# Leading three-baryon forces from SU(3) chiral effective field theory

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Leading three-baryon forces are derived within SU(3) chiral effective field theory. Three classes of irreducible diagrams contribute: three-baryon contact terms, one-meson exchange, and two-meson exchange diagrams. We provide the minimal nonrelativistic terms of the chiral Lagrangian that contribute to these diagrams. SU(3) relations are given for the strangeness $S = 0$ and $-1$ sectors. In the strangeness-zero sector we recover the well-known three-nucleon forces from chiral effective field theory. Explicit expressions for the $\Lambda NN$ chiral potential in isospin space are presented.

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## I. INTRODUCTION

Solving nuclear few- and many-body problems based on microscopic interactions has been a continuous challenge in nuclear physics. Nowadays the nucleon-nucleon ($NN$) interaction can be treated to high accuracy using phenomenological models [1–3] or potentials derived from chiral effective field theory ($\chi$EFT) [4,5]. However, few-body systems such as the triton cannot be described satisfactorily with two-body forces only. Substantial improvements result from the consideration of three-nucleon forces (3NFs) [6,7]. These 3NFs are introduced either phenomenologically, such as the families of Tuscon-Melbourne [8,9], Brazilian [10], or Urbana-Illinois [11,12] 3NFs, or deduced from more basic principles using $\chi$EFT [7,13–21]. Effective field theory approaches have the advantage that 3NFs can be derived consistently with the underlying $NN$ interaction and that theoretical error estimates are possible.

The situation in strangeness nuclear physics is less clear. Owing to the lack of high-precision experimental data, the hyperon-nucleon ($YN$) interaction cannot be sufficiently well constrained. Different models describe the empirical scattering data equivalently [22–25], but differ considerably from each other. Nonetheless, three-baryon forces (3BFs), in particular a repulsive $\Lambda NN$ interaction, appear to be essential for the description of hypernuclei and hypernuclear matter [26–34]. Empirical facts about dense neutron-star matter favor such considerations. The recent observation of two-solar-mass neutron stars [35,36] sets strong stiffness constraints for the equation of state (EoS) of dense baryonic matter [37–39]. A naive introduction of $\Lambda$ hyperons as an additional baryonic degree of freedom in neutron-star matter would soften the EoS [40] such that it is not possible to stabilize two-solar-mass neutron stars against gravitational collapse. The introduction of strongly repulsive $YNN$ forces is one possible suggestion to improve the situation [41–43].

So far, baryonic three-body forces involving hyperons have been investigated only by employing phenomenological interactions, and a more systematic approach is desirable. Chiral effective field theory is an appropriate tool for such considerations. It exploits the symmetries of quantum chromodynamics together with the appropriate low-energy degrees of freedom. The description of the low-energy interaction of hadrons can be improved systematically by going to higher order in the power counting in small momenta. Furthermore, the hierarchy of baryonic forces, from long-range to intermediate- and short-range interactions, emerges naturally within this framework. Two- and three-baryon forces can be described in a consistent way.

Recently, the $YN$ interaction has been studied up to next-to-leading order (NLO) in $\chi$EFT. The $YN$ scattering data [25], as well as the self-energies of hyperons in nuclear matter [44,45], can be well described within this framework. The irreducible chiral 3BFs appear formally at next-to-next-to-leading order (NNLO) [4]. However, e.g., the low-energy constants of the 3NFs at NNLO are unnaturally large and cause effects comparable to those one would expect at the NLO level. These large values are connected with the excitation of the low-lying $\Delta(1232)$ resonance and can be understood in terms of the so-called resonance saturation. Indeed, the inclusion of the $\Delta$ isobar as an explicit degree of freedom in EFT promotes parts of the 3NFs to NLO [4,46]. In systems with strangeness $S = -1$, resonances such as the $\Sigma^*(1385)$ could play a similar role as the $\Delta$ in the $NNN$ system. It is therefore likewise compelling to treat their effects in 3BFs together with the NLO $YN$ interaction.

In the standard power counting scheme of the baryonic forces in $\chi$EFT (cf. Refs. [4,5]) the chiral dimension $\nu$ of a given Feynman diagram is determined by

$$\nu = -4 + 2\mathcal{B} + 2L + \sum_i v_i \Delta_i,$$
$$\Delta_i = d_i + \frac{1}{2} b_i - 2, \tag{1}$$

where $\mathcal{B}$ is the number of external baryons and $L$ the number of Goldstone boson loops. The number of vertices with vertex dimension $\Delta_i \geqslant 0$ is denoted by $v_i$. The symbol $d_i$ stands

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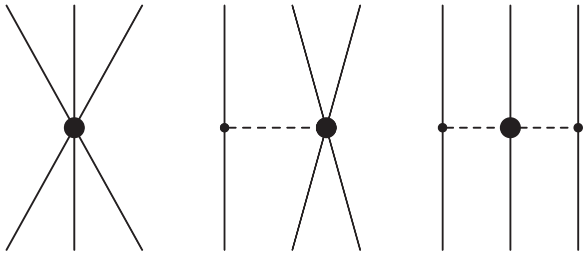

FIG. 1. Leading three-baryon interactions: (from left to right) contact term, one-meson exchange, and two-meson exchange. Solid circles and solid dots denote vertices with $\Delta_i = 1$ and $\Delta_i = 0$, respectively.

for the number of derivatives or pseudoscalar-meson mass insertions at the vertex and $b_i$ is the number of internal baryon lines at the considered vertex. Following Eq. (1), one obtains at NNLO with $\nu = 3$ the leading three-baryon diagrams of Fig. 1 in complete analogy to the leading 3NFs. Note that a two-meson exchange diagram, like in Fig. 1, with a (leading-order) Weinberg-Tomozawa vertex in the middle, would formally be a NLO contribution. However, as in the nucleonic sector, this contribution is kinematically suppressed to higher order. In SU(3) $\chi$EFT nucleons and strange baryons ($\Lambda$, $\Sigma$, $\Xi$) are treated on equal footing. Accordingly, reducible diagrams involving those baryons do not constitute genuine 3BFs. These diagrams must not be included into the chiral potential, as they will be generated automatically when solving the Faddeev or Yakubovsky equations consistently within a coupled-channel approach. This differs from typical phenomenological calculations with $\Lambda NN$ 3BFs, where reducible diagrams like the one with two one-meson exchanges and an intermediate $\Sigma NN$ state are often used. In our approach such diagrams do not correspond to a 3BF, but to an iterated two-baryon force.

In this work we construct the potentials for the leading 3BFs relevant for few- and many-body calculations, within the framework of SU(3) $\chi$EFT. The present paper is organized as follows. In Sec. II we show the minimal effective Lagrangian for six-baryon contact terms and its construction principles. We explain how antisymmetrized potentials can be obtained from the contact Lagrangian. Furthermore, we investigate the group-theoretical classification of the interactions and provide SU(3) relations for the strangeness 0 and $-1$ sectors. In Sec. III the minimal chiral Lagrangian for the four-baryon contact vertex involving one pseudoscalar meson is given and applied to the 3BF with one-meson exchange. Section IV is devoted to the two-meson exchange potentials. In Sec. V we provide explicit expressions for the potentials of the $\Lambda NN$ interaction for the contact term and the pion-exchange components. For comparison the three-body potentials in the nucleonic sector are reproduced. Conclusions and an outlook are given in Sec. VI.

## II. CONTACT INTERACTION

In the following we consider the three-baryon contact interaction. We construct the minimal Lagrangian, demonstrate how to derive the antisymmetrized potentials, and investigate their group-theoretical classification.

### A. Overcomplete contact Lagrangian

The terms of the effective Lagrangian have to fulfill the symmetries of quantum chromodynamics and are constructed to obey invariance under charge conjugation, parity transformation, Hermitian conjugation, and the local chiral symmetry group $\mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$. The baryon fields are collected in the traceless matrix

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}. \tag{2}$$

To obtain the most general contact Lagrangian in flavor SU(3), we follow the same procedure as used for the four-baryon contact terms in Ref. [47]. Generalizing these construction rules straightforwardly to six-baryon contact terms, we end up with a (largely) overcomplete set of terms for the leading covariant Lagrangian,

$$\mathcal{L} = \sum_{f=1}^{11} \sum_{a=1}^{5} t^{f,a} \mathcal{T}^{f,a}, \tag{3}$$

where the index $f$ runs over 11 possible flavor structures. These are given by

$$\begin{aligned}
\mathcal{T}^{1,a} &= \langle \bar{B}_\alpha \bar{B}_\beta \bar{B}_\gamma (\Gamma^{1,a} B)_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\gamma \bar{B}_\beta \bar{B}_\alpha (\Gamma^{3,a} B)_\gamma (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{2,a} &= \langle \bar{B}_\alpha \bar{B}_\beta (\Gamma^{1,a} B)_\alpha \bar{B}_\gamma (\Gamma^{2,a} B)_\beta (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\gamma \bar{B}_\beta (\Gamma^{3,a} B)_\gamma \bar{B}_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{3,a} &= \langle \bar{B}_\alpha \bar{B}_\beta (\Gamma^{1,a} B)_\alpha (\Gamma^{2,a} B)_\beta \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\beta \bar{B}_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle, \\
\mathcal{T}^{4,a} &= \langle \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \bar{B}_\beta (\Gamma^{2,a} B)_\beta \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \bar{B}_\beta (\Gamma^{2,a} B)_\beta \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{5,a} &= \langle \bar{B}_\alpha \bar{B}_\beta (\Gamma^{1,a} B)_\alpha (\Gamma^{2,a} B)_\beta \rangle \, \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\beta \bar{B}_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle \, \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle, \\
\mathcal{T}^{6,a} &= \langle \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \bar{B}_\beta (\Gamma^{2,a} B)_\beta \rangle \, \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \bar{B}_\beta (\Gamma^{2,a} B)_\beta \rangle \, \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle, \\
\mathcal{T}^{7,a} &= \langle \bar{B}_\alpha \bar{B}_\beta \bar{B}_\gamma (\Gamma^{1,a} B)_\alpha \rangle \, \langle (\Gamma^{2,a} B)_\beta (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\gamma \bar{B}_\beta \rangle \, \langle \bar{B}_\alpha (\Gamma^{3,a} B)_\gamma (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{8,a} &= \langle \bar{B}_\alpha \bar{B}_\beta \bar{B}_\gamma \rangle \, \langle (\Gamma^{1,a} B)_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\gamma \bar{B}_\beta \bar{B}_\alpha \rangle \, \langle (\Gamma^{3,a} B)_\gamma (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{9,a} &= \langle \bar{B}_\alpha \bar{B}_\beta (\Gamma^{1,a} B)_\alpha \rangle \, \langle (\Gamma^{2,a} B)_\beta \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\beta \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \, \langle \bar{B}_\alpha (\Gamma^{2,a} B)_\beta (\Gamma^{1,a} B)_\alpha \rangle, \\
\mathcal{T}^{10,a} &= \langle \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \rangle \, \langle \bar{B}_\beta (\Gamma^{2,a} B)_\beta \rangle \, \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle \\
&\quad + (-1)^{c_a} \langle \bar{B}_\alpha (\Gamma^{1,a} B)_\alpha \rangle \langle \bar{B}_\beta (\Gamma^{2,a} B)_\beta \rangle \langle \bar{B}_\gamma (\Gamma^{3,a} B)_\gamma \rangle,
\end{aligned}$$

TABLE I. Dirac structures $\Gamma^1,\Gamma^2,\Gamma^3$. Only structures with independent potential contributions are considered.

| $a$ | $c_a$ | $\Gamma^{1,a}$ | $\Gamma^{2,a}$ | $\Gamma^{3,a}$ | $V^a_{ijk} = (\bar{u}\Gamma^{1,a}u)_i(\bar{u}\Gamma^{2,a}u)_j(\bar{u}\Gamma^{3,a}u)_k$ |
|---|---|---|---|---|---|
| 1 | 0 | $\mathbb{1}$ | $\mathbb{1}$ | $\mathbb{1}$ | $\mathbb{1}$ |
| 2 | 0 | $-\mathbb{1}$ | $\gamma_5\gamma^\mu$ | $\gamma_5\gamma_\mu$ | $\vec{\sigma}_j\cdot\vec{\sigma}_k$ |
| 3 | 0 | $\gamma_5\gamma^\mu$ | $-\mathbb{1}$ | $\gamma_5\gamma_\mu$ | $\vec{\sigma}_i\cdot\vec{\sigma}_k$ |
| 4 | 0 | $\gamma_5\gamma^\mu$ | $\gamma_5\gamma_\mu$ | $-\mathbb{1}$ | $\vec{\sigma}_i\cdot\vec{\sigma}_j$ |
| 5 | 1 | $\gamma_5\gamma_\mu$ | $-i\sigma^{\mu\nu}$ | $\gamma_5\gamma_\nu$ | $i\vec{\sigma}_i\cdot(\vec{\sigma}_j\times\vec{\sigma}_k)$ |

$$\mathcal{T}^{11,a} = \langle\bar{B}_\alpha\bar{B}_\beta\rangle\,\langle\bar{B}_\gamma(\Gamma^{1,a}B)_\alpha\rangle\,\langle(\Gamma^{2,a}B)_\beta(\Gamma^{3,a}B)_\gamma\rangle + (-1)^{c_a}\langle\bar{B}_\gamma\bar{B}_\beta\rangle\langle\bar{B}_\alpha(\Gamma^{3,a}B)_\gamma\rangle\langle(\Gamma^{2,a}B)_\beta(\Gamma^{1,a}B)_\alpha\rangle, \tag{4}$$

where the indices $\alpha,\beta,\gamma$ are Dirac indices. The index $a = 1,\dots,5$ in Eq. (3) labels the three combined Dirac structures $\Gamma^{1,a},\Gamma^{2,a},\Gamma^{3,a}$ that have to be inserted into each flavor structure $f = 1,\dots,11$. The allowed Dirac structures are given in Table I. Note that we start with a covariant Lagrangian, but in the end are only interested in the minimal nonrelativistic Lagrangian. Therefore, only Dirac structures that lead to independent (nonrelativistic) spin operators are considered in Table I. The corresponding spin-dependent potentials $V^a_{ijk}$ (shown in the last column of Table I) are defined by the Dirac structures sandwiched between Dirac spinors in spin spaces $i$, $j$, and $k$. The overcomplete set of terms in the Lagrangian Eq. (3) contains 55 low-energy constants $t^{f,a}$. One observes that some combinations of Dirac and flavor structures do not even contribute at the leading order. Nevertheless, this set is a good starting point to obtain the minimal nonrelativistic contact Lagrangian.

It is advantageous to rewrite the Lagrangian in the particle basis, which gives

$$\mathcal{L} = \sum_{f=1}^{11}\sum_{a=1}^{5}\tilde{t}^{f,a}\sum_{i,j,k,l,m,n} \times N^{f,a}_{\substack{ikm\\ jln}}(\bar{B}_i\Gamma^{1,a}B_j)(\bar{B}_k\Gamma^{2,a}B_l)(\bar{B}_m\Gamma^{3,a}B_n). \tag{5}$$

where $B_i$ are the baryon fields in the particle basis and the indices $i,j,k,l,m,n$ label the six occurring baryon fields, $B_i \in \{n,p,\Lambda,\Sigma^+,\Sigma^0,\Sigma^-,\Xi^0,\Xi^-\}$. The SU(3) factors $N$ can be obtained easily by employing Eq. (2), multiplying the respective flavor matrices, and taking traces. Note that the constants $\tilde{t}^{f,a}$ are equal to $t^{f,a}$, but with an additional minus sign for $f = 1,3,5,7,8,9,11$, coming from the interchange of anticommuting baryon fields.

### B. Derivation of the contact potential

Let us now consider the process $B_1B_2B_3 \to B_4B_5B_6$, where the $B_i$ are again baryons in the particle basis. The aim is to derive a potential operator $V$ in the (multiple) spin space for this process. We define the operators in spin space 1 to act between the two-component Pauli spinors of $B_1$ and $B_4$. Similarly, spin space 2 belongs to $B_2$ and $B_5$ and spin space 3 to $B_3$ and $B_6$. The potential for a fixed spin configuration is then obtained as

$$\chi^{(1)\dagger}_{B_4}\chi^{(2)\dagger}_{B_5}\chi^{(3)\dagger}_{B_6}\,V\,\chi^{(1)}_{B_1}\chi^{(2)}_{B_2}\chi^{(3)}_{B_3}, \tag{6}$$

where the superscript of a spinor denotes the spin space and the subscript denotes the baryon to which the spinor belongs.

The potential is given by $V = -\langle B_4B_5B_6|\ \mathcal{L}\ |B_1B_2B_3\rangle$, where the appropriate terms of $\mathcal{L}$ in Eq. (5) have to be inserted, and the 36 Wick contractions have to be performed. First, each of the 55 terms in the Lagrangian (labeled by $f,a$) provides six so-called direct terms,

$$\begin{aligned}
&\tilde{t}^{f,a}N^{f,a}_{\substack{456\\123}}(\bar{B}_4\Gamma^{1,a}B_1)(\bar{B}_5\Gamma^{2,a}B_2)(\bar{B}_6\Gamma^{3,a}B_3)\\
&+\tilde{t}^{f,a}N^{f,a}_{\substack{564\\231}}(\bar{B}_5\Gamma^{1,a}B_2)(\bar{B}_6\Gamma^{2,a}B_3)(\bar{B}_4\Gamma^{3,a}B_1)\\
&+\tilde{t}^{f,a}N^{f,a}_{\substack{645\\312}}(\bar{B}_6\Gamma^{1,a}B_3)(\bar{B}_4\Gamma^{2,a}B_1)(\bar{B}_5\Gamma^{3,a}B_2)\\
&+\tilde{t}^{f,a}N^{f,a}_{\substack{465\\132}}(\bar{B}_4\Gamma^{1,a}B_1)(\bar{B}_6\Gamma^{2,a}B_3)(\bar{B}_5\Gamma^{3,a}B_2)\\
&+\tilde{t}^{f,a}N^{f,a}_{\substack{654\\321}}(\bar{B}_6\Gamma^{1,a}B_3)(\bar{B}_5\Gamma^{2,a}B_2)(\bar{B}_4\Gamma^{3,a}B_1)\\
&+\tilde{t}^{f,a}N^{f,a}_{\substack{546\\213}}(\bar{B}_5\Gamma^{1,a}B_2)(\bar{B}_4\Gamma^{2,a}B_1)(\bar{B}_6\Gamma^{3,a}B_3),
\end{aligned} \tag{7}$$

where the baryon bilinears combine the baryon pairs 1-4, 2-5, and 3-6 in the form as set up in Eq. (6). Keeping in mind that baryons $B_1$, $B_2$, $B_3$ are in spin spaces 1, 2, 3, respectively, one obtains by performing the (six direct) Wick contractions the direct potential[1]

$$V^D = -\sum_{f=1}^{11}\sum_{a=1}^{5}\tilde{t}^{f,a}\Big(N^{f,a}_{\substack{456\\123}}V^a_{123} + N^{f,a}_{\substack{564\\231}}V^a_{231} + N^{f,a}_{\substack{645\\312}}V^a_{312} + N^{f,a}_{\substack{465\\132}}V^a_{132} + N^{f,a}_{\substack{654\\321}}V^a_{321} + N^{f,a}_{\substack{546\\213}}V^a_{213}\Big). \tag{8}$$

The spin operators $V^a_{ijk}$ arise from the Dirac structures $\Gamma^{1,a}\otimes\Gamma^{2,a}\otimes\Gamma^{3,a}$ and can be found in Table I. The indices $i,j,k$ of $V^a_{ijk}$ denote the spin spaces of the three baryon bilinears.

One has not only these six direct Wick contractions, but in total 36 Wick contractions that contribute to the potential. This number corresponds to the $3!\times 3!$ possibilities to arrange the three initial and three final baryons into Dirac bilinears. For example, a term

$$\tilde{t}^{f,a}N^{f,a}_{\substack{546\\312}}(\bar{B}_5\Gamma^{1,a}B_3)(\bar{B}_4\Gamma^{2,a}B_1)(\bar{B}_6\Gamma^{3,a}B_2) \tag{9}$$

gives rise to a potential contribution

$$\tilde{t}^{f,a}N^{f,a}_{\substack{546\\312}}V^a_{312}, \tag{10}$$

[1] One observes that Eq. (8) holds independently of whether some of the baryons are identical or not.

where the sign, reverted in comparison to Eq. (8), originates from the exchange of baryon fields. However, this potential is not in accordance with the form of Eq. (6), as baryon pairs 1-4, 2-6, 3-5 are each connected in a separate spin space. Hence, an exchange of the spin-wave functions $\chi^{(2)}_{B_5}$ and $\chi^{(3)}_{B_6}$ in the final state has to be performed and this is achieved by multiplying the potential with $P^{(\sigma)}_{23}$,

$$\tilde{t}^{f,a} N^{f,a}_{\substack{546\\312}} P^{(\sigma)}_{23} V^{a}_{312}, \tag{11}$$

where $P^{(\sigma)}_{ij} = \frac{1}{2}(\mathbb{1} + \vec{\sigma}_i \cdot \vec{\sigma}_j)$ is the well-known spin-exchange operator.

Employing the above considerations to all Wick contractions, the full potential including 36 contributions is derived. For a shorter notation we express the remaining 30 contributions in terms of the six direct contributions in Eq. (8), with the declared replacement of the labels. The full potential is thus given by

$$\begin{aligned} V = V^D &+ P^{(\sigma)}_{23} P^{(\sigma)}_{13} (V^D)_{\substack{4\to5\\5\to6\\6\to4}} + P^{(\sigma)}_{23} P^{(\sigma)}_{12} (V^D)_{\substack{4\to6\\5\to4\\6\to5}} \\ &- P^{(\sigma)}_{23} (V^D)_{\substack{4\to4\\5\to6\\6\to5}} - P^{(\sigma)}_{13} (V^D)_{\substack{4\to6\\5\to5\\6\to4}} - P^{(\sigma)}_{12} (V^D)_{\substack{4\to5\\5\to4\\6\to6}}. \end{aligned} \tag{12}$$

The procedure described above automatically incorporates the generalized Pauli principle and leads to an antisymmetrized potential.

### C. Minimal contact Lagrangian

Now we are in the position to determine a minimal and complete contact Lagrangian for the leading three-baryon contact interaction. We have derived the potential according to Eq. (12) and decomposed it with respect to the following operators in the three-body spin space,

$$\mathbb{1}, \quad \vec{\sigma}_1 \cdot \vec{\sigma}_2, \quad \vec{\sigma}_1 \cdot \vec{\sigma}_3, \quad \vec{\sigma}_2 \cdot \vec{\sigma}_3, \quad i\vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3). \tag{13}$$

A minimal set of Lagrangian terms in the nonrelativistic limit is obtained by leaving out terms until the rank of the final potential matrix matches the number of terms in the Lagrangian. Redundant terms have been deleted in such a way that one obtains a maximal number of Lagrangian terms with a single flavor trace. The minimal six-baryon contact Lagrangian in the nonrelativistic limit is then given by

$$\begin{aligned} \mathcal{L} = &- C_1 \langle \bar{B}_a \bar{B}_b \bar{B}_c B_a B_b B_c \rangle + C_2 \langle \bar{B}_a \bar{B}_b B_a \bar{B}_c B_b B_c \rangle \\ &- C_3 \langle \bar{B}_a \bar{B}_b B_a B_b \bar{B}_c B_c \rangle + C_4 \langle \bar{B}_a B_a \bar{B}_b B_b \bar{B}_c B_c \rangle \\ &- C_5 \langle \bar{B}_a \bar{B}_b B_a B_b \rangle \langle \bar{B}_c B_c \rangle \\ &- C_6 [\langle \bar{B}_a \bar{B}_b \bar{B}_c B_a (\sigma^i B)_b (\sigma^i B)_c \rangle \\ &\quad + \langle \bar{B}_c \bar{B}_b \bar{B}_a (\sigma^i B)_c (\sigma^i B)_b B_a \rangle] \\ &+ C_7 [\langle \bar{B}_a \bar{B}_b B_a \bar{B}_c (\sigma^i B)_b (\sigma^i B)_c \rangle \\ &\quad + \langle \bar{B}_c \bar{B}_b (\sigma^i B)_c \bar{B}_a (\sigma^i B)_b B_a \rangle] \\ &- C_8 [\langle \bar{B}_a \bar{B}_b B_a (\sigma^i B)_b \bar{B}_c (\sigma^i B)_c \rangle \\ &\quad + \langle \bar{B}_b \bar{B}_a (\sigma^i B)_b B_a \bar{B}_c (\sigma^i B)_c \rangle] \\ &+ C_9 \langle \bar{B}_a B_a \bar{B}_b (\sigma^i B)_b \bar{B}_c (\sigma^i B)_c \rangle \\ &- C_{10} [\langle \bar{B}_a \bar{B}_b B_a (\sigma^i B)_b \rangle \langle \bar{B}_c (\sigma^i B)_c \rangle \\ &\quad + \langle \bar{B}_b \bar{B}_a (\sigma^i B)_b B_a \rangle \langle \bar{B}_c (\sigma^i B)_c \rangle] \\ &- C_{11} \langle \bar{B}_a \bar{B}_b \bar{B}_c (\sigma^i B)_a B_b (\sigma^i B)_c \rangle \\ &+ C_{12} \langle \bar{B}_a \bar{B}_b (\sigma^i B)_a \bar{B}_c B_b (\sigma^i B)_c \rangle \\ &- C_{13} \langle \bar{B}_a \bar{B}_b (\sigma^i B)_a (\sigma^i B)_b \bar{B}_c B_c \rangle \\ &- C_{14} \langle \bar{B}_a \bar{B}_b (\sigma^i B)_a (\sigma^i B)_b \rangle \langle \bar{B}_c B_c \rangle \\ &- i\epsilon^{ijk} C_{15} \langle \bar{B}_a \bar{B}_b \bar{B}_c (\sigma^i B)_a (\sigma^j B)_b (\sigma^k B)_c \rangle \\ &+ i\epsilon^{ijk} C_{16} \langle \bar{B}_a \bar{B}_b (\sigma^i B)_a \bar{B}_c (\sigma^j B)_b (\sigma^k B)_c \rangle \\ &- i\epsilon^{ijk} C_{17} \langle \bar{B}_a \bar{B}_b (\sigma^i B)_a (\sigma^j B)_b \bar{B}_c (\sigma^k B)_c \rangle \\ &+ i\epsilon^{ijk} C_{18} \langle \bar{B}_a (\sigma^i B)_a \bar{B}_b (\sigma^j B)_b \bar{B}_c (\sigma^k B)_c \rangle. \end{aligned} \tag{14}$$

The indices $a,b,c$ are two-component spinor indices and the indices $i,j,k$ are vector indices. One ends up with 18 low-energy constants $C_1,\ldots,C_{18}$. The minus signs in front of some terms have been included to compensate minus signs from fermion exchange, arising from reordering baryon bilinears into the form of Eq. (5).

Various checks have been performed. In particular, we verified conservation of strangeness $S$, isospin $I$, and isospin projection $I_3$ and the independence of the resulting potentials from $I_3$. The Lagrangian has been constructed to fulfill $C$ and $P$ symmetry. Time-reversal symmetry follows via the $CPT$ theorem, and we explicitly confirmed $T$ invariance for all potentials.

### D. Group-theoretical considerations

Let us now consider the three-baryon contact terms from a group-theoretical point of view. In flavor space the three octet baryons form the tensor product $\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8}$, which decomposes into the irreducible SU(3) representations

$$\begin{aligned} \mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8} = \mathbf{64} \oplus (\mathbf{35} \oplus \overline{\mathbf{35}})_2 \oplus \mathbf{27}_6 \\ \oplus (\mathbf{10} \oplus \overline{\mathbf{10}})_4 \oplus \mathbf{8}_8 \oplus \mathbf{1}_2, \end{aligned} \tag{15}$$

where the subscripts denote the multiplicity of a representation. In spin space the tensor product of three doublets decomposes as

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2}_2 \oplus \mathbf{4}. \tag{16}$$

Transitions are only allowed between irreducible representations of the same type. In analogy to Ref. [48] for the two-baryon sector, we determine which of the irreducible representations in Eq. (15) can contribute to a particular three-baryon multiplet, characterized by hypercharge $Y = S + 3$ (with strangeness $S$) and isospin $I$. Table II gives for the relevant SU(3) representations the $(Y,I)$ multiplets that they contain. From this table one can read off which representations are involved in the various three-baryon states, presented in Table III. At leading order the potentials are momentum independent and therefore only $S$ waves are present. Owing to

TABLE II. Hypercharge $Y$ and isospin $I$ for irreducible SU(3) representations of dimension $D$.

| $D$ | Allowed $(Y, I)$ |
|---|---|
| $\mathbf{1}$ | (0,0) |
| $\mathbf{8}$ | $(1,\frac{1}{2})$,(0,0),(0,1),$(-1,\frac{1}{2})$ |
| $\mathbf{10}$ | $(1,\frac{3}{2})$,(0,1),$(-1,\frac{1}{2})$,(−2,0) |
| $\overline{\mathbf{10}}$ | (2,0),$(1,\frac{1}{2})$,(0,1),$(-1,\frac{3}{2})$ |
| $\mathbf{27}$ | (2,1),$(1,\frac{1}{2})$,$(1,\frac{3}{2})$,(0,0),(0,1), (0,2),$(-1,\frac{1}{2})$,$(-1,\frac{3}{2})$,(−2,1) |
| $\mathbf{35}$ | (2,2),$(1,\frac{3}{2})$,$(1,\frac{5}{2})$,(0,1),(0,2),$(-1,\frac{1}{2})$, $(-1,\frac{3}{2})$,(−2,0),(−2,1),$(-3,\frac{1}{2})$ |
| $\overline{\mathbf{35}}$ | $(3,\frac{1}{2})$,(2,0),(2,1),$(1,\frac{1}{2})$,$(1,\frac{3}{2})$,(0,1), (0,2),$(-1,\frac{3}{2})$,$(-1,\frac{5}{2})$,(−2,2) |
| $\mathbf{64}$ | $(3,\frac{3}{2})$,(2,1),(2,2),$(1,\frac{1}{2})$,$(1,\frac{3}{2})$,$(1,\frac{5}{2})$, (0,0),(0,1),(0,2),(0,3),$(-1,\frac{1}{2})$, $(-1,\frac{3}{2})$,$(-1,\frac{5}{2})$,(−2,1),(−2,2),$(-3,\frac{3}{2})$ |

the Pauli principle the totally symmetric spin quartet **4** must combine with the totally antisymmetric part of $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}$ in flavor space,

$$\mathrm{Alt}_3(\mathbf{8}) = \mathbf{56}_a = \mathbf{27}_a + \mathbf{10}_a + \overline{\mathbf{10}}_a + \mathbf{8}_a + \mathbf{1}_a. \tag{17}$$

Therefore, these totally antisymmetric representations are present only in partial waves with total spin 3/2. Furthermore, the totally symmetric part of $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}$ decomposes as

$$\mathrm{Sym}_3(\mathbf{8}) = \mathbf{120}_s = \mathbf{64}_s + \mathbf{27}_s + \mathbf{10}_s + \overline{\mathbf{10}}_s + \mathbf{8}_s + \mathbf{1}_s. \tag{18}$$

Because it has no totally antisymmetric counterpart in spin space, it cannot contribute. This is especially true for the highest dimensional **64** representation, which appears only once in the decomposition $\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}$. In Table III we have already included these exclusion criteria that follow from the generalized Pauli principle.

In the next step, we can derive the potentials for transitions between the three-baryon states and redefine the 18 constants such that they belong to transitions between irreducible representations. It is a highly nontrivial check of our results that this redefinition meets the restrictions of Table III. For example, in the $NNN$ interaction and the $\Xi\Xi\Sigma$ (−2,2) interaction the same constant associated with the $\overline{\mathbf{35}}$ representation has to be present.

To obtain a representation of the potentials in the isospin basis, we use the relation[2]

$$\begin{aligned}
&\langle (i_4 i_5) i_{\rm out} (i_{\rm out} i_6) I_{\rm out} M_{\rm out} | \hat{\mathcal{O}} | (i_1 i_2) i_{\rm in} (i_{\rm in} i_3) I_{\rm in} M_{\rm in} \rangle \\
&= \sum_{\substack{m_1,m_2,m_3,m_{\rm in},\\ m_4,m_5,m_6,m_{\rm out}}} \delta_{m_{\rm out}, m_4+m_5} \delta_{M_{\rm out}, m_{\rm out}+m_6} \\
&\quad \times \delta_{m_{\rm in}, m_1+m_2} \delta_{M_{\rm in}, m_{\rm in}+m_3} \\
&\quad \times C^{i_4 i_5 i_{\rm out}}_{m_4 m_5 m_{\rm out}} C^{i_{\rm out} i_6 I_{\rm out}}_{m_{\rm out} m_6 M_{\rm out}} C^{i_1 i_2 i_{\rm in}}_{m_1 m_2 m_{\rm in}} C^{i_{\rm in} i_3 I_{\rm in}}_{m_{\rm in} m_3 M_{\rm in}} \\
&\quad \times \langle i_4 m_4; i_5 m_5; i_6 m_6 | \hat{\mathcal{O}} | i_1 m_1; i_2 m_2; i_3 m_3 \rangle,
\end{aligned} \tag{19}$$

where $i$ stands for the isospin and $C$ are the Clebsch-Gordan coefficients. To be consistent with the Condon-Shortley convention for the Clebsch-Gordan coefficients, we use the baryon matrix as defined in Eq. (2) and make the following sign changes in the identification of the particle states $|i,m\rangle$: $\Sigma^+ = -|1,+1\rangle$, $\Xi^- = -|1/2,-1/2\rangle$. In Eq. (19) we have

[2]To obtain Table IV, we strictly employ Eq. (19); i.e., no further combinatorial factors, such as $1/\sqrt{2}$ for a $\Lambda NN$ state, are included. They can be included by just multiplying the corresponding row in Table IV with that factor.

TABLE III. Irreducible representations for three-baryon states with hypercharge $Y$ and isospin $I$ in partial waves.

| States | $(Y,I)$ | $^2S_{1/2}$ | $^4S_{3/2}$ |
|---|---|---|---|
| $NNN$ | $(3,\frac{1}{2})$ | $\overline{\mathbf{35}}$ | |
| $\Lambda NN,\Sigma NN$ | (2,0) | $\overline{\mathbf{10}},\overline{\mathbf{35}}$ | $\overline{\mathbf{10}}_a$ |
| $\Lambda NN,\Sigma NN$ | (2,1) | $\mathbf{27},\overline{\mathbf{35}}$ | $\mathbf{27}_a$ |
| $\Sigma NN$ | (2,2) | $\mathbf{35}$ | |
| $\Lambda\Lambda N,\Sigma\Lambda N,\Sigma\Sigma N,\Xi NN$ | $(1,\frac{1}{2})$ | $\mathbf{8},\overline{\mathbf{10}},\mathbf{27},\overline{\mathbf{35}}$ | $\mathbf{8}_a,\overline{\mathbf{10}}_a,\mathbf{27}_a$ |
| $\Sigma\Lambda N,\Sigma\Sigma N,\Xi NN$ | $(1,\frac{3}{2})$ | $\mathbf{10},\mathbf{27},\mathbf{35},\overline{\mathbf{35}}$ | $\mathbf{10}_a,\mathbf{27}_a$ |
| $\Sigma\Sigma N$ | $(1,\frac{5}{2})$ | $\mathbf{35}$ | |
| $\Lambda\Lambda\Lambda,\Sigma\Sigma\Lambda,\Sigma\Sigma\Sigma,\Xi\Lambda N,\Xi\Sigma N$ | (0,0) | $\mathbf{8},\mathbf{27}$ | $\mathbf{1}_a,\mathbf{8}_a,\mathbf{27}_a$ |
| $\Sigma\Lambda\Lambda,\Sigma\Sigma\Lambda,\Sigma\Sigma\Sigma,\Xi\Lambda N,\Xi\Sigma N$ | (0,1) | $\mathbf{8},\mathbf{10},\overline{\mathbf{10}},\mathbf{27},\mathbf{35},\overline{\mathbf{35}}$ | $\mathbf{8}_a,\mathbf{10}_a,\overline{\mathbf{10}}_a,\mathbf{27}_a$ |
| $\Sigma\Sigma\Lambda,\Sigma\Sigma\Sigma,\Xi\Sigma N$ | (0,2) | $\mathbf{27},\mathbf{35},\overline{\mathbf{35}}$ | $\mathbf{27}_a$ |
| $\Xi\Lambda\Lambda,\Xi\Sigma\Lambda,\Xi\Sigma\Sigma,\Xi\Xi N$ | $(-1,\frac{1}{2})$ | $\mathbf{8},\mathbf{10},\mathbf{27},\mathbf{35}$ | $\mathbf{8}_a,\mathbf{10}_a,\mathbf{27}_a$ |
| $\Xi\Sigma\Lambda,\Xi\Sigma\Sigma,\Xi\Xi N$ | $(-1,\frac{3}{2})$ | $\overline{\mathbf{10}},\mathbf{27},\mathbf{35},\overline{\mathbf{35}}$ | $\overline{\mathbf{10}}_a,\mathbf{27}_a$ |
| $\Xi\Sigma\Sigma$ | $(-1,\frac{5}{2})$ | $\overline{\mathbf{35}}$ | |
| $\Xi\Xi\Lambda,\Xi\Xi\Sigma$ | (−2,0) | $\mathbf{10},\mathbf{35}$ | $\mathbf{10}_a$ |
| $\Xi\Xi\Lambda,\Xi\Xi\Sigma$ | (−2,1) | $\mathbf{27},\mathbf{35}$ | $\mathbf{27}_a$ |
| $\Xi\Xi\Sigma$ | (−2,2) | $\overline{\mathbf{35}}$ | |
| $\Xi\Xi\Xi$ | $(-3,\frac{1}{2})$ | $\mathbf{35}$ | |

chosen to couple the isospin of the first two particles in the initial state $i_1$, $i_2$ to $i_{\rm in}$ and then to couple $i_{\rm in}$ with the isospin $i_3$ of the third particle to total isospin $I_{\rm in}$. The same procedure is applied to the final state. Other coupling schemes can be obtained by recoupling with the help of Racah $W$ coefficients or equivalently with Wigner's 6-$j$ symbols.

It is advantageous to present the three-body potentials not only in terms of the spin operators in Eq. (13), but to project them also onto partial-wave contributions. For a general operator

$$\hat{\mathcal{O}} = a_1\,\mathbb{1} + a_2\,\vec\sigma_1\cdot\vec\sigma_2 + a_3\,\vec\sigma_1\cdot\vec\sigma_3 + a_4\,\vec\sigma_2\cdot\vec\sigma_3 + a_5\,i\vec\sigma_1\times\vec\sigma_2\cdot\vec\sigma_3, \tag{20}$$

with coefficients $a_i$, the partial-wave decomposition leads to the nonvanishing transitions (between $S$-waves)

$$\begin{aligned}
\langle 0\,{}^2S_{1/2}|\hat{\mathcal{O}}|0\,{}^2S_{1/2}\rangle &= a_1 - 3a_2,\\
\langle 1\,{}^2S_{1/2}|\hat{\mathcal{O}}|0\,{}^2S_{1/2}\rangle &= \sqrt{3}(-a_3 + a_4 - 2a_5),\\
\langle 0\,{}^2S_{1/2}|\hat{\mathcal{O}}|1\,{}^2S_{1/2}\rangle &= \sqrt{3}(-a_3 + a_4 + 2a_5),\\
\langle 1\,{}^2S_{1/2}|\hat{\mathcal{O}}|1\,{}^2S_{1/2}\rangle &= a_1 + a_2 - 2a_3 - 2a_4,\\
\langle 1\,{}^4S_{3/2}|\hat{\mathcal{O}}|1\,{}^4S_{3/2}\rangle &= a_1 + a_2 + a_3 + a_4,
\end{aligned} \tag{21}$$

where a state $|s\;{}^{2S+1}L_J\rangle$ is characterized by the total spin $S = \frac{1}{2},\frac{3}{2}$, the angular momentum $L = 0$, and the total angular momentum $J = \frac{1}{2},\frac{3}{2}$. Here we have chosen to couple the spins of the first two baryons to $s = 0,1$ and to couple this with the spin $\frac{1}{2}$ of the third baryon to $S$ [in complete analogy to the isospin coupling in Eq. (19)]. After this partial-wave decomposition it is trivial to identify the combinations of constants belonging to the totally antisymmetric flavor representations, because these act only in the $1\;{}^4S_{3/2}$ states owing to the generalized Pauli principle.

Finally, we give the SU(3) relations for the strangeness 0 and $-1$ sectors in Table IV. The constants associated with the irreducible SU(3) representations are related to the low-energy constants of the minimal Lagrangian by

$$\begin{aligned}
c_{\overline{35}} &= 6(-C_4 + C_9),\\
c_{35} &= 3(C_4 - C_9 + 6C_{18}),\\
c_{\overline{10}} &= \tfrac{3}{4}(2C_2 + C_3 - C_4 + C_5 - 6C_8 + C_9 - 6C_{10} - 6C_{12} + 3C_{13} + 3C_{14} + 6C_{17} - 6C_{18}),\\
c_{27^1} &= -\tfrac{37}{294}C_2 + \tfrac{769}{588}C_3 - \tfrac{473}{392}C_4 + \tfrac{769}{588}C_5 - \tfrac{74}{49}C_7 - \tfrac{429}{98}C_8 + \tfrac{473}{392}C_9\\
&\quad - \tfrac{429}{98}C_{10} + \tfrac{185}{98}C_{12} + \tfrac{89}{196}C_{13} + \tfrac{89}{196}C_{14} + \tfrac{244}{49}C_{16} - \tfrac{207}{98}C_{17} + \tfrac{57}{14}C_{18},\\
c_{27^2} &= \tfrac{1}{24}(-4C_2 - 22C_3 + 57C_4 - 22C_5 - 48C_7 - 12C_8 - 57C_9 - 12C_{10}\\
&\quad + 60C_{12} + 78C_{13} + 78C_{14} - 96C_{16} + 60C_{17} - 252C_{18}),\\
c_{27^3} &= \tfrac{1}{8}(20C_2 - 2C_3 - 21C_4 - 2C_5 - 16C_7 + 28C_8 + 21C_9 + 28C_{10} - 44C_{12}\\
&\quad - 22C_{13} - 22C_{14} + 32C_{16} - 76C_{17} + 12C_{18}),\\
c_{\overline{10}a} &= 6(-C_2 + C_3 - C_4 + C_5 - 2C_7 + 2C_8 - C_9 + 2C_{10} - C_{12} + C_{13} + C_{14}),\\
c_{27a} &= \tfrac{2}{3}(C_2 + C_3 + 3C_4 + C_5 + 2C_7 + 2C_8 + 3C_9 + 2C_{10} + C_{12} + C_{13} + C_{14}).
\end{aligned} \tag{22}$$

The SU(3) relations have not been obtained by group-theory considerations directly, but by rewriting our results such that they fulfill the group-theoretical constraints of Table III. The three constants $C_{27^1}$, $C_{27^2}$, $C_{27^3}$ are associated to the irreducible representations of dimension **27**. We have chosen a particular definition for them in Eq. (22). Note that other linear combinations of $C_{27^1}$, $C_{27^2}$, $C_{27^3}$ would work equally well. The SU(3) relations in Table IV have been derived from the most general SU(3) symmetric Lagrangian. Therefore, any three-baryon potential that fulfills flavor SU(3) symmetry has to fulfill these relations. These relations provide also a valuable check for the SU(3) decomposition of the $S$-wave contributions from three-baryon interactions generated by one- or two-meson exchange (with all meson masses set equal).

## III. ONE-MESON EXCHANGE

For the one-meson exchange diagram in Fig. 1 we employ the standard chiral effective Lagrangian for meson-baryon couplings [49]

$$\mathcal{L} = \frac{D}{2}\langle\bar{B}\gamma^\mu\gamma_5\{u_\mu,B\}\rangle + \frac{F}{2}\langle\bar{B}\gamma^\mu\gamma_5[u_\mu,B]\rangle, \tag{23}$$

with the axial vector coupling constants $D \approx 0.8$ and $F \approx 0.5$ and $u_\mu = -\frac{1}{f_0}\partial_\mu\phi + \mathcal{O}(\phi^3)$, where the pseudoscalar-meson fields are collected in the traceless Hermitian matrix

$$\phi = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2\eta}{\sqrt{3}} \end{pmatrix}. \tag{24}$$

Here $f_0$ is the pion decay constant (in the chiral limit). As done in Eq. (5) it is advantageous to express this Lagrangian in the particle basis,

$$\mathcal{L} = -\sum_{i,j,k}\frac{1}{2f_0}N_{B_iB_j\phi_k}(\bar{B}_i\gamma^\mu\gamma_5 B_j)(\partial_\mu\phi_k), \tag{25}$$

TABLE IV. SU(3) relations of three-baryon contact terms with strangeness 0 and −1 in nonvanishing partial waves.

| Transition | $I$ | $i_{\rm in}$ | $i_{\rm out}$ | $V_{0^2S_{1/2}\to 0^2S_{1/2}}$ | $V_{0^2S_{1/2}\to 1^2S_{1/2}}$ | $V_{1^2S_{1/2}\to 0^2S_{1/2}}$ | $V_{1^2S_{1/2}\to 1^2S_{1/2}}$ | $V_{1^4S_{3/2}\to 1^4S_{3/2}}$ |
|---|---|---|---|---|---|---|---|---|
| $NNN\to NNN$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | $3\,c_{\overline{35}}$ | 0 |
| $NNN\to NNN$ | $\frac{1}{2}$ | 1 | 0 | 0 | $-3c_{\overline{35}}$ | 0 | 0 | 0 |
| $NNN\to NNN$ | $\frac{1}{2}$ | 0 | 1 | 0 | 0 | $-3c_{\overline{35}}$ | 0 | 0 |
| $NNN\to NNN$ | $\frac{1}{2}$ | 1 | 1 | $3c_{\overline{35}}$ | 0 | 0 | 0 | 0 |
| $\Lambda NN\to\Lambda NN$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $c_{\overline{10}}+c_{\overline{35}}$ | $\frac{c_{\overline{10}}}{\sqrt{3}}+\frac{c_{\overline{35}}}{\sqrt{3}}$ | $\frac{c_{\overline{10}}}{\sqrt{3}}+\frac{c_{\overline{35}}}{\sqrt{3}}$ | $\frac{c_{\overline{10}}}{3}+\frac{c_{\overline{35}}}{3}$ | $c_{\overline{10}a}$ |
| $\Lambda NN\to\Lambda NN$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $c_{27^1}+\frac{12c_{27^2}}{49}+\frac{3c_{\overline{35}}}{16}$ | $-\sqrt{3}c_{27^1}-\frac{12\sqrt{3}c_{27^2}}{49}-\frac{3\sqrt{3}c_{\overline{35}}}{16}$ | $-\sqrt{3}c_{27^1}-\frac{12\sqrt{3}c_{27^2}}{49}-\frac{3\sqrt{3}c_{\overline{35}}}{16}$ | $3c_{27^1}+\frac{36c_{27^2}}{49}+\frac{9c_{\overline{35}}}{16}$ | 0 |
| $\Sigma NN\to\Sigma NN$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $c_{\overline{10}}$ | $-\sqrt{3}c_{\overline{10}}$ | $-\sqrt{3}c_{\overline{10}}$ | $3c_{\overline{10}}$ | 0 |
| $\Sigma NN\to\Sigma NN$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $c_{27^2}+\frac{c_{\overline{35}}}{48}$ | $\frac{c_{27^3}}{\sqrt{3}}+\frac{\sqrt{3}c_{\overline{35}}}{16}$ | $\frac{c_{27^3}}{\sqrt{3}}+\frac{\sqrt{3}c_{\overline{35}}}{16}$ | $\frac{4c_{27^1}}{3}-\frac{c_{27^2}}{147}-\frac{2c_{27^3}}{3}+\frac{9c_{\overline{35}}}{16}$ | $c_{27a}$ |
| $\Sigma NN\to\Sigma NN$ | 1 | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{5c_{27^2}}{4\sqrt{2}}-\frac{3c_{27^3}}{4\sqrt{2}}-\frac{c_{\overline{35}}}{6\sqrt{2}}$ | $-\sqrt{\frac{3}{2}}c_{27^1}+\frac{1}{196}\sqrt{\frac{3}{2}}c_{27^2}+\frac{c_{27^3}}{4\sqrt{6}}-\frac{1}{2}\sqrt{\frac{3}{2}}c_{\overline{35}}$ | $\frac{c_{27^3}}{4\sqrt{6}}-\frac{3}{4}\sqrt{\frac{3}{2}}c_{27^2}$ | $\frac{c_{27^1}}{3\sqrt{2}}-\frac{c_{27^2}}{588\sqrt{2}}-\frac{11c_{27^3}}{12\sqrt{2}}$ | $-\sqrt{2}c_{27a}$ |
| $\Sigma NN\to\Sigma NN$ | 1 | $\frac{1}{2}$ | $\frac{3}{2}$ | $-\frac{5c_{27^2}}{4\sqrt{2}}-\frac{3c_{27^3}}{4\sqrt{2}}-\frac{c_{\overline{35}}}{6\sqrt{2}}$ | $\frac{c_{27^3}}{4\sqrt{6}}-\frac{3}{4}\sqrt{\frac{3}{2}}c_{27^2}$ | $-\sqrt{\frac{3}{2}}c_{27^1}+\frac{1}{196}\sqrt{\frac{3}{2}}c_{27^2}+\frac{c_{27^3}}{4\sqrt{6}}-\frac{1}{2}\sqrt{\frac{3}{2}}c_{\overline{35}}$ | $\frac{c_{27^1}}{3\sqrt{2}}-\frac{c_{27^2}}{588\sqrt{2}}-\frac{11c_{27^3}}{12\sqrt{2}}$ | $-\sqrt{2}c_{27a}$ |
| $\Sigma NN\to\Sigma NN$ | 1 | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{9c_{27^1}}{8}+\frac{38c_{27^2}}{49}+\frac{3c_{27^3}}{8}+\frac{2c_{\overline{35}}}{3}$ | $-\frac{\sqrt{3}c_{27^1}}{8}+\frac{23\sqrt{3}c_{27^2}}{49}+\frac{7c_{27^3}}{8\sqrt{3}}$ | $-\frac{\sqrt{3}c_{27^1}}{8}+\frac{23\sqrt{3}c_{27^2}}{49}+\frac{7c_{27^3}}{8\sqrt{3}}$ | $\frac{c_{27^1}}{24}+\frac{124c_{27^2}}{147}-\frac{5c_{27^3}}{24}$ | $2c_{27a}$ |
| $\Sigma NN\to\Sigma NN$ | 2 | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{c_{\overline{35}}}{2}$ | $-\frac{\sqrt{3}c_{\overline{35}}}{2}$ | $-\frac{\sqrt{3}c_{\overline{35}}}{2}$ | $\frac{3c_{\overline{35}}}{2}$ | 0 |
| $\Lambda NN\to\Sigma NN$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $c_{\overline{10}}-\frac{c_{\overline{35}}}{2}$ | $\frac{\sqrt{3}c_{\overline{35}}}{2}-\sqrt{3}c_{\overline{10}}$ | $\frac{c_{\overline{10}}}{\sqrt{3}}-\frac{c_{\overline{35}}}{2\sqrt{3}}$ | $\frac{c_{\overline{35}}}{2}-c_{\overline{10}}$ | 0 |
| $\Lambda NN\to\Sigma NN$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{c_{27^2}}{2}+\frac{c_{27^3}}{2}-\frac{c_{\overline{35}}}{16}$ | $\frac{2c_{27^1}}{\sqrt{3}}-\frac{c_{27^2}}{98\sqrt{3}}-\frac{c_{27^3}}{2\sqrt{3}}-\frac{3\sqrt{3}c_{\overline{35}}}{16}$ | $-\frac{\sqrt{3}c_{27^2}}{2}-\frac{\sqrt{3}c_{27^3}}{2}+\frac{\sqrt{3}c_{\overline{35}}}{16}$ | $-2c_{27^1}+\frac{c_{27^2}}{98}+\frac{c_{27^3}}{2}+\frac{9c_{\overline{35}}}{16}$ | 0 |
| $\Lambda NN\to\Sigma NN$ | 1 | $\frac{1}{2}$ | $\frac{3}{2}$ | $-\frac{3c_{27^1}}{2\sqrt{2}}-\frac{121c_{27^2}}{196\sqrt{2}}-\frac{c_{27^3}}{4\sqrt{2}}+\frac{c_{\overline{35}}}{2\sqrt{2}}$ | $\frac{c_{27^1}}{2\sqrt{6}}-\frac{221c_{27^2}}{196\sqrt{6}}-\frac{5c_{27^3}}{4\sqrt{6}}$ | $\frac{3}{2}\sqrt{\frac{3}{2}}c_{27^1}+\frac{121}{196}\sqrt{\frac{3}{2}}c_{27^2}+\frac{1}{4}\sqrt{\frac{3}{2}}c_{27^3}-\frac{1}{2}\sqrt{\frac{3}{2}}c_{\overline{35}}$ | $-\frac{c_{27^1}}{2\sqrt{2}}+\frac{221c_{27^2}}{196\sqrt{2}}+\frac{5c_{27^3}}{4\sqrt{2}}$ | 0 |

with the baryon fields as defined before $B_i \in \{n,p,\Lambda,\Sigma^+,\Sigma^0,\Sigma^-,\Xi^0,\Xi^-\}$ and the pseudoscalar-meson fields $\phi_i \in \{\pi^0,\pi^+,\pi^-,K^+,K^-,K^0,\bar{K}^0,\eta\}$.

The second vertex, necessary for the one-meson-exchange three-body interaction, involves four baryon fields and one pseudoscalar-meson field. An overcomplete set of terms for the corresponding relativistic Lagrangian can be found in our earlier work, Ref. [47]. To obtain a complete minimal set of terms in the nonrelativistic limit, we consider the matrix elements of the process $B_1B_2 \to B_3B_4\phi_1$ and proceed as in Sec. II. The transition matrix element is expressed in terms of the spin operators

$$\vec{\sigma}_1\cdot\vec{q},\quad \vec{\sigma}_2\cdot\vec{q},\quad i(\vec{\sigma}_1\times\vec{\sigma}_2)\cdot\vec{q}, \tag{26}$$

where $\vec{q}$ denotes the momentum of the emitted meson. The minimal Lagrangian is obtained by eliminating redundant terms until the rank of the matrix formed by all transitions matches the number of terms in the Lagrangian. As before, redundant terms are deleted in such a way that one obtains a maximal number of terms with a single flavor trace. The minimal nonrelativistic chiral Lagrangian for the four-baryon vertex including one meson is given by

$$\begin{aligned}\mathcal{L} =\;& D_1/f_0\langle\bar{B}_a(\nabla^i\phi)B_a\bar{B}_b(\sigma^iB)_b\rangle + D_2/f_0[\langle\bar{B}_aB_a(\nabla^i\phi)\bar{B}_b(\sigma^iB)_b\rangle + \langle\bar{B}_aB_a\bar{B}_b(\sigma^iB)_b(\nabla^i\phi)\rangle] + D_3/f_0\langle\bar{B}_b(\nabla^i\phi)(\sigma^iB)_b\bar{B}_aB_a\rangle\\ &- D_4/f_0[\langle\bar{B}_a(\nabla^i\phi)\bar{B}_bB_a(\sigma^iB)_b\rangle + \langle\bar{B}_b\bar{B}_a(\sigma^iB)_b(\nabla^i\phi)B_a\rangle] - D_5/f_0[\langle\bar{B}_a\bar{B}_b(\nabla^i\phi)B_a(\sigma^iB)_b\rangle + \langle\bar{B}_b\bar{B}_a(\nabla^i\phi)(\sigma^iB)_bB_a\rangle]\\ &- D_6/f_0[\langle\bar{B}_b(\nabla^i\phi)\bar{B}_a(\sigma^iB)_bB_a\rangle + \langle\bar{B}_a\bar{B}_bB_a(\nabla^i\phi)(\sigma^iB)_b\rangle] - D_7/f_0[\langle\bar{B}_a\bar{B}_bB_a(\sigma^iB)_b(\nabla^i\phi)\rangle + \langle\bar{B}_b\bar{B}_a(\sigma^iB)_bB_a(\nabla^i\phi)\rangle]\\ &+ D_8/f_0\langle\bar{B}_a(\nabla^i\phi)B_a\rangle\langle\bar{B}_b(\sigma^iB)_b\rangle + D_9/f_0\langle\bar{B}_aB_a(\nabla^i\phi)\rangle\langle\bar{B}_b(\sigma^iB)_b\rangle + D_{10}/f_0\langle\bar{B}_b(\nabla^i\phi)(\sigma^iB)_b\rangle\langle\bar{B}_aB_a\rangle\\ &+ i\epsilon^{ijk}D_{11}/f_0\langle\bar{B}_a(\sigma^iB)_a(\nabla^k\phi)\bar{B}_b(\sigma^jB)_b\rangle - i\epsilon^{ijk}D_{12}/f_0[\langle\bar{B}_a(\nabla^k\phi)\bar{B}_b(\sigma^iB)_a(\sigma^jB)_b\rangle - \langle\bar{B}_b\bar{B}_a(\sigma^jB)_b(\nabla^k\phi)(\sigma^iB)_a\rangle]\\ &- i\epsilon^{ijk}D_{13}/f_0\langle\bar{B}_a\bar{B}_b(\nabla^k\phi)(\sigma^iB)_a(\sigma^jB)_b\rangle - i\epsilon^{ijk}D_{14}/f_0\langle\bar{B}_a\bar{B}_b(\sigma^iB)_a(\sigma^jB)_b(\nabla^k\phi)\rangle.\end{aligned} \tag{27}$$

Here the indices $a$ and $b$ are two-component spinor indices and the indices $i$, $j$, and $k$ are vector indices. There are in total 14 low-energy constants $D_1,\dots,D_{14}$ for all five strangeness sectors $S=-4,\dots,0$. As before, the minus signs in front of some terms have been included to compensate minus signs from fermion exchange, arising from reordering baryon bilinears [see Eq. (28) below]. Let us note that the conservation of strangeness $S$, isospin $I$ and isospin projection $I_3$, independence of $I_3$, and time-reversal symmetry have been checked for the $BB\to BB\phi$ transition matrix elements resulting from Eq. (27). Moreover, several tests employing group-theoretical methods have been performed.

As done in Sec. II A, we write the Lagrangian in the particle basis

$$\begin{aligned}\mathcal{L} =& \sum_{f=1}^{10}\frac{D_f}{f_0}\sum_{i,j,k,l,m=1}^{8}N^f_{\substack{ik\\ jl}\,\phi_m}(\bar{B}_iB_j)(\bar{B}_k\vec{\sigma}B_l)\cdot\vec{\nabla}\phi_m\\ &+\sum_{f=11}^{14}\frac{D_f}{f_0}\sum_{i,j,k,l,m=1}^{8}N^f_{\substack{ik\\ jl}\,\phi_m}\,i[(\bar{B}_i\vec{\sigma}B_j)\times(\bar{B}_k\vec{\sigma}B_l)]\cdot\vec{\nabla}\phi_m,\end{aligned} \tag{28}$$

where in each term the first bilinear comes from the summation over spin index $a$ and the second bilinear from the summation over spin index $b$ in Eq. (27). The indices $i,j,k,l$ label octet baryons.

Let us now consider the generic one-meson exchange diagram in Fig. 2. It involves the baryons $i,j,k$ in the initial state, the baryons $l,m,n$ in the final state, and an exchanged meson $\phi$. The four-baryon contact vertex is separated into two parts to indicate which baryons are in the same bilinear. The indices $A,B,C$ label the spin spaces related to the baryon bilinears.

Using standard Feynman rules for the vertices and the meson propagator one obtains the three-body potential

$$V = \frac{1}{2f_0^2}\frac{\vec{\sigma}_A\cdot\vec{q}_{li}}{\vec{q}_{li}^{\,2}+m_\phi^2}[N_1\vec{\sigma}_C\cdot\vec{q}_{li} + N_2 i(\vec{\sigma}_B\times\vec{\sigma}_C)\cdot\vec{q}_{li}], \tag{29}$$

with the momentum transfer $\vec{q}_{li}=\vec{p}_l-\vec{p}_i$ carried by the exchanged meson and the constants

$$\begin{aligned}N_1 &= N_{B_lB_i\phi}\sum_{f=1}^{10}D_fN^f_{\substack{mn\\ jk}\,\bar{\phi}},\\ N_2 &= N_{B_lB_i\phi}\sum_{f=11}^{14}D_fN^f_{\substack{mn\\ jk}\,\bar{\phi}},\end{aligned} \tag{30}$$

where $\bar{\phi}$ denotes the charge-conjugated meson of meson $\phi$, in particle basis (e.g., $\pi^+\leftrightarrow\pi^-$).

The full one-meson exchange three-body potential for the process $B_1B_2B_3\to B_4B_5B_6$ is obtained easily by summing up for a fixed meson the 36 permutations of initial and

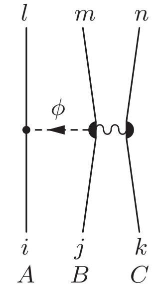


FIG. 2. Generic one-meson exchange diagram. The wiggly line symbolized the four-baryon contact vertex, to illustrate the baryon bilinears.

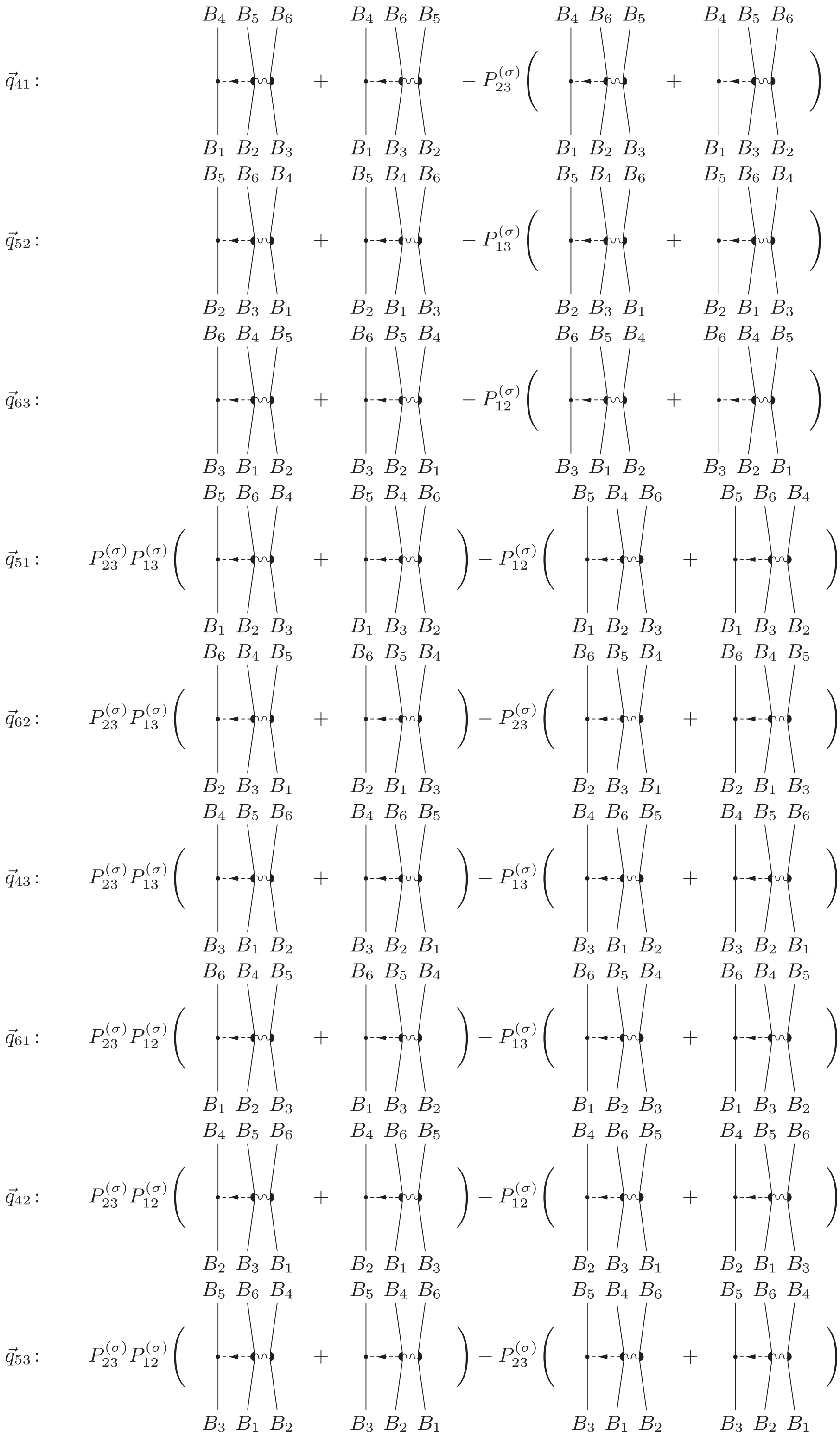


FIG. 3. Feynman diagrams contributing to the one-meson exchange three-body potential for $B_1B_2B_3 \to B_4B_5B_6$.

final baryons, shown diagrammatically in Fig. 3, and summing over all mesons $\phi \in \{\pi^0,\pi^+,\pi^-,K^+,K^-,K^0,\bar{K}^0,\eta\}$. Of course, many of these contributions will vanish for a particular process. The Feynman diagrams fall into nine classes, where in each class the same momentum transfer $\vec{q}_{li}$ is present. In Fig. 3 each row corresponds to such a class and the corresponding momentum transfer is written on the left of the row. Furthermore, additional minus signs from interchanging fermions have to be included and some diagrams need to be multiplied from the left by spin-exchange operators (as indicated in Fig. 3) to be in accordance with the form set up in Eq. (6). As before, the baryons $B_1$, $B_2$, and $B_3$ belong to the spin spaces 1, 2, and 3, respectively.

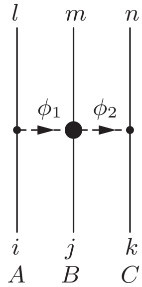


FIG. 4. Generic two-meson exchange diagram.

## IV. TWO-MESON EXCHANGE

For the two-meson exchange diagram of Fig. 1 we need in addition to the Lagrangian in Eq. (23) the well-known $\mathcal{O}(q^2)$ meson-baryon Lagrangian [50]. We use the version given in Ref. [51] and display here only the terms relevant for our purpose,

$$\begin{aligned}\mathcal{L} = {} & b_D\langle \bar{B}\{\chi_+,B\}\rangle + b_F\langle \bar{B}[\chi_+,B]\rangle + b_0\langle \bar{B}B\rangle\,\langle \chi_+\rangle \\ & + b_1\langle \bar{B}[u^\mu,[u_\mu,B]]\rangle + b_2\langle \bar{B}\{u^\mu,\{u_\mu,B\}\}\rangle \\ & + b_3\langle \bar{B}\{u^\mu,[u_\mu,B]\}\rangle + b_4\langle \bar{B}B\rangle\,\langle u^\mu u_\mu\rangle \\ & + id_1\langle \bar{B}\{[u^\mu,u^\nu],\sigma_{\mu\nu}B\}\rangle + id_2\langle \bar{B}[[u^\mu,u^\nu],\sigma_{\mu\nu}B]\rangle \\ & + id_3\langle \bar{B}u^\mu\rangle\langle u^\nu\sigma_{\mu\nu}B\rangle, \end{aligned} \tag{31}$$

with $u_\mu = -\frac{1}{f_0}\partial_\mu\phi + \mathcal{O}(\phi^3)$ and $\chi_+ = 2\chi - \frac{1}{4f_0^2}\{\phi,\{\phi,\chi\}\} + \mathcal{O}(\phi^4)$, where

$$\chi = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2 - m_\pi^2 \end{pmatrix}. \tag{32}$$

Note that the terms proportional to $b_D,b_F,b_0$ break explicitly SU(3) flavor symmetry, through different meson masses $m_K \neq m_\pi$. Rewriting the Lagrangian in the particle basis as in the previous sections, one obtains

$$\begin{aligned}\mathcal{L} = {} & -\sum_{c^f=b_D,b_F,b_0}\frac{c^f}{4f_0^2}\sum_{i,j,k,l=1}^{8} N^f_{\phi_k {}^i_j \phi_l}(\bar{B}_iB_j)\phi_k\phi_l \\ & + \sum_{c^f=b_1,b_2,b_3,b_4}\frac{c^f}{f_0^2}\sum_{i,j,k,l=1}^{8} N^f_{\phi_k {}^i_j \phi_l}(\bar{B}_iB_j)\partial_\mu\phi_k\partial^\mu\phi_l \\ & + \sum_{c^f=d_1,d_2,d_3}\frac{ic^f}{f_0^2}\sum_{i,j,k,l=1}^{8} N^f_{\phi_k {}^i_j \phi_l}(\bar{B}_i\sigma_{\mu\nu}B_j)\partial^\mu\phi_k\partial^\nu\phi_l. \end{aligned} \tag{33}$$

Let us now consider the generic two-meson exchange diagram depicted in Fig. 4. It includes the baryons $i,j,k$ in the initial state, the baryons $l,m,n$ in the final state, and two virtual mesons $\phi_1$ and $\phi_2$ are exchanged. The indices $A,B,C$ label the spin spaces related to the baryon bilinears and they are defined by the three initial baryons. The momentum transfers carried by the virtual mesons are $\vec{q}_{li} = \vec{p}_l - \vec{p}_i$ and $\vec{q}_{nk} = \vec{p}_n - \vec{p}_k$. One obtains the following transition amplitude from the generic two-meson exchange diagram

$$\begin{aligned} V = {} & -\frac{1}{4f_0^4}\frac{\vec{\sigma}_A\cdot\vec{q}_{li}\;\vec{\sigma}_C\cdot\vec{q}_{nk}}{\left(\vec{q}_{li}^{\,2}+m_{\phi_1}^2\right)\left(\vec{q}_{nk}^{\,2}+m_{\phi_2}^2\right)} \\ & \times [N_1' + N_2'\,\vec{q}_{li}\cdot\vec{q}_{nk} + N_3'\,i(\vec{q}_{li}\times\vec{q}_{nk})\cdot\vec{\sigma}_B], \end{aligned} \tag{34}$$

with the combinations of parameters

$$\begin{aligned} N_1' &= N_{B_lB_i\bar{\phi}_1}N_{B_nB_k\phi_2}\sum_{c^f=b_D,b_F,b_0}\frac{c^f}{4}\left(N^f_{\phi_1 {}^m_j \bar{\phi}_2} + N^f_{\bar{\phi}_2 {}^m_j \phi_1}\right), \\ N_2' &= -\,N_{B_lB_i\bar{\phi}_1}N_{B_nB_k\phi_2}\sum_{c^f=b_1,b_2,b_3,b_4} c^f\left(N^f_{\phi_1 {}^m_j \bar{\phi}_2} + N^f_{\bar{\phi}_2 {}^m_j \phi_1}\right), \\ N_3' &= N_{B_lB_i\bar{\phi}_1}N_{B_nB_k\phi_2}\sum_{c^f=d_1,d_2,d_3} c^f\left(N^f_{\phi_1 {}^m_j \bar{\phi}_2} - N^f_{\bar{\phi}_2 {}^m_j \phi_1}\right). \end{aligned} \tag{35}$$

The complete three-body potential for a transition $B_1B_2B_3 \to B_4B_5B_6$ is finally obtained by summing up the contributions of the 18 Feynman diagrams in Fig. 5 and by summing over all possible exchanged mesons. Obviously, additional (negative) spin-exchange operators need to be applied if the baryon lines are not in the configuration 1-4, 2-5, and 3-6, as illustrated in Fig. 5.

## V. $NNN$ AND $\Lambda NN$ THREE-BARYON POTENTIALS

To give a concrete example, we present in this section the explicit expressions for the $\Lambda NN$ three-body interaction in spin, isospin, and momentum space. Moreover, the leading-order chiral $NNN$ interaction is rederived, and consistency with the conventional expression is shown. The potentials are calculated in particle basis (as shown in the previous sections) and afterwards reexpressed with isospin operators.

By adding up all 36 contributions [coming from Eqs. (8) and (12)], one obtains the form of the $NNN$ contact potential

$$\begin{aligned} V_{\rm ct}^{NNN} = {} & -\tfrac{3}{8}E[(3\,\mathbb{1} - \vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{\sigma}_1\cdot\vec{\sigma}_3 - \vec{\sigma}_2\cdot\vec{\sigma}_3)\,\mathbb{1} \\ & + (-\mathbb{1} - \vec{\sigma}_1\cdot\vec{\sigma}_2 + \vec{\sigma}_1\cdot\vec{\sigma}_3 + \vec{\sigma}_2\cdot\vec{\sigma}_3)\,\vec{\tau}_1\cdot\vec{\tau}_2 \\ & + (-\mathbb{1} + \vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{\sigma}_1\cdot\vec{\sigma}_3 + \vec{\sigma}_2\cdot\vec{\sigma}_3)\,\vec{\tau}_1\cdot\vec{\tau}_3 \\ & + (-\mathbb{1} + \vec{\sigma}_1\cdot\vec{\sigma}_2 + \vec{\sigma}_1\cdot\vec{\sigma}_3 - \vec{\sigma}_2\cdot\vec{\sigma}_3)\,\vec{\tau}_2\cdot\vec{\tau}_3 \\ & - \vec{\sigma}_1\times\vec{\sigma}_2\cdot\vec{\sigma}_3\;\vec{\tau}_1\times\vec{\tau}_2\cdot\vec{\tau}_3], \end{aligned} \tag{36}$$

with the low-energy constant $E = 2(C_4 - C_9) = -c_{\underline{35}}/3$ and where $\vec{\sigma}$, $\vec{\tau}$ denote the usual Pauli matrices in spin and isospin space. This is exactly the $NNN$ contact potential of Ref. [7] in its antisymmetrized form,

$$V_{\rm ct}^{NNN} = \frac{1}{2}E\,\mathcal{A}\sum_{j\neq k}\vec{\tau}_j\cdot\vec{\tau}_k, \tag{37}$$

where $\mathcal{A}$ denotes the three-body antisymmetrization operator, $\mathcal{A} = (\mathbb{1} - \mathcal{P}_{12})(\mathbb{1} - \mathcal{P}_{13} - \mathcal{P}_{23})$. Each two-particle exchange operator $\mathcal{P}_{ij} = P_{ij}^{(\sigma)}P_{ij}^{(\tau)}P_{ij}^{(p)}$ is the product of an exchange operator in spin space $P_{ij}^{(\sigma)} = \frac{1}{2}(\mathbb{1} + \vec{\sigma}_i\cdot\vec{\sigma}_j)$, in isospin space $P_{ij}^{(\tau)} = \frac{1}{2}(\mathbb{1} + \vec{\tau}_i\cdot\vec{\tau}_j)$, and in momentum space $P_{ij}^{(p)}$. Note that the leading-order $NNN$ contact potential is momentum independent, and therefore $P_{ij}^{(p)}$ has no effect. We remind

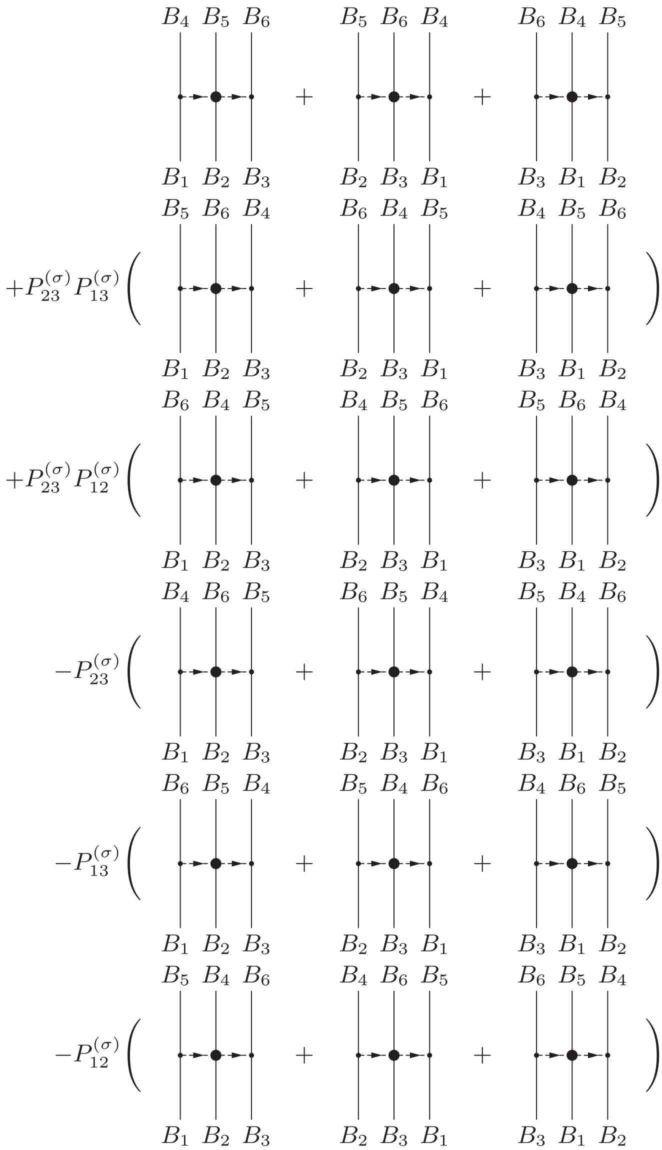


FIG. 5. Feynman diagrams contributing to the two-meson exchange three-body potential for $B_1B_2B_3 \to B_4B_5B_6$.

the reader that in our calculation the generalized Pauli principle is automatically built in by performing all Wick contractions.

For the $\Lambda NN$ contact interaction we obtain the expression

$$\begin{aligned} V_{\rm ct}^{\Lambda NN} =& C_1' \,(\mathbb{1} - \vec\sigma_2\cdot\vec\sigma_3)(3+\vec\tau_2\cdot\vec\tau_3) \\ &+ C_2'\,\vec\sigma_1\cdot(\vec\sigma_2+\vec\sigma_3)\,(\mathbb{1}-\vec\tau_2\cdot\vec\tau_3) \\ &+ C_3'\,(3+\vec\sigma_2\cdot\vec\sigma_3)(\mathbb{1}-\vec\tau_2\cdot\vec\tau_3), \end{aligned} \tag{38}$$

where the primed constants are given by

$$\begin{aligned} C_1' =& -\tfrac{1}{48}(2C_2 - 13C_3 + 21C_4 - 13C_5 + 24C_7 \\ &+ 54C_8 - 21C_9 + 54C_{10} - 30C_{12} - 15C_{13} \\ &- 15C_{14} - 48C_{16} + 18C_{17} - 18C_{18}), \\ C_2' =& -\tfrac{1}{24}(8C_2 - 5C_3 - 3C_4 - 5C_5 + 12C_7 - 18C_8 \\ &+ 15C_9 - 18C_{10} - 3C_{13} - 3C_{14} + 6C_{17} - 6C_{18}), \\ C_3' =& -\tfrac{1}{48}(10C_2 - 13C_3 + 21C_4 - 13C_5 + 24C_7 \\ &- 18C_8 + 3C_9 - 18C_{10} + 18C_{12} - 15C_{13} \\ &- 15C_{14} - 6C_{17} + 6C_{18}). \end{aligned} \tag{39}$$

The constants $C_1,\ldots,C_{18}$ originate from the minimal contact Lagrangian in Eq. (14). Note that the constant $C_1'$ belongs exclusively to the transition with total isospin $I=1$, whereas the constants $C_2'$ and $C_3'$ appear for total isospin $I=0$. Interestingly, none of these three constants can be substituted by the constant $E$ of the purely nucleonic sector. Thus, the strength of the $\Lambda NN$ three-body contact interaction is not related to the one for $NNN$ via SU(3) symmetry.

The one-pion exchange $NNN$ potential reads (in antisymmetrized form)

$$\begin{aligned} V_{\rm OPE}^{NNN} =& \big(X_{123}^{456} + X_{231}^{564} + X_{312}^{645}\big) \\ &+ P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{13}^{(\sigma)}P_{13}^{(\tau)}\big(X_{123}^{564}+X_{231}^{645}+X_{312}^{456}\big) \\ &+ P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{12}^{(\sigma)}P_{12}^{(\tau)}\big(X_{123}^{645}+X_{231}^{456}+X_{312}^{564}\big), \end{aligned} \tag{40}$$

where we have defined the abbreviation[3]

$$\begin{aligned} X_{ijk}^{lmn} =& -\frac{g_A}{16 f_0^2} d' \frac{\vec\sigma_i\cdot\vec q_{li}}{\vec q_{li}^{\,2}+m_\pi^2}[(\vec\tau_j-\vec\tau_k)\cdot\vec\tau_i\,(\vec\sigma_j-\vec\sigma_k)\cdot\vec q_{li} \\ &+ (\vec\tau_j\times\vec\tau_k)\cdot\vec\tau_i\,(\vec\sigma_j\times\vec\sigma_k)\cdot\vec q_{li}], \end{aligned} \tag{41}$$

with $g_A = D+F$ and $d' = 4(D_1 - D_3 + D_8 - D_{10})$. Each term in Eq. (40) corresponds to a complete row in Fig. 3. We have verified that this result is equal to the antisymmetrization of the expression given in Ref. [7]

$$V_{\rm OPE}^{NNN} = -\frac{g_A}{8 f_\pi^2} d' \mathcal{A} \sum_{i\neq j\neq k} \frac{\vec\sigma_j\cdot\vec q_j}{\vec q_j^{\,2}+m_\pi^2}\vec\tau_i\cdot\vec\tau_j\,\vec\sigma_i\cdot\vec q_j, \tag{42}$$

inserting the momentum transfers $\vec q_1 = \vec q_{41} = \vec p_4 - \vec p_1$, $\vec q_2 = \vec q_{52} = \vec p_5 - \vec p_2$, $\vec q_3 = \vec q_{63} = \vec p_6 - \vec p_3$. In this case the momentum part of each two-body exchange operator, $P_{ij}^{(p)}$, exchanges also the momenta in the final state.[4]

Let us continue with the $\Lambda NN$ one-pion exchange three-body potentials. Many diagrams are absent owing to the vanishing of the $\Lambda\Lambda\pi$ vertex (by isospin symmetry). We find the following result for the $\Lambda NN$ three-body interaction

[3]We have used the symbol $d'$ instead of the conventional $D$ to avoid confusion with the axial vector constant in Eq. (23).

[4]For example, $P_{23}^{(p)}$ leads to the replacements $q_{41},q_{52},q_{63} \to q_{41},q_{62},q_{53}$ and $P_{12}^{(p)}P_{13}^{(p)}$ to $q_{41},q_{52},q_{63} \to q_{61},q_{42},q_{53}$.

mediated by one-pion exchange,

$$
\begin{aligned}
V_{\mathrm{OPE}}^{\Lambda NN} = &-\frac{g_A}{2f_0^2}\Bigg\{\frac{\vec{\sigma}_2\cdot\vec{q}_{52}}{\vec{q}_{52}^{\,2}+m_\pi^2}\vec{\tau}_2\cdot\vec{\tau}_3[(D_1'\vec{\sigma}_1+D_2'\vec{\sigma}_3)\cdot\vec{q}_{52}] \\
&+\frac{\vec{\sigma}_3\cdot\vec{q}_{63}}{\vec{q}_{63}^{\,2}+m_\pi^2}\vec{\tau}_2\cdot\vec{\tau}_3[(D_1'\vec{\sigma}_1+D_2'\vec{\sigma}_2)\cdot\vec{q}_{63}] \\
&+P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{13}^{(\sigma)}\frac{\vec{\sigma}_2\cdot\vec{q}_{62}}{\vec{q}_{62}^{\,2}+m_\pi^2}\vec{\tau}_2\cdot\vec{\tau}_3\left[-\frac{D_1'+D_2'}{2}(\vec{\sigma}_1+\vec{\sigma}_3)\cdot\vec{q}_{62}+\frac{D_1'-D_2'}{2}i(\vec{\sigma}_3\times\vec{\sigma}_1)\cdot\vec{q}_{62}\right] \\
&+P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{12}^{(\sigma)}\frac{\vec{\sigma}_3\cdot\vec{q}_{53}}{\vec{q}_{53}^{\,2}+m_\pi^2}\vec{\tau}_2\cdot\vec{\tau}_3\left[-\frac{D_1'+D_2'}{2}(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{q}_{53}-\frac{D_1'-D_2'}{2}i(\vec{\sigma}_1\times\vec{\sigma}_2)\cdot\vec{q}_{53}\right]\Bigg\},
\end{aligned}
\tag{43}
$$

where we have defined the two linear combinations of constants

$$
\begin{aligned}
D_1' &= \tfrac{1}{6}(-3D_1+D_2+D_3+5D_4+9D_5+D_6-6D_8+D_{11}+2D_{12}-3D_{13}),\\
D_2' &= \tfrac{1}{6}(D_1+D_2-3D_3+D_4+9D_5+5D_6-6D_{10}-D_{11}-2D_{12}+3D_{13}).
\end{aligned}
\tag{44}
$$

The four lines in Eq. (43) correspond to the four rows in Fig. 3 that have no $\Lambda$ hyperon at the baryon-baryon-meson vertex, i.e., the diagrams involving the momentum transfers $\vec{q}_{52},\vec{q}_{63},\vec{q}_{62},\vec{q}_{53}$.

Finally, we obtain for the $NNN$ interaction mediated by two-pion exchange

$$
\begin{aligned}
V_{\mathrm{TPE}}^{NNN} =&\big(Y_{123}^{456}+Y_{231}^{564}+Y_{312}^{645}\big)+P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{13}^{(\sigma)}P_{13}^{(\tau)}\big(Y_{123}^{564}+Y_{231}^{645}+Y_{312}^{456}\big)+P_{23}^{(\sigma)}P_{23}^{(\tau)}P_{12}^{(\sigma)}P_{12}^{(\tau)}\big(Y_{123}^{645}+Y_{231}^{456}+Y_{312}^{564}\big)\\
&-P_{23}^{(\sigma)}P_{23}^{(\tau)}\big(Y_{123}^{465}+Y_{231}^{654}+Y_{312}^{546}\big)-P_{13}^{(\sigma)}P_{13}^{(\tau)}\big(Y_{123}^{654}+Y_{231}^{546}+Y_{312}^{465}\big)-P_{12}^{(\sigma)}P_{12}^{(\tau)}\big(Y_{123}^{546}+Y_{231}^{465}+Y_{312}^{654}\big),
\end{aligned}
\tag{45}
$$

where the 18 terms follow the ordering displayed in Fig. 5 and we have introduced the abbreviation

$$
Y_{ijk}^{lmn} = \frac{g_A^2}{4f_\pi^4}\frac{\vec{\sigma}_i\cdot\vec{q}_{li}\,\vec{\sigma}_k\cdot\vec{q}_{nk}}{\big(\vec{q}_{li}^{\,2}+m_\pi^2\big)\big(\vec{q}_{nk}^{\,2}+m_\pi^2\big)}\big[\vec{\tau}_i\cdot\vec{\tau}_k\big(-4c_1m_\pi^2+2c_3\vec{q}_{li}\cdot\vec{q}_{nk}\big)+c_4\vec{\tau}_j\cdot(\vec{\tau}_i\times\vec{\tau}_k)\,\vec{\sigma}_j\cdot(\vec{q}_{li}\times\vec{q}_{nk})\big],
\tag{46}
$$

with the constants (see also Refs. [52,53])

$$
c_1=\tfrac{1}{2}(2b_0+b_D+b_F),\quad c_3=b_1+b_2+b_3+2b_4,\quad c_4=4(d_1+d_2).
\tag{47}
$$

Again, the result in Eq. (45) is equal to the antisymmetrization of the expression given in Ref. [7],

$$
V_{\mathrm{TPE}}^{NNN} = \frac{g_A^2}{8f_\pi^2}\mathcal{A}\sum_{i\neq j\neq k}\frac{\vec{\sigma}_i\cdot\vec{q}_i\,\vec{\sigma}_j\cdot\vec{q}_j}{\big(\vec{q}_i^{\,2}+m_\pi^2\big)\big(\vec{q}_j^{\,2}+m_\pi^2\big)}F_{ijk}^{\alpha\beta}\tau_i^\alpha\tau_j^\beta,
\tag{48}
$$

with

$$
F_{ijk}^{\alpha\beta} = \frac{\delta^{\alpha\beta}}{f_\pi^2}\big(-4c_1m_\pi^2+2c_3\vec{q}_i\cdot\vec{q}_j\big)+\sum_\gamma\frac{c_4}{f_\pi^2}\epsilon^{\alpha\beta\gamma}\tau_k^\gamma\,\vec{\sigma}_k\cdot(\vec{q}_i\times\vec{q}_j).
\tag{49}
$$

The $\Lambda NN$ three-body interaction generated by two-pion exchange takes the form

$$
\begin{aligned}
V_{\mathrm{TPE}}^{\Lambda NN} =& \frac{g_A^2}{3f_0^4}\frac{\vec{\sigma}_3\cdot\vec{q}_{63}\,\vec{\sigma}_2\cdot\vec{q}_{52}}{\big(\vec{q}_{63}^{\,2}+m_\pi^2\big)\big(\vec{q}_{52}^{\,2}+m_\pi^2\big)}\vec{\tau}_2\cdot\vec{\tau}_3\big[-(3b_0+b_D)m_\pi^2+(2b_2+3b_4)\,\vec{q}_{63}\cdot\vec{q}_{52}\big]\\
&-P_{23}^{(\sigma)}P_{23}^{(\tau)}\frac{g_A^2}{3f_0^4}\frac{\vec{\sigma}_3\cdot\vec{q}_{53}\,\vec{\sigma}_2\cdot\vec{q}_{62}}{\big(\vec{q}_{53}^{\,2}+m_\pi^2\big)\big(\vec{q}_{62}^{\,2}+m_\pi^2\big)}\vec{\tau}_2\cdot\vec{\tau}_3\big[-(3b_0+b_D)m_\pi^2+(2b_2+3b_4)\,\vec{q}_{53}\cdot\vec{q}_{62}\big].
\end{aligned}
\tag{50}
$$

Note that only those two diagrams in Fig. 5 contribute, where the (final and initial) $\Lambda$ hyperon are associated to the central baryon line. All other diagrams are simply zero owing to the vanishing of the $\Lambda\Lambda\pi$ vertex.

## VI. SUMMARY AND OUTLOOK

In this work we have derived the leading contributions to the three-baryon interaction from SU(3) $\chi$EFT. First, we have established the minimal nonrelativistic Lagrangian for contact terms of six octet-baryons, leading to 18 constants. Using this foundation, general SU(3) relations among the three-baryon channels with strangeness 0 and $-1$ have been derived. Furthermore, the four-baryon contact Lagrangian with one Goldstone boson has been given in its minimal form, in which it involves 14 constants. The irreducible three-body potentials have been constructed at NNLO in the chiral power counting based on the effective chiral Lagrangians. Contributions arise from contact terms, from one-meson exchange and from two-meson exchange diagrams. The three-body potential for the $\Lambda NN$ interaction has been presented as an explicit example.

The large number of unknown low-energy constants is related to the variety of three-baryon multiplets, with strangeness ranging from 0 to −6. For selected processes only a small subset of these constants contributes as has been exemplified for the $\Lambda NN$ three-body interaction. Estimates of the predominant low-energy constants can be made by using decuplet-baryon saturation. An example for that is the $\Sigma^*(1385)$ excitation in case of the $\Lambda NN$ two-pion exchange three-body interaction. Owing to the small decuplet-octet mass splitting, such effects are promoted to NLO in the chiral power counting, in analogy to the role played by the $\Delta$ resonance in the nucleonic sector [4]. Work along this direction is in progress [54]. We anticipate that the chiral potentials derived in this work will shed light on the role of 3BFs in hypernuclei. In particular, their application in studies of light hypernuclei will be very instructive because such systems can be treated within reliable few-body techniques [55,56]. Furthermore, one expects that the present investigations can help pave the way for more systematic studies on the role of three-baryon interactions in hyperonic neutron-star matter.

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