Quasiparticle theory of transport coefficients for hadronic matter at finite temperature and baryon density

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We develop a flexible quasiparticle theory of transport coefficients of hot hadronic matter at finite baryon density. We begin with a hadronic quasiparticle model which includes a scalar and a vector mean field. Quasiparticle energies and the mean fields depend on temperature and baryon chemical potential. Starting with the quasiparticle dispersion relation, we derive the Boltzmann equation and use the Chapman-Enskog expansion to derive formulas for the shear and bulk viscosities and thermal conductivity. We obtain both relaxation-time approximation formulas and more general integral equations. Throughout the work, we explicitly enforce the Landau-Lifshitz conditions of fit and ensure the theory is thermodynamically self-consistent. The derived formulas should be useful for predicting the transport coefficients of the hadronic phase of matter produced in heavy-ion collisions at the Relativistic Heavy Ion Collider and at other accelerators.

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I. INTRODUCTION

A central challenge in nuclear physics is elucidating the structure of the quantum chromodynamics (QCD) phase diagram. Based on theoretical models, it is widely believed that the phase diagram contains a line of first-order phase transition that ends at a point of second-order phase transition: the critical point [1,2]. Despite a dedicated search for the critical point with the first beam energy scan at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Lab and possible hints of the critical point [3], the location of the critical point remains a mystery. From lattice QCD calculations [4–8], we know that the transition from hadrons to quarks and gluons is an analytic crossover near temperature $T \approx 150$ MeV at zero baryon chemical potential μ_B. Hence, the critical point is generally thought to be located at $T < 160$ MeV and μ_B equal to several hundreds of MeV.

A second, future beam energy scan at RHIC will search for the critical point with greatly increased statistics and upgraded detectors [9]. To maximize the discovery potential, experimental efforts must be accompanied by complementary improvements in theoretical modeling of QCD matter at moderate temperatures and large baryon chemical potentials. In previous papers, we investigated the equation of state at finite baryon chemical potential [10,11]. In this work, we derive new formulas to compute the shear and bulk viscosities and thermal conductivity of hot hadronic matter with $\mu_B > 0$. We employ a flexible, thermodynamically consistent framework of hadronic quasiparticles with medium-dependent quasiparticle masses and with a scalar and vector mean field. This may be considered a natural extension of Ref. [12] to include nonzero baryon chemical potential and the concomitant vector mean field.

Transport coefficients like the shear and bulk viscosities and thermal conductivity are especially interesting quantities to study for several reasons. First, the temperature and chemical potential dependence of transport coefficients may reveal the location of phase transitions: In many physical systems, the shear viscosity is a minimum and the bulk viscosity a maximum at the phase transition [13]. A second motivation is investigating the Kovtun-Son-Starinets lower bound [14] on the shear viscosity to entropy density $\eta/s \geqslant 1/4\pi$ for strongly coupled conformal theories and its implications for QCD. Furthermore, low viscosity may lead to observable turbulent instabilities [15] in heavy-ion collisions. Finally, transport coefficients are essential theoretical inputs for hydrodynamic simulations. While hydrodynamic simulations may neglect fundamental features of heavy-ion collisions, such as rotation and turbulent instabilities, such simulations have nevertheless proven themselves invaluable tools for interpreting heavy-ion collision data. In hydrodynamic simulations, the shear and bulk viscosities influence various observables, such as the elliptic flow coefficients v_n and the hadron transverse momentum (p_T) spectrum [16–19].

In principle, the transport coefficients can be computed directly from QCD using the Kubo formulas [20]. However, QCD is strongly coupled at energies accessible to heavy-ion collision experiments, complicating first-principles calculations. There were some early attempts to employ lattice QCD [21,22], but even today it is challenging to achieve a large-enough grid with a small-enough grid spacing to accurately compute transport coefficients. Furthermore, lattice QCD simulations are currently very difficult at finite baryon chemical potential owing to the well-known fermion sign problem. Hence, many of the early works [23–26] computed transport coefficients of quark-gluon plasmas or hadronic gases with a few species of particles using the Boltzmann equation in the relaxation-time approximation. These early works did not include mean fields or medium-dependent masses.

Later on, Jeon [27] and Jeon and Yaffe [28] computed the shear and bulk viscosities of a hot, weakly coupled scalar field theory using perturbation theory. Amazingly, they showed that their complicated perturbative calculation of transport

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coefficients was reproduced by a simpler kinetic theory of quasiparticles with temperature-dependent masses and a scalar mean field. The same conclusion was found for hot, weakly coupled QCD and QED [29–36]. This was also consistent with an earlier analysis of transport in a nucleon plus σ meson system, which similarly found that renormalized quasiparticle masses were required [37]. Though astounding, this makes intuitive sense: Kinetic theory is widely used to model nonequilibrium systems, and renormalized particle masses are ubiquitous in finite-temperature field theories. (They are also present in Fermi liquid theory [38].) Also, temperature- and chemical potential-dependent masses allow quasiparticle models to generate more realistic, nonideal-gas, equations of state [39]. Furthermore, as Gorenstein and Yang pointed out [40], the scalar mean field is essential for maintaining thermodynamic self-consistency when masses depend on temperature and/or chemical potential. Hence, it seems that kinetic theories of quasiparticles with medium-dependent masses and mean fields are powerful theoretical tools, though thermodynamic consistency must be carefully maintained.

More recently, the conjecture of a lower bound on η/s by Kovtun, Son, and Starinets from the anti-de Sitter/conformal field theory correspondence [14] ignited a flurry of additional work. There were several more lattice calculations [41–44]. There were also many studies with Boltzmann equations, most of them without medium-dependent masses or mean fields. Shear viscosity was computed for pion-nucleon gases at low temperatures and varying chemical potentials in Refs. [45,46]. The bulk viscosity of cool pion gases was computed using chiral perturbation theory in Refs. [46,47]. Shear viscosity in mixtures of hadrons with excluded volumes were calculated in Refs. [48–50].

There were a few attempts to employ the more powerful quasiparticle models with medium-dependent masses to compute transport coefficients. In an early work, Sasaki and Redlich applied kinetic theory and the relaxation-time approximation to a quasiparticle model to compute the bulk viscosity near a chiral phase transition [51]. Later, Chakraborty and one of us developed a comprehensive theory of shear and bulk viscosities in hadronic gases [12]. That work included multiple hadron species with temperature-dependent masses and a scalar mean field in a thermodynamically self-consistent way. In that work, formulas for shear and bulk viscosity were derived, and both relaxation-time approximation formulas and more general integral equations were given. However, that work did not include chemical potentials; hence, thermal conductivity was not considered. Bluhm, Kämpfer, and Redlich used a similar quasiparticle formalism to study the shear and bulk viscosity of gluon matter in Ref. [52] (also without chemical potentials). Thus, a natural question is as follows: How does the formalism of Ref. [12] generalize to finite baryon chemical potential? Also, what is the formula for thermal conductivity?

Several papers have tried different *Ansätze* for generalized viscosity formulas (in the relaxation-time approximation) when the baryon chemical potential is nonzero. Chen *et al.* calculated the shear and bulk viscosities of weakly coupled quark-gluon plasma at finite temperature and chemical potential in Ref. [53] using a quasiparticle model with medium-dependent masses and a scalar mean field. Khvorostukhin, Toneev, and Voskresensky compared three *Ansätze* for the generalized bulk viscosity formula [54] of a hadron gas with medium-dependent masses and a scalar mean field; see also Refs. [55,56]. Interestingly, Khvorostukhin's quasiparticle model also included a vector (ω) mean field [54,55]; as is well known, they are important to account for repulsive forces in hadronic matter with large baryon densities. This type of model is quite relevant for studying the moderate-temperature hadronic matter formed in the beam energy scan at RHIC. It is also relevant for experiments at the Super Proton Synchrotron Heavy Ion and Neutrino Experiment (SHINE) at CERN and at the future Facility for Antiproton and Ion Research (FAIR). Given the usefulness of this kind of model, it is desirable to put the results on a firmer theoretical foundation and (ideally) determine which of the *Ansätze* presented in Refs. [53] and [54] are correct.

In this work, we present detailed derivations of the formulas for the shear and bulk viscosities and thermal conductivity of a gas of hadronic quasiparticles. We include a scalar and a vector mean field, where the mean fields and the quasiparticle masses depend on temperature and baryon chemical potential. Generalization to multiple scalar and vector fields is straightforward but not included here for clarity of presentation. Starting from the quasiparticle dispersion relation, we obtain the Boltzmann equation, and then use the Chapman-Enskog expansion to derive formulas for the transport coefficients. At each step we ensure that thermodynamic self-consistency is maintained, and we carefully enforce the conditions of fit associated with the Landau-Lifshitz local rest frame; we later show that this is vital to obtaining the correct results. We derive both relaxation-time approximation formulas and more general integral equations. Finally, we show that the formulas for shear and bulk viscosities are straightforward generalizations of previous results [12,28] if one recalls that entropy per baryon is conserved in ideal hydrodynamics (neglecting viscous effects). Classical statistics are used in the main text for ease of presentation, but results which include quantum statistics are presented in the Appendix, albeit without detailed derivations.

II. QUASIPARTICLES

In this section we discuss quasiparticle dispersion relations for baryons and mesons. In the simplest mean-field approach all hadrons acquire effective masses in the medium. In addition, baryons acquire effective chemical potentials. We focus attention on baryons because the inclusion of the baryon chemical potential is the new feature of this work compared to Ref. [12].

The piece of the Lagrangian involving baryons is

$$\mathcal{L}_{\text{baryon}} = \sum_j \bar{\psi}_j (i\partial\!\!\!/ - m_j + g_{\sigma j}\sigma - g_{\omega j}\omega\!\!\!/)\psi_j. \qquad (1)$$

Here j refers to the species of baryon. For simplicity of presentation we include only a generic scalar meson σ and a generic vector meson ω. When evaluating the partition function there enters an additional term of the form $\mu_B \bar{\psi}_j \gamma^0 \psi_j$, where μ_B is the baryon chemical potential. Because we are using Dirac spinors, both particles and antiparticles are included. Particles have chemical potential μ_B, while antiparticles have chemical potential $-\mu_B$.

For a uniform medium in thermal equilibrium the meson fields acquire space-time independent nonzero mean values denoted by $\bar{\sigma}$ and $\bar{\omega}^\mu$; in the rest frame of the medium the spatial part of the vector field vanishes on account of rotational symmetry, $\bar{\boldsymbol{\omega}} = 0$, but in a general frame of reference it does not. The dispersion relation for particles is

$$E_j^+(\mathbf{p}) = \sqrt{(\mathbf{p} - g_{\omega j}\bar{\boldsymbol{\omega}})^2 + m_j^{*2}} + g_{\omega j}\bar{\omega}^0 \quad (2)$$

and for antiparticles

$$E_j^-(\mathbf{p}) = \sqrt{(\mathbf{p} + g_{\omega j}\bar{\boldsymbol{\omega}})^2 + m_j^{*2}} - g_{\omega j}\bar{\omega}^0. \quad (3)$$

The kinetic momentum $\mathbf{p}^*$ is related to the canonical momentum $\mathbf{p}$ by

$$\mathbf{p}_j^* = \mathbf{p} - g_{\omega j}\bar{\boldsymbol{\omega}} \quad (4)$$

for particles and by

$$\mathbf{p}_j^* = \mathbf{p} + g_{\omega j}\bar{\boldsymbol{\omega}} \quad (5)$$

for antiparticles. Particles and antiparticles have a common mass m_j^*. In this mean-field approach it is given by $m_j^* = m_j - g_{\sigma j}\bar{\sigma}$.

A more convenient way to think about the dispersion relations is to recognize a shift in both the mass and the chemical potential of quasiparticles and antiquasiparticles. They both have energy

$$E_j^{*\pm}(\mathbf{p}^*) = \sqrt{\mathbf{p}^{*2} + m_j^{*2}}, \quad (6)$$

while their chemical potentials are opposite in sign,

$$\mu_j^{*\pm} = \pm(\mu_B - g_{\omega j}\bar{\omega}^0), \quad (7)$$

as befits particles and antiparticles.

Mesons do not have a baryon chemical potential. They could have chemical potentials for electric charge or strangeness, but we do not consider that possibility here for simplicity. Hence, their dispersion relations in mean-field approximation are of the form

$$E^*(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^{*2}}. \quad (8)$$

Note that the kinetic and canonical momenta are the same for mesons. The effective masses and effective chemical potentials can be found self-consistently once one fixes the Lagrangian.

In equilibrium the phase-space density for a particle (or antiparticle) of type a is given by

$$f_a(\mathbf{x},\mathbf{p}^*,t) = \frac{1}{e^{(E_a^* - \mu_a^*)/T} - (-1)^{2s_a}}. \quad (9)$$

Here s_a denotes the spin. There are Fermi-Dirac and Bose-Einstein distributions. Later we simplify our results by using classical statistics, although that approximation is not necessary. Results including quantum statistics are given in the Appendix. Momentum space integration will be abbreviated as

$$d\Gamma_a^* = (2s_a + 1)\frac{d^3 p_a^*}{(2\pi)^3}, \quad (10)$$

indicating that the kinetic momentum is chosen as the independent variable, and the spin degeneracy is included.

III. BOLTZMANN EQUATION

The general form of the Boltzmann equation for the distribution function $f_a(\mathbf{x},\mathbf{p}^*,t)$ is

$$\frac{df_a}{dt}(\mathbf{x},\mathbf{p}^*,t) = \frac{\partial f_a}{\partial t} + \frac{\partial f_a}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_a}{\partial p^{*i}}\frac{dp^{*i}}{dt} = \mathcal{C}_a. \quad (11)$$

The right-hand side is the collision term, which is discussed later. Here we focus on the left-hand side. It involves the trajectory $\mathbf{x}(t)$ and $\mathbf{p}^*(t)$ between collisions. This trajectory is, in general, not a straight line because the particle is moving in a mean field which can be space and time dependent.

The velocity is

$$\frac{dx^i}{dt} = \frac{\partial E_a}{\partial p_a^i} = \frac{p^{*i}}{E_a^*}. \quad (12)$$

The relativistic version of Newton's second law is

$$\frac{dp_a^i}{dt} = -\left(\frac{\partial E_a}{\partial x^i}\right)_p. \quad (13)$$

Note that it is p that is held fixed, not p^*. The right-hand side is

$$\left(\frac{\partial E_a}{\partial x^i}\right)_p = \frac{m_a^*}{E_a^*}\frac{\partial m_a^*}{\partial x^i} - g_{\omega a}\frac{\partial \bar{\omega}^j}{\partial x^i}\frac{p^{*j}}{E_a^*} + g_{\omega a}\frac{\partial \bar{\omega}^0}{\partial x^i}. \quad (14)$$

The left-hand side of Newton's second law can be written in terms of the kinetic momentum as

$$\frac{dp_a^i}{dt} = \frac{dp^{*i}}{dt} + g_{\omega a}\frac{d\bar{\omega}^i}{dt} = \frac{dp^{*i}}{dt} + g_{\omega a}\left(\frac{\partial \bar{\omega}^i}{\partial t} + \frac{p^{*j}}{E_a^*}\frac{\partial \bar{\omega}^i}{\partial x^j}\right). \quad (15)$$

The time derivatives of $\mathbf{x}$ and $\mathbf{p}^*$ can now be replaced in Eq. (11) to put the Boltzmann equation in the form

$$\frac{df_a}{dt}(\mathbf{x},\mathbf{p}^*,t) = \frac{\partial f_a}{\partial t} + \frac{p^{*i}}{E_a^*}\frac{\partial f_a}{\partial x^i}$$

$$- \frac{\partial f_a}{\partial p^{*i}}\left\{\frac{m_a^*}{E_a^*}\frac{\partial m_a^*}{\partial x^i} + g_{\omega a}\left[\frac{\partial \bar{\omega}^0}{\partial x^i} + \frac{\partial \bar{\omega}^i}{\partial t} + \frac{p^{*j}}{E_a^*}\left(\frac{\partial \bar{\omega}^i}{\partial x^j} - \frac{\partial \bar{\omega}^j}{\partial x^i}\right)\right]\right\}$$

$$= \mathcal{C}_a. \quad (16)$$

This can be simplified by making use of the kinetic 4-momentum

$$p_a^{*\mu} = (E_a^*,\mathbf{p}^*) \quad (17)$$

and the field strength tensor

$$\omega^{\alpha\beta} \equiv \partial^\alpha \omega^\beta - \partial^\beta \omega^\alpha. \quad (18)$$

The final form is

$$\frac{df_a}{dt}(\mathbf{x},\mathbf{p}^*,t) = \frac{p^{*\mu}}{E_a^*}\partial_\mu f_a - \left[\frac{m_a^*}{E_a^*}\frac{\partial m_a^*}{\partial x^i} + g_{\omega a}\frac{p_\mu^*}{E_a^*}\bar{\omega}^{\mu i}\right]\frac{\partial f_a}{\partial p^{*i}}$$

$$= \mathcal{C}_a. \quad (19)$$

IV. ENERGY-MOMENTUM TENSOR AND BARYON CURRENT

In this section we present the structure of the energy-momentum tensor $T^{\mu\nu}$ and of the baryon current J_B^μ. In terms of temperature, chemical potential, and flow velocity u^μ they are

$$T^{\mu\nu} = -Pg^{\mu\nu} + wu^\mu u^\nu + \Delta T^{\mu\nu} \tag{20}$$

and

$$J_B^\mu = n_B u^\mu + \Delta J_B^\mu, \tag{21}$$

where $P(T,\mu_B)$ is the pressure, $s = \partial P/\partial T$ is the entropy density, $n_B = \partial P/\partial \mu_B$ is the baryon density, $\epsilon = -P + Ts + \mu_B n_B$ is the energy density, and $w = \epsilon + P$ is the enthalpy density. In the Landau-Lifshitz approach, which we use, u^μ is the velocity of energy transport. The $\Delta T^{\mu\nu}$ and ΔJ_B^μ are dissipative parts given by

$$\Delta T^{\mu\nu} = \eta\big(D^\mu u^\nu + D^\nu u^\mu + \tfrac{2}{3}\Delta^{\mu\nu}\partial_\rho u^\rho\big) - \zeta\Delta^{\mu\nu}\partial_\rho u^\rho \tag{22}$$

and

$$\Delta J_B^\mu = \lambda\left(\frac{n_B T}{w}\right)^2 D^\mu\left(\frac{\mu_B}{T}\right). \tag{23}$$

Here η, ζ, and λ are the shear viscosity, bulk viscosity, and thermal conductivity, respectively. The other symbols are

$$D = u^\rho\partial_\rho, \tag{24}$$

$$D^\mu = \partial^\mu - u^\mu D, \tag{25}$$

$$\Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}. \tag{26}$$

Our metric is (+, − , − ,−). Additionally, the entropy current is

$$s^\mu = su^\mu - \frac{\mu_B}{T}\Delta J_B^\mu. \tag{27}$$

Now we need to express $T^{\mu\nu}$ and J_B^μ in terms of the quasiparticles and mean fields. One expression for the former is

$$T^{\mu\nu} = \sum_a \int d\Gamma_a^* \frac{p_a^{*\mu} p_a^{*\nu}}{E_a^*} f_a + g^{\mu\nu}U(\bar\sigma,\bar\omega^\rho\bar\omega_\rho) + m_\omega^2\bar\omega^\mu\bar\omega^\nu. \tag{28}$$

The first term is familiar as the kinetic contribution. The second term U is the usual meson-field potential energy; it includes the mass terms $\frac{1}{2}m_\sigma^2\bar\sigma^2$ and $-\frac{1}{2}m_\omega^2\bar\omega^\rho\bar\omega_\rho$, plus any interaction terms which are more than two powers of the fields. Note that kinetic terms for the mean meson fields are not included because they are second order in space-time gradients and are not included in first-order viscous fluid dynamics. The last term is not obviously of the form of Eq. (20). However, when one remembers that T^{0i} is the energy flux in the direction i, and that E_a is the complete quasiparticle energy and not E_a^*, then one would write

$$T^{\mu\nu} = \sum_a \int d\Gamma_a^* \frac{p_a^{\mu} p_a^{*\nu}}{E_a^*} f_a + g^{\mu\nu}U(\bar\sigma,\bar\omega^\rho\bar\omega_\rho). \tag{29}$$

Using $p_a^\mu = p_a^{*\mu} + g_{\omega a}\bar\omega^\mu$ we get

$$T^{\mu\nu} = \sum_a \int d\Gamma_a^* \frac{p_a^{*\mu} p_a^{*\nu}}{E_a^*} f_a + g^{\mu\nu}U(\bar\sigma,\bar\omega^\rho\bar\omega_\rho) + \bar\omega^\mu\sum_a g_{\omega a}\int d\Gamma_a^*\frac{p_a^{*\nu}}{E_a^*}f_a. \tag{30}$$

The vector mean field is determined by its equation of motion. Assuming an interaction only with the baryons (this assumption is easily relaxed), it is

$$\big(\partial^2 + m_\omega^2\big)\bar\omega^\nu = \sum_j g_{\omega j}\langle\bar\psi_j\gamma^\nu\psi_j\rangle, \tag{31}$$

where the averaging refers to the quasiparticle distribution. Recognizing that the summation index j refers to both baryons and antibaryons, and dropping the d'Alembertian because of first-order viscous fluid dynamics, we have

$$m_\omega^2\bar\omega^\nu = \sum_a g_{\omega a}\int d\Gamma_a^*\frac{p_a^{*\nu}}{E_a^*}f_a. \tag{32}$$

(We remind the reader that the coupling $g_{\omega a}$ is opposite in sign for baryons and antibaryons.) Hence, Eqs. (28) and (29) are the same.

In a similar way the scalar mean field is determined by its equation of motion. This turns out to be

$$\frac{\partial U(\bar\sigma,\bar\omega^\rho\bar\omega_\rho)}{\partial\bar\sigma} = \sum_a g_{\sigma a}\int d\Gamma_a^*\frac{m_a^*}{E_a^*}f_a. \tag{33}$$

The coupling to scalar mesons of baryons and antibaryons has the same sign, unlike the coupling to vector mesons.

The structure of the baryon current is readily deduced to be

$$J_B^\mu = \sum_a b_a\int d\Gamma_a^*\frac{p_a^{*\mu}}{E_a^*}f_a, \tag{34}$$

where b_a denotes the baryon number of a.

It can be shown that energy and momentum are conserved, namely,

$$\partial_\mu T^{\mu\nu} = 0, \tag{35}$$

and so is baryon number

$$\partial_\mu J_B^\mu = 0. \tag{36}$$

These conservation laws follow from the requirement that

$$\sum_a\int d\Gamma_a^*\chi_a\mathcal{C}_a = 0. \tag{37}$$

The χ_a represents the contribution from quasiparticle a to any conserved quantity, such as energy, momentum, or baryon number. The calculations are straightforward but very lengthy and tedious. (See, for example, de Groot, van Leeuwen, and van Weert [57] for details.) We have performed the derivations, but they are not reproduced here. It is also straightforward, and much less tedious, to show that the mean-field equation of state follows from the above expressions for $T^{\mu\nu}$ and J_B^μ when the system is uniform, time independent, and in thermal and chemical equilibrium.

V. DEPARTURES FROM EQUILIBRIUM OF THE QUASIPARTICLE DISTRIBUTION FUNCTION

To first order in departures from equilibrium, we can express the quasiparticle distribution function as

$$f_a = f_a^{\rm eq}(1+\phi_a), \tag{38}$$

where $f_a^{\rm eq}$ is the distribution function in thermal and chemical equilibrium. The nonequilibrium part ϕ_a leads to the nonequilibrium contributions $\Delta T^{\mu\nu}$ and ΔJ_B^μ, so ϕ_a must contain the same space-time gradients as found in them. Therefore, ϕ_a must have the form

$$\begin{aligned}\phi_a =& - A_a \partial_\rho u^\rho - B_a p_a^\nu D_\nu\left(\frac{\mu_B}{T}\right)\\ &+ C_a p_a^\mu p_a^\nu\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right).\end{aligned} \tag{39}$$

The functions A_a, B_a, and C_a only depend on momentum p, while u^μ only depends on space-time coordinate x.

The departure from equilibrium of the quasiparticle distributions can be used to compute the departure from equilibrium of the energy-momentum tensor. It is convenient to work in the local rest frame. The variation of the space-space part of expression (28) is

$$\delta T^{ij} = \sum_a \int d\Gamma_a^* \frac{p_a^{*i} p_a^{*j}}{E_a^*}\left(\delta f_a - f_a^{\rm eq}\frac{\delta E_a^*}{E_a^*}\right) + g^{ij}\delta U. \tag{40}$$

To obtain the variation in the mean-field potential we start with the expression for the pressure $P(T,\mu_B) = P_0 - U$. Here P_0 is the kinetic contribution to the pressure from the quasiparticles. The entropy density is obtained from $s = \partial P(T,\mu_B)/\partial T$. This has three contributions: The first is from s_0 which is the same functional form as for particles with T- and μ_B-independent energies, the second is from the variation of the quasiparticle energies owing to variations in T and μ_B, and finally there is the contribution $-\partial U/\partial T$ at fixed μ_B. The mean field carries no entropy; therefore, the second and third terms must cancel. Using classical statistics for simplicity we have

$$P_0 = T\sum_a \int d\Gamma_a^* f_a^{\rm eq} \tag{41}$$

and thus

$$\frac{\partial U}{\partial T} = -\sum_a \int d\Gamma_a^* \left(\frac{\partial E_a}{\partial T}\right)_{\mu_B} f_a^{\rm eq}. \tag{42}$$

The same argument applies to differentiation with respect to μ_B, which gives the baryon density. The mean field carries no baryon number, so, similarly,

$$\frac{\partial U}{\partial \mu_B} = -\sum_a \int d\Gamma_a^* \left(\frac{\partial E_a}{\partial \mu_B}\right)_T f_a^{\rm eq}. \tag{43}$$

Hence,

$$\delta U = -\sum_a \int d\Gamma_a^* \delta E_a f_a^{\rm eq}, \tag{44}$$

where $E_a = E_a^* + g_{\omega a}\bar\omega^0$ and

$$\delta E_a = \frac{m_a^*}{E_a^*}\delta m_a^* + g_{\omega a}\delta\bar\omega^0. \tag{45}$$

Now we come to the deviation in the quasiparticle distribution function. The f_a in general will have departures from the equilibrium form, but it can also change because the quasiparticle energy departs from its equilibrium value. Let us denote E_a^0 the equilibrium value and E_a the total nonequilibrium energy; it is the latter which is conserved in the particle collisions. Similarly, we denote T^0 and μ_B^0 the equilibrium values. Then we write

$$\begin{aligned} f_a(E_a,T,\mu_B) &= f_a^{\rm eq}(E_a^0,T^0,\mu_B^0) + \delta f_a,\\ f_a(E_a,T,\mu_B) &= f_a^{\rm eq}(E_a,T^0,\mu_B^0) + \delta \tilde f_a.\end{aligned} \tag{46}$$

The deviations are related to each other by

$$\delta f_a = \delta\tilde f_a + \left(\frac{\partial f_a^{\rm eq}}{\partial E_a}\right)_{T^0,\,\mu_B^0}\delta E_a = \delta\tilde f_a - \frac{\delta E_a}{T}f_a^{\rm eq}, \tag{47}$$

where the second equality follows when using classical statistics.

It is always the $\delta\tilde f_a$ which determine the transport coefficients. Therefore, we express δT^{ij} in terms of $\delta\tilde f_a$ instead of δf_a:

$$\begin{aligned}\delta T^{ij} =& \sum_a\int d\Gamma_a^* \frac{p_a^{*i}p_a^{*j}}{E_a^*}\delta\tilde f_a\\ &- \sum_a\int d\Gamma_a^* \frac{p_a^{*i}p_a^{*j}}{E_a^*}\left(\frac{\delta E_a}{T} + \frac{\delta E_a^*}{E_a^*}\right)f_a^{\rm eq}\\ &+ \delta^{ij}\sum_a\int d\Gamma_a^*\delta E_a f_a^{\rm eq}.\end{aligned} \tag{48}$$

The integrand of the second term depends only on the magnitude of $\mathbf{p}_a^*$, apart from the factor $p_a^{*i}p_a^{*j}$. Therefore, one may effectively make the replacement $p_a^{*i}p_a^{*j} \to \frac{1}{3}|\mathbf{p}_a^*|^2\delta^{ij}$. Then the terms not involving $\delta\tilde f_a$ all have a factor of δ^{ij}. They can be written as a sum of

$$\frac{\delta\bar\omega^0}{T}\sum_a g_{\omega a}\int d\Gamma_a^*\left(T - \frac{|\mathbf{p}_a^*|^2}{3E_a^*}\right)f_a^{\rm eq}$$

and

$$\sum_a \delta m_a^* \int d\Gamma_a^* \frac{m_a^*}{E_a^*}\left(1 - \frac{|\mathbf{p}_a^*|^2}{3TE_a^*} - \frac{|\mathbf{p}_a^*|^2}{3E_a^{*2}}\right)f_a^{\rm eq}.$$

It can be shown that both of these integrate to zero (using classical statistics). Hence, we find

$$\delta T^{ij} = \sum_a\int d\Gamma_a^*\frac{p_a^{*i}p_a^{*j}}{E_a^*}\delta\tilde f_a \tag{49}$$

as our final result.

The variation in the time-time component of the energy-momentum tensor, starting with either Eq. (28) or

Eq. (29), is

$$\delta T^{00} = \sum_a \int d\Gamma_a^* E_a \delta f_a. \tag{50}$$

We use Eq. (47) for δf_a. The variation of the local energy E_a,

$$\delta E_a = \frac{\delta m_a^{*2}}{2E_a^*} + g_{\omega a}\delta\bar{\omega}^0, \tag{51}$$

can be expressed in terms of the variations in temperature and chemical potential

$$\delta m_a^{*2} = \left(\frac{\partial m_a^{*2}}{\partial T}\right)_{\mu_B}\delta T + \left(\frac{\partial m_a^{*2}}{\partial \mu_B}\right)_T \delta\mu_B, \tag{52}$$

$$\delta\bar{\omega}^0 = \left(\frac{\partial \bar{\omega}^0}{\partial T}\right)_{\mu_B}\delta T + \left(\frac{\partial \bar{\omega}^0}{\partial \mu_B}\right)_T \delta\mu_B. \tag{53}$$

The variations δT and $\delta\mu_B$ are not independent. They are related by the hydrodynamic flow of the matter which to this order occurs at constant entropy per baryon $\sigma = s/n_B$. Dissipation should not be included because it would lead to second-order effects which are consistently neglected in first-order viscous fluid dynamics. To keep the formulas compact, it is helpful to define the susceptibilities:

$$\chi_{xy} = \frac{\partial^2 P(T,\mu)}{\partial x \partial y}. \tag{54}$$

Then one finds several equivalent expressions:

$$\begin{aligned}\left(\frac{\partial \mu_B}{\partial T}\right)_\sigma &= \frac{\mu_B}{T}\frac{v_s^2}{v_n^2} = \frac{1}{T}\left[\mu_B + \frac{1}{v_n^2}\left(\frac{\partial P}{\partial n_B}\right)_\epsilon\right] \\ &= \frac{\chi_{TT} - \sigma\chi_{\mu T}}{\sigma\chi_{\mu\mu} - \chi_{\mu T}}.\end{aligned} \tag{55}$$

Here $v_x^2 = (\partial P/\partial\epsilon)_x$ is the speed of sound at constant x. It is easily shown that

$$\begin{aligned} v_n^2 &= \frac{s\chi_{\mu\mu} - n_B\chi_{\mu T}}{T\left(\chi_{TT}\chi_{\mu\mu} - \chi_{\mu T}^2\right)}, \\ v_s^2 &= \frac{n_B\chi_{TT} - s\chi_{\mu T}}{\mu_B\left(\chi_{TT}\chi_{\mu\mu} - \chi_{\mu T}^2\right)}, \\ v_\sigma^2 &= \frac{v_n^2 Ts + v_s^2\mu n_B}{w}, \end{aligned} \tag{56}$$

relationships that are independent of the specific equation of state. Of course, waves do not physically propagate at constant n or s, only at constant σ, but these definitions are useful for various intermediate steps in various applications. Rather than thinking of m_a^* and $\bar{\omega}^0$ as functions of T and μ_B we can think of them as functions of T and σ. Then

$$\delta m_a^{*2} = \left(\frac{\partial m_a^{*2}}{\partial T}\right)_\sigma \delta T, \tag{57}$$

$$\delta\bar{\omega}^0 = \left(\frac{\partial\bar{\omega}^0}{\partial T}\right)_\sigma \delta T. \tag{58}$$

Next, we need to relate the variations in T and μ_B to the variation $\delta\tilde{f}_a$. The latter variation is done at fixed E_a and is

$$\delta\tilde{f}_a = f_a^{\rm eq}\left[E_a - \mu_a + T\left(\frac{\partial\mu_a}{\partial T}\right)_\sigma\right]\frac{\delta T}{T^2}. \tag{59}$$

(Recall that $\mu_a = b_a\mu_B$.) The term from Eq. (47) which needs to be rewritten is

$$\begin{aligned}\frac{\delta E_a}{T}f_a^{\rm eq} &= \frac{1}{E_a^*}\left[\frac{T^2\left(\partial m_a^{*2}/\partial T^2\right)_\sigma + g_{\omega a}T(\partial\bar{\omega}^0/\partial T)_\sigma E_a^*}{E_a - \mu_a + T(\partial\mu_a/\partial T)_\sigma}\right]\delta\tilde{f}_a \\ &= \left[\frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial\mu_a/\partial T)_\sigma}\right]\delta\tilde{f}_a.\end{aligned} \tag{60}$$

We reiterate that the temperature derivative of a function F depending on T and μ_B, taken at fixed entropy per baryon, is

$$\begin{aligned}\left(\frac{\partial F}{\partial T}\right)_\sigma &= \left(\frac{\partial F}{\partial T}\right)_{\mu_B} + \left(\frac{\partial F}{\partial \mu_B}\right)_T\left(\frac{\partial \mu_B}{\partial T}\right)_\sigma \\ &= \left(\frac{\partial F}{\partial T}\right)_{\mu_B} + \frac{\mu_B}{T}\frac{v_s^2}{v_n^2}\left(\frac{\partial F}{\partial \mu_B}\right)_T.\end{aligned} \tag{61}$$

The final expression is therefore

$$\delta T^{00} = \sum_a \int d\Gamma_a^* E_a\left\{1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial\mu_a/\partial T)_\sigma}\right\}\delta\tilde{f}_a. \tag{62}$$

When the baryon density goes to zero this reduces to the formula known in the literature.

The time-space component has the very natural form

$$\delta T^{0j} = \sum_a \int d\Gamma_a^* \frac{p_a^{*j}}{E_a^*}E_a\delta f_a. \tag{63}$$

To express this in terms of $\delta\tilde{f}_a$, we note that the last term on the right-hand side of Eq. (47) is spherically symmetric in momentum space and therefore that term integrates to zero. This is not true of the other term because the deviation ϕ_a does have terms that depend on the direction of the momentum. Therefore,

$$\delta T^{0j} = \sum_a \int d\Gamma_a^* \frac{p_a^{*j}}{E_a^*}E_a\delta\tilde{f}_a. \tag{64}$$

Last we need the variations in the baryon current. The steps are by now very familiar. The results are

$$\delta J_B^0 = \sum_a b_a \int d\Gamma_a^*\left\{1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial\mu_a/\partial T)_\sigma}\right\}\delta\tilde{f}_a \tag{65}$$

and

$$\delta J_B^i = \sum_a b_a \int d\Gamma_a^* \frac{p_a^{*i}}{E_a^*}\delta\tilde{f}_a. \tag{66}$$

VI. GENERAL FORMULAS FOR THE TRANSPORT COEFFICIENTS

Suppose that we know the scalars A_a, B_a, and C_a in Eq. (39) as functions of the magnitude of the momentum $\mathbf{p}_a^*$. Then in the local rest frame we should equate the hydrodynamic expression ΔT^{ij} from Eq. (22) with the quasiparticle expression δT^{ij} from Eq. (49), the latter being

$$\delta T^{ij} = \sum_a \int d\Gamma_a^* \frac{p_a^{*i} p_a^{*j}}{E_a^*}\left[- A_a \partial_\rho u^\rho - B_a p_a^\nu D_\nu\left(\frac{\mu_B}{T}\right) + C_a p_a^\mu p_a^\nu \left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right)\right] f_a^{\rm eq}. \tag{67}$$

The B_a integrates to zero by symmetry. In the local rest frame the derivative $\partial_k u_0 = 0$, so the the summation over μ and ν is a sum over spatial indices kl only. In the A_a term we can use

$$p_a^{*i} p_a^{*j} \to \tfrac{1}{3}|\mathbf{p}_a^*|^2 \delta_{ij}$$

and in the C_a term we can use

$$p_a^{*i} p_a^{*j} p_a^{*k} p_a^{*l} \to \tfrac{1}{15}|\mathbf{p}_a^*|^4(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

because in the local rest frame $\mathbf{p} = \mathbf{p}^*$. Equating the tensorial structures then gives us the shear viscosity

$$\eta = \frac{2}{15}\sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^4}{E_a^*} f_a^{\rm eq} C_a \tag{68}$$

and the bulk viscosity

$$\zeta = \frac{1}{3}\sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} f_a^{\rm eq} A_a. \tag{69}$$

For the baryon current we compare the ΔJ_B^i from Eq. (23) with the dissipative part of Eq. (34) in the local rest frame. The latter is

$$\delta J_B^i = \sum_a b_a \int d\Gamma_a^* \frac{p_a^{*i}}{E_a^*}\left[- B_a p_a^\nu D_\nu\left(\frac{\mu_B}{T}\right)\right] f_a^{\rm eq}. \tag{70}$$

Obviously, the A_a and C_a terms integrate to zero on account of symmetry. After some manipulation this results in an expression for the thermal conductivity,

$$\lambda = \frac{1}{3}\left(\frac{w}{n_B T}\right)^2 \sum_a b_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} f_a^{\rm eq} B_a. \tag{71}$$

To solve for the functions A_a, B_a, and C_a, we turn to the Chapman-Enskog method. This entails expanding both sides of the Boltzmann equation (19) to first order in the ϕ_a. It leads to integral equations, which, in general, must be solved numerically.

Here we follow the notation of Ref. [12]. Including 2-to-2, 2-to-1, and 1-to-2 processes and using classical statistics (these restrictions are easily relaxed) the collision integral is

$$\mathcal{C}_a = \sum_{bcd} \frac{1}{1+\delta_{ab}} \int d\Gamma_b^* d\Gamma_c^* d\Gamma_d^* W(a,b|c,d)\{f_c f_d - f_a f_b\} + \sum_{cd} \int d\Gamma_c^* d\Gamma_d^* W(a|c,d)\{f_c f_d - f_a\} + \sum_{bc} \int d\Gamma_b^* d\Gamma_c^* W(c|a,b)\{f_c - f_a f_b\}. \tag{72}$$

The W's are given as

$$W(a,b|c,d) = \frac{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}{2E_a^* 2E_b^* 2E_c^* 2E_d^*} \overline{|\mathcal{M}(a,b|c,d)|^2} \tag{73}$$

and

$$W(a|c,d) = \frac{(2\pi)^4 \delta^4(p_a - p_c - p_d)}{2E_a^* 2E_c^* 2E_d^*} \overline{|\mathcal{M}(a|c,d)|^2}. \tag{74}$$

The use of E_a^* instead of E_a in the denominators ensures that the phase-space integration is Lorentz covariant. Also note that, following Larionov *et al.* [58], we use dimensionless matrix elements $\mathcal{M}$ averaged over spin in both initial and final states. This is necessary to balance the degeneracy factors in the $d\Gamma_a^*$. We use chemical equilibrium (for example, $a+b \leftrightarrow c+d$ gives $f_a^{\rm eq} f_b^{\rm eq} = f_c^{\rm eq} f_d^{\rm eq}$). Then the collision integral becomes

$$\mathcal{C}_a = f_a^{\rm eq}\sum_{bcd} \frac{1}{1+\delta_{ab}} \int d\Gamma_b^* d\Gamma_c^* d\Gamma_d^* f_b^{\rm eq} W(a,b|c,d) \times [\phi_c + \phi_d - \phi_a - \phi_b] + f_a^{\rm eq}\sum_{cd}\int d\Gamma_c^* d\Gamma_d^* W(a|c,d)[\phi_c + \phi_d - \phi_a] + f_a^{\rm eq}\sum_{bc}\int d\Gamma_b^* d\Gamma_c^* f_b^{\rm eq} W(c|a,b)[\phi_c - \phi_a - \phi_b]. \tag{75}$$

This constitutes the right-hand side of the Boltzmann equation.

The left-hand side of the Boltzmann equation (19) is computed using the local equilibrium form of the distribution function,

$$f_a^{\rm eq}(x,\mathbf{p}^*) = \exp\left[-\frac{u_\alpha(x) p_a^\alpha}{T(x)}\right]\exp\left[\frac{\mu_a(x)}{T(x)}\right] = \exp\left[-\frac{u_\alpha(x) p_a^{*\alpha}}{T(x)}\right]\exp\left[\frac{\mu_a^*(x)}{T(x)}\right]. \tag{76}$$

Here the flow velocity, temperature, and chemical potential all depend on x. Although not explicitly indicated, p_a^α depends on x via the dependence of m_a^* and $\bar{\omega}^\alpha$ on x, while E_a^* depends on x via m_a^* only. The left-hand side must be expressed in terms of the same space-time gradients as ϕ_a, namely $\partial_\rho u^\rho$, $D_\nu(\mu_B/T)$, and $(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho)$. The calculation is long and tedious. Space-time derivatives of T and μ_B are expressed in terms of the relevant tensor structures by using the perfect fluid equations for conservation of energy, momentum, and baryon number. Some useful intermediate results are

$$DT = -v_n^2 T\, \partial_\rho u^\rho, \qquad D\mu_B = -v_s^2 \mu_B\, \partial_\rho u^\rho. \tag{77}$$

One form of the left-hand side (in the local rest frame) is

$$\frac{df_a^{\rm eq}}{dt} = f_a^{\rm eq}\left[\frac{|\mathbf{p}_a^*|^2}{3TE_a^*} + v_n^2 T\frac{\partial}{\partial T}\left(\frac{E_a-\mu_a}{T}\right)_\sigma\right]\partial_\rho u^\rho + f_a^{\rm eq}\left(b_a - \frac{n_B E_a}{w}\right)\frac{p_a^\mu}{E_a^*}D_\mu\left(\frac{\mu_B}{T}\right) - f_a^{\rm eq}\frac{p_a^\mu p_a^\nu}{2TE_a^*}\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right). \tag{78}$$

Now $E_a - \mu_a$ in the first line could be replaced with $E_a^* - \mu_a^*$, and E_a in the second line could be replaced by $E_a^* + g_{\omega a}\bar{\omega}^0$. With a little manipulation this can be shown to be equivalent to Sasaki and Redlich who, however, did not include a vector field or the $D_\mu(\mu_B/T)$ term. Another form is to write out the derivatives in the first line explicitly. This results in

$$\frac{df_a^{\rm eq}}{dt} = f_a^{\rm eq}\frac{1}{3TE_a^*}\left\{|\mathbf{p}_a^*|^2 - 3v_n^2\left[E_a^{*2} - T^2\left(\frac{\partial m_a^{*2}}{\partial T^2}\right)_\sigma + T^2\frac{\partial}{\partial T}\left(\frac{\mu_a^*}{T}\right)_\sigma E_a^*\right]\right\}\partial_\rho u^\rho + f_a^{\rm eq}\left(b_a - \frac{n_B E_a}{w}\right)\frac{p_a^\mu}{E_a^*}D_\mu\left(\frac{\mu_B}{T}\right) - f_a^{\rm eq}\frac{p_a^\mu p_a^\nu}{2TE_a^*}\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right). \tag{79}$$

In the limit that the chemical potential goes to zero this reproduces the results of Jeon and Yaffe [28] and of Chakraborty and Kapusta [12].

Now we subtract the right-hand side from the left-hand side and set the resulting expression to zero. This leads to

$$\mathcal{A}_a(\partial_\rho u^\rho) + \mathcal{B}_a^\mu D_\mu\left(\frac{\mu_B}{T}\right) - \mathcal{C}_a^{\mu\nu}\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right) = 0, \tag{80}$$

where

$$\begin{aligned}\mathcal{A}_a =& \frac{1}{3TE_a^*}\left\{|\mathbf{p}_a^*|^2 - 3v_n^2\left[E_a^{*2} - T^2\left(\frac{\partial m_a^{*2}}{\partial T^2}\right)_\sigma + T^2\frac{\partial}{\partial T}\left(\frac{\mu_a^*}{T}\right)_\sigma E_a^*\right]\right\}\\ &+ \sum_{bcd}\frac{1}{1+\delta_{ab}}\int d\Gamma_b^*\, d\Gamma_c^*\, d\Gamma_d^*\, f_b^{\rm eq}\, W(a,b|c,d)[A_c + A_d - A_a - A_b]\\ &+ \sum_{cd}\int d\Gamma_c^*\, d\Gamma_d^*\, W(a|c,d)[A_c + A_d - A_a] + \sum_{bc}\int d\Gamma_b^*\, d\Gamma_c^*\, f_b^{\rm eq}\, W(c|a,b)[A_c - A_a - A_b]\end{aligned} \tag{81}$$

and

$$\begin{aligned}\mathcal{B}_a^\mu =& \left(b_a - \frac{n_B E_a}{w}\right)\frac{p_a^\mu}{E_a^*} + \sum_{bcd}\frac{1}{1+\delta_{ab}}\int d\Gamma_b^* d\Gamma_c^* d\Gamma_d^* f_b^{\rm eq} W(a,b|c,d)\big[B_c p_c^\mu + B_d p_d^\mu - B_a p_a^\mu - B_b p_b^\mu\big]\\ &+ \sum_{cd}\int d\Gamma_c^* d\Gamma_d^* W(a|c,d)\big[B_c p_c^\mu + B_d p_d^\mu - B_a p_a^\mu\big] + \sum_{bc}\int d\Gamma_b^* d\Gamma_c^* f_b^{\rm eq} W(c|a,b)\big[B_c p_c^\mu - B_a p_a^\mu - B_b p_b^\mu\big]\end{aligned} \tag{82}$$

and

$$\begin{aligned}\mathcal{C}_a^{\mu\nu} =& \frac{p_a^\mu p_a^\nu}{2E_a^* T} + \sum_{bcd}\frac{1}{1+\delta_{ab}}\int d\Gamma_b^* d\Gamma_c^* d\Gamma_d^* f_b^{\rm eq} W(a,b|c,d)\big[C_c p_c^\mu p_c^\nu + C_d p_d^\mu p_d^\nu - C_a p_a^\mu p_a^\nu - C_b p_b^\mu p_b^\nu\big]\\ &+ \sum_{cd}\int d\Gamma_c^* d\Gamma_d^* W(a|c,d)\big[C_c p_c^\mu p_c^\nu + C_d p_d^\mu p_d^\nu - C_a p_a^\mu p_a^\nu\big]\\ &+ \sum_{bc}\int d\Gamma_b^* d\Gamma_c^* f_b^{\rm eq} W(c|a,b)\big[C_a p_a^\mu p_a^\nu + C_b p_b^\mu p_b^\nu - C_c p_c^\mu p_c^\nu\big].\end{aligned} \tag{83}$$

Owing to the tensorial structure of these equations the solution requires that $\mathcal{A}_a = 0$, $\mathcal{B}_a^\mu = 0$, and $\mathcal{C}_a^{\mu\nu} = 0$. These are integral equations for the functions A_a, B_a, and C_a which depend on the magnitude of the momentum $\mathbf{p}^*$.

VII. LANDAU-LIFSHITZ CONDITIONS OF FIT

The set of Eqs. (81)–(83) are integral equations for the functions A_a, B_a, and C_a. Consider the equation for A_a. If we have a particular solution $A_a^{\rm par}$ we can generate another solution $A_a = A_a^{\rm par} - a_E E_a - a_B b_a$, where the constant coefficients a_E and a_B are independent of particle type a. The reason is that energy and baryon number are conserved in the collision, decay, and fusion processes. This arbitrariness exists because of the freedom to define the local rest frame or, equivalently, the flow velocity u^μ. In the Eckart frame, u^μ gives the flow of baryon number, while in the Landau-Lifshitz frame u^μ gives the flow of energy. To remove the arbitrariness and pick one specific frame, one enforces equations called *conditions of fit* [57]. To pick the Landau-Lifshitz frame, the conditions of fit in the local rest frame require $\delta T^{0\nu} = 0$ and $\delta J_B^0 = 0$. Enforcing $\delta T^{00} = 0$ results in

$$a_E \sum_a \int d\Gamma_a^* E_a^2 \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] f_a^{\rm eq} + a_B \sum_a b_a \int d\Gamma_a^* E_a \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] f_a^{\rm eq}$$
$$= \sum_a \int d\Gamma_a^* E_a \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] A_a^{\rm par} f_a^{\rm eq}. \tag{84}$$

Requiring that $\delta J_B^0 = 0$ in the local rest frame results in

$$a_E \sum_a b_a \int d\Gamma_a^* E_a \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] f_a^{\rm eq} + a_B \sum_a b_a^2 \int d\Gamma_a^* \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] f_a^{\rm eq}$$
$$= \sum_a b_a \int d\Gamma_a^* \left[1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a - \mu_a + T(\partial \mu_a/\partial T)_\sigma} \right] A_a^{\rm par} f_a^{\rm eq}. \tag{85}$$

Let us express the integrals in Eq. (84) as X_E, X_B, and Z_E and in Eq. (85) as Y_E, Y_B, and Z_B. Then Eqs. (84) and (85) become

$$a_E X_E + a_B X_B = Z_E, \quad a_E Y_E + a_B Y_B = Z_B. \tag{86}$$

The solutions are

$$a_B = \frac{Y_E Z_E - X_E Z_B}{Y_E X_B - X_E Y_B}, \quad a_E = \frac{X_B Z_B - Y_B Z_E}{Y_E X_B - X_E Y_B}. \tag{87}$$

When these are substituted into the expression (69) for the bulk viscosity we get

$$\zeta = \frac{1}{3} \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} f_a^{\rm eq} A_a - T n_B a_B - T w a_E. \tag{88}$$

First consider the case where there are no mean fields, only on-shell particles traveling in vacuum and undergoing localized collisions. In this case $\delta f_a = \delta \tilde{f}_a$, and one finds

$$X_E = T\big(T^2 \chi_{TT} + 2\mu_B T \chi_{\mu T} + \mu_B^2 \chi_{\mu\mu}\big), \quad X_B = T(T\chi_{\mu T} + \mu_B \chi_{\mu\mu}), \quad Y_E = T(T\chi_{\mu T} + \mu_B \chi_{\mu\mu}), \quad Y_B = T\chi_{\mu\mu}. \tag{89}$$

The combination of a_E and a_B which is needed for the bulk viscosity is

$$T n_B a_B + T w a_E = v_n^2 Z_E + \big(v_s^2 - v_n^2\big)\mu_B Z_B = \sum_a \int d\Gamma_a^* \big[v_n^2 E_a + \big(v_s^2 - v_n^2\big) b_a \mu_B\big] A_a^{\rm par} f_a^{\rm eq}. \tag{90}$$

Here $E_a = E_a^* = \sqrt{p^2 + m_a^2}$ because of the absence of mean fields. The bulk viscosity is then

$$\zeta = \frac{1}{3} \sum_a \int d\Gamma_a^* \left\{ \frac{|\mathbf{p}_a^*|^2}{E_a^*} - 3\big[v_n^2 E_a^* + \big(v_s^2 - v_n^2\big) b_a \mu_B\big] \right\} A_a^{\rm par} f_a^{\rm eq}. \tag{91}$$

This is a limiting form of

$$\zeta = \frac{1}{3} \sum_a \int d\Gamma_a^* \left[\frac{|\mathbf{p}_a^*|^2}{E_a^*} + 3 v_n^2 T^2 \frac{\partial}{\partial T} \left(\frac{E_a - \mu_a}{T} \right)_\sigma \right] A_a^{\rm par} f_a^{\rm eq} \tag{92}$$

once one recognizes Eq. (55). This makes perfect sense because the modification of the integrand compared to Eq. (69) matches the structure of the source of A_a in Eq. (81).

It is not easy to find simple expressions for X_E, X_B, Y_E, Y_B when mean fields are included; hence, there are no simple expressions for a_E and a_B. Fortunately, the individual expressions for a_E and a_B are not needed to find a simple expression for the bulk viscosity. Returning to Eq. (69) we have

$$\zeta = \frac{1}{3} \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} f_a^{\rm eq} \big(A_a^{\rm par} - a_E E_a - a_B b_a\big). \tag{93}$$

Now the trick is to take a judicious linear combination of the conditions of fit. Add $T(\partial \mu_B/\partial T)_\sigma - \mu_B$ times (85) to (84). This gives

$$a_B \sum_a b_a \int d\Gamma_a^* \left(\frac{\partial f_a^{\rm eq}}{\partial T} \right)_\sigma + a_E \sum_a \int d\Gamma_a^* E_a \left(\frac{\partial f_a^{\rm eq}}{\partial T} \right)_\sigma = -\sum_a \int d\Gamma_a^* f_a^{\rm eq} A_a^{\rm par} \frac{\partial}{\partial T} \left(\frac{E_a - \mu_a}{T} \right)_\sigma. \tag{94}$$

The coefficient of a_B is just $(\partial n_B/\partial T)_\sigma$, and from Eq. (50) the coefficient of a_E is just $(\partial\epsilon/\partial T)_\sigma$. Therefore, we have

$$a_B\left(\frac{\partial n_B}{\partial T}\right)_\sigma + a_E\left(\frac{\partial\epsilon}{\partial T}\right)_\sigma = -\sum_a \int d\Gamma_a^* f_a^{\rm eq} A_a^{\rm par} \frac{\partial}{\partial T}\left(\frac{E_a-\mu_a}{T}\right)_\sigma. \tag{95}$$

Because v_n^2 enters into Eq. (92) it is useful to derive the thermodynamic relations

$$T v_n^2 = \frac{w}{(\partial\epsilon/\partial T)_\sigma} = \frac{n_B}{(\partial n_B/\partial T)_\sigma}. \tag{96}$$

First we derive the relation between the derivatives appearing in the above equations. Using $d\epsilon = Tds + \mu_B dn_B$ and $ds = n_B d\sigma + \sigma dn_B$, we obtain

$$\left(\frac{\partial\epsilon}{\partial T}\right)_\sigma = \frac{w}{n_B}\left(\frac{\partial n_B}{\partial T}\right)_\sigma. \tag{97}$$

Now for $(\partial n_B/\partial T)_\sigma$ we use Eq. (61), the third equality of Eq. (55), and the first equality of Eq. (56) to obtain

$$T\left(\frac{\partial n_B}{\partial T}\right)_\sigma = \frac{n_B}{v_n^2}. \tag{98}$$

Together with the previous equation we obtain the desired result (96). Using these results in Eq. (95) we have

$$T n_B a_B + T w a_E = -v_n^2 T^2 \sum_a \int d\Gamma_a^* f_a^{\rm eq} A_a^{\rm par} \frac{\partial}{\partial T}\left(\frac{E_a-\mu_a}{T}\right)_\sigma. \tag{99}$$

Making this substitution in Eq. (88) we obtain the expression (92).

A similar arbitrariness arises in Eq. (82). Owing to energy-momentum conservation, if we have a particular solution $B_a^{\rm par}$ we can generate another solution as $B_a = B_a^{\rm par} - b$, where b is a constant independent of particle species a. This freedom is resolved by the Landau-Lifshitz condition of fit which requires that $\delta T^{0j} = 0$ in the local rest frame. Starting with expression (64) we have

$$\delta T^{0j} = \sum_a \int d\Gamma_a^* \frac{p_a^{*j}}{E_a^*} E_a \left[-\left(B_a^{\rm par} - b\right) p_a^{*i} D_i\left(\frac{\mu_B}{T}\right)\right] f_a^{\rm eq}. \tag{100}$$

Factoring out the spatial derivative, and making use of the momentum space isotropy, we require that

$$b \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} E_a f_a^{\rm eq} = \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} E_a B_a^{\rm par} f_a^{\rm eq}. \tag{101}$$

The integral multiplying b is just $3Tw$ so that

$$b = \frac{1}{3Tw}\int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*} E_a B_a^{\rm par} f_a^{\rm eq}. \tag{102}$$

Substitution into expression (71) gives

$$\lambda = \frac{1}{3}\left(\frac{w}{n_B T}\right)^2 \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*}\left(b_a - \frac{n_B E_a}{w}\right) B_a^{\rm par} f_a^{\rm eq}. \tag{103}$$

There is no ambiguity in the solution to Eq. (83) for C_a, so the expression for the shear viscosity (68) is unchanged.

VIII. RELAXATION-TIME APPROXIMATION

At this point, it is convenient to derive the relaxation-time approximation formulas for the shear and bulk viscosities and thermal conductivity. We start with the Boltzmann equation with the Chapman-Enskog expansion:

$$\frac{df_a^{\rm eq}}{dt} = \mathcal{C}_a. \tag{104}$$

The left-hand side of Eq. (104) is given by Eq. (78) while $\mathcal{C}_a$ can be found in Eq. (75). In the energy-dependent relaxation-time approximation [12], we assume particle species a is out of equilibrium ($\phi_a \neq 0$), while all other particle species are in equilibrium ($\phi_b = \phi_c = \phi_d = 0$). Using Eq. (75), the collision integral $\mathcal{C}_a$ greatly simplifies, and the Boltzmann equation becomes

$$\frac{df_a^{\rm eq}}{dt} = \mathcal{C}_a = -\frac{f_a^{\rm eq}\phi_a}{\tau_a}, \tag{105}$$

where the relaxation time $\tau_a(E_a^*)$ for species a is given by

$$\frac{1}{\tau_a(E_a^*)} = \sum_{bcd} \frac{1}{1+\delta_{ab}} \int d\Gamma_b^* \, d\Gamma_c^* \, d\Gamma_d^* \, f_b^{\rm eq} W(a,b|c,d) + \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d) + \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, f_b^{\rm eq} W(c|a,b). \quad (106)$$

Next we replace the left-hand side of Eq. (105) using Eq. (78). Into the right-hand side we substitute ϕ_a using Eq. (39). Then we equate terms on the left- and right-hand sides by matching tensor structures, and we obtain particular solutions for the functions A_a, B_a, and C_a from ϕ_a:

$$A_a^{\rm par} = \frac{\tau_a}{3T}\left[\frac{|\mathbf{p}_a^*|^2}{E_a^*} + 3v_n^2 T^2 \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right], \quad (107)$$

$$B_a^{\rm par} = \frac{\tau_a}{E_a^*}\left(b_a - \frac{n_B E_a}{w}\right), \quad (108)$$

$$C_a^{\rm par} = \frac{\tau_a}{2T E_a^*}. \quad (109)$$

Finally, we substitute Eqs. (107)–(109) into Eqs. (68), (92), and (103) and obtain the desired relaxation-time formulas:

$$\eta = \frac{1}{15T} \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^4}{E_a^{*2}} \tau_a(E_a^*) f_a^{\rm eq}, \quad (110)$$

$$\zeta = \frac{1}{9T} \sum_a \int d\Gamma_a^* \frac{\tau_a(E_a^*)}{E_a^{*2}} \left[|\mathbf{p}_a^*|^2 + 3v_n^2 T^2 E_a^* \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right]^2 f_a^{\rm eq}, \quad (111)$$

$$\lambda = \frac{1}{3}\left(\frac{w}{n_B T}\right)^2 \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^{*2}} \tau_a(E_a^*) \left(b_a - \frac{n_B E_a}{w}\right)^2 f_a^{\rm eq}. \quad (112)$$

A few observations are in order. First, the transport coefficients computed with Eqs. (110)–(112) are strictly non-negative, as they must be. Second, this non-negativity is ensured by the squares in the integrands which came from enforcing the Landau-Lifshitz conditions of fit. [Recall the derivation of Eqs. (92) and (103).] This shows that it is absolutely vital that the Landau-Lifshitz conditions are carefully enforced to obtain the correct results. A third point is that Eqs. (110) and (111) are obvious generalizations of the formulas obtained in previous works [12,28] to finite baryon chemical potential. The crucial insight is that entropy per baryon ($\sigma = s/n_B$) is conserved in zeroth-order (ideal) hydrodynamics, so that variable must be held fixed when deriving the variations from equilibrium.

IX. CONCLUSION

In this paper, we developed a flexible relativistic quasiparticle theory of transport coefficients in hot and dense hadronic matter. A major goal was the simultaneous inclusion of temperature- and baryon chemical potential-dependent quasiparticle masses with scalar and vector mean fields, all in a thermodynamically self-consistent way. Classical statistics were used throughout to simplify the presentation, although complete results with quantum statistics are given in the Appendix. From the dispersion relations for the quasiparticles, we derived the Boltzmann equation and then the transport coefficients using the Chapman-Enskog expansion. Next we derived compact analytic expressions for the shear and bulk viscosities and thermal conductivity. These formulas can be used with the relaxation-time approximation; alternatively, we have provided integral equations which may be solved for greater accuracy. We have shown that the transport coefficients are non-negative in the relaxation-time approximation (as they must be) which is a direct consequence of carefully enforcing the Landau-Lifshitz conditions of fit.

We also showed that previous bulk viscosity formulas (derived assuming zero baryon chemical potential) generalize straightforwardly to finite baryon chemical potential if one recalls that entropy per baryon is conserved in ideal hydrodynamics. This was the crucial detail that allowed us to compute the variations from equilibrium and use them to derive the bulk viscosity and thermal conductivity formulas.

It is a trivial matter to include a variety of scalar and vector fields; that is simply a matter of book-keeping. The same is true of additional conserved charges beyond baryon number. The next step is to study specific hadronic Lagrangians whose parameters are fit to known nuclear properties, such as described in Chap. 11 of Ref. [59]. Then the equation of state, shear and bulk viscosities, and thermal conductivity would all be obtained from the same hadronic Lagrangian. However, the numerical effort required is significantly more than in Ref. [12] and so is left to future publications.

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APPENDIX

This Appendix has two goals. The first is to summarize the important results derived in the main body of the paper. The second is to include the effects of quantum statistics. All results presented here include quantum statistics. The limit of classical statistics is attained when $|f_a| \ll 1$. Departures from local kinetic and chemical equilibrium for particle species a are expressed in terms of the function ϕ_a as

$$f_a = f_a^{\rm eq}(1+\phi_a). \tag{A1}$$

We let δf_a represent the deviation expressed in terms of the equilibrium energy E_a^0 while $\delta\tilde{f}_a$ represents the deviation expressed in terms of the total nonequilibrium energy E_a; it is the latter which is conserved in local collisions and the one relevant for transport coefficients. The deviations are related to each other by

$$\delta f_a = \delta\tilde{f}_a + \left(\frac{\partial f_a^{\rm eq}}{\partial E_a}\right)_{T^0,\,\mu_B^0}\delta E_a = \delta\tilde{f}_a - \frac{\delta E_a}{T} f_a^{\rm eq}\big(1+d_a f_a^{\rm eq}\big). \tag{A2}$$

Here the notation is $d_a = (-1)^{2s_a}$. We need to relate the variations in T and μ_B to the variation $\delta\tilde{f}_a$. The latter variation is done at fixed E_a and is

$$\delta\tilde{f}_a = f_a^{\rm eq}\left[E_a - \mu_a + T\left(\frac{\partial\mu_a}{\partial T}\right)_\sigma\right]\big(1+d_a f_a^{\rm eq}\big)\frac{\delta T}{T^2}. \tag{A3}$$

Here in what follows the derivative is carried out at fixed entropy per baryon σ. The factor from Eq. (A2) which needs to be rewritten is

$$\frac{\delta E_a}{T} f_a^{\rm eq}\big(1+d_a f_a^{\rm eq}\big) = \left[\frac{T(\partial E_a/\partial T)_\sigma}{E_a-\mu_a+T(\partial\mu_a/\partial T)_\sigma}\right]\delta\tilde{f}_a. \tag{A4}$$

In terms of $\delta\tilde{f}_a$ the deviations in the energy-momentum tensor and baryon current are as follows:

$$\delta T^{ij} = \sum_a \int d\Gamma_a^* \frac{p_a^{*i} p_a^{*j}}{E_a^*}\delta\tilde{f}_a, \tag{A5}$$

$$\delta T^{0j} = \sum_a \int d\Gamma_a^* \frac{p_a^{*j}}{E_a^*} E_a \delta\tilde{f}_a, \tag{A6}$$

$$\delta T^{00} = \sum_a \int d\Gamma_a^* E_a \left\{1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a-\mu_a+T(\partial\mu_a/\partial T)_\sigma}\right\}\delta\tilde{f}_a, \tag{A7}$$

$$\delta J_B^i = \sum_a b_a \int d\Gamma_a^* \frac{p_a^{*i}}{E_a^*}\delta\tilde{f}_a, \tag{A8}$$

$$\delta J_B^0 = \sum_a b_a \int d\Gamma_a^* \left\{1 - \frac{T(\partial E_a/\partial T)_\sigma}{E_a-\mu_a+T(\partial\mu_a/\partial T)_\sigma}\right\}\delta\tilde{f}_a. \tag{A9}$$

The collision term on the right side of the Boltzmann equation reads

$$\begin{aligned}\mathcal{C}_a =& \sum_{bcd}\frac{1}{1+\delta_{ab}}\int d\Gamma_b^*\, d\Gamma_c^*\, d\Gamma_d^*\, W(a,b|c,d)\{f_c f_d(1+d_a f_a)(1+d_b f_b) - f_a f_b(1+d_c f_c)(1+d_d f_d)\}\\ &+ \sum_{cd}\int d\Gamma_c^*\, d\Gamma_d^*\, W(a|c,d)\{f_c f_d(1+d_a f_a) - f_a(1+d_c f_c)(1+d_d f_d)\} + \sum_{bc}\int d\Gamma_b^*\, d\Gamma_c^*\, W(c|a,b)\\ &\times\{f_c(1+d_a f_a)(1+d_b f_b) - f_a f_b(1+d_c f_c)\}.\end{aligned} \tag{A10}$$

This expression explicitly includes $2 \leftrightarrow 2$ and $2 \leftrightarrow 1$ reactions. Higher-order reactions are included in an obvious way.

We now consider small departures from equilibrium, meaning that we keep terms only linear in the ϕ_a. We use chemical equilibrium; for example, $a+b \leftrightarrow c+d$ gives

$$f_c^{\rm eq} f_d^{\rm eq}\big(1+d_a f_a^{\rm eq}\big)\big(1+d_b f_b^{\rm eq}\big) = f_a^{\rm eq} f_b^{\rm eq}\big(1+d_c f_c^{\rm eq}\big)\big(1+d_d f_d^{\rm eq}\big).$$

Then the collision integral becomes

$$\begin{aligned}\mathcal{C}_a =& \sum_{bcd}\frac{1}{1+\delta_{ab}}\int d\Gamma_b^*\, d\Gamma_c^*\, d\Gamma_d^*\, W(a,b|c,d)\big\{f_a^{\rm eq} f_b^{\rm eq}\big[\big(1+d_d f_d^{\rm eq}\big)\phi_c + \big(1+d_c f_c^{\rm eq}\big)\phi_d\big]\\ &- f_c^{\rm eq} f_d^{\rm eq}\big[\big(1+d_b f_b^{\rm eq}\big)\phi_a + \big(1+d_a f_a^{\rm eq}\big)\phi_b\big]\big\}\end{aligned}$$

$$+ \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d)\big\{ f_a^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)\phi_c + \big(1 + d_c f_c^{\rm eq}\big)\phi_d\big] - f_c^{\rm eq} f_d^{\rm eq}\phi_a\big\}$$

$$+ \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, W(c|a,b)\big\{ - f_c^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)\phi_a + \big(1 + d_a f_a^{\rm eq}\big)\phi_b\big] + f_a^{\rm eq} f_b^{\rm eq}\phi_c\big\}. \tag{A11}$$

The left-hand side of the Boltzmann equation is computed using the local equilibrium form of the distribution function. One form of the left-hand side (in the local rest frame) is

$$\frac{df_a^{\rm eq}}{dt} = f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big)\left[\frac{|\mathbf{p}_a^*|^2}{3T E_a^*} + v_n^2 T \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right]\partial_\rho u^\rho + f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big)\left(b_a - \frac{n_B E_a}{w}\right)\frac{p_a^\mu}{E_a^*} D_\mu\left(\frac{\mu_B}{T}\right)$$

$$- f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big)\frac{p_a^\mu p_a^\nu}{2T E_a^*}\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right). \tag{A12}$$

Now we subtract the right-hand side from the left-hand side and set the resulting expression to zero. This leads to

$$\mathcal{A}_a(\partial_\rho u^\rho) + \mathcal{B}_a^\mu D_\mu\left(\frac{\mu_B}{T}\right) - \mathcal{C}_a^{\mu\nu}\left(D_\mu u_\nu + D_\nu u_\mu + \frac{2}{3}\Delta_{\mu\nu}\partial_\rho u^\rho\right) = 0, \tag{A13}$$

where

$$\mathcal{A}_a = \left[\frac{|\mathbf{p}_a^*|^2}{3T E_a^*} + v_n^2 T \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right] f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big) + \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int d\Gamma_b^* \, d\Gamma_c^* \, d\Gamma_d^* \, W(a,b|c,d)$$

$$\times \big\{ f_a^{\rm eq} f_b^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)A_c + \big(1 + d_c f_c^{\rm eq}\big)A_d\big] - f_c^{\rm eq} f_d^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)A_a + \big(1 + d_a f_a^{\rm eq}\big)A_b\big]\big\}$$

$$+ \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d)\big\{ f_a^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)A_c + \big(1 + d_c f_c^{\rm eq}\big)A_d\big] - f_c^{\rm eq} f_d^{\rm eq} A_a\big\}$$

$$+ \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, W(c|a,b)\big\{ - f_c^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)A_a + \big(1 + d_a f_a^{\rm eq}\big)A_b\big] + f_a^{\rm eq} f_b^{\rm eq} A_c\big\}, \tag{A14}$$

$$\mathcal{B}_a^\mu = \left(b_a - \frac{n_B E_a}{w}\right)\frac{p_a^\mu}{E_a^*} f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big) + \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int d\Gamma_b^* \, d\Gamma_c^* \, d\Gamma_d^* \, W(a,b|c,d)$$

$$\times \big\{ f_a^{\rm eq} f_b^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)B_c p_c^\mu + \big(1 + d_c f_c^{\rm eq}\big)B_d p_d^\mu\big] - f_c^{\rm eq} f_d^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)B_a p_a^\mu + \big(1 + d_a f_a^{\rm eq}\big)B_b p_b^\mu\big]\big\}$$

$$+ \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d)\big\{ f_a^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)B_c p_c^\mu + \big(1 + d_c f_c^{\rm eq}\big)B_d p_d^\mu\big] - f_c^{\rm eq} f_d^{\rm eq} B_a p_a^\mu\big\}$$

$$+ \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, W(c|a,b)\big\{ - f_c^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)B_a p_a^\mu + \big(1 + d_a f_a^{\rm eq}\big)B_b p_b^\mu\big] + f_a^{\rm eq} f_b^{\rm eq} B_c p_c^\mu\big\}, \tag{A15}$$

$$\mathcal{C}_a^{\mu\nu} = \frac{p_a^\mu p_a^\nu}{2E_a^* T} f_a^{\rm eq}\big(1 + d_a f_a^{\rm eq}\big) + \sum_{bcd} \frac{1}{1 + \delta_{ab}} \int d\Gamma_b^* \, d\Gamma_c^* \, d\Gamma_d^* \, W(a,b|c,d)\big\{ f_a^{\rm eq} f_b^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)C_c p_c^\mu p_c^\nu + \big(1 + d_c f_c^{\rm eq}\big)C_d p_d^\mu p_d^\nu\big]$$

$$- f_c^{\rm eq} f_d^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)C_a p_a^\mu p_a^\nu + \big(1 + d_a f_a^{\rm eq}\big)C_b p_b^\mu p_b^\nu\big]\big\}$$

$$+ \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d)\big\{ f_a^{\rm eq}\big[\big(1 + d_d f_d^{\rm eq}\big)C_c p_c^\mu p_c^\nu + \big(1 + d_c f_c^{\rm eq}\big)C_d p_d^\mu p_d^\nu\big] - f_c^{\rm eq} f_d^{\rm eq} C_a p_a^\mu p_a^\mu\big\}$$

$$+ \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, W(c|a,b)\big\{ - f_c^{\rm eq}\big[\big(1 + d_b f_b^{\rm eq}\big)C_a p_a^\mu p_a^\nu + \big(1 + d_a f_a^{\rm eq}\big)C_b p_b^\mu p_b^\nu\big] + f_a^{\rm eq} f_b^{\rm eq} C_c p_c^\mu p_c^\nu\big\}. \tag{A16}$$

Owing to the tensorial structure of these equations the solution requires that $\mathcal{A}_a = 0$, $\mathcal{B}_a^\mu = 0$, and $\mathcal{C}_a^{\mu\nu} = 0$. These are integral equations for the functions A_a, B_a, and C_a which depend on the magnitude of the momentum $\mathbf{p}^*$.

The solutions for A_a and B_a are not unique. It is necessary to specify whether u^μ represents the flow of energy (Landau-Lifshitz) or baryon number (Eckart). We enforce the Landau-Lifshitz condition, sometimes known as the condition of fit, using any particular solutions. The results for the transport coefficients are

$$\zeta = \frac{1}{3} \sum_a \int d\Gamma_a^* \left[\frac{|\mathbf{p}_a^*|^2}{E_a^*} + 3 v_n^2 T^2 \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right] A_a^{\rm par} f_a^{\rm eq}, \tag{A17}$$

$$\lambda = \frac{1}{3}\left(\frac{w}{n_B T}\right)^2 \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^*}\left(b_a - \frac{n_B E_a}{w}\right) B_a^{\text{par}} f_a^{\text{eq}}, \tag{A18}$$

$$\eta = \frac{2}{15} \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^4}{E_a^*} C_a^{\text{par}} f_a^{\text{eq}}. \tag{A19}$$

The particular solutions need not even satisfy the Boltzmann equation to satisfy the condition of fit.

A common approximation is the energy-dependent relaxation-time approximation. It assumes that only one ϕ_a is nonzero and the others vanish. Then the Boltzmann equation is approximated by

$$\frac{df_a^{\text{eq}}}{dt} = \mathcal{C}_a = -\frac{f_a^{\text{eq}}\phi_a}{\tau_a}, \tag{A20}$$

where the relaxation time $\tau_a(E_a^*)$ for species a is given by

$$\frac{1 + d_a f_a^{\text{eq}}}{\tau_a(E_a^*)} = \sum_{bcd} \frac{1}{1+\delta_{ab}} \int d\Gamma_b^* \, d\Gamma_c^* \, d\Gamma_d^* \, W(a,b|c,d) f_b^{\text{eq}}\big(1 + d_c f_c^{\text{eq}}\big)\big(1 + d_d f_d^{\text{eq}}\big)$$
$$+ \sum_{cd} \int d\Gamma_c^* \, d\Gamma_d^* \, W(a|c,d)\big(1 + d_c f_c^{\text{eq}}\big)\big(1 + d_d f_d^{\text{eq}}\big) + \sum_{bc} \int d\Gamma_b^* \, d\Gamma_c^* \, W(c|a,b) f_b^{\text{eq}}\big(1 + d_c f_c^{\text{eq}}\big). \tag{A21}$$

The particular solutions are

$$A_a^{\text{par}} = \frac{\tau_a}{3T}\left[\frac{|\mathbf{p}_a^*|^2}{E_a^*} + 3v_n^2 T^2 \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right]\big(1 + d_a f_a^{\text{eq}}\big), \tag{A22}$$

$$B_a^{\text{par}} = \frac{\tau_a}{E_a^*}\left(b_a - \frac{n_B E_a}{w}\right)\big(1 + d_a f_a^{\text{eq}}\big), \tag{A23}$$

$$C_a^{\text{par}} = \frac{\tau_a}{2T E_a^*}\big(1 + d_a f_a^{\text{eq}}\big). \tag{A24}$$

Substitution gives the transport coefficients

$$\eta = \frac{1}{15T} \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^4}{E_a^{*2}} \tau_a(E_a^*) f_a^{\text{eq}}\big(1 + d_a f_a^{\text{eq}}\big), \tag{A25}$$

$$\zeta = \frac{1}{9T} \sum_a \int d\Gamma_a^* \frac{\tau_a(E_a^*)}{E_a^{*2}}\left[|\mathbf{p}_a^*|^2 + 3v_n^2 T^2 E_a^* \frac{\partial}{\partial T}\left(\frac{E_a - \mu_a}{T}\right)_\sigma\right]^2 f_a^{\text{eq}}\big(1 + d_a f_a^{\text{eq}}\big), \tag{A26}$$

$$\lambda = \frac{1}{3}\left(\frac{w}{n_B T}\right)^2 \sum_a \int d\Gamma_a^* \frac{|\mathbf{p}_a^*|^2}{E_a^{*2}} \tau_a(E_a^*)\left(b_a - \frac{n_B E_a}{w}\right)^2 f_a^{\text{eq}}\big(1 + d_a f_a^{\text{eq}}\big). \tag{A27}$$

These are clearly positive definite.

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