Skyrme models and nuclear matter equation of state

C. Adam,[1] M. Haberichter,[2] and A. Wereszczynski[3]

[1]*Departamento de Física de Partículas, Universidad de Santiago de Compostela and Instituto Galego de Física de Altas Enerxias (IGFAE)*
E-15782 Santiago de Compostela, Spain
[2]*School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NF, United Kingdom*
[3]*Institute of Physics, Jagiellonian University, Lojasiewicza 11, Kraków, Poland*

We investigate the role of pressure in a class of generalized Skyrme models. We introduce pressure as the trace of the spatial part of the energy-momentum tensor and show that it obeys the usual thermodynamical relation. Then, we compute analytically the mean-field equation of state in the high- and medium-pressure regimes by applying topological bounds on compact domains. The equation of state is further investigated numerically for the charge-one Skyrmions. We identify which term in a generalized Skyrme model is responsible for which part in the equation of state. Further, we compare our findings with the corresponding results in the Walecka model.

I. INTRODUCTION

The Skyrme model [1] is a low-energy effective model of QCD where baryons (atomic nuclei) are realized as emergent objects in a mesonic fluid. In modern language, they are solitons in a field theory based entirely on some mesonic field U, in the simplest version, U ∈ SU(2). The pertinent topological charge carried by a soliton (Skyrmion) may be identified with the baryon charge.

The most general Lagrange density respecting Poincare invariance and leading to a standard Hamiltonian reads

$$\mathcal{L}_{0246} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6, \tag{1.1}$$

where

$$\mathcal{L}_2 = -\lambda_2 \, \text{Tr} \, (L_\mu L^\mu), \quad \mathcal{L}_4 = \lambda_4 \, \text{Tr} \, ([L_\mu, L_\nu]^2),$$
$$\mathcal{L}_6 = -(24\pi^2)^2 \lambda_6 \mathcal{B}_\mu \mathcal{B}^\mu. \tag{1.2}$$

Here, the left invariant current L_μ is given by

$$L_\mu = U^\dagger \partial_\mu U \tag{1.3}$$

and $\mathcal{L}_0$ is a nonderivative part, i.e., a potential. In its simplest version $\mathcal{L}_{024}$, where the sextic part is neglected, the model was shown to describe baryons and light nuclei with a very good accuracy [2–7]. However, this particular proposal leads to results that are difficult to reconcile with two important qualitative properties of atomic nuclei and nuclear matter. It gives rather large binding energies and describes crystal-like nuclear matter [8], whereas nuclear matter is only weakly bound in atomic nuclei and behaves more like a fluid. These unwanted properties of the usual Skyrme model can be resolved if one adds a larger amount of the sextic term. This is related to the observation that there exists a submodel within the family of Skyrme type models, which has the BPS property (classical zero binding energies [9]) and describes a fluid (has the energy-momentum tensor of a perfect fluid [10] and is, in fact, a field theoretical realization of the Eulerian fluid [11]). This submodel is called the BPS Skyrme model [9] and reads

$$\mathcal{L}_{\text{BPS}} \equiv \mathcal{L}_6 + \tilde{\mathcal{L}}_0 \tag{1.4}$$

where $\tilde{\mathcal{L}}_0$ is a further potential. Then, an improved proposal, which might also be valid for the high baryon charge regime, is given by the near-BPS Skyrme model [12]

$$\mathcal{L} = \epsilon(\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4) + \mathcal{L}_{\text{BPS}}, \tag{1.5}$$

where the usual Skyrme model only provides rather small corrections to masses (binding energies), as ϵ is assumed to be a small parameter. Let us remark that the sextic term $\mathcal{L}_6$ can be effectively introduced by a coupling with the ω meson [13–18]. Here we consider the general Skyrme theory with the SU(2) fields only, where some effects induced by further mesons emerge due to a particular form of the action rather than by the inclusion of some new explicit degrees of freedom.

Although the BPS limit is physically very well motivated and provides rather accurate results for binding energies for larger nuclei [after a careful treatment of semiclassical quantisation of (iso)rotational modes, inclusion of the Coulomb interaction and isospin breaking [12] (see also Ref. [19])], it is still necessary to investigate the full near-BPS version. The inclusion of the non-BPS part leads to the appearance of physical pions and determines the proper geometry (shape) of Skyrmions [20], which is of high importance as it results in particular patterns for the isorotational excitations. Hopefully, one can get a model that unifies the convincing results for baryons and light nuclei with the crucial properties of the BPS model. This is, in principle, a difficult task since the solvability property of the BPS model is lost once the perturbative part (usual Skyrme action) is added. Moreover, such a perturbation is nonanalytical as the non-BPS part is the dominating part close to the vacuum. Hence, no small ϵ expansion exists at the level of the field equations. Some recent findings for solitons with first few baryon charges, however in a not-so-near-BPS regime, can be found in Ref. [21] (see also Ref. [22]).

The aim of the present work is to investigate the issue of the equation of state (EoS) in the general Skyrme model with a special focus on the near-BPS model. The BPS submodel allows for a rather complete analysis of its thermodynamics at zero temperature, as a consequence of the BPS property and of its perfect fluid form which, however, no longer hold for the general (non-BPS) model. The first difficulty, therefore, is to

properly define the pressure, since in the absence of a perfect fluid description no obvious definition exists. This is analyzed in Sec. II. In Sec. III, using some topological energy bounds on a compact domain, an asymptotic equation of state is derived. Then, in Sec. IV, we make a conjecture about subleading terms in the EoS by exploiting some properties of generalized BPS Skyrme models. In Sec. V, the medium and large pressure regimes are checked numerically for the charge-one, hedgehog configuration. Section VI is devoted to a comparison with the equation of state (EoS) of the Walecka model. Finally, we summarize our findings. We shall frequently use the standard Skyrme field parametrization

$$
\begin{aligned}
U &= \cos\xi + i\sin\xi\,\vec{n}\cdot\vec{\sigma}, \\
\vec{n} &= (\sin\chi\cos\Phi, \sin\chi\sin\Phi, \cos\chi)
\end{aligned}
\tag{1.6}
$$

where ξ, χ, Φ are real field variables and $\vec{\sigma}$ are the Pauli matrices. Further, we shall always assume that $U = \mathbf{1}$, i.e., $\xi = 0$, is the vacuum value of the Skyrme field, which it must approach at spatial infinity.

II. AVERAGE PRESSURE AND AVERAGE CHEMICAL POTENTIAL

The energy-momentum tensor in the BPS Skyrme model takes the form of a perfect fluid energy-momentum tensor and, therefore, it directly defines the field theoretical (microscopic) pressure and baryon chemical potential (i.e., the pressure and chemical potential densities). For a generic Skyrme Lagrangian, there is no such fluid description and one usually has a nontrivial energy-momentum tensor with different stresses T^{ij} in different directions. However, a pressure density $p = \frac{1}{3}\sum_i T^{ii}$ and the corresponding average pressure P, i.e., the volume average of $\frac{1}{3}T^{ii}$ (summation assumed), as well as an average chemical potential $\bar{\mu}$ may still be defined as proper thermodynamic (macroscopic) variables, i.e., obeying the required thermodynamical relations

$$
\left(\frac{\partial E}{\partial V}\right)_B = -P,
\tag{2.1}
$$

$$
\left(\frac{\partial E}{\partial B}\right)_V = \bar{\mu}.
\tag{2.2}
$$

Here $E = E(V, P, B, \bar{\mu})$ is the static energy of the Skyrme model, depending on the volume V, pressure P, baryon charge B, and chemical potential $\bar{\mu}$. For this purpose, we first define the following generalized step function

$$
\tilde{\Theta}(U) = \begin{cases} 1 & \text{for} \quad U \neq 1 \\ 0 & \text{for} \quad U = 1 \end{cases},
\tag{2.3}
$$

[for the standard parametrization (1.6), $\tilde{\Theta}(U)$ may be replaced by the standard step function $\Theta(\xi)$]. Further, we define the locus function of a static Skyrmion configuration $U = U_0(\vec{x})$ as the pullback $U_0^*[\tilde{\Theta}(U)]$ of $\tilde{\Theta}(U)$ under U_0, and the locus set of the Skyrmion $U_0(\vec{x})$,

$$
\Omega = \{\vec{x} \in \mathbb{R}^3 \mid U_0^*[\tilde{\Theta}(U)] \equiv \tilde{\Theta}[U_0(\vec{x})] = 1\},
\tag{2.4}
$$

i.e., the set $\Omega \subset \mathbb{R}^3$ where the Skyrmion is located (deviates from the vacuum). Now, let us consider the general Skyrme

static energy functional

$$
E(V, P, B, \bar{\mu}) = \int d^3x\,\varepsilon[U] + P\left(\int d^3x\,\tilde{\Theta}(U(\vec{x})) - V\right) \\
- \bar{\mu}\left(\int d^3x\,\mathcal{B}_0 - B\right),
\tag{2.5}
$$

where $\varepsilon[U]$ is the energy density and P and $\bar{\mu}$ are Lagrange multipliers. In particular, P imposes the condition that all possible solutions of the variational problem (2.5) must have volume V, i.e., $\int d^3x\,\tilde{\Theta}[U(\vec{x})] = \int_\Omega d^3x = V$. Obviously, P and $\bar{\mu}$ obey the thermodynamical relations (2.1) and (2.2) by construction. We still have to show that the Lagrange multiplier P is indeed the average pressure as defined above. For that we consider scaling transformations $x^i \to e^\lambda x^i = (1 + \lambda)x^i$. Then $\delta_\lambda U = \lambda x^i \partial_i U$. For the energy functional (2.5) we get off shell, in first order in λ,

$$
\delta_\lambda E = \lambda \int T_{ii} d^3x - 3\lambda P \int d^3x\,\tilde{\Theta}[U(\vec{x})],
\tag{2.6}
$$

where we use the fact that the variation of the energy $\int d^3x\,\varepsilon[U]$ under the scaling transformation gives the integral of the trace of the spatial part of the energy-momentum tensor [20]. But any solution of the Euler-Lagrange equations is a stationary point and, therefore, $\delta_\lambda E = 0$ on-shell and

$$
P = \frac{\frac{1}{3}\int_\Omega T_{ii}d^3x}{\int_\Omega d^3x},
\tag{2.7}
$$

which is exactly the average pressure definition. We remark that the Lagrange multiplier imposing a fixed value for the volume does not uniquely determine the compact domain Ω. In fact, using the volume-preserving diffeomorphisms we get infinitely many different sets Ω, which result in different values of the trace integral (i.e., the pressure P). In other words, one has to fix the compact domain Ω and, therefore, the Skyrmion solution up to target space symmetries, to get a unique average pressure. In general, this is a very difficult task. However, for a spherically symmetric unit charge Skyrmion, the obvious choice is such that the symmetry of the equilibrium solution is preserved while squeezed. Then, Ω is a three-dimensional ball. Moreover, in the BPS limit the static energy functional is invariant under the volume preserving diffeomorphisms of the base space and therefore all SDiff related Ω give exactly the same pressure—as is the case for a perfect fluid.

Let us also remark that the proper (thermodynamical) volume appearing in relation (2.1) is the geometrical volume of a topological soliton. It means that in a typical case such a volume tends to infinity in the equilibrium (zero pressure) limit since typical Skyrmions are infinitely extended solutions. This leads to some difficulties if one wants to consider the mean-field energy density $\bar{\varepsilon} = E/V$ at the equilibrium. Obviously, one gets $\bar{\varepsilon} = 0$, which is the same as for the vacuum configuration. Note, however, that compact Skyrmions with finite volumes are known (for example, they are quite common in the BPS Skyrme model).

III. ASYMPTOTIC EQUATION OF STATE

Now we will use these definitions to compute thermodynamical functions in the asymptotic regime, i.e., in the limit of high pressure (density) or equivalently in the limit of small volume. In order to do that we need an expression that shows how the energy of Skyrmions changes with volume and baryon density. This is a quite difficult problem as, in principle, it requires the solution of the full three-dimensional problem. However, in the asymptotic regime we can use some previously found topological bounds.

The relevant bounds for the quartic (Skyrme) and sextic terms are [23,24]

$$E_4 = \frac{1}{16} \int d^3x \, \text{Tr} \, [L_i, L_j]^2 \geqslant 3(2\pi^2)^{4/3} \frac{B^{4/3}}{V^{1/3}}, \quad (3.1)$$

$$E_6 = \int d^3x \, (\epsilon^{ijk} \, \text{Tr} L_i L_j L_k)^2 \geqslant \frac{B^2}{V}. \quad (3.2)$$

There is no bound for the quadratic part

$$E_2 = \int d^3x \, \text{Tr} \, L_i L_i, \quad (3.3)$$

or for a general potential. Therefore, the asymptotic static energy for any Skyrme model can be approximated as [$\lambda_6 = \lambda^2/(24)^2$ and $\lambda_0 = \mu^2$]

$$E = \lambda_2 E_2 + \lambda_4 E_4 + \lambda^2 \pi^4 E_6 + \lambda_0 E_0$$
$$\geqslant \lambda_4 3(2\pi^2)^{4/3} \frac{B^{4/3}}{V^{1/3}} + \pi^4 \lambda^2 \frac{B^2}{V} \geqslant \pi^4 \lambda^2 \frac{B^2}{V}. \quad (3.4)$$

In the small volume limit, the quartic term provides a subleading contribution. The same holds for the potential term and the σ model part, which affect the total energy even less. However, since in the small volume limit the sextic term governs the masses of Skyrmions, one recovers the BPS limit. Hence, the bound is, in fact, saturated for sufficiently small volumes (high pressures). This is not the case for the quartic term. As we will see, the corresponding bound is not saturated. All this allows us to predict the asymptotic formula for the energy as

$$E = \pi^4 \lambda^2 \frac{B^2}{V} + \alpha \frac{B^{4/3}}{V^{1/3}} + o(V^{-1/3}), \quad V \to 0, \quad (3.5)$$

with

$$\alpha \geqslant 3(2\pi^2)^{4/3} \lambda_4. \quad (3.6)$$

Then, the average pressure is

$$P = \pi^4 \lambda^2 \bar{\rho}_B^2 + \frac{\alpha}{3} \bar{\rho}_B^{4/3} + o(\bar{\rho}_B^{4/3}), \quad (3.7)$$

and the average baryon chemical potential

$$\bar{\mu} = 2\pi^4 \lambda^2 \bar{\rho}_B + \frac{4\alpha}{3} \bar{\rho}_B^{1/3} + o(\bar{\rho}_B^{1/3}), \quad (3.8)$$

where $\bar{\rho}_B = \frac{B}{V}$ is the average baryon (particle) density. The average energy density is

$$\bar{\varepsilon} = \pi^4 \lambda^2 \bar{\rho}_B^2 + \alpha \bar{\rho}_B^{4/3} + o(\bar{\rho}_B^{4/3}). \quad (3.9)$$

In the leading approximation, we reobtain the BPS Skyrme model asymptotic behavior

$$P = \pi^4 \lambda^2 \bar{\rho}_B^2, \quad (3.10)$$

$$\bar{\mu} = 2\pi^4 \lambda^2 \bar{\rho}_B, \quad (3.11)$$

$$\bar{\varepsilon} = \pi^4 \lambda^2 \bar{\rho}_B^2. \quad (3.12)$$

The resulting equation of state reads

$$\bar{\varepsilon} = P. \quad (3.13)$$

Let us summarize the main findings.

(i) The sextic term gives the main contribution in the high-pressure limit. This means that this term should not be omitted if dense nuclear matter is considered. The asymptotic equation of state always has a universal (potential independent) form $\bar{\varepsilon} = P$.

(ii) The quartic, usual Skyrme term, gives a subleading contribution, which modifies the equation of state at moderate pressures. The functional dependence is known, which is not the case for the multiplicative constant α, for which we have derived a lower bound.

(iii) The potential and the σ model part give contributions, which are even subleading in comparison to the E_4 contribution. On the other hand, they may be significant close to nuclear saturation density.

For a deeper insight into the role played by the σ model part as well as by the potential term, we need to perform numerical computations. Nonetheless, one can get some help from the BPS Skyrme model, which, as we will see below, can give some understanding on the functional mass-volume relation originating from the E_2 and E_0 terms.

IV. BPS FLUID TOY MODELS

Each part of the generalized Skyrme model has a specific number of spatial derivatives. This is a main—but clearly not the only—difference between them. As we have previously commented, the sextic term is special as it leads to a perfect fluid model. Here, we want to learn how a specific number of derivatives can change the equation of state, assuming that all terms are based on the same thermodynamical quantity—here the baryon density. Within this approach, we replace the σ model term and the quartic Skyrme term by energy functionals based only on the baryon density, which is a natural quantity for the sextic term (note that similar nonlinear models have been recently considered in Ref. [25] in the context of the baby Skyrme model in 2 + 1 dimensions).

We start with the Skyrme (quartic) term

$$\mathcal{L}_4 = \lambda_4 \, \text{Tr} \, ([L_\mu, L_\nu]^2) \longrightarrow \mathcal{L}_4^f = \lambda_4 (\mathcal{B}^\mu \mathcal{B}_\mu)^{2/3}, \quad (4.1)$$

which now is represented by a four-derivative term (in the sense that under Derrick scaling $x^\mu \to \Lambda x^\mu$ it scales like Λ^{-4}) constructed from the baryon density (current). Such a new term leads to the following energy density (for static solutions)

$$\varepsilon = \lambda_4 \rho_B^{4/3}, \quad (4.2)$$

and pressure

$$P = \frac{1}{3}\lambda_4 \rho_B^{4/3}, \quad (4.3)$$

where we use that, for a general energy density based on baryon (particle) density, the pressure reads

$$P = \rho_B \frac{\partial \varepsilon}{\partial \rho_B} - \varepsilon. \quad (4.4)$$

The corresponding equation of state reads

$$P = \frac{1}{3}\bar{\varepsilon}, \quad (4.5)$$

which exactly coincides with the mean-field equation of state for the usual Skyrme term. Hence, the substitution completely reproduces the thermodynamical properties of the E_4 model. This leads to the conjecture that, at least to some extent, thermodynamical properties related to each term of the generalized Skyrme model can be similar to properties generated by the BPS fluid counterpart, i.e, by terms built out of the baryon density elevated to a certain power. Now, let us apply the same strategy to extract information about possible equations of state that may emerge from E_2 and E_0. In the case of the σ model term we get

$$\mathcal{L}_2 = -\lambda_2 \, \mathrm{Tr}\, (L_\mu L^\mu) \longrightarrow \mathcal{L}_2^f = -\lambda_2 (\mathcal{B}_\mu \mathcal{B}^\mu)^{1/3}. \quad (4.6)$$

Thus, we find

$$\varepsilon = \lambda_2 \rho_B^{2/3}, \quad (4.7)$$

$$P = -\frac{\lambda_2}{3}\rho_B^{2/3}, \quad (4.8)$$

and the equation of state

$$P = -\frac{1}{3}\varepsilon. \quad (4.9)$$

As expected, this part is responsible for a negative pressure, i.e., an attractive force (which is balanced by the inclusion of other stabilizing terms).

The issue of the potential is more subtle as it does not depend on the baryon density but, in general, on the Skyrme field. One finds that

$$\varepsilon = \mathcal{U}, \quad P = -\mathcal{U}, \quad (4.10)$$

and

$$P = -\varepsilon. \quad (4.11)$$

However, for the step-function potential $\mathcal{U} = \tilde{\Theta}(U)$, it only contributes as a numerical constant to thermodynamical quantities. Then, for example, the energy density is shifted by such a constant. Since in a first approximation, we can always model a potential by the step-function potential, we may expect that a constant density term shows up.

Hence, finally we conjecture a theoretically motivated energy density-baryon density relation, which can be valid not only for asymptotically high pressure but also in a medium-pressure regime

$$\bar{\varepsilon} = \pi^4 \lambda^2 \bar{\rho}_B^2 + \alpha \bar{\rho}_B^{4/3} + \beta + \tilde{\beta}\,\bar{\rho}_B^{2/3} + O(1). \quad (4.12)$$

This leads, for the charge one sector, to the following energy-volume formula

$$E = \pi^4 \lambda^2 \frac{1}{V} + \alpha \frac{1}{V^{1/3}} + \beta V + \tilde{\beta}\, V^{1/3} + O(V). \quad (4.13)$$

In the subsequent sections we will compare formula (4.13) with numerical computations. One should keep in mind that, in contrast to the first two leading terms, which have been proven to emerge in the high-pressure regime, the last two subleading terms are conjectured (or at best heuristically motivated) by analyzing similar BPS models.

V. NUMERICAL RESULTS IN THE CHARGE-ONE SECTOR

The equation of state as well as the equivalence of the two pressure definitions can be tested by direct numerical computations in the charge one sector where the hedgehog ansatz can be assumed

$$U = \cos\xi + i\sin\xi \vec{n}\cdot\vec{\sigma}, \quad (5.1)$$

where $\xi = \xi(r)$, $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector covering the $\mathbb{S}^2$ target subspace once, and $\vec{\sigma}$ are the Pauli matrices. Then, the energy contributions are given by

$$E_0 = 4\pi \int r^2 dr\, 2(1 - \cos\xi), \quad (5.2)$$

$$E_2 = 4\pi \int r^2 dr\, 2\left(\xi_r^2 + \frac{2\sin^2\xi}{r^2}\right), \quad (5.3)$$

$$E_4 = 4\pi \int r^2 dr\left(2\frac{\sin^2\xi}{r^2}\xi_r^2 + \frac{\sin^4\xi}{r^4}\right), \quad (5.4)$$

$$E_6 = 4\pi \int r^2 dr\, \frac{1}{4r^4}\sin^4\xi \xi_r^2. \quad (5.5)$$

The aim is not only to verify the asymptotic formula for the EoS but also to get some insight into the role that is played by the quadratic and potential parts, for which we do not have any analytical (asymptotic) expression. For $B = 1$ Skyrmions we solve a one-dimensional ODE with boundary conditions, which guarantee the nontrivial topology and pressure

$$\xi(r = 0) = \pi, \quad \xi(r = R) = 0. \quad (5.6)$$

In other words, we enclose the Skyrmion in a finite volume $V = (4\pi/3)R^3$.

We use the collocation method [26] to determine the profile function ξ, which minimizes the Skyrme energy functional E with boundary conditions $\xi(0) = \pi$ and $\xi(R) = 0$. The associated Euler-Lagrange equations are solved on the interval $0 < r < R$. For the high-pressure regime, we consider the interval $0.1 < R < 1$. We start with $R = 0.1$ as right boundary point and then increase in each collocation run the right boundary value by $dr = 0.0001$. The increment dr is chosen to be sufficiently small, which allows us to check numerically the equivalence of field-theoretical and thermodynamical pressure. For the low-pressure regime, we run simulations up to a maximal boundary point value of 10 with $dr = 0.001$.

We use the nonlinear curve-fitting function OPTI-MIZE.CURVE_FIT from Python's SCIPY package [27] to fit the

asymptotic energy and pressure formulas to our numerical data in the high-pressure regime.

The energy and length units have to be fixed by comparison with experimental nuclear physics data. In this article, we calibrate the usual Skyrme model without the sextic part using the approach of Refs. [2]. The energy and length units are tuned to match the experimental nucleon ($M_N = 939$ MeV) and Δ ($M_\Delta = 1231$ MeV) masses. This fixes the energy and length scale as follows

$$12\pi^2 \times \left[\frac{F_\pi}{4e_{\text{Sky}}}\right] = 12\pi^2 \times 5.58 \,\text{MeV},$$
$$\left[\frac{2}{e_{\text{Sky}}F_\pi}\right] = 0.755 \,\text{fm}, \tag{5.7}$$

and gives the well-known parameter values

$$e_{\text{Sky}} = 4.84, \quad F_\pi = 108 \,\text{MeV}, \quad \hbar = 46.8,$$
$$\text{and} \quad m_\pi = 138 \,\text{MeV}. \tag{5.8}$$

In this article, if not explicitly stated otherwise, the rescaled pion mass μ is set to 1. Note that the limitations of the parameter set (5.8) have been discussed previously [28–30] and different parameter choices have been explored in the literature, e.g., Refs. [6,7,28,31].

Our numerical results in Sec. V B on BPS and near-BPS Skyrme models are expressed in terms of the same energy and length units (5.7). However, note that for the full near-BPS Skyrme model E_{0246}, discussed in Sec. V B 3, the coupling parameters of the BPS part are fixed by matching the BPS Skyrmion mass to one fourth of the helium nucleus mass and the Skyrmion radius to the nucleon radius (see Sec. V B 3 and Ref. [32] for a detailed discussion).

A. Perturbative Skyrme model

We start with the usual, old Skyrme model without the sextic part. The reason for this is to find a clear signal from the quartic term, which now gives the leading high pressure behavior and then understand the influence of the potential and sigma model terms as first subleading effects. The energy functional we minimize reads

$$E_{024} = \lambda_2 E_2 + \lambda_4 E_4 + \lambda_0 E_0, \tag{5.9}$$

where the constants are

$$\lambda_2 = \frac{1}{24\pi^2}, \quad \lambda_4 = \frac{1}{12\pi^2}, \quad \lambda_0 = \frac{1}{12\pi^2}, \tag{5.10}$$

or zero in the case where the pertinent term is omitted. Here we consider the usual Skyrme potential, which is relevant for the conventional (perturbative) Skyrme model

$$E_0 = \text{Tr}\,(1 - U). \tag{5.11}$$

For all possible submodels we first checked the equivalence between the average field-theoretical pressure and the thermodynamical pressure. We always found a perfect agreement between the pressures computed by Eqs. (2.1) and (2.7). It is also a precision test for our numerical computations.

1. E_4 model

We begin with the simplest submodel, which contains only the quartic part. Such a limit of the Skyrme model has been considered before, however, from the instanton point of view [33]. Obviously, due to the Derrick theorem there are no stable solitonic solutions at equilibrium (zero pressure), i.e., in the full infinite $\mathbb{R}^3$. However, if we close the system in a finite volume V, then Skyrmions do exist. For a charge one soliton we have the following energy integral

$$E_4 = \lambda_4 \mathcal{E}_4 = \frac{1}{12\pi^2} 4\pi \int_0^R dr\, r^2 \left(2\frac{\sin^2 \xi}{r^2}\xi_r^2 + \frac{\sin^4 \xi}{r^4}\right). \tag{5.12}$$

This leads to a relatively simple equation of motion

$$\frac{d^2}{dr^2}\cos\xi + \frac{1}{r^2}\sin^2\xi\cos\xi = 0 \tag{5.13}$$

accompanied by the pertinent boundary conditions. Unfortunately, we were not able to solve it analytically. It is straightforward to notice that a solution on the segment $[0, R]$ can be related to a solution on a different segment $[0, \Lambda R]$ by the scale transformation $r \to \Lambda r$. Then, the energy scales with the factor Λ^{-1}, which leads to

$$E_4[V] = \frac{1}{V^{1/3}}E_4[V=1] \equiv \alpha\frac{1}{V^{1/3}}, \tag{5.14}$$

where the coefficient α can be understood as the energy in the unit volume. Hence, the pressure is a simple function of the volume

$$P = \frac{\alpha}{3}\frac{1}{V^{4/3}}. \tag{5.15}$$

In Fig. 1, such an equation is confirmed by numerical computations. Indeed, we find that

$$\alpha = 1.853 \quad \text{or} \quad \alpha = 924.4 \,\text{MeV fm}, \tag{5.16}$$

where the first value is in Skyrme units while the second is in physical units. It is worth to compare this with the previously derived bound. Namely, $3(2\pi^2)^{4/3}/(12\pi^2) = 1.351$. This shows that the bound is not saturated. The true solution is significantly above the bound.

Finally, we get an exact density-pressure equation of state, which is valid for any value of the pressure

$$\bar{\varepsilon} = 3P. \tag{5.17}$$

This result can be easily generalized to any topological sector and to any given compact Ω. Indeed, let us consider the energy functional

$$E_4[\Omega] = \frac{1}{12\pi^2}\frac{1}{16}\int_\Omega d^3x\,\text{Tr}\,[L_i, L_j]^2. \tag{5.18}$$

Obviously, it scales homogeneously under scaling transformations. Then again

$$E_4[V; \Omega] = \frac{\alpha_{B,\Omega}}{V^{1/3}}, \tag{5.19}$$

where the constant α depends on the topological charge and a particular form of the compact manifold (assumed now to have unit volume). From bound (3.1) we know that

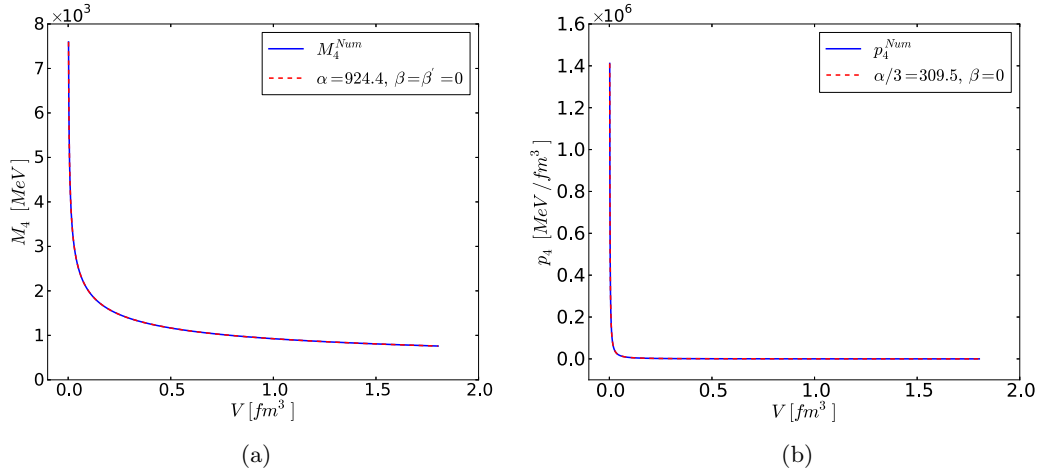$$\alpha_{B,\Omega} \geqslant 3(2\pi^2)^{4/3}B^{4/3}. \tag{5.20}$$

FIG. 1. (Color online) (a) Energy and (b) pressure of the $B = 1$ Skyrmion as a function of the volume for E_4 model.

However, as the energy functional is not invariant under the volume preserving transformations this constant should depend on the choice of Ω. In any case, it proves that the equation of state (5.17) is an off-shell universal equation of state for the pure quartic model.

Finally, let us remark that the bound is saturated if the geometries of the base and target spaces coincide. Indeed, if the base space is a three-sphere with a radius R then we get

$$E_4^{\mathbb{S}^3} = \frac{1}{12\pi^2} 4\pi \frac{1}{R} \int_0^\pi d\psi \left(2\sin^2\xi \xi_\psi^2 + \frac{\sin^4\xi}{\psi^2} \right), \quad (5.21)$$

where ψ is the third angular coordinate on the sphere. This variational problem has a solution $\xi = \psi$. Hence, the charge-one Skyrmionic solution is just the identity map between the base and target spheres. One can easily verify that in this case the bound is equal to the value of the last integral. This agrees with a classical result of Manton [34] where he shows that the identity map is a stable solution for the E_{24} Skyrme model for a sufficiently small radius R of the base space sphere. The geometrical reason why the $\mathbb{R}^3$ and $\mathbb{S}^3$ cases are so distinct is the following. The bound can be saturated if and only if the three eigenvalues of the corresponding strain tensor are (i) all equal and (ii) constant, which means that the corresponding map must be an isometry [23]. This is a very restrictive condition which, for the target space $\mathbb{S}^3$, may be fulfilled only if the base space is also a three-sphere.

2. E_{04} model

The pure quartic Skyrme model can be stabilized by the addition of a potential term. Then, the Derrick argument is evaded and an equilibrium solution exists. One peculiarity of this submodel (with the usual Skyrme potential) is that one gets a compact Skyrmion, i.e., a topological soliton, which achieves its vacuum value at a finite distance. The size of the compact charge one Skyrmion at equilibrium is

$$R = 2.07 \quad \text{or} \quad R = 1.563 \text{ fm},$$

where the first number is in Skyrme units while the second in physical units. This sets the maximum volume (i.e., the volume at zero pressure) to $V_{\text{max}} = 15.9$ fm^3.

The general asymptotic energy formula

$$E = \alpha \frac{1}{V^{1/3}} + \tilde{\beta} V^{1/3} + \beta V + o(V), \quad (5.22)$$

gives the following values for the parameters

$$\alpha_{04} = 924.6 \text{ MeV fm}, \quad \beta_{04} = 13.4 \text{ MeV fm}^{-3}, \quad \tilde{\beta}_{04} = 0. \quad (5.23)$$

In Fig. 2 we plot the numerically computed energy (mass) and pressure as a function of the volume together with the theoretical formulas. We find a perfect agreement. We can also use the theoretical formula (5.22) to get the mass of the equilibrium Skyrmion. Then, we get $M_{04} = 580$ MeV, which is only 3.5% above the true mass. Undoubtedly, the theoretical formula works very well for this submodel.
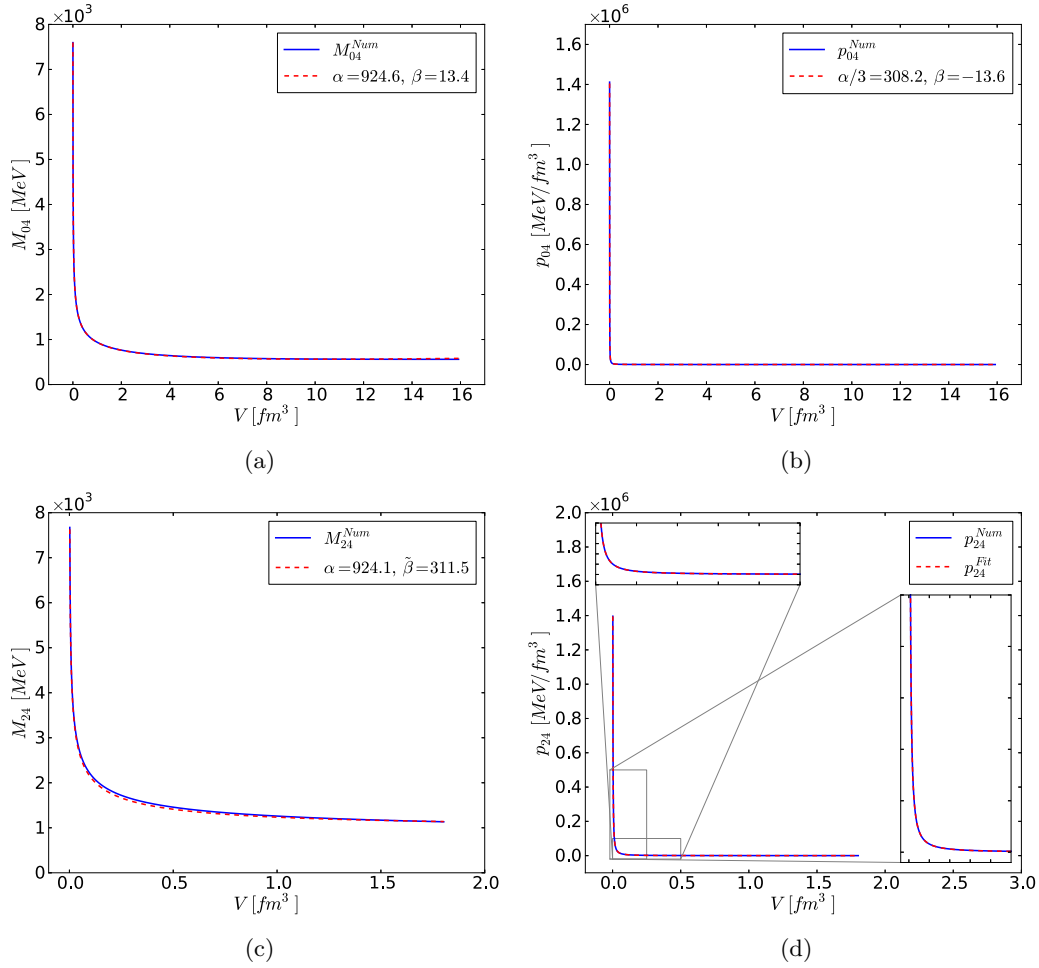
3. E_{24} model

The conventional way for stabilizing the E_4 model is to add the σ model part. This is the usual massless Skyrme model. Now the equilibrium solution in the charge one sector is an infinitely extended Skyrmion. Again, we want to fit the theoretical energy formula. The first issue is that we get a slightly smaller value of the leading term, namely, $\alpha_{24} = 915.9$ MeV fm. This means that we are still a bit away from the exact asymptotic regime. Apparently, one has to squeeze the Skyrmion to higher pressures to reach the proper value for α. It shows that the σ model part provides a significantly stronger attractive force than the potential term, which influences the squeezed configurations and has a stronger impact on the energy-volume relation.

In the fitting procedure we take this into account by imposing that $\alpha_{24} = 924.1$ MeV fm. Then,

$$\alpha_{24} = 924.1 \text{ MeV fm}, \quad \beta_{24} = 0, \quad \tilde{\beta}_{24} = 311.5 \text{ MeV fm}^{-1}. \quad (5.24)$$

In Fig. 2 we compare the numerical results and the theoretical curve. Again we find nice agreement in the high and medium pressure regime. For sufficiently small pressure (large volume) the theoretical formula is not valid any longer as the $V^{1/3}$ term gives a diverging contribution.

FIG. 2. (Color online) Energy and pressure of the $B = 1$ Skyrmion as a function of the volume for E_{04} and E_{24} model.

4. E_{024} model

Finally, one can include all three terms. Then, after fitting the theoretical curve we get (with the same remark on the α constant, which again does not take precisely its asymptotic value but $\alpha = 915.9$ MeV)

$$\alpha_{024} = 924.1 \text{ MeV fm}, \quad \beta_{024} = -106.1 \text{ MeV fm}^{-3},$$
$$\tilde{\beta}_{024} = 458.0 \text{ MeV fm}^{-1}. \tag{5.25}$$

Note that the constant β, which is related to the potential part of the action, takes a negative value. In fact, the potential and σ model term contribute together to the subleading part of the energy-volume relation, which leads to this effect. It also suggests that the theoretical formula (the last two terms motivated by the BPS model analysis) is not quite adequate for this model. In fact, if we subtract the (corrected) leading term from the numerical energy then the best fit is found for V^a where $a \approx 1/5$, see Fig. 3. This leads to the following effective mass-volume formula

$$E_{024} = \alpha \frac{1}{V^{1/3}} + \gamma V^{1/5} + o(V^{1/5}), \tag{5.26}$$

where $\gamma = 346.1$ MeV fm$^{-1/5}$. This shows that in the full perturbative Skyrme model there is a rather strong mixing

between the σ model and potential part (and probable also with the quartic term), which effectively provides new subleading terms in the energy-volume relation. Therefore the theoretical curve proposed previously does not seem to work quite well in the full perturbative Skyrme model.

In Fig. 4 we plot numerical results describing how the energy (mass) and pressure depend on the average baryon density $n_B = 1/V$. Furthermore, we plot the resulting equation of state relating the energy and pressure.

It is also instructive to compare the mean-field energy density $\bar{\varepsilon}$ with the true energy density distribution. This comparison makes no sense at the equilibrium ($P = 0$) where the soliton is infinitely extended. This means that the geometric volume is infinity $V = \infty$ and the corresponding mean-field energy density is simply zero. However, for squeezed configuration the volume is always finite. Then, at least for the medium- and high-pressure regime the mean-field values obtained here for the charge-one sector should coincide with their counterparts derived for a Skyrmion on a torus (crystal). In Fig. 5 we plot the local energy density for different values of the pressure together with the corresponding mean-field energy density. It is visible that the energy density has a global maximum whose value is significantly bigger than the average mean-field energy density. For small pressure,
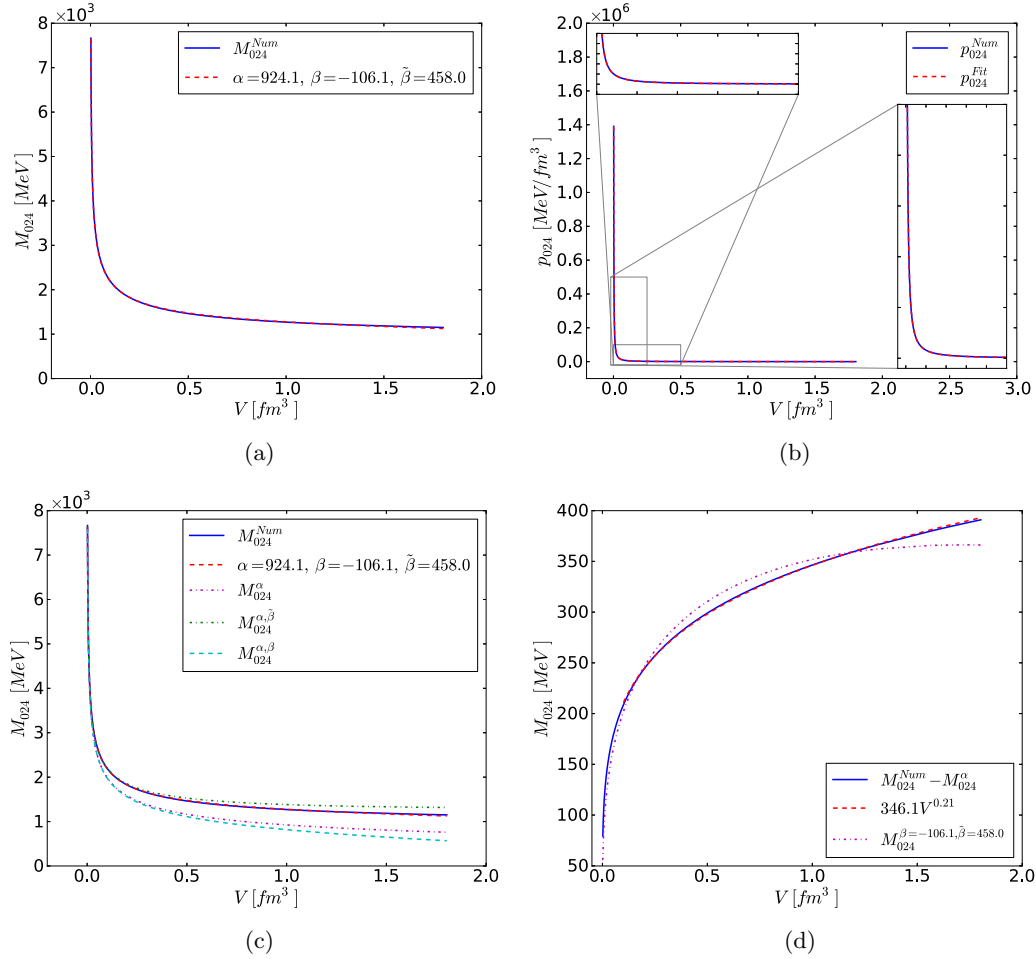
FIG. 3. (Color online) Energy and pressure of the $B = 1$ Skyrmion as a function of the volume for E_{024} model. (a)–(c) with the theoretical fit. (d) with the best fitted curve.

this is not surprising, as the corresponding solution has a rather big volume. However, even for very large pressure ($P = 1.432 \times 10^6$ MeV fm^{-3}) the maximum of the energy density is approximately five times bigger that the average value. For medium pressure this difference increases. For example for $P = 9.231$ MeV fm^{-3} the maximum energy density is 20 times bigger than the average value.

Let us finally remark that the leading term coefficient found in our numerical computations, $\alpha = 1.853$ in Skyrme units (or $\alpha = 924.4$ MeV in physical units), provides a bigger number (a better bound) not only than the analytical bound but also than the value found by Kutschera *et al.* for a multi-Skyrmion configuration being a cubic lattice of Skyrmions, which in the Skyrme units is $\alpha_{\text{kut}} = 1.837$ [35]. As in the asymptotic regime the multi-Skyrmion should behave identical to the charge-one Skyrmion, we think that this constant gives the proper bound for high pressure.

B. Near BPS Skyrme model

In this subsection, we switch on the sextic term which, as we already know, governs the high-density regime. The total energy has the following form

$$E_{0246} = \epsilon(\lambda_2 E_2 + \lambda_4 E_4 + \lambda_0 E_0) + \lambda_6 E_6 + \tilde{\lambda}_0 \tilde{E}_0. \quad (5.27)$$

The constants are

$$\lambda_2 = \frac{1}{24\pi^2}, \quad \lambda_4 = \frac{1}{12\pi^2}, \quad \lambda_0 = \frac{1}{12\pi^2},$$
$$\lambda_6 = \lambda^2 \pi^4 \frac{1}{12\pi^2}, \quad \tilde{\lambda}_0 = \frac{\mu^2}{12\pi^2}, \quad (5.28)$$

where λ and μ^2 will be fixed later (or set to zero if the corresponding term is omitted). For the usual (perturbative) Skyrme model part, this corresponds to the same choice as before, but which the overall multiplication by the parameter ϵ, which is assumed to have three values $\epsilon = 1, 0.1, 0.01$. Again, the potential entering the perturbative Skyrme part is the standard potential (providing the physical mass for pions)

$$E_0 = \text{Tr}\,(1 - U), \quad (5.29)$$

while the BPS part of the potential is $\tilde{E}_0 = E_0^2$.

1. BPS model: E_{06} model

In order to test the numerics we begin with the BPS Skyrme model, which can be solved analytically. The field equations of motion can be integrated to the following first-order
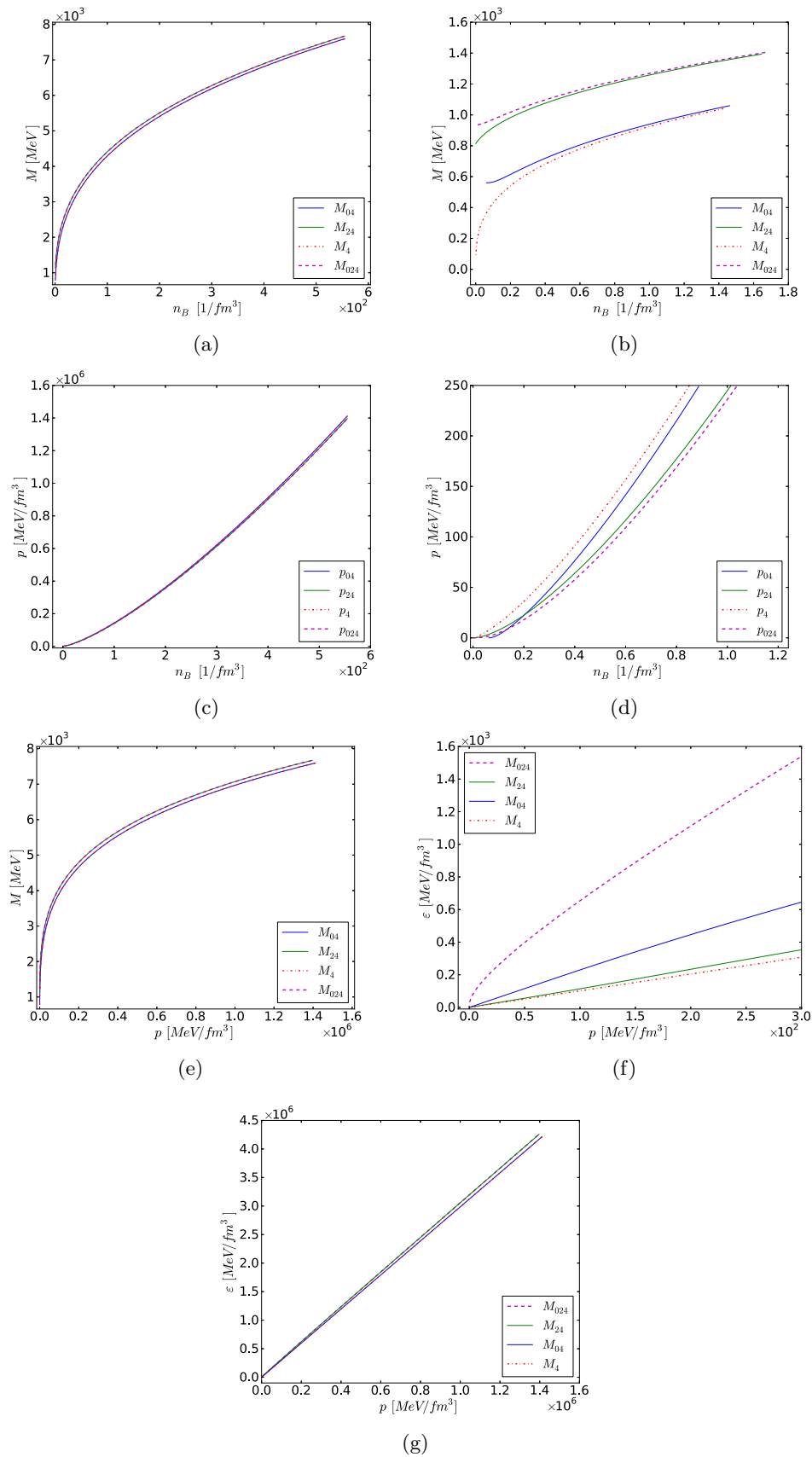
FIG. 4. (Color online) Comparison plots for the $B = 1$ Skyrmion: (a)–(d) mass and pressure as a function of the average baryon density; (e) mass as a function of pressure; (f)–(g) equation of state.
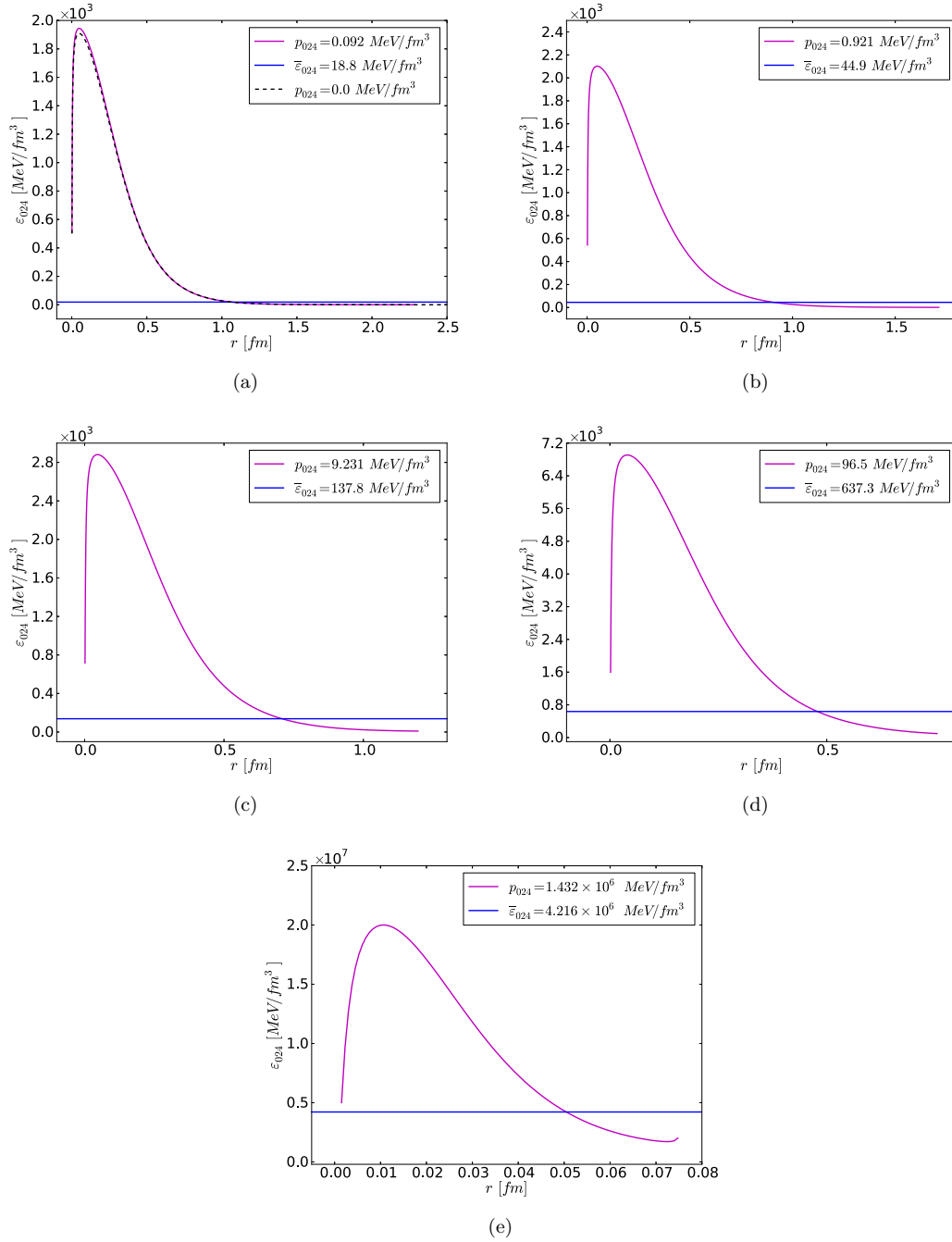
FIG. 5. (Color online) Local energy density distribution for the $B = 1$ Skyrmion in the E_{024} model for different values of the pressure together with the corresponding mean-field energy density.

equation [10]

$$\frac{\lambda}{2r^2} \sin^2 \xi \, \xi_r = -\mu \sqrt{\tilde{E}_0 + P \frac{12\pi^2}{\mu^2}}, \qquad (5.30)$$

where we assumed the hedgehog ansatz for the charge one soliton. Of course, the BPS equation can be found as the zero pressure condition [36]. The potential reads $\tilde{E}_0 = 4(1 - \cos \xi)^2$. Using this equation, it is possible to compute the total energy integral and find the volume of the solution with a given

pressure. Indeed, we find [11]

$$E(P) = \frac{2\pi}{12\pi^2} \lambda \mu \int_0^\pi d\xi \sin^2 \xi \frac{8(1 - \cos \xi)^2 + P \frac{12\pi^2}{\mu^2}}{\sqrt{4(1 - \cos \xi)^2 + P \frac{12\pi^2}{\mu^2}}}, \qquad (5.31)$$

$$V(P) = 2\pi \frac{\lambda}{\mu} \int_0^\pi d\xi \sin^2 \xi \frac{1}{\sqrt{4(1 - \cos \xi)^2 + P \frac{12\pi^2}{\mu^2}}}. \qquad (5.32)$$

This gives the following mean-field equation of state

$$\bar{\varepsilon} = P + \tilde{\mu}^2 \left(\frac{5}{2} \frac{{}_3F_2\left[\left\{\frac{1}{2},\frac{7}{4},\frac{9}{4}\right\},\left\{\frac{5}{2},3\right\},-\frac{4\tilde{\mu}^2}{P}\right]}{{}_3F_2\left[\left\{\frac{1}{2},\frac{3}{4},\frac{5}{4}\right\},\left\{\frac{3}{2},2\right\},-\frac{4\tilde{\mu}^2}{P}\right]} \right), \quad (5.33)$$

where ${}_pF_q[\{a_1,\ldots,a_p\},\{b_1,\ldots,b_q\},z]$ is a generalized hypergeometric function, and $\tilde{\mu}^2 = 1/3\pi^2$. Note, however, that the true energy density is not spatially constant and, in fact, differs quite significantly from its mean-field approximation. This is of some importance, e.g., for the application to neutron stars. For example, the mass-radius relation changes significantly if we switch from the true non-mean-field energy density to the mean-field average energy density [11].

The mean-field equation of state results, in the asymptotic region, in a simple energy-volume formula

$$E = \tilde{\alpha}\frac{1}{V} + \beta_\infty V + o(V), \quad (5.34)$$

where

$$\tilde{\alpha} = \frac{1}{12\pi^2}\lambda^2\pi^4 = \frac{\pi^2}{3} = 935.6 \text{ Mev fm}^3,$$

$$\beta_\infty = \frac{5}{6\pi^2} = 64.8 \text{ MeV fm}^{-3}. \quad (5.35)$$

in Skyrme units and physical units (with the ANW calibration assumed). Here we chose $\lambda = 2$ and $\mu = 1$. This asymptotical offset can be compared with the equilibrium energy density

$$\bar{\varepsilon}_0 = \frac{1}{3\pi^3} = 26.0 \text{ MeV fm}^{-3}. \quad (5.36)$$

The nonzero value is due to the compact nature of the equilibrium solution. In fact, it has a finite size and a volume

$$R = \left(\frac{3\pi}{2}\right)^{1/3} \Rightarrow V = 2\pi^2 = 8.49 \text{ fm}^3. \quad (5.37)$$

Let us comment that for the most interesting full near-BPS Skyrme model we apply a different choice for parameters and different calibration scheme, which was previously used for neutron stars.

In numerical computation we reproduce the mean-field equation of state for the BPS Skyrme model. For instance, we find

$$\tilde{\alpha} = 938.2 \text{ MeV fm}^3 \quad \text{and} \quad \beta = 61.5 \text{ Mev fm}^{-3}, \quad (5.38)$$

which is in agreement with the analytical results within 0.3% and 5% respectively.

2. E_{026} model

Now we consider a submodel for which the attractive force is provided entirely by the sextic term, i.e., the quartic term is absent. Additionally, we assume $\lambda = 2$ and $\mu = 1$. The leading term in the energy-volume relation is found in the numerical computations. Specifically, we reach a regime where

$$\tilde{\alpha}_{\epsilon=0.01} = 938.4, \quad \tilde{\alpha}_{\epsilon=0.1} = 939.0, \quad \tilde{\alpha}_{\epsilon=1} = 940.6. \quad (5.39)$$

Obviously, the leading part more correctly describes also a medium volume regime if we are closer to the BPS limit, i.e., if we reduce the ϵ parameter (see Fig. 6). We also get the

subleading (linear in the volume) term. For the nearest BPS Skyrme considered here ($\epsilon = 0.01$) we find

$$\beta_{\epsilon=0.01} = 62.2 \text{ MeV fm}^{-3}, \quad (5.40)$$

with, however, a nonzero value for the off-set. It is also possible to fit $\tilde{\beta}V^{1/3} + \beta V$ curve. For $\epsilon = 0.1$ and $\epsilon = 1$ it leads to a negative value for β, which seems to indicate that such a curve is rather not the right one.

3. E_{0246} model

Finally, we consider the full near-BPS Skyrme model. As we are close to the BPS limit (for $\epsilon = 0.1$ and $\epsilon = 0.01$) it is reasonable to apply a different choice for the calibration. Namely, we fit the parameters of the BPS part of the model to the mass of the helium nucleon $m_{He}/4 = 931.75$ MeV and size of the nucleon $r_N = 1.25$ fm. For the BPS Skyrme model we have

$$E_{\text{BPS}} = \frac{2}{12\pi^2}2\pi^2\lambda\mu, \quad (5.41)$$

$$R = \left(\frac{3\pi}{2}\right)^{1/3}\left(\frac{\lambda}{2\mu}\right)^{1/3}. \quad (5.42)$$

Then, $\lambda^2 = 2317$ Mev fm^3, $\mu^2 = 3372$ MeV fm^{-3} in physical units or $\lambda^2 = 8.14$ and $\mu^2 = 2.19$ in the Skyrme units. In our numerical simulations we find the following values of the leading term in the energy-mass relation

$$\tilde{\alpha}_{\epsilon=1} = 1929 \text{ MeV fm}^3, \quad \tilde{\alpha}_{\epsilon=0.1} = 1908 \text{ MeV fm}^3,$$

$$\tilde{\alpha}_{\epsilon=0.01} = 1905 \text{ MeV fm}^3, \quad (5.43)$$

which can be compared with the theoretical value $\tilde{\alpha} = \frac{\lambda^2\pi^2}{12} = 1905$ MeV fm^3.

As expected, the sextic term gives the leading behavior for the mass-volume formula in the asymptotic regime. The subleading contribution, emerging from the quartic part of the action, has the form $\alpha V^{-1/3}$, with the following values for the constant Fig. 7

$$\alpha_{\epsilon=1} = 924.7 \text{ MeV fm}, \quad \alpha_{\epsilon=0.1} = 92.7 \text{ MeV fm},$$

$$\alpha_{\epsilon=0.01} = 9.8 \text{ MeV fm}, \quad (5.44)$$

which to a reasonable precision satisfies an obvious relation $\alpha_\epsilon = \epsilon \, \alpha_{\epsilon=1}$. This can be treated as a test for our numerics.

In Fig. 8 we compare the near-BPS Skyrme model (for three values of ϵ) with the usual (perturbative) Skyrme model. Of course, for a sufficiently large value of the average particle density the total energy (mass) as well as the pressure obey the relation, which follows from the pure BPS model (sextic part), see Figs. 8(a), 8(c). For smaller values of n_B (less squeezed configurations), i.e., closer to the equilibrium the relations are modified by a subleading behavior related to other terms of the action. In general, taking smaller ϵ results in a flatter $M - n_B$ curve for small n_B. This is perhaps related to the fact that the quartic term is more important for small densities and its strength is increased or reduced by a particular value of ϵ. Furthermore, the $M - n_B$ and $p - n_B$ curves grow much faster for the full model ($M \sim n_B$ and $p \sim n_B^2$) than in the perturbative Skyrme model ($M \sim n_B^{1/3}$ and $p \sim n_B^{4/3}$).
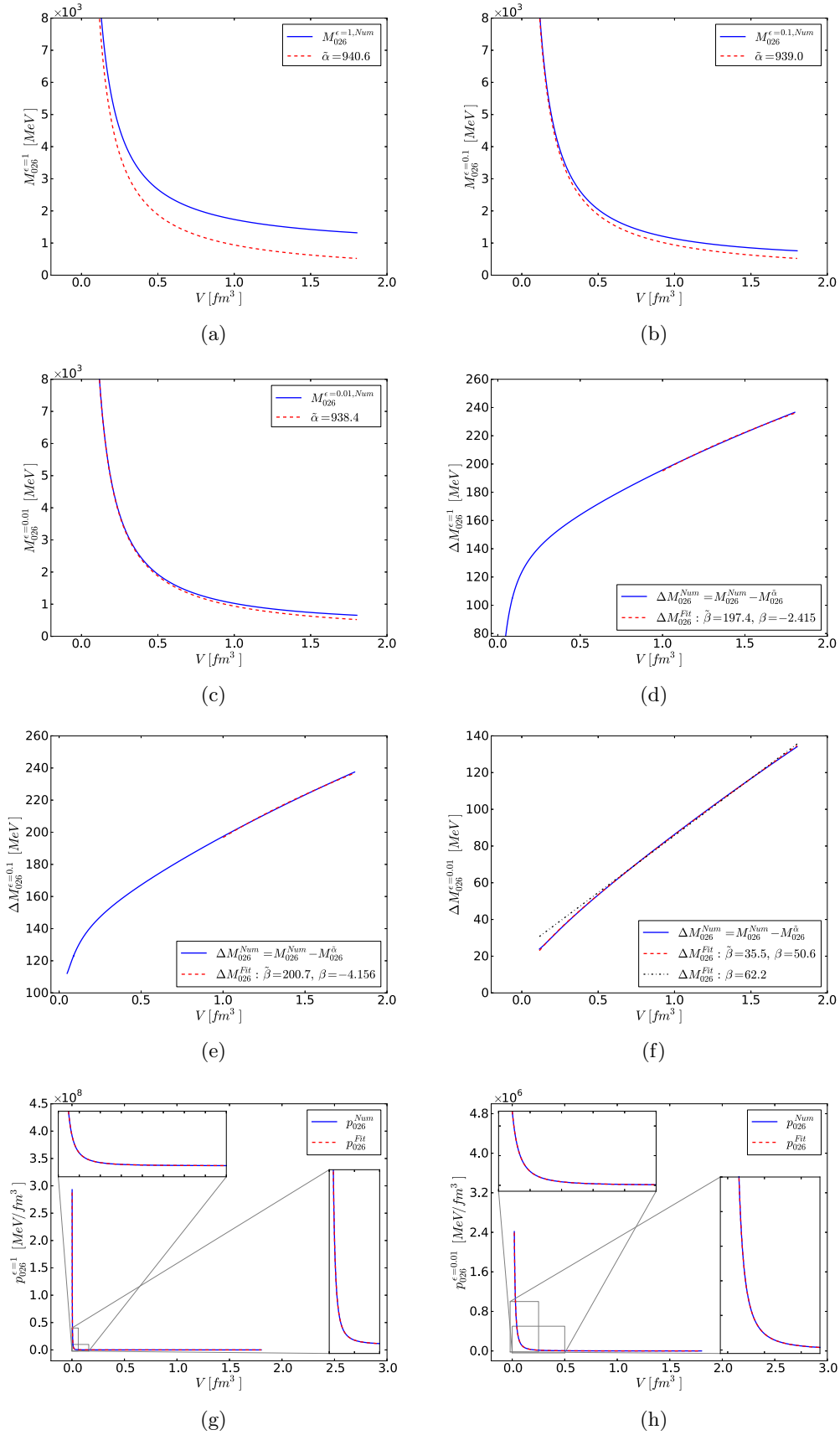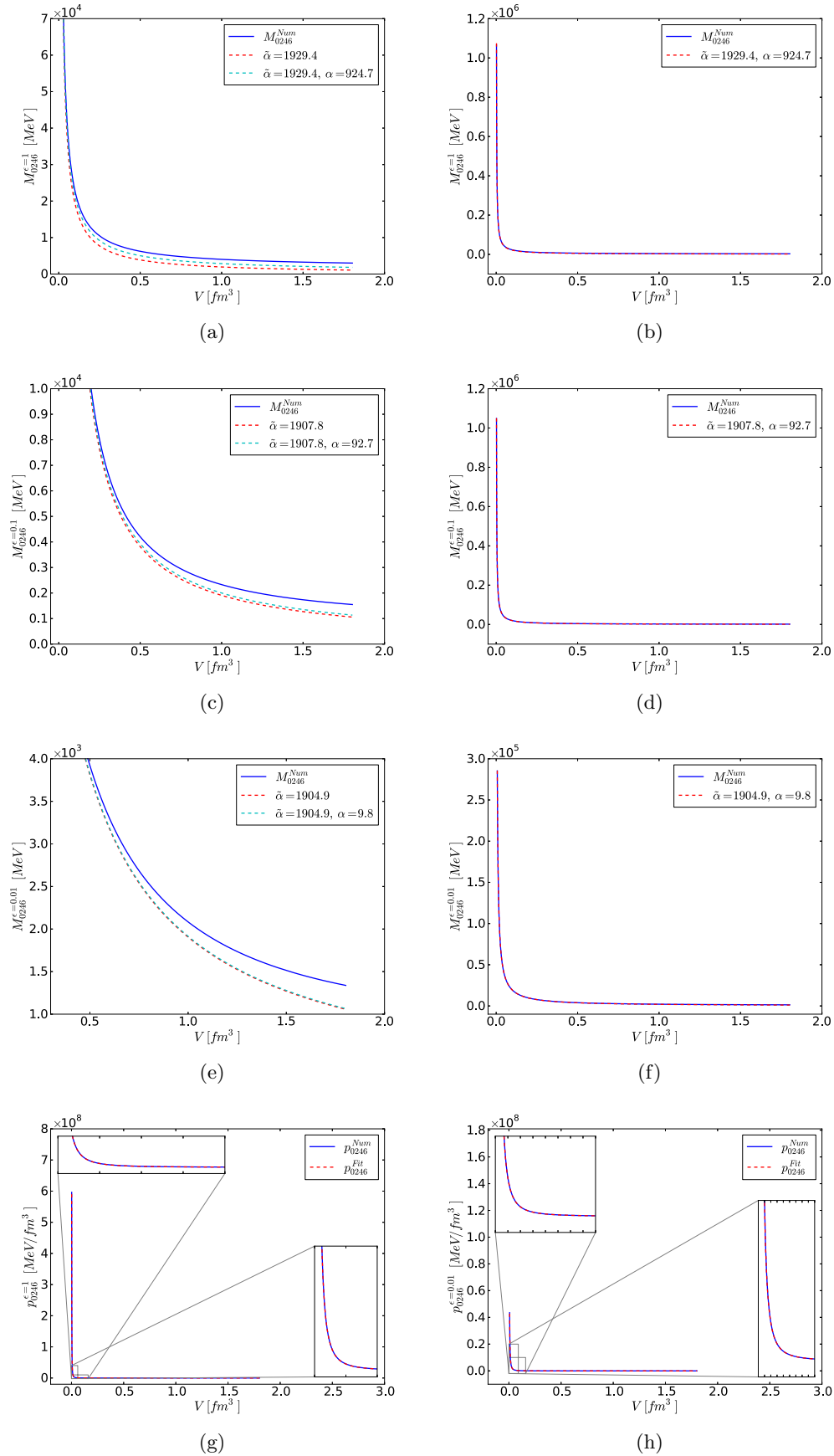
FIG. 6. (Color online) Energy and pressure of the $B = 1$ Skyrmions as a function of the volume for E_{026} model.

FIG. 7. (Color online) Energy and pressure of $B = 1$ Skyrmion as a function of the volume for E_{0246} model.
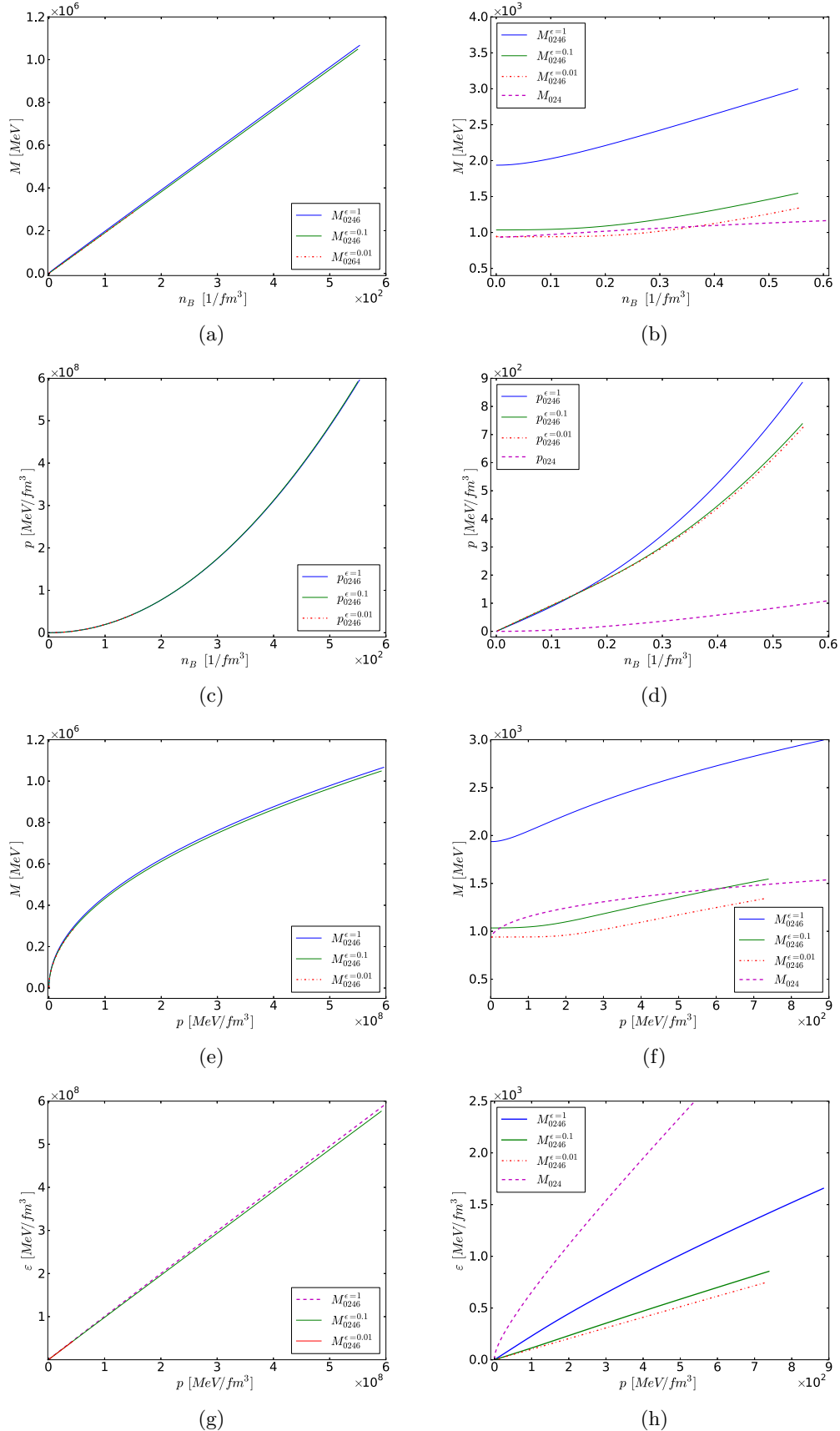
FIG. 8. (Color online) Comparison plots for the $B = 1$ Skyrmion: (a)–(d) mass and pressure as a function of the average baryon density; (e)–(f) mass as a function of pressure; (g)–(h) equation of state.

VI. COMPARISON TO THE WALECKA MODEL

Let us now understand the results from the point of view of the Walecka effective model [37,38]. The pressure and energy density are given as

$$P = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\bar{\rho}_B^2 - \frac{1}{2}\frac{g_\sigma^2}{m_\sigma^2}n_s^2 + \frac{1}{4\pi^2}\left[\left(\frac{2}{3}k_F^3 - (m^*)^2 k_F\right)E_F^*\right.$$
$$\left. + (m^*)^4 \ln\frac{k_F + E_F^*}{m^*}\right], \tag{6.1}$$

$$\bar{\varepsilon} = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\bar{\rho}_B^2 + \frac{1}{2}\frac{g_\sigma^2}{m_\sigma^2}n_s^2 + \frac{1}{4\pi^2}\left[\left(2k_F^3 + (m^*)^2 k_F\right)E_F^*\right.$$
$$\left. - (m^*)^4 \ln\frac{k_F + E_F^*}{m^*}\right], \tag{6.2}$$

where the in-medium Fermi energy (chemical potential) and in-medium nucleon mass are

$$E_F^* = \sqrt{k_F^2 + (m^*)^2}, \quad m^* = m - \frac{g_\sigma^2}{m_\sigma^2}n_s. \tag{6.3}$$

The baryon and scalar densities at $T = 0$ are

$$\bar{\rho}_B = \frac{2k_F^3}{3\pi^2}, \quad n_s = \frac{m^*}{\pi^2}\left[k_F E_F^* - (m^*)^2 \ln\frac{k_F + E_F^*}{m^*}\right]. \tag{6.4}$$

It is straightforward to find the leading behavior of the mean-field energy density as a function of the mean-field baryon density

$$\bar{\varepsilon} = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}\bar{\rho}_B^2 + \frac{3}{4}\left(\frac{3\pi^2}{2}\right)^{1/3}\bar{\rho}_B^{4/3} + \frac{1}{2}\frac{m_N^2 m_\sigma^2}{g_\sigma^2}, \tag{6.5}$$

where the leading part comes from the ω meson repulsion, the first subleading term is due to the free Fermion gas while the second subleading term comes from the scalar meson (more specifically, we use that in the asymptotic regime the scalar condensate tends to a constant value),

$$n_s = \frac{m_N m_\sigma^2}{g_\sigma^2}. \tag{6.6}$$

The results can be summarized as follows.

(i) The asymptotic formulas for the equations of state in the full (i.e., containing the BPS part) Skyrme model and in the Walecka model coincide. Their functional dependence is exactly the same. The leading part implying $\bar{\varepsilon} \simeq P$ comes from the baryon density current. In the Skyrme model it enters via the sextic term while in the Walecka model it appears due to the ω meson repulsive interaction dominating at high density. So, we may detect the ω meson in the Skyrme action—as an emergent object hidden in the sextic term.

(ii) Furthermore, there is a coincidence between the first subleading terms, which behave as $\bar{\rho}_B^{4/3}$. In the Skyrme model it is generated by the usual Skyrme (quartic) part of the action while in the Walecka it is due to the ultrarelativistic free fermion (nucleon) behavior.

(iii) Moreover, we got a constant term whose origin is found to be the potential (the Skyrme model) or scalar density condensation (the Walecka model). This

gives further support to the idea that the potential corresponds to the σ meson and provides the attractive long-distance force.

(iv) Let us note that there are no further terms in the large density limit of the Walecka model that could correspond to the term generated by the σ model part of the Skyrme model. Namely, a term behaving as $\bar{\rho}^a$ with $4/3 > a > 0$.

The observed similarity between the equations of state can be used to get some quantitative insight into ω and σ mesons within the Skyrme model framework, where such low-energy particles are not explicitly included. If we compare the coefficients in the expression above we find

$$\pi^4 \lambda^2 = \frac{1}{2}\frac{g_\omega^2}{m_\omega^2}, \tag{6.7}$$

$$\alpha = \frac{3}{4}\left(\frac{3\pi^2}{2}\right)^{1/3}, \tag{6.8}$$

$$\tilde{\beta} = \frac{1}{2}\frac{m_N^2 m_\sigma^2}{g_\sigma^2}. \tag{6.9}$$

The typical parameter values in the Walecka model are

$$m_N = 939 \text{ MeV}, \quad m_\omega = 783 \text{ MeV}, \quad m_\sigma = (500\text{–}600) \text{ MeV}, \tag{6.10}$$

$$\frac{g_\omega^2}{4\pi} = 14.717, \quad \frac{g_\sigma^2}{4\pi} = 9.537. \tag{6.11}$$

This gives (after taking into account the Plank constant $\hbar = 197.3$ MeV fm)

$$\frac{1}{2\pi^4}\frac{g_\omega^2}{m_\omega^2} = 12 \text{ MeV fm}^3, \quad \frac{3}{4}\left(\frac{3\pi^2}{2}\right)^{1/3} = 362 \text{ MeV fm},$$

$$\frac{1}{2}\frac{m_N^2 m_\sigma^2}{g_\sigma^2} = (120\text{–}175) \text{ MeV fm}^{-3}. \tag{6.12}$$

This can be compared with values derived for the generalized Skyrme model. For the BPS model (or near-BPS version with small ϵ, for example equals to 0.01) λ^2 and μ^2 for the pure BPS Skyrme model in the case of the step-function potential (with the parameter fit to the nuclear saturation density and the binding energy per nucleon of infinite nuclear matter)

$$\lambda^2 = 31 \text{ MeV fm}^3, \quad \beta = 71 \text{ MeV fm}^{-3}. \tag{6.13}$$

As we know, in the case of the usual (perturbative) Skyrme model, with the calibration assumed here one gets

$$\alpha = 924 \text{ MeV fm}^3. \tag{6.14}$$

Thus, for near-BPS Skyrme models this should be multiplied by the ϵ constant, which leads to a rather small number in comparison to the Walecka model.

VII. CONCLUSIONS

In the present work we have analyzed thermodynamical properties of Skyrmionic matter at zero temperature for the

most general Skyrme theory that possesses both Poincaré invariance and a standard Hamiltonian.

The main result is that the sextic term is responsible for the leading behavior at high pressure and therefore is unavoidable for any realistic application of the Skyrme model for a description of nuclear matter at high densities. As we found, the dependence of all global characteristics (total energy, pressure, average energy density) on the geometric volume tends to analytical relations derived for the BPS Skyrme model as the volume decreases. The same happens with the mean-field equation of state, which approaches a much stiffer form, $\bar{\varepsilon} = P$, instead of $\bar{\varepsilon} = 3P$ for the usual Skyrme model without the sextic part. Similarly, the mean-field energy density grows much more rapidly with the average baryon density—the previously found $\bar{\varepsilon} \sim \bar{\rho}_B^{4/3}$ relation [35] is replaced by $\bar{\varepsilon} \sim \bar{\rho}_B^2$. This resolves a longstanding discrepancy between the Skyrme model and other conventional models of nuclear matter. Indeed, the behavior generated by the sextic term coincides with the corresponding relation derived by Bethe and Johnson [39] (nonrelativistic) as well as Walecka [37] (relativistic). Moreover, the inclusion of this term might not only modify the phase diagram of cold nuclear matter or neutron stars, but can also influence the scattering of nuclei [40]. It is also an intriguing observation that an effective low-energy model of QCD, which here is represented by the Skyrme model, has a solvable and integrable limit (in the sense of the generalized integrability) at asymptotically high density (pressure).

The first subleading contribution is provided by the Skyrme term, i.e., by the quartic part of the action, and it can modify the thermodynamical properties at medium densities (pressure). Although we analytically derived the functional dependence for this contribution, we found only a lower bound for the true value of the proportionality constant α. The true value was obtained numerically. The next subleading terms were found to be related to the attractive channels of the model, which emerge from the σ model term and the potential. However, due to the strong nonlinearity of the theory, there is a significant mixing in these subleading contributions, which results in the appearance of some new, effective terms in the energy-volume relation. In particular, in the full E_{0246} model, instead of terms $V^{1/3}$ and V—which are clearly visible in the E_{24} and E_{04} submodels, we find an effective V^a, $a \in (0, 1)$ dependence.

Let us also note that, from the point of view of nuclear matter properties, our results are relevant for the high- and medium-pressure regimes. Definitely, the obtained mean-field equations of states are not applicable at the saturation density, where we got zero energy density. In fact, it is a common feature of Skyrme-type models (with the exception of the BPS Skyrme model and the E_{04} submodel) that $B = 1$ solitons are infinitely extended at equilibrium ($P = 0$). Then the equilibrium volume is infinite, which leads to a vanishing mean-field energy density. This is not the expected behavior for infinite B. In the original Skyrme model, e.g, at infinite B a crystal-like configuration with a finite volume per baryon number is conjectured to be the true minimizer even for small P. This does not mean that one cannot use finite-charge Skyrmions to understand thermodynamical properties of nuclear matter or neutron stars. It simply means that one has to be very careful with the mean-field approximation.

The full (non-mean-field) densities are still fine. We want to underline that in our computations we solve the full field equations, which provide us with full (spatially dependent) densities. Then they are averaged, leading to the mean-field equation of state. This is an alternative approach to mean-field computations previously done for Skyrme-related models [41].

If compared with another widely used effective model, i.e., the Walecka model, we found a correspondence between the ω meson interaction and the sextic term in the Skyrme model. Both generate the same mean-field equation of state, $\bar{\varepsilon} = P$. This again supports the crucial role played by the sextic term (and the BPS Skyrme model part) in the Skyrme framework. There is also a clear correspondence between the main subleading terms related to the ultrarelativistic free fermion gas (Walecka) and the quartic Skyrme term (Skyrme model). The next subleading term in the mean-field equation of state in the Walecka model emerges due to the attractive scalar meson interaction. A similar constant term was found in the Skyrme model as well and it is generated by the potential. This provides some support to the natural expectation that the potential in the Skyrme model corresponds to the σ meson in the Walecka model. However, the situation is much more involved as in the Skyrme model the $\bar{\rho}_B^0$ term is accompanied by a $\bar{\rho}_B^{2/3}$ term, which is generated by another source of the attractive force, namely the σ model part. Furthermore, there is no such term in the Walecka model. In addition, the observed effective mixing between these attractive channels does not allow for a clear understanding of a possible emergence of the hidden scalar meson in the Skyrme model, in a similar fashion as it has been recently understood in the case of the emergent ω meson [42,43]. In any case, it is an interesting feature of the Skyrme model that it does include some effects related to low-lying mesons (here the ω and σ) although they are not introduced at the level of fields.

There are several natural directions in which the present investigation can and should be further developed. First of all, one should find the mean-field equation of state in the limit of infinite nuclear matter. This can be achieved by considering the general Skyrme model, with the sextic term included, on a torus. The main advantage of this setup follows from the fact that the minimum energy solution occurs for a particular size (volume) of the torus. Therefore, the equilibrium solutions (at zero pressure) do not correspond to infinite geometrical volume which, as we already commented, is a drawback of finite-charge solutions. As a consequence, equilibrium solutions possess finite mean-field energy and particle densities. Besides the derivation of the equation of state and understanding how it differs from the charge one results (which should be negligible in the high-pressure limit), one can expect the appearance of some new phases (as, for example, a half-Skyrmion phase). It would be desirable to fully discover the Skyrme phase diagram with the coupling constants as free parameters.

Second, one can ask the more mathematical question regarding what the energy minimizer in a given topological sector on a given compact manifold with a certain volume is (see for example [44]). Obviously, the form of the solution does not depend only on the volume of the manifold but also

on its geometry (shape), although, as has been shown here, the asymptotic (small volume) behavior is geometry independent.

Third, it would also be desirable to extend our results to nonzero temperature, $T > 0$. The simplest way to accomplish this is to apply the caloron approach, i.e., to use Euclidean periodic solutions in four dimensions [45]. Alternatively, at least in the BPS limit, one can try to compute the partition function, which is essentially given by an integration over the moduli space [46]. Unfortunately, for the BPS Skyrme model this group has infinite dimension, which makes the problem quite involved.

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