.

The nonvanishing electromagnetic transition amplitudes are

$$\begin{aligned}
&\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{2M'1,0}\rangle \\
&\quad = e\,Q_{20}\frac{1}{\sqrt{7}}\,C^{3M}_{2M'2\mu}\langle\varphi_{10}|\varphi_{10}\rangle\Big(r_{\rm I} - \frac{5}{4}r_{\rm II}\Big), \\
&\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{1M'1,0}\rangle \\
&\quad = e\,Q_{20}\sqrt{\frac{3}{7}}\,C^{3M}_{1M'2\mu}\langle\varphi_{10}|\varphi_{10}\rangle(0.63\,r_{\rm I} - 0.5\,r_{\rm II}), \\
&\langle\Psi_{3M1,0}|\mathcal{M}(M3;\mu)|\Psi_{0,0,0,0}\rangle \\
&\quad = i\mathcal{M}(M3)\sqrt{\frac{2}{7}}\,C^{3M}_{003\mu}\,C^{31}_{0031}\langle\varphi_{10}|\nabla|\varphi_{00}\rangle\Big(r_{\rm I} - \frac{1}{4}\,r_{\rm II}\Big),
\end{aligned} \tag{54}$$

where the expression of

$$\mathcal{M}(M3) = -\frac{3}{20}\sqrt{\frac{42}{\pi}}R_3^2\left[1 - \frac{1}{3}\left(\frac{R_1}{R_3}\right)^3\right]\left[1 - \left(\frac{R_1}{R_3}\right)^2\right] \tag{55}$$

is derived in the Appendix.

V. OVERTONES

In previous papers [9,11] I studied the states of intrinsic energy $2E_{\rm scissors}$, called first overtones because of the harmonic approximation. I know that, in general, in nuclear physics one can trust collective models at most for the lowest excitation. Nevertheless, I considered it worthwhile to investigate the first overtones for two reasons. First, their excitation energy falls below the threshold for neutron emission and, therefore, their width is of purely electromagnetic nature, which might make their existence plausible, in spite of the fragmentation of the scissors mode. Second, their electric quadrupole transition amplitude is of zero order [9] in θ_0 and therefore much greater than that of the $J = 2$ member of the scissors rotational band that is of order θ_0.

I reconsider now these states by using the present method of solving the constraint (12).

The state $I = 0 = m = 0$, $n = 1$ cannot be excited by electromagnetic radiation, and for this reason it was called the elusive overtone [11]. The same is true for the state $I = 1$, $m = 0$, $n = 1$.

The states $I = 2$, $m = 0$, $n = 1$ and $I = 2$, $m = 2$, $n = 0$ are degenerate and their wave functions in regions I and II are

$$s_{\rm I}\,\Psi_{2,M,0,1} = \mathcal{F}^2_{M0}\,\varphi_{0,1},$$
$$s_{\rm I}\,\Psi_{2,M,2,0} = \mathcal{F}^2_{M2}\,\varphi_{2,0}, \tag{56}$$

$$s_{\rm II}\,\Psi^{\rm II}_{2M2,0} = \mathcal{G}^2_{M2}\,\mathring{\varphi}_{2,0},$$
$$s_{\rm II}\,\Psi^{\rm II}_{2M0,1} = \mathcal{G}^2_{M0}\,\mathring{\varphi}_{0,1}, \tag{57}$$

where

$$\mathcal{G}^2_{M0} = \frac{1}{2}\big(\mathcal{F}^2_{M0} + \sqrt{3}\,\mathcal{F}^2_{M2}\big),$$
$$\mathcal{G}^2_{M2} = \frac{1}{2}\big(\mathcal{F}^2_{M0} - \mathcal{F}^2_{M2}\big). \tag{58}$$

The eigenfunctions of $\hat{I}^2_\xi$ with eigenvalues 0,4, respectively, are

$$\mathcal{K}^2_{M0} = \frac{1}{2}\big(\mathcal{F}^2_{M0} - \sqrt{3}\,\mathcal{F}^2_{M2}\big),$$
$$\mathcal{K}^2_{M2} = \frac{1}{2}\big(\sqrt{3}\,\mathcal{F}^2_{M0} + \mathcal{F}^2_{M2}\big). \tag{59}$$

Expressing the $\mathcal{F}^2_{MK}$ and $\mathcal{G}^2_{MK}$ in terms of $\mathcal{K}^2_{M0}$ and $\mathcal{K}^2_{M2}$ and imposing the constraint (12), I find

$$\Phi^2_{0,0,1} = \varphi_{0,1} - \frac{1}{2}\,\mathring{\varphi}_{0,1},$$
$$\Phi^2_{0,2,1} = -\frac{\sqrt{3}}{2}\,\mathring{\varphi}_{0,1},$$
$$\Phi^2_{2,0,0} = -\frac{\sqrt{3}}{2}\,\mathring{\varphi}_{2,0}, \tag{60}$$
$$\Phi^2_{2,2,0} = \varphi_{2,0} + \frac{1}{2}\,\mathring{\varphi}_{2,0}.$$

The state $\Psi_{2M0,1}$ can be regarded as a member of the rotational band over the elusive overtone $\Psi_{0,0,0,1}$. Its quadrupole transition amplitude to the ground state vanishes. Because, as the bandhead, it cannot be excited from the ground state, I do not discuss it any further (even though if reached from above, it could decay to the scissors mode).

The nonvanishing electromagnetic transition amplitudes of the state $\Psi_{2M2,0}$ are

$$\begin{aligned}
&\langle\Psi_{2M2,0}|\mathcal{M}(E2;\mu)|\Psi_{0,0,0,0}\rangle \\
&\quad = eQ_{20}\frac{1}{4}\sqrt{\frac{3}{10}}\,C^{2M}_{002\mu}r_{\rm II}, \\
&\langle\Psi_{2M2,0}|\mathcal{M}(M1;\mu)|\Psi_{1,M'1,0}\rangle \\
&\quad = i\sqrt{\frac{3}{5}}\frac{1}{4\theta_0}\mathcal{M}(M1)C^{2M}_{1M'1\mu}(r_{\rm I} + r_{\rm II}).
\end{aligned} \tag{61}$$

The transition strengths are

$$B(E2)\uparrow_{\rm overtone} = \frac{1}{32\,\theta_0^2}\,\frac{4r_{\rm II}^2}{(r_{\rm I} + r_{\rm II})^2}\,B(E2)\uparrow_{\rm scissors},$$
$$B(M1;\text{overtone} \to \text{scissors}) = \frac{1}{7}\,B(M1)\uparrow_{\rm scissors}. \tag{62}$$

In the quoted investigation of overtones [9] I did not find the present (60), most general solution of the constraint (12), but the particular solution

$$\frac{1}{\sqrt{2}}(\Psi_{2M2,0}+\Psi_{2M0,1}), \tag{63}$$

which was incorrectly assumed to be unique, and the em transition amplitudes were evaluated accordingly. I notice that with the TRM values of the parameters $r_{\rm I}$,$r_{\rm II}$, the electric quadrupole transition strength of the overtone $\Psi_{2M2,0}$ is a factor 2 larger than that of the above state while the magnetic dipole transition strength is a factor $4/7$ smaller.

VI. ENTANGLEMENT

The TRM gives distinctive predictions that should enable us to reach a definite conclusion concerning the existence of entanglement in atomic nuclei.

To be definite, I compare the predictions of the TRM with those of a reference model that does not have entanglement. This reference model is what is often regarded to be *the* TRM as derived from microscopic models [12–14]. It is the intrinsic Hamiltonian $H_{\rm I}$ with the understanding that it acts on intrinsic wave functions defined and normalized in the whole range $0<\theta<\pi/2$. The reference model has the same eigenvalues as the TRM. Unlike the TRM, it has axial symmetry and obviously describes a precession around the ζ axis only. Its eigenfunctions are pure K states and can be obtained from the eigenfunctions of the TRM setting $\mathring{\varphi}_{Kn}=0$ and $\langle\varphi_{Kn}|\varphi_{Kn}\rangle=1$. The transition amplitudes can be obtained from the expressions relative to the TRM by setting

$$r_{\rm I}=2,\ \ r_{\rm II}=0\,,\ \text{in the reference model.} \tag{64}$$

I discuss the entanglement separately for the different states.

A. Entanglement in the scissors rotational band

1. *Entanglement in the $J=1,2$ members of the band*

The em transition amplitudes of these states are the same in the TRM and in the reference model, because they are proportional to $r_{\rm I}+r_{\rm II}$, a quantity that takes the same value in both models. Therefore, one cannot learn anything about entanglement from their comparison with experiment.

2. *Entanglement in the $J=3$ member of the band*

Let us denote by RM and TRM the transition amplitudes for the reference model and the TRM, respectively. Then

$$\begin{aligned}
&\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{2M'1,0}\rangle^{\rm RM}\\
&\quad=-8\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{2M'1,0}\rangle^{\rm TRM},\\
&\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{1M'1,0}\rangle^{\rm RM}\\
&\quad=9.7\langle\Psi_{3M1,0}|\mathcal{M}(E2;\mu)|\Psi_{1M'1,0}\rangle^{\rm TRM},\\
&\langle\Psi_{3M1,0}|\mathcal{M}(M3;\mu)|\Psi_{0,0,0,0}\rangle^{\rm RM}\\
&\quad=\frac{8}{3}\langle\Psi_{3M1,0}|\mathcal{M}(M3;\mu)|\Psi_{0,0,0,0}\rangle^{\rm TRM}.
\end{aligned} \tag{65}$$

One can see that the amplitudes for decay of the $J=3$, $m=1$, $n=0$ state to the lower members of the band are depressed by large factors in the TRM with respect to the reference model. Equations (54) show that this is attributable to destructive interference between the contributions from regions I and II. The difference in strengths is so large that if this member of the band can be observed, one should be able to reach a definite conclusion about entanglement.

B. Entanglement in the first overtones

The electric quadrupole amplitude for decay of the overtone $\Psi_{2M,2,0}$ to the ground state *vanishes in the absence of entanglement*. The relation between the magnetic dipole transition amplitudes in the reference model and the TRM is

$$\begin{aligned}
&\langle\Psi_{2M2,0}|\mathcal{M}(M1;\mu)|\Psi_{0,0,0,0}\rangle^{\rm RM}\\
&\quad=1.7\langle\Psi_{2M2,0}|\mathcal{M}(M1;\mu)|\Psi_{0,0,0,0}\rangle^{\rm TRM}.
\end{aligned} \tag{66}$$

Observation of the magnetic transition in the absence of the electric decay would give strong support to the absence of entanglement. Obviously on the contrary, observation of both transitions with the strengths (62) would be evidence in favor of it.

In a recent experiment the deformed nucleus ^{156}Gd, where the scissors mode has been discovered initially [2], has been studied by a high-resolution nuclear resonance fluorescence experiment at the superconducting Darmstadt linear electron accelerator up to 7 MeV of excitation energy. "A single candidate with the following characteristics a) a ground state decay indicating a quadrupole radiation, and b) simultaneously a significant branch to the main fragment of the scissors mode at 3 MeV has not been found above the detection limit" [15].

For an assessment of the realization in nature of the first overtone and its entanglement, it is crucial to put the above findings in relation with the present estimate of its decay strength to the scissors mode. Indeed, such a strength is not so large and in the comparison with experiment it should be reduced by a factor equal to the percentage of the total strength carried by the main fragment of the scissors mode.

VII. OTHER THEORETICAL APPROACHES

There is a copious literature on the scissors modes, in which, however, entanglement never appears explicitly. Therefore, it is sufficient for me to examine schematically how one could investigate entanglement in the different approaches. For this purpose, I can schematically divide them into two categories.

In the first one, following different procedures, one derives a collective Hamiltonian that has an eigenstate with the quantum numbers of the scissors mode and approximately the same excitation energy. However, there can be some important subtleties that I illustrate by two examples. One is provided by the interacting boson model [16]. It has been shown [12] that in the coherent states approximation, for small vibrations of the rotor axes around the ζ axis, it reproduces the intrinsic part $H_{\rm I}$ of the TRM Hamiltonian. I think that the vibrations around the η axis are also present in the interacting boson approximation Hamiltonian and that in the coherent-state approximation they should provide the Hamiltonian $H_{\rm II}$, but this remains to be verified. I must notice, however, that

in calculations done with the interacting boson model one does not use the coherent-state approximation, but rather other approximations assuming explicit symmetries of the wave functions. In a comparison with the TRM one has to check whether and how the invariance under inversion of the orientation of the rotors axes has been implicitly implemented and whether the assumed symmetries imply, for instance, axial symmetry, which would eliminate the entanglement altogether.

Another relevant example is the recent analytical approach to rotational states [14], in which the TRM Hamiltonian has been derived in the form (1). This paper is especially interesting in this context, because in the derivation of the collective Hamiltonian, as far as I understand, entanglement has not been enforced explicitly, and then also the Hamiltonian of the reference model should be a possible outcome. A clarification of this point is of the highest consequence for a strict connection between a many-body Hamiltonian and the TRM.

In conclusion, one must be sure of which conditions concerning invariance under inversion of the rotor axes are explicitly or implicitly set on the wave functions in the course of the derivation.

The second category includes model or microscopic calculations in which a collective state appears that can be interpreted as the scissors mode. The RPA, for instance, reproduces at a semiquantitative level the eigenvalues and the em strengths of the TRM for scissors modes [17]. A recent approach, the Wigner function moments method [13], also belongs to this class.

In all the works belonging to this category, however, to my knowledge, the resulting collective modes have not been analyzed in relation to the entanglement.

All the theoretical approaches of which I am aware are restricted to the lowest scissors excitation. This is justified by the fact that, in general, collective models in nuclear physics can be trusted at most for the first excited state. I notice, however, that this does not need to be an absolute rule, and indeed it is not true for all systems. For instance, in the evaluation of the magnetic susceptibility of single-domain magnetic nanoparticles using the TRM, all the excited states appear and contribute [6]. The important point is whether the rotors actually behave as rigid bodies at the energy of the collective state of interest, namely whether the coupling between intrinsic and collective degrees of freedom is or is not important. A general criterion can be found in Ref. [18]. However, for higher states this point can be more efficiently investigated in a constructive way, introducing in a many-body Hamiltonian a number of collective variables with an equal number of constraints to not change the effective number of degrees of freedom. In a variant of such a method one can avoid explicit constraints that make the calculations awkward by modifying the microscopic Hamiltonian in such a way as to push the spurious excitations associated with the redundant variables out of the part of the spectrum one is interested in. Such a method was used long ago to enforce translational invariance [19] in shell-model calculations and exploited to introduce collective rotations [20]. The latter application might be extended to the physics of the TRM by introducing the collective variable θ in addition to the Euler angles.

VIII. CONCLUSION

The wave functions of the TRM have a peculiar entanglement. In applications of the model to many-body systems this becomes a coherent entanglement of many particles of which I do not know other examples.

In nuclear physics with the present data there is no evidence in favor or against it, and the only check I can envisage is to compare the mass density distribution of the states in which the scissors mode is fragmented with that predicted by the TRM.

I have shown, however, that significant pieces of information can be obtained from the study of higher excited states. I hope that a definitive assessment concerning the first overtone will come soon [15]. The other crucial investigation concerns the $J = 3$ member of the scissors rotational band. If such a state is realized in nature and can be observed, one has enough distinctive predictions to identify it.

It is interesting to consider the application of the TRM to single-domain magnetic nanoparticles. These objects consist of a magnetic structure, called macrospin, that rotates with respect to a nonmagnetic lattice. They have been represented as a couple of rigid rotors, one associated with the nonmagnetic lattice, and the other one, with a spin attached, with the macrospin [6]. The macrospin has usually two stable orientations antiparallel to each other, separated by an energy barrier. At finite temperature there is a finite probability for the magnetization to flip and reverse its orientation. The double-well potential, at variance with the case of atomic nuclei in which it might appear an artifact, is in this case at the basis of the dynamics. There is a strong, even though indirect evidence of the validity of the TRM for nanoparticles stuck in rigid matrices [6]. I think that a direct check of the entanglement predicted by the TRM is possible by measuring the magnetic susceptibility of free nanoparticles at temperatures of the order of 1 K.

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APPENDIX: ELECTROMAGNETIC OPERATORS

The magnetic multipole operator in the intrinsic frame is

$$\mathcal{M}'(Ml,\mu) = \frac{e}{mc}\frac{1}{V_{\text{nucleus}}}\int d\vec{r}\, S'_k \frac{\partial}{\partial x_k}(r^l Y_{l\mu}), \qquad \text{(A1)}$$

where V_{nucleus} is the nuclear volume and the operators S'_k in the intrinsic frame are given in Eqs. (20). I found that the terms S'_η, S'_ζ do not contribute to the transitions of the states I consider, and I ignore them. Working out the above equation I then get the expression of the magnetic dipole (already well

known) and octupole operators in the laboratory frame

$$\mathcal{M}(M1,\mu) = -\mathcal{M}(M1)\frac{1}{\sqrt{2}}\left(D^1_{\mu 1} - D^1_{\mu -1}\right) i\left(\nabla_\theta s_{\rm I} - \nabla_{\mathring{\theta}} s_{\rm II}\right),$$
$$\mathcal{M}(M3,\mu) = \mathcal{M}(M3)\frac{1}{\sqrt{2}}\left(D^3_{\mu 1} - D^3_{\mu -1}\right) i\left(\nabla_\theta s_{\rm I} - \nabla_{\mathring{\theta}} s_{\rm II}\right), \tag{A2}$$

where

$$\mathcal{M}(M1) = \sqrt{\frac{3}{4\pi}}\frac{e}{2m},$$
$$\mathcal{M}(M3) = -\frac{3}{20}\sqrt{\frac{42}{\pi}}R_3^2\left[1-\frac{1}{3}\left(\frac{R_1}{R_3}\right)^3\right]\left[1-\left(\frac{R_1}{R_3}\right)^2\right]\frac{e}{2mc}. \tag{A3}$$

R_3, R_1 are the lengths of the semiaxes of the ellipsoids. In the evaluation of transition amplitudes I will need the matrix elements

$$\langle\varphi_{20}|\nabla_\theta|\varphi_{10}\rangle = -\frac{1}{2\sqrt{2}}\frac{1}{\theta_0}, \quad \langle\varphi_{10}|\nabla_\theta|\varphi_{00}\rangle = -\frac{1}{2}\frac{1}{\theta_0}. \tag{A4}$$

The electric quadrupole operator in the laboratory frame was evaluated in [1,9]

$$\mathcal{M}(E2,\mu) = e\,Q_{20}\left[\mathcal{D}^2_{\mu 0}\left(s_{\rm I} - \frac{1}{2}s_{\rm II}\right) + \frac{1}{2}\sqrt{\frac{3}{2}}\left(\mathcal{D}^2_{\mu 2} + \mathcal{D}^2_{\mu -2}\right)s_{\rm II}\right]$$
$$- i\frac{\sqrt{5}}{2}e\,Q_{20}\left(\theta s_{\rm I} + \mathring{\theta}\, s_{\rm II}\right)\left(\mathcal{D}^2_{\mu 1} + \mathcal{D}^2_{\mu -1}\right), \tag{A5}$$

where $e\,Q_{20}$ is the electric quadrupole moment in the intrinsic frame. Notice that the first line is of zero order in θ while the second line is of order θ. In the evaluation of transition amplitudes I need the matrix element

$$\langle\varphi_{20}|\varphi_{00}\rangle = \frac{1}{2\sqrt{2}}. \tag{A6}$$

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