



Numerical assessment of post-prior equivalence for inclusive breakup reactions

Jin Lei* and Antonio M. Moro†

Departamento de FAMN, Universidad de Sevilla, Apartado 1065, 41080 Sevilla, Spain

(Received 10 November 2015; published 14 December 2015)

We address the problem of the post-prior equivalence in inclusive breakup reactions induced by weakly bound nuclei. The problem is studied within the distorted-wave Born approximation (DWBA) model of Ichimura, Austern, and Vincent [*Phys. Rev. C* **32**, 431 (1985)]. The post and prior formulas obtained in this model are briefly recalled and applied to several breakup reactions induced by deuterons and ${}^6\text{Li}$ projectiles, to test their actual numerical equivalence. The different contributions of the prior-form formula are also discussed. A critical comparison with the prior-form DWBA model of Udagawa and Tamura [*Phys. Rev. C* **24**, 1348 (1981)] is also provided.

DOI: [10.1103/PhysRevC.92.061602](https://doi.org/10.1103/PhysRevC.92.061602)

PACS number(s): 24.10.Eq, 25.70.Mn, 25.45.-z

Introduction. Breakup is an important mechanism occurring in nuclear reactions involving weakly bound nuclei. The analysis of these reactions has provided useful information on the structure of weakly bound nuclei (such as separation energies, angular momenta and parities, electric responses to the continuum, etc.). A detailed understanding of these processes is also necessary for a number of applications, such as in (d, pf) surrogate reactions [1] or the production of radioisotopes for medical purposes [2].

For two-body projectiles, these reactions can be represented as $a + A \rightarrow b + x + A$, where $a = b + x$. If the three outgoing particles are observed in a definite final state, the reaction is said to be *exclusive*. This problem can be treated as an effective scattering problem with three particles interacting via some effective two-body interactions. Although the exact, rigorous solution of this problem can in principle be obtained by solving the so-called Faddeev equations [3], in practice the complexity of this method limits so far its applicability to specific situations. For this reason, alternative approaches, such as the popular continuum-discretized coupled-channels (CDCC) method [4], have been used. At higher energies, semiclassical approaches become an efficient and appealing alternative (see, e.g., Refs. [5,6]).

A qualitatively different scenario occurs when the final state is not fully specified. For example, this is the case of reactions of the form $A(a, bX)$, in which only one of the projectile constituents (say, b) is observed. In this case the reaction is said to be *inclusive* with respect to the unobserved particle(s). The simplest process contributing to the inclusive cross section is that in which the three outgoing particles remain in their ground states, which receives the name of *elastic breakup* (EBU). However, more complicated processes are possible; for example, breakup accompanied by x or A excitation, by particle transfer between x and A , or by fusion of x with A (incomplete fusion, ICF). The sum of these contributions is referred to as nonelastic breakup (NEB).

Due to the large number of accessible states, a detailed calculation of the NEB part, in which all these processes

are included explicitly, is in general not possible. For that reason, in the 1980s several groups developed closed-form expressions in which the sum over final states was done in a formal way, using completeness of the $x + A$ states [7–15]. Here, we focus on the models proposed by Udagawa and Tamura (UT hereafter) [10,11] and by Ichimura, Austern, and Vincent (IAV hereafter) [12,13,15]. The main difference between these models is that, whereas UT use the prior-form DWBA, IAV employ the post-form representation. Although the final expressions for these models have the same formal structure (see Sec. II), they lead to different predictions for the NEB cross sections. This is in contrast to the DWBA formula for transfer between bound states, where it is well known that the post and prior formulas are fully equivalent. This discrepancy led to a long-standing controversy between these two groups, which lasted for more than a decade. At the heart of the discussion was the fact that the transformation of the post-form DWBA expression of IAV to its prior form gave rise to additional terms, not present in the UT prior formula. These additional terms guaranteed the post-prior equivalence for NEB, but they were nevertheless regarded as unphysical by UT. To support their conclusions, UT performed calculations for several inclusive reactions [16,17], in which they showed that the IAV calculations largely overestimated the data.

The IAV model has been recently reexamined and implemented by several groups [18–20]. Contrary to the referred results of Udagawa, Tamura, and collaborators, the comparison of these recent calculations with available data has shown very encouraging results. These calculations have been performed using either the original post-form formulation [18,20] or its (in principle) equivalent prior form [19]. However, a consistent comparison between the post and prior results has not been made to our knowledge. One of the reasons is that a direct evaluation of the post-form formula is not feasible, owing to the marginal convergence of the post-form breakup amplitudes. To overcome this problem, several regularization procedures have been suggested, such as the integration in the complex plane of Vincent and Fortune [21], the introduction of a convergence damping factor [22,23] or the replacement of the oscillatory distorted waves of the outgoing b fragments by some averaged wave packets [24]. The convergence and stability of these procedures need to be carefully examined.

*jinlei@us.es

†moro@us.es

The goal of this work is manifold. First, we aim to assess, in a quantitative way, the actual equivalence of the post and prior NEB formulas of the IAV model. For that, we apply these formulas to specific cases. Furthermore, this study serves to test the validity of the regularization procedure of the post-form integrals invoked in Refs. [20,24]. In each case, we also compare with the UT model and with available data in order to assess the validity of these models against the data. Finally, we aim at examining the relative importance of the different terms entering the prior-form expression. For that, we have performed calculations for $^{62}\text{Ni}(d, pX)$ at $E = 25.5$ MeV and $^{209}\text{Bi}(^6\text{Li}, \alpha X)$ at $E = 36$ MeV.

The paper is organized as follows: In Sec. II we summarize the main formulas of the IAV and UT models and outline the relation between them. In Sec. III, the formalism is applied to several inclusive reactions induced by deuterons and ^6Li . Finally, in Sec. IV we summarize the main results.

Post and prior formulas for inclusive breakup. In this section we briefly review the main results of the UT and IAV models. Further details can be found in the referred works as well as in our preceding paper [20]. We write the process under study as

$$a(= b + x) + A \rightarrow b + B^*. \quad (1)$$

We assume that the experiment is inclusive with respect to the particle x . Consequently, only b is observed and the corresponding experimental cross sections will correspond to a sum over all possible final states of the $x + A$ system. This includes the EBU as well as the NEB components mentioned in the introduction.

The IAV model, as well as the UT model, treats the b particle as an *spectator*, meaning that its interaction with the target nucleus is described with an optical potential U_{bA} .

Using the post-form DWBA, the inclusive breakup differential cross section, as a function of the detected angle and energy of the fragment b , is given by

$$\frac{d^2\sigma}{d\Omega_b E_b} = \frac{2\pi}{\hbar v_a} \rho(E_b) \sum_c |\langle \chi_b^{(-)} \Psi_{xA}^{c,(-)} | V_{\text{post}} | \chi_a^{(+)} \phi_a \phi_a^0 \rangle|^2 \times \delta(E - E_b - E^c), \quad (2)$$

where v_a is the projectile-target relative velocity, $V_{\text{post}} \equiv V_{bx} + U_{bA} - U_{bB}$ is the post-form transition operator,¹ $\rho_b(E_b) = k_b \mu_b / [(2\pi)^3 \hbar^2]$ (with μ_b being the reduced mass of $b + B$ and k_b being their relative wave number), $\phi_a(\vec{r}_{bx})$ and ϕ_A^0 are the projectile and target ground-state wave functions, $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are distorted waves describing the $a - A$ and $b - B$ relative motion, respectively, and $\Psi_{xA}^{c,(-)}$ are the eigenstates of the $x + A$ system, with $c = 0$ denoting the x and A ground states. Thus, for $c = 0$ this expression gives the EBU part, whereas the terms $c \neq 0$ give the NEB contribution.

¹In their original papers [15], IAV usually make the approximation $V_{\text{post}} \approx V_{bx}$, thus neglecting the so-called remnant term, $U_{bA} - U_{bB}$. In Ref. [20] we showed that this is a good approximation for deuterons on heavy targets, but not for ^6Li reactions. In this work we retain the full transition operator, for an accurate comparison between the post and prior results.

The theory of IAV allows us to perform the sum in a formal way, making use of the Feshbach projection formalism and the optical model reduction, leading to a closed form for the NEB differential cross section:

$$\frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{IAV}} = -\frac{2}{\hbar v_a} \rho_b(E_b) \langle \psi_x^{\text{post}} | W_x | \psi_x^{\text{post}} \rangle, \quad (3)$$

where W_x is the imaginary part of the optical potential U_x , which describes $x + A$ elastic scattering. The function $\psi_x^{\text{post}}(\vec{r}_x)$ (the x -channel wave function hereafter) describes the $x - A$ relative motion when the target is in the ground state and the b particle scatters with momentum \vec{k}_b , and is obtained by solving the inhomogeneous equation

$$(E_x^+ - K_x - U_x) \psi_x^{\text{post}}(\vec{r}_x) = (\chi_b^{(-)} | V_{\text{post}} | \phi_a \chi_a^{(+)}). \quad (4)$$

where $E_x = E - E_b$.

Note that the result (3) bears some resemblance with the well-known optical theorem, which provides the total reaction (absorption) cross section in two-body scattering. This analogy was in fact exploited in Ref. [20] to derive Eq. (3), using a *generalized optical theorem* [25].

Udagawa and Tamura [10] derived a very similar formula for the same problem, but making use of the prior-form DWBA. Their final result is formally identical to Eq. (3), but with the x -channel wave function given by ψ_x^{prior} , which is a solution of

$$(E_x^+ - K_x - U_x) \psi_x^{\text{prior}}(\vec{r}_x) = (\chi_b^{(-)} | V_{\text{prior}} | \chi_a^{(+)} \phi_a), \quad (5)$$

with $V_{\text{prior}} \equiv U_{xA} + U_{bA} - U_{aA}$.

Despite their formal analogy, the UT and IAV expressions lead to different predictions for the NEB cross sections. An important result to understand the connection between these two expressions is the relation [16]

$$\psi_x^{\text{post}} = \psi_x^{\text{prior}} + \psi_x^{\text{NO}}, \quad (6)$$

where

$$\psi_x^{\text{NO}}(\vec{r}_x) = (\chi_b^{(-)} | \chi_a^{(+)} \phi_a), \quad (7)$$

is the so-called nonorthogonality (NO) overlap.

Replacing Eq. (7) into Eq. (3), one gets

$$\frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{IAV}} = \frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{UT}} + \frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{NO}} + \frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{IN}}, \quad (8)$$

where we have introduced the nonorthogonality (NO) *cross section*

$$\frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{NO}} = -\frac{2}{\hbar v_a} \rho_b(E_b) \langle \psi_x^{\text{NO}} | W_x | \psi_x^{\text{NO}} \rangle, \quad (9)$$

and the interference (IN) term

$$\frac{d^2\sigma}{dE_b d\Omega_b} \Big|_{\text{NEB}}^{\text{IN}} = -\frac{4}{\hbar v_a} \rho_b(E_b) \text{Re} \langle \psi_x^{\text{prior}} | W_x | \psi_x^{\text{NO}} \rangle. \quad (10)$$

Equation (8) represents the post-prior equivalence of the NEB cross sections in the IAV model, with the right-hand side (RHS) corresponding to the prior-form expression of

this model. The first term is just the UT formula, which is formally analogous to the IAV post-form formula (3), but with the x -channel wave function given by $\psi_x^{\text{prior}}(\vec{r}_x)$. The two additional terms, which are responsible for the discrepancy of the IAV and UT results, arise from the NO overlap. These terms ensure the post-prior equivalence of the NEB cross sections. However, UT considered that these two additional terms are unphysical and hence that the post-prior equivalence does not hold for the NEB. We note here that this problem does not arise for the EBU part, for which the post and prior formulas are well known to give identical results [15]. To support their interpretation, Mastroleo, Udagawa, and Tamura [17] performed calculations for the reactions $^{58}\text{Ni}(\alpha, pX)$ at $E_\alpha = 80$ MeV and $^{62}\text{Ni}(d, pX)$ at $E_d = 25.5$ MeV. In both cases, they found that the sum of the EBU (calculated with DWBA) and the NEB (calculated with the IAV model) overestimates the data. This result was interpreted as evidence for the failure of the IAV model, and support for the UT theory.

This interpretation was later questioned in subsequent works by Ichimura *et al.* [26–28] and also by Hussein and coworkers [29]. These works clearly demonstrated that the UT formula provides only the so-called *elastic breakup fusion* component, which corresponds to breakup without simultaneous excitation of the target A by the interaction V_{xA} , and that the prior-post equivalence does indeed hold for inclusive processes as well.

Despite these intense formal developments, the post-prior equivalence for NEB, represented by Eq. (8), has never been numerically tested to our knowledge. One of the reasons is that the direct solution of Eq. (4) is not possible due to the oscillatory behavior of the source term. This, in turn, is a consequence of the oscillatory behavior of the scattering wave function $\chi_b^{(-)}$, which is not damped asymptotically by either the initial state wave function ϕ_a or by the transition operator V_{post} . Notice that this problem does not arise in the prior form because, in this case, the transition operator (V_{prior}) makes the source term short-ranged. As noted in the introduction, some regularization procedures have been proposed in the literature to overcome this problem. Here, we adopt the method proposed in Ref. [24], which consists of averaging the distorted waves $\chi_b^{(-)}$ over small momentum intervals (bins). The resulting averaged functions become square integrable and the source term of Eq. (4) vanishes at large distances. This procedure was successfully applied in our previous work [20] to several reactions.

In the following section, we apply the IAV and UT models to specific reactions comparing, in the former, the prior and post results.

Calculations. As a first example, we consider the reaction $^{62}\text{Ni}(d, pX)$ at $E_d = 25.5$ MeV, which will allow to compare our results with those from Ref. [17].

In our calculations, the deuteron ground-state wave function was generated with the simple Gaussian potential of Ref. [4]. The deuteron and proton distorted waves are generated with the same optical potentials used in Ref. [17]. As noted in the previous section, to evaluate the post-form formula the distorted waves χ_b are averaged over small momentum intervals. Although this procedure is not required for the prior-form formula, to have consistent ingredients in both

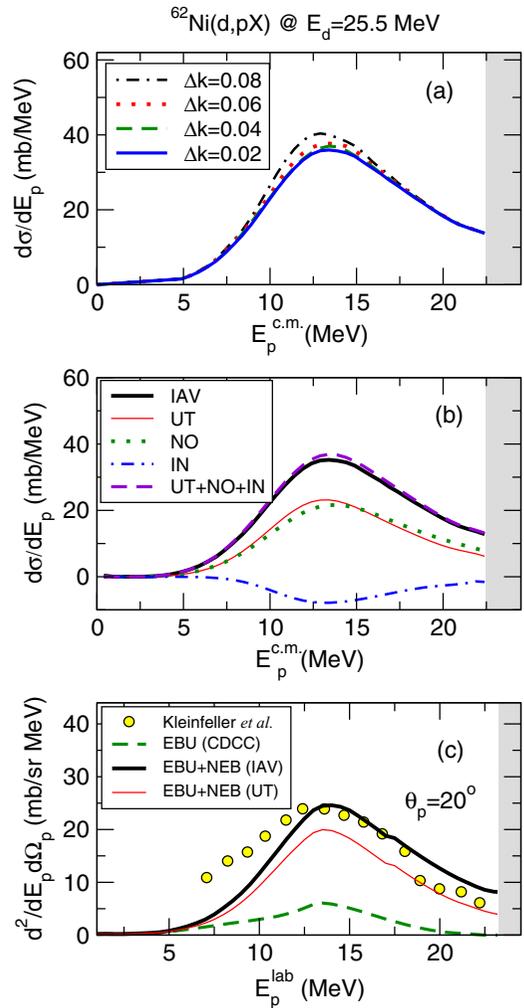


FIG. 1. (Color online) Proton energy spectra for $^{62}\text{Ni}(d, pX)$ at $E_d = 25.5$ MeV. (a) Convergence of the angle-integrated energy distribution (in c.m.) of the NEB cross sections with respect to the bin width (in fm^{-1}) used for the b distorted waves. (b) Comparison of the post and prior results. (c) Comparison of IAV and UT models with the data from Refs. [17,30], corresponding to the double differential cross section as a function of the proton energy in the laboratory frame, for $\theta_p = 20^\circ$.

calculations, the same averaged distorted waves were used in that case.

Before comparing the post and prior results, we investigate the convergence of the post-form formula with respect to the bin size, Δk_b . This is shown in Fig. 1(a) for the angle-integrated NEB differential cross section as a function of the proton energy in the center-of-mass (c.m.) frame. The shaded region corresponds to negative energies of the neutron, that is, transfer to bound states. Although these contributions could be accounted for by using the procedure of Ref. [31], they have not been considered here for simplicity. It is seen that, as the bin width decreases, the results stabilize and for $\Delta k \approx 0.04 \text{ fm}^{-1}$ they are well converged.

In Fig. 1(b) we compare the converged post-form IAV calculation (thick solid line) with the prior calculation (dashed line), for the same observable. The agreement between the

prior and post calculations is seen to be very satisfactory, with only small differences possibly due to numerical inaccuracies. This agreement corroborates the post-prior equivalence at the numerical level. The choice of one or another representation becomes therefore a matter of numerical convenience. We show also in this figure the separate contributions of the prior-form calculation (i.e., UT, NO, and IN), according to Eq. (8). It is seen that the full IAV calculation and the UT result (thin solid line) are in clear disagreement, as anticipated in the introduction.

In Fig. 1(c) we compare the calculations with the experimental data from Refs. [17,30], corresponding to the double differential cross section as a function of the proton energy and for a proton detection angle of $\theta_p = 20^\circ$ in the laboratory frame. We note that, in this experiment, compound-nucleus contributions were estimated and subtracted so the data should mainly correspond to the direct breakup modes considered here. The EBU contribution was calculated with the CDCC formalism, which goes beyond DWBA since it treats Coulomb and nuclear couplings to all orders. For the NEB part, we display the results obtained with the IAV and UT models. It is seen that the sum EBU + NEB(UT), represented by the thin solid line, largely underpredicts the data. In contrast, the sum EBU + NEB(IAV) (thick solid line) reproduces reasonably well the magnitude and shape of the data, except for some underestimation at the smaller energies and some overestimation at the larger ones. We note that the low-energy tail will be mostly affected by the compound-nucleus subtraction and hence some uncertainty is expected at these energies. Our results are in contrast with those reported in Ref. [17], who found an overestimation of the IAV model.

As a second example, we consider the reaction $^{209}\text{Bi}(^6\text{Li},\alpha X)$, which was also analyzed in our previous work [20], by using the post-form IAV model. These calculations reproduced rather well the experimental angular distributions of α particles for a wide range of incident energies above and below the Coulomb barrier. To test the post-prior equivalence, we consider the incident energy of $E = 36$ MeV. For the calculations presented here, we use the potentials employed in Ref. [20].

The results are shown in Fig. 2(a) for the angle-integrated α energy distribution (in the c.m. frame), with the same meaning for the lines as in Fig. 1. The results are qualitatively similar to those found in the deuteron case; namely, (i) the post-form IAV model and the prior-form UT model yield significantly different results, and (ii) the sum UT + NO + IN gives a result very close to the post-form IAV model. Thus, the post-prior equivalence is also well fulfilled in this case.

In Fig. 2(b) we compare these calculations with the data from Ref. [32], which correspond to the angular distribution of α particles in the laboratory frame. The EBU cross section corresponds to the CDCC calculation performed in Ref. [20], so we refer the reader to this reference for further details on this calculation. The EBU+NEB(IAV) calculation (thick solid line) reproduces remarkably well the shape and magnitude of the data. In contrast, the EBU+NEB(UT) calculation, represented by the thin solid line, clearly underestimates the data. This result reinforces the reliability of the IAV model.

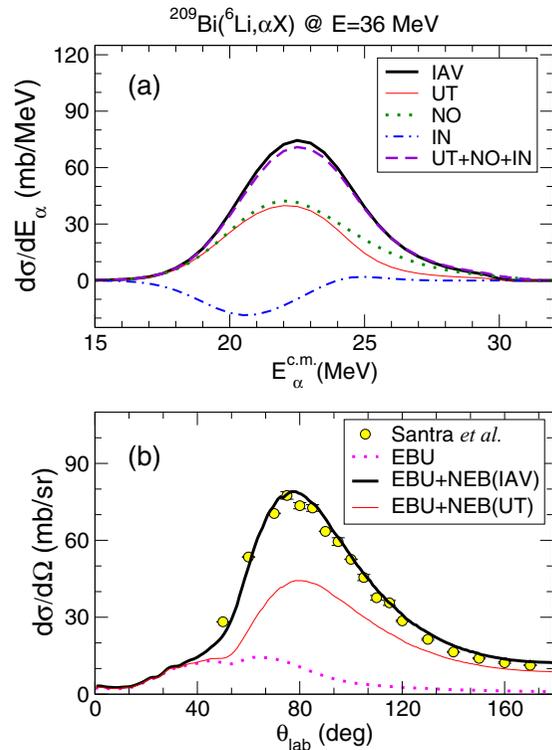


FIG. 2. (Color online) (a) Angle-integrated energy differential cross section, as a function of the α c.m. energy, for the $^{209}\text{Bi}(^6\text{Li},\alpha X)$ reaction at 36 MeV. (b) Experimental and calculated angular distribution of α particles, in the laboratory frame, for the same reaction. The data are from Ref. [32].

Summary and conclusions. In summary, we addressed the problem of the post-prior equivalence in the calculation of NEB cross sections within the closed-form DWBA models proposed in the 1980s by Ichimura, Austern, and Vincent [12,13,15] and by Udagawa and Tamura [10].

We performed calculations for the $^{62}\text{Ni}(d,pX)$ and $^{209}\text{Bi}(^6\text{Li},\alpha X)$ reactions at 25.5 and 36 MeV, respectively. In both cases, we find an excellent agreement between the post and prior expressions of the IAV model, confirming this equivalence at a numerical level. Moreover, the IAV model reproduces rather well the data in both reactions. In contrast, the UT model has been found to underestimate the experimental cross sections. In the $^{62}\text{Ni}(d,pX)$ case, our results disagree with those of Ref. [17], which were used to criticize the theory of IAV.

The results presented in this work, along with those presented in related works [18–20], indicate that the IAV model provides a reliable framework to calculate NEB cross sections in reactions induced by deuteron and ^6Li projectiles. Possible applications to other systems and problems are currently under study.

Acknowledgments. We are grateful to B.V. Carlson and M. Gómez-Ramos for a critical reading of the manuscript. This work has been partially supported by the Spanish Ministerio de Economía y Competitividad, under Grant No. FIS2013-41994-

P, by the Spanish Consolider–Ingenio 2010 Programme CPAN (CSD2007-00042), and by Junta de Andalucía (FQM160, P07-

FQM-02894). J.L. is partially supported by a grant funded by the China Scholarship Council.

-
- [1] J. E. Escher, J. T. Burke, F. S. Dietrich, N. D. Scielzo, I. J. Thompson, and W. Younes, *Rev. Mod. Phys.* **84**, 353 (2012).
- [2] E. Beták *et al.*, IAEA Technical Reports Series.
- [3] L. D. Faddeev, *Zh. Eksp. Teor. Fiz.* **39**, 1459 (1960) [*Sov. Phys. JETP* **12**, 1014 (1961)].
- [4] N. Austern, Y. Iseri, M. Kamimura, M. Kawai, G. Rawitscher, and M. Yahiro, *Phys. Rep.* **154**, 125 (1987).
- [5] S. Typel and G. Baur, *Phys. Rev. C* **50**, 2104 (1994).
- [6] H. Esbensen and G. F. Bertsch, *Nucl. Phys. A* **600**, 37 (1996).
- [7] A. Budzanowski, G. Baur, C. Alderliesten, J. Bojowald, C. Mayer-Böricke, W. Oelert, P. Turek, F. Rösel, and D. Trautmann, *Phys. Rev. Lett.* **41**, 635 (1978).
- [8] G. Baur, R. Shyam, F. Rosel, and D. Trautmann, *Phys. Rev. C* **21**, 2668 (1980).
- [9] R. Shyam, G. Baur, F. Rosel, and D. Trautmann, *Phys. Rev. C* **22**, 1401 (1980).
- [10] T. Udagawa and T. Tamura, *Phys. Rev. C* **24**, 1348 (1981).
- [11] T. Udagawa, X. H. Li, and T. Tamura, *Phys. Lett. B* **135**, 333 (1984).
- [12] N. Austern and C. M. Vincent, *Phys. Rev. C* **23**, 1847 (1981).
- [13] A. Kasano and M. Ichimura, *Phys. Lett. B* **115**, 81 (1982).
- [14] M. Hussein and K. McVoy, *Nucl. Phys. A* **445**, 124 (1985).
- [15] M. Ichimura, N. Austern, and C. M. Vincent, *Phys. Rev. C* **32**, 431 (1985).
- [16] X.-H. Li, T. Udagawa, and T. Tamura, *Phys. Rev. C* **30**, 1895 (1984).
- [17] R. C. Mastroleo, T. Udagawa, and T. Tamura, *J. Phys. G* **15**, 473 (1989).
- [18] B. V. Carlson, R. Capote, and M. Sin, [arXiv:1508.01466](https://arxiv.org/abs/1508.01466).
- [19] G. Potel, F. M. Nunes, and I. J. Thompson, *Phys. Rev. C* **92**, 034611 (2015).
- [20] J. Lei and A. M. Moro, *Phys. Rev. C* **92**, 044616 (2015).
- [21] C. M. Vincent and H. T. Fortune, *Phys. Rev. C* **2**, 782 (1970).
- [22] R. Huby and J. R. Mines, *Rev. Mod. Phys.* **37**, 406 (1965).
- [23] C. M. Vincent, *Phys. Rev.* **175**, 1309 (1968).
- [24] I. J. Thompson, *J. Phys.: Conf. Ser.* **312**, 082041 (2011).
- [25] S. R. Cotanch, *Nucl. Phys. A* **842**, 48 (2010).
- [26] M. Ichimura, N. Austern, and C. M. Vincent, *Phys. Rev. C* **34**, 2326 (1986).
- [27] M. Ichimura, N. Austern, and C. M. Vincent, *Phys. Rev. C* **37**, 2264 (1988).
- [28] M. Ichimura, *Phys. Rev. C* **41**, 834 (1990).
- [29] M. Hussein, T. Frederico, and R. Mastroleo, *Nucl. Phys. A* **511**, 269 (1990).
- [30] J. Kleinfeller *et al.*, *Nucl. Phys. A* **370**, 205 (1981).
- [31] T. Udagawa, Y. J. Lee, and T. Tamura, *Phys. Lett. B* **196**, 291 (1987).
- [32] S. Santra, S. Kailas, K. Ramachandran, V. V. Parkar, V. Jha, B. J. Roy, and P. Shukla, *Phys. Rev. C* **83**, 034616 (2011).