Anisotropic hydrodynamics for a mixture of quark and gluon fluids

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A system of equations for anisotropic hydrodynamics is derived that describes a mixture of anisotropic quark and gluon fluids. The consistent treatment of the zeroth, first, and second moments of the kinetic equations allows us to construct a new framework with more general forms of the anisotropic phase-space distribution functions than used before. In this way, the main deficiencies of the previous formulations of anisotropic hydrodynamics for mixtures are overcome and a good agreement with the exact kinetic-theory results is obtained.

I. INTRODUCTION

The successful description of relativistic heavy-ion collisions at RHIC and at the LHC in terms of relativistic dissipative fluid dynamics (for a recent review see Ref. [1]) has focused a lot of attention on studies aiming to construct the most adequate hydrodynamic framework. One way to achieve this task is to compare the results of various hydrodynamic approaches [2–12], which differ by the number of terms included in the formalism and by the values of the transport coefficients, with the results of the underlying microscopic kinetic theory [13–18]. The latter is very often used as a starting point to derive the specific form of the evolution equations of relativistic hydrodynamics, however, several approximations done in such procedures may result in differences between the predictions of the kinetic theory and the hydrodynamic models constructed directly with its help.

As a lot of work has already been done in this context for simple (i.e., one-component) fluids, the analysis of mixtures has so far been quite limited [19–21]; for some recent developments, see Refs. [22–24]. One of the problems of the previous approaches using the concept of anisotropic hydrodynamics [19–21] was that they were based only on the zeroth and first moments of the kinetic equations. Assuming that the distribution functions used in anisotropic hydrodynamics are described by the original Romatschke-Strickland form [25], one finds an underdetermined set of equations, where the number of unknown parameters is larger than the number of equations. Consequently, to close the system of equations, in Refs. [19–21] the transverse-momentum scale parameters for quarks and gluons were taken to be equal.[1]

In this work we develop the approach presented in Refs. [19–21], see also [26–29]. We use the zeroth, first, and second moments of the kinetic equations for quarks, antiquarks, and gluons. This allows us to use the Romatschke-Strickland form with more independent parameters compared to previous works. Our selection of the equations is presented and discussed in greater detail below. Here we only mention

that our equations include the baryon number and energy-momentum conservation laws. This is achieved by the use of two types of the Landau matching condition: the first one fixes the effective chemical potential μ, while the second one (more commonly used) fixes the effective temperature T. Moreover, we use a special combination of equations coming from the second moment, which guarantees the agreement with the Israel-Stewart theory for a system approaching local equilibrium [30] (see also Refs. [31–33]).

Our approach is restricted to one-dimensional, boost-invariant systems [34], denoted $(0+1)$D systems. We show that the evolution equations of anisotropic hydrodynamics for quark and gluon fluids derived herein yield a good agreement with the results of the kinetic theory. Several options for the selection of the zeroth-moment equations have been studied and the best choice is indicated. The new approach eliminates solutions with exponentially damped anisotropy parameters, found in Ref. [19], which do not agree with the kinetic-theory solutions [21]. Such solutions appeared in cases where the initial conditions corresponded to oblate-prolate or prolate-prolate initial quark and gluon momentum distribution functions. The new solutions have the same qualitative character for all types of initial conditions (oblate-oblate, oblate-prolate, and prolate-prolate), do not exhibit exponential damping, and closely follow the kinetic-theory solutions for systems far from and close to local thermal equilibrium.

The paper is organized as follows: In Sec. II the kinetic equations for quarks, antiquarks, and gluons in the relaxation time approximation (RTA) are introduced. In Sec. III the zeroth moments of the quark and antiquark equations are discussed in the context of the baryon number conservation. In Sec. IV we discuss the first moment of the kinetic equations and analyze the energy-momentum conservation. The second moments are discussed in Sec. V and our results are presented in Sec. VI. We conclude in Sec. VII. Throughout the paper we use natural units and the metric tensor's signature is $(+,-,-,-)$.

II. KINETIC EQUATIONS

We start our analysis with the kinetic equations for quarks, antiquarks, and gluons written in the RTA [26–28]. They read

$$p^\mu \partial_\mu Q^\pm(x,p) = -p^\mu U_\mu \frac{Q^\pm(x,p) - Q^\pm_{\rm eq}(x,p)}{\tau_{\rm eq}}, \quad (1)$$

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[1]The transverse-momentum scale parameters can be interpreted also as transverse temperatures: parameters characterizing transverse-momentum distributions.

$$p^\mu \partial_\mu G(x,p) = -p^\mu U_\mu \frac{G(x,p) - G_{\text{eq}}(x,p)}{\tau_{\text{eq}}}, \qquad (2)$$

where $Q^+(x,p)$ [$Q^-(x,p)$] is the quark [antiquark] phase-space distribution function, $G(x,p)$ is the gluon distribution function, and τ_{eq} is the relaxation time. The four-vector U describes the hydrodynamic flow in the Landau frame (i.e., U is defined as the eigen four-vector of the energy-momentum tensor).

The quark and gluon distribution functions are assumed to have a generic structure [25],

$$Q^\pm(x,p) = \exp\left(\frac{\pm\lambda_q - \sqrt{(p \cdot U)^2 + \xi_q(p \cdot Z)^2}}{\Lambda_q}\right),$$

$$G(x,p) = \exp\left(-\frac{\sqrt{(p \cdot U)^2 + \xi_g(p \cdot Z)^2}}{\Lambda_g}\right), \qquad (3)$$

where the parameters Λ_q and Λ_g define the transverse momentum scale, λ_q is the nonequilibrium baryon chemical potential of quarks, and ξ_q and ξ_g are the anisotropy parameters. In local equilibrium, the two anisotropy parameters vanish and Eqs. (3) are reduced to the standard equilibrium distributions,

$$Q_{\text{eq}}^\pm(x,p) = \exp\left(\frac{\pm\mu - p \cdot U}{T}\right),$$

$$G_{\text{eq}}(x,p) = \exp\left(-\frac{p \cdot U}{T}\right), \qquad (4)$$

where T is the temperature and μ is the baryon chemical potential. For the sake of simplicity, we assume here the classical Boltzmann statistics. Generalization of the present results to the case of the quantum Bose-Einstein and Fermi-Dirac statistics is straightforward [16,29].

The equilibrium distribution functions of the form (4) are used to define the RTA collision terms in (1) and (2). In this case μ and T should be treated as the effective baryon chemical potential and effective temperature, which are determined by the appropriate Landau matching conditions.

In addition to the flow vector U, which can be parametrized in terms of the three-velocity in the standard way as

$$U^\mu = \gamma(1, v_x, v_y, v_z), \quad \gamma = (1 - v^2)^{-1/2}, \qquad (5)$$

we introduce the four-vector Z, defined as [11]

$$Z^\mu = \gamma_z(v_z, 0, 0, 1), \quad \gamma_z = (1 - v_z^2)^{-1/2}. \qquad (6)$$

The appearance of Z is connected with the privileged direction of the beam axis.

The four vectors U and Z satisfy the following normalization conditions:

$$U^2 = 1, \quad Z^2 = -1, \quad U \cdot Z = 0. \qquad (7)$$

In the local rest frame of the fluid element, U^μ and Z^μ have simple forms:

$$U^\mu = (1, 0, 0, 0), \quad Z^\mu = (0, 0, 0, 1). \qquad (8)$$

In the $(0+1)$D case, we may further use

$$U^\mu = (t/\tau, 0, 0, z/\tau), \quad Z^\mu = (z/\tau, 0, 0, t/\tau), \qquad (9)$$

where τ is the (longitudinal) proper time,

$$\tau = \sqrt{t^2 - z^2}. \qquad (10)$$

III. ZEROTH MOMENTS OF THE KINETIC EQUATIONS

Integrating Eqs. (1) and (2) over the three-momentum and including the internal degrees of freedom we obtain the three scalar equations

$$\partial_\mu(n_q^\pm U^\mu) = \frac{n_{q,\text{eq}}^\pm - n_q^\pm}{\tau_{\text{eq}}}, \qquad (11)$$

$$\partial_\mu(n_g U^\mu) = \frac{n_{g,\text{eq}} - n_g}{\tau_{\text{eq}}}, \qquad (12)$$

where we have introduced the nonequilibrium and equilibrium particle densities defined by the expressions

$$n_q^\pm = \frac{g_q}{\pi^2} \frac{e^{\pm\lambda_q/\Lambda_q} \Lambda_q^3}{\sqrt{1 + \xi_q}}, \quad n_{q,\text{eq}}^\pm = \frac{g_q}{\pi^2} e^{\pm\mu/T} T^3, \qquad (13)$$

$$n_g = \frac{g_g}{\pi^2} \frac{\Lambda_g^3}{\sqrt{1 + \xi_g}}, \quad n_{g,\text{eq}} = \frac{g_g}{\pi^2} T^3. \qquad (14)$$

Note that for the $(0+1)$D system we have $U^\mu \partial_\mu = d/d\tau$ and $\partial_\mu U^\mu = 1/\tau$.

Instead of using Eqs. (11) and (12) we use the difference in the equations for quarks vs antiquarks appearing in (11),

$$\frac{d}{d\tau}(n_q^+ - n_q^-) + \frac{n_q^+ - n_q^-}{\tau} = \frac{n_{q,\text{eq}}^+ - n_{q,\text{eq}}^- - (n_q^+ - n_q^-)}{\tau_{\text{eq}}}, \qquad (15)$$

and the following linear combination of Eqs. (11) and (12):

$$\alpha\left(\frac{dn_q}{d\tau} + \frac{n_q}{\tau}\right) + (1 - \alpha)\left(\frac{dn_g}{d\tau} + \frac{n_g}{\tau}\right)$$

$$= \alpha\frac{n_{q,\text{eq}} - n_q}{\tau_{\text{eq}}} + (1 - \alpha)\frac{n_{g,\text{eq}} - n_g}{\tau_{\text{eq}}}. \qquad (16)$$

In Eq. (16) we have introduced the notation n_q and $n_{q,\text{eq}}$ for the sum of the quark and antiquark densities:

$$n_q = n_q^+ + n_q^- = \frac{2g_q}{\pi^2} \frac{\cosh(\lambda_q/\Lambda_q)\Lambda_q^3}{\sqrt{1 + \xi_q}},$$

$$n_{q,\text{eq}} = n_{q,\text{eq}}^+ + n_{q,\text{eq}}^- = \frac{2g_q}{\pi^2} \cosh(\mu/T) T^3. \qquad (17)$$

It is important to note that, in contrast to Eq. (15), which leads directly to the fundamental law of baryon number conservation, the use of Eq. (16) is not as well motivated. A straightforward treatment of Eqs. (11) and (12) suggests that Eq. (15) should be supplemented by two extra equations, for example, one equation for the sum of the quark and antiquark distributions and the other for the gluon distribution. It turns out, however, that the use of three equations obtained from the zeroth moment leads, finally, to an overdetermined system of equations. Therefore, we have decided to use the linear combination defined by Eq. (16), where $\alpha = $ const. is a parameter taken from the range $0 \leqslant \alpha \leqslant 1$.

One may check *a posteriori* which value of α is the best by comparing the hydrodynamic results with the kinetic-theory

results. In this way we have found that the best agreement is obtained for $\alpha = 1$. A similar agreement is also obtained for the case $\alpha = 0$. We note that the cases $\alpha = 0$ and $\alpha = 1$ do not introduce the coupling between the quark and the gluon sectors at the level of the zeroth moment. This seems to be a desirable situation, since the kinetic equations, (1) and (2), treated exactly include the coupling between the quark and the gluon sectors *only* through the energy-momentum conservation [that is, within the first moment of Eqs. (1) and (2)] with the corresponding Landau matching condition (see Ref. [21]). We return to the discussion of this point following Eq. (43).[2]

A. Baryon number conservation and the corresponding Landau matching

Equation (15) divided by a factor of 3 gives the constraint on the baryon number density $b = (n_q^+ - n_q^-)/3$, namely,

$$\frac{db}{d\tau} + \frac{b}{\tau} = \frac{b_{\rm eq} - b}{\tau_{\rm eq}}. \tag{18}$$

In order to have the baryon number conserved both the left- and the right-hand sides of (18) should vanish. This yields

$$b(\tau) = \frac{b_0 \tau_0}{\tau} = \frac{2g_q}{3\pi^2} \sinh\left(\frac{\lambda_q}{\Lambda_q}\right) \frac{\Lambda_q^3}{\sqrt{1+\xi_q}} \tag{19}$$

and

$$\sinh\left(\frac{\mu}{T}\right)T^3 = \sinh\left(\frac{\lambda_q}{\Lambda_q}\right) \frac{\Lambda_q^3}{\sqrt{1+\xi_q}}, \tag{20}$$

respectively. We note that the solution $b(\tau) = b_0 \tau_0/\tau$ has the scaling form known from the Bjorken model [34]. The quantity $b_0 \equiv b(\tau = \tau_0)$ on the right-hand side of Eq. (19) is the baryon number density at the initial proper time.

Equations (19) and (20) can be solved for the chemical potentials λ_q and μ [19]. We find

$$\frac{\lambda_q}{\Lambda_q} = \sinh^{-1}(D) = \ln[D + \sqrt{1+D^2}], \tag{21}$$

where

$$D(\tau, \Lambda_q, \xi_q) = \left(\frac{3\pi^2 b_0 \tau_0 \sqrt{1+\xi_q}}{2g_q \tau \Lambda_q^3}\right), \tag{22}$$

and

$$\frac{\mu}{T} = \sinh^{-1}\left(\frac{D}{\kappa_q}\right) = \ln\left[\frac{D}{\kappa} + \sqrt{1 + \frac{D^2}{\kappa_q^2}}\right], \tag{23}$$

where

$$\kappa_q(T, \Lambda_q, \xi_q) = \frac{T^3 \sqrt{1+\xi_q}}{\Lambda_q^3}. \tag{24}$$

One should note here that the ratio $D/\kappa_q = (3\pi^2 b_0 \tau_0)/(2g_q \tau T^3)$ no longer depends on Λ_q and ξ_q. The condition

$$b_{\rm eq} = b, \tag{25}$$

resulting in Eq. (20), should be treated as the Landau matching condition that guarantees baryon number conservation. It defines the effective baryon chemical potential μ [see Eqs. (20) and (23)] in terms of the variables τ, T, Λ_q, and ξ_q. Similarly, the baryon conservation equation allows us to express λ_q in terms of τ, T, Λ_q, and ξ_q. In this way, in the following expressions we may completely eliminate both μ and λ_q (the expressions depend, however, in an explicit way on τ_0 and b_0).

B. Sum of the zeroth-order moments

Using Eqs. (21) and (23) and the mathematical identity

$$\cosh[\sinh^{-1}(x)] = \sqrt{1+x^2}, \tag{26}$$

we find the expressions for the quark nonequilibrium and equilibrium densities,

$$n_q = \frac{2g_q}{\pi^2} \frac{\sqrt{1+D^2}\Lambda_q^3}{\sqrt{1+\xi_q}},$$
$$n_{q,\rm eq} = \frac{2g_q}{\pi^2} \sqrt{1+D^2/\kappa_q^2}\ T^3, \tag{27}$$

respectively. Using this notation we rewrite Eq. (16) in the form

$$\frac{d}{d\tau}\left(\alpha\frac{\sqrt{1+D^2}\Lambda_q^3}{\sqrt{1+\xi_q}} + (1-\alpha)\frac{\tilde{r}\Lambda_g^3}{\sqrt{1+\xi_g}}\right)$$
$$+ \left(\frac{1}{\tau} + \frac{1}{\tau_{\rm eq}}\right)\left(\alpha\frac{\sqrt{1+D^2}\Lambda_q^3}{\sqrt{1+\xi_q}} + (1-\alpha)\frac{\tilde{r}\Lambda_g^3}{\sqrt{1+\xi_g}}\right)$$
$$= \frac{T^3}{\tau_{\rm eq}}\left(\alpha\sqrt{1+D^2/\kappa_q^2} + (1-\alpha)\tilde{r}\right), \tag{28}$$

where $\tilde{r}$ is the ratio of the internal degrees of freedom for gluons and quarks,

$$\tilde{r} = \frac{g_g}{2g_q}. \tag{29}$$

In the numerical calculations we use the value $\tilde{r} = 2/3$.

IV. ENERGY-MOMENTUM CONSERVATION LAW

A. Landau matching for the energy density

The first moment of the sum of the kinetic equations, (1) and (2), gives the divergence of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = \frac{1}{\tau_{\rm eq}} U_\mu \left(T_{q,\rm eq}^{\mu\nu} + T_{g,\rm eq}^{\mu\nu} - \left(T_q^{\mu\nu} + T_g^{\mu\nu}\right)\right), \tag{30}$$

where the energy-momentum tensor $T^{\mu\nu}$ includes the quark and gluon contributions,

$$T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}. \tag{31}$$

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[2]The choice $\alpha = 1$ eliminates Eq. (12) from our considerations. As this may be unexpected at first glance, we note that Eq. (12) does not reflect any basic physics rule. It describes the gluon production or gluon entropy production. The latter is also included in the second-moment equations that we take into account.

The Lorentz structure of the distribution functions for quarks and gluons implies the following forms of the energy-momentum tensors for quarks and gluons [11]:

$$T_i^{\mu\nu} = (\varepsilon_i + P_{i,T})U^\mu U^\nu - P_{i,T}g^{\mu\nu}$$
$$- (P_{i,T} - P_{i,L})Z^\mu Z^\nu, \qquad (32)$$

where ε_i, $P_{i,L}$, and $P_{i,T}$ denote the energy density, longitudinal pressure, and transverse pressure, respectively, and the index i stands for quarks ($i = q$) or gluons ($i = g$). Here the energy densities of quarks and gluons are given by the expressions

$$\varepsilon_q = \frac{6g_q\Lambda_q^4}{\pi^2}\sqrt{1 + D^2}\,\mathcal{R}(\xi_q),$$
$$\varepsilon_g = \frac{3g_g\Lambda_g^4}{\pi^2}\,\mathcal{R}(\xi_g), \qquad (33)$$

where the function $\mathcal{R}(\xi)$ is defined as [10]

$$\mathcal{R}(\xi) = \frac{1}{2(1 + \xi)}\left[1 + \frac{(1 + \xi)\tan^{-1}\sqrt{\xi}}{\sqrt{\xi}}\right]. \qquad (34)$$

Correspondingly, for the equilibrium part we find

$$T_{\mathrm{eq}}^{\mu\nu} = T_{q,\mathrm{eq}}^{\mu\nu} + T_{g,\mathrm{eq}}^{\mu\nu}, \qquad (35)$$

where

$$T_{i,\mathrm{eq}}^{\mu\nu} = (\varepsilon_{i,\mathrm{eq}} + P_{i,\mathrm{eq}})U^\mu U^\nu - P_{i,\mathrm{eq}}g^{\mu\nu}, \qquad (36)$$

with $\varepsilon_{i,\mathrm{eq}}$ and $P_{i,\mathrm{eq}}$ being the equilibrium energy density and pressure, respectively. The equilibrium energy densities for quarks and gluons are given by the expressions

$$\varepsilon_{q,\mathrm{eq}} = \frac{6g_q T^4}{\pi^2}\sqrt{1 + D^2/\kappa_q^2},$$
$$\varepsilon_{g,\mathrm{eq}} = \frac{3g_g T^4}{\pi^2}. \qquad (37)$$

The energy-momentum conservation law $\partial_\mu T^{\mu\nu} = 0$ requires that the right-hand side of Eq. (30) vanishes. This is nothing but the Landau matching condition for energy-momentum conservation. This matching requires that the energy determined from the nonequilibrium distribution functions is the same as the energy obtained with the equilibrium distribution functions:

$$\varepsilon = \varepsilon_q + \varepsilon_g = \varepsilon_{\mathrm{eq}} = \varepsilon_{q,\mathrm{eq}} + \varepsilon_{g,\mathrm{eq}}. \qquad (38)$$

This leads directly to the constraint on the effective temperature T:

$$T^4 = \frac{\Lambda_q^4\sqrt{1 + D^2}\,\mathcal{R}(\xi_q) + \Lambda_g^4\tilde{r}\,\mathcal{R}(\xi_g)}{\sqrt{1 + D^2/\kappa_q^2} + \tilde{r}}. \qquad (39)$$

B. Energy and momentum conservation

In the (0 + 1)D case considered here the energy and momentum conservation takes the form

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + P_L}{\tau}, \qquad (40)$$

where P_L is the sum of the longitudinal pressures for quarks and gluons,

$$P_{q,L} = \frac{6g_q\Lambda_q^4}{\pi^2}\sqrt{1 + D^2}\,\mathcal{R}_L(\xi_q),$$
$$\qquad (41)$$
$$P_{g,L} = \frac{3g_g\Lambda_g^4}{\pi^2}\,\mathcal{R}_L(\xi_g),$$

with $\mathcal{R}_L$ defined through the formula

$$\mathcal{R}_L(\xi) = -[2(1 + \xi)R'(\xi) + R(\xi)]. \qquad (42)$$

This leads directly to the formula

$$\frac{d}{d\tau}\left[\Lambda_q^4\sqrt{1 + D^2}\,\mathcal{R}(\xi_q) + \tilde{r}\Lambda_g^4\mathcal{R}(\xi_g)\right]$$
$$= \frac{2}{\tau}\left[\Lambda_q^4\sqrt{1 + D^2}(1 + \xi_q)\mathcal{R}'(\xi_q) + \tilde{r}\Lambda_g^4(1 + \xi_g)\mathcal{R}'(\xi_g)\right].$$
$$\qquad (43)$$

We note that Eqs. (39) and (43) couple the quark and gluon parameters similarly to the way these two sectors are coupled in the exact treatment of the kinetic equations, (1) and (2). Hence, the best agreement between the hydrodynamic equations and the kinetic theory *may* be expected if no other coupling is incorporated in the hydrodynamic approach. This, in turn, suggests using the value $\alpha = 1$ or $\alpha = 0$ in Eq. (16). This conjecture has been supported by our numerical calculations done with various values of α taken within the range $0 \leqslant \alpha \leqslant 1$.

V. SECOND MOMENT OF THE KINETIC EQUATION

So far we have constructed three equations [see Eqs. (28), (39), and (43)] for five unknown functions: Λ_q, Λ_g, ξ_q, ξ_g, and T. In order to close the system of equations we need to include two extra equations. We construct them using the second moment of the kinetic equation.

The second moment of the kinetic equation in the RTA was studied in Ref. [30], where boost-invariant and cylindrically symmetric systems were analyzed. In that case, it was shown that it is convenient to select the following equations as the basis for the hydrodynamic approximation:

$$\frac{d}{d\tau}\ln\Theta_I + \theta - 2\theta_I - \frac{1}{3}\sum_J\left[\frac{d}{d\tau}\ln\Theta_J + \theta - 2\theta_J\right]$$
$$= \frac{1}{\tau_{\mathrm{eq}}}\left[\frac{\Theta_{\mathrm{eq}}}{\Theta_I} - 1\right] - \frac{1}{3}\sum_J\left\{\frac{1}{\tau_{\mathrm{eq}}}\left[\frac{\Theta_{\mathrm{eq}}}{\Theta_J} - 1\right]\right\}. \qquad (44)$$

Here $I = X,Y,Z$ and $J = X,Y,Z$ are space indices. In the (0 + 1)D case the coefficients θ_I have the form $\theta_X = \theta_Y = 0$, $\theta_Z = -1/\tau$, and $\theta = 1/\tau$. The three functions Θ_I are obtained as contractions of the third moment of the distribution function with the tensor $U \otimes I \otimes I$ (where I is now the four-vector corresponding to the index I). The function Θ_{eq} is obtained by the contraction of the third moment of the equilibrium distribution function with any of the tensors $U \otimes I \otimes I$ (the result is independent of I). The four-vectors U and Z are defined above. The four vectors X and Y in our case, where the transverse expansion is neglected, are given simply

by the formulas [35]

$$X^\mu = (0,1,0,0), \quad Y^\mu = (0,0,1,0). \tag{45}$$

It is important to emphasize that in our (0+1)D case, only one of three equations in (44) is independent.[3] It may be taken as

$$\frac{d}{d\tau} \ln \Theta_X - \frac{d}{d\tau} \ln \Theta_Z - \frac{2}{\tau} = \frac{\Theta_{eq}}{\tau_{eq}} \left[\frac{1}{\Theta_X} - \frac{1}{\Theta_Z} \right]. \tag{46}$$

Since Eqs. (44) turned out to be very successful in the construction of hydrodynamic models [33]—in particular, they are consistent with the Israel-Stewart theory for systems close to equilibrium—we use this form separately for quarks and gluons. See also our remarks following Eq. (43).

A. Quarks and antiquarks

Following the method of Ref. [31] one can derive the following formulas for the sum of the quark and antiquark distributions:

$$\Theta_X^q = \Theta_Y^q = \frac{8g_q \Lambda_q^5}{\pi^2 (1+\xi_q)^{1/2}} \sqrt{1+D^2},$$

$$\Theta_Z^q = \frac{8g_q \Lambda_q^5}{\pi^2 (1+\xi_q)^{3/2}} \sqrt{1+D^2}. \tag{47}$$

Similarly, for the equilibrium quark functions one gets

$$\Theta_{X,eq}^q = \Theta_{Y,eq}^q = \Theta_{Z,eq}^q = \frac{8g_q T^5}{\pi^2} \sqrt{1+D^2/\kappa_q^2}. \tag{48}$$

Using Eqs. (47) and (49) in (46) we find

$$\frac{d}{d\tau} \ln \left(\frac{\Lambda_q^5}{(1+\xi_q)^{1/2}} \sqrt{1+D^2} \right)$$

$$- \frac{d}{d\tau} \ln \left(\frac{\Lambda_q^5}{(1+\xi_q)^{3/2}} \sqrt{1+D^2} \right) - \frac{2}{\tau}$$

$$= \frac{T^5}{\tau_{eq}\Lambda_q^5} \xi_q (1+\xi_q)^{1/2} \frac{\sqrt{1+D^2/\kappa_q^2}}{\sqrt{1+D^2}}. \tag{49}$$

B. Gluons

In the case of gluons, the analogous expressions are

$$\Theta_X^g = \Theta_Y^g = \frac{4g_g \Lambda_g^5}{\pi^2 (1+\xi_g)^{1/2}},$$

$$\Theta_Z^g = \frac{4g_g \Lambda_g^5}{\pi^2 (1+\xi_g)^{3/2}}, \tag{50}$$

and the equilibrium functions are

$$\Theta_{X,eq}^g = \Theta_{Y,eq}^g = \Theta_{Z,eq}^g = \frac{4g_g T^5}{\pi^2}. \tag{51}$$

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[3]By construction, only two equations in (44) are independent. In the case where the transverse flow is neglected, these two become degenerate.

Using Eqs. (50) and (51) in (46) one obtains

$$\frac{d}{d\tau} \ln \left(\frac{\Lambda_g^5}{(1+\xi_g)^{1/2}} \right) - \frac{d}{d\tau} \ln \left(\frac{\Lambda_g^5}{(1+\xi_g)^{3/2}} \right) - \frac{2}{\tau}$$

$$= \frac{T^5}{\tau_{eq}\Lambda_g^5} \xi_g (1+\xi_g)^{1/2}. \tag{52}$$

Equations (28), (39), and (43) together with Eqs. (49) and (52) represent five independent equations that allow us to determine five unknown functions of the proper time: Λ_q, Λ_g, ξ_q, ξ_g, and T.

VI. RESULTS

In this section we present our numerical results for four types of initial conditions. In the first case the initial conditions correspond to the oblate quark and gluon distribution functions (oblate-oblate configuration). In the second case the initial distribution of quarks is prolate, while the gluon distribution is oblate (prolate-oblate configuration), and in the third case the two distributions are prolate (prolate-prolate configuration). In the first three cases we set the baryon number density equal to 0. The effect of the nonvanishing baryon density is studied in the fourth case, where the quark and gluon distributions are both oblate.

We note that the oblate (prolate) distributions correspond to positive (negative) values of the anisotropy parameter ξ and, consequently, to a transverse pressure higher (lower) than the longitudinal pressure. The results of the microscopic calculations indicate that the initial conditions in relativistic heavy-ion collisions correspond to the (oblate) case where $P_T > P_L$, as soon as the coherent longitudinal color fields disappear [36–38]. This suggests that the oblate-oblate configuration is probably the most realistic one.

In all the considered cases the initial starting (proper) time is $\tau = \tau_0 = 0.1$ fm/c and we continue the evolution to $\tau = 10$ fm/c. The initial transverse-momentum parameters $\Lambda_{0,i}$ for quarks and gluons have been set equal to 1 GeV. From the Landau matching condition for the energy [see Eq. (39)], we determine the initial temperature T_0, which is different for the cases with different initial anisotropies. The value of the relaxation time used in this work is constant, $\tau_{eq} = 0.25$ fm/c. A temperature-dependent τ_{eq} can also be used, as described in Refs. [13,14]. However, to check the agreement with the kinetic theory it is enough to use a constant value.

In the four cases presented here, the results of anisotropic hydrodynamics are compared with the exact solutions of the kinetic equations, (1) and (2), that were constructed for (0 + 1)D systems in Ref. [21]. We refer to this paper for all details connected with the exact treatment of Eqs. (1) and (2). Here we only emphasize that the initial distribution functions used in the kinetic-theory calculations are specified by the same initial parameters as used in the hydrodynamic calculations.

The results for the oblate-oblate initial configuration with $\xi_{0,q} = 1$ and $\xi_{0,g} = 10$ are shown in Fig. 1. Figure 1(a) shows the ratio of the total transverse pressure to the total energy density, P_T/ε. The solid line describes the result of the kinetic theory, while the dashed line is the result of
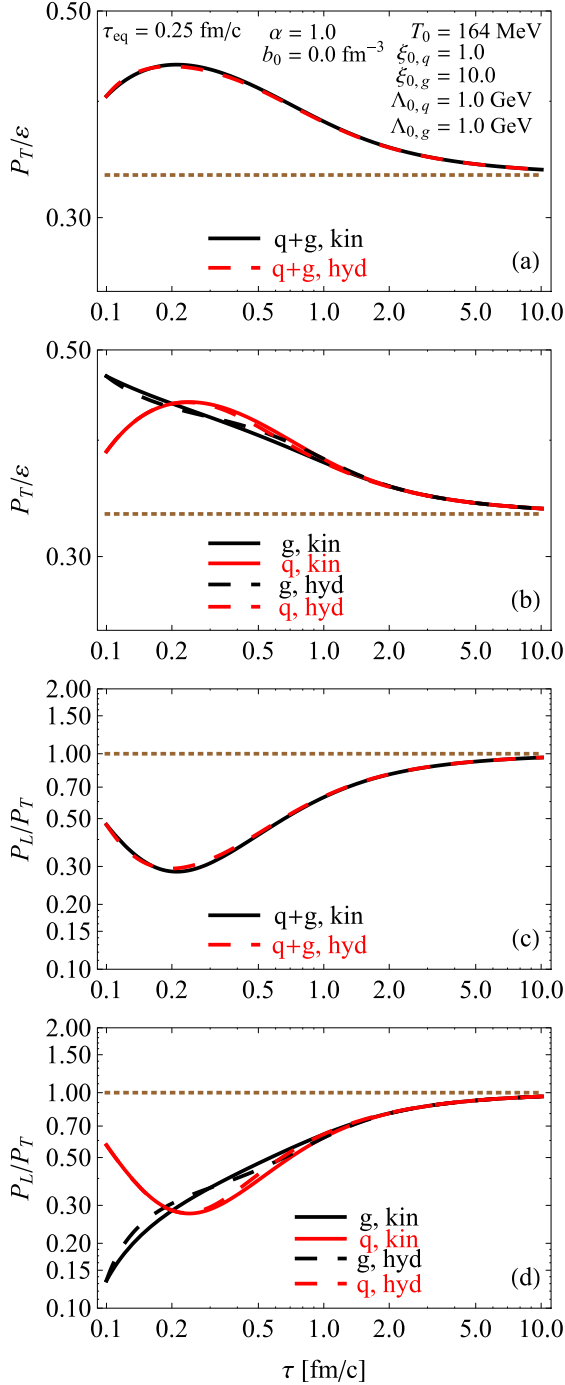
FIG. 1. (Color online) Comparison of the hydrodynamic and kinetic-theory results for the initial oblate-oblate configurations. A detailed description is given in the text.

FIG. 2. (Color online) The same as Fig. 1, but for the prolate-oblate configuration.

anistropic hydrodynamics. Figure 1(b) shows the same ratio but for the individual quark (red lines) and gluon (black lines) components. Figures 1(c) and 1(d) show the P_L/P_T ratio for the whole system and the individual components, respectively. Figures 2 and 3 show the same ratios as presented in Fig. 1, but for the prolate-oblate ($\xi_{0,q} = -0.5$, $\xi_{0,g} = 10$) and prolate-prolate ($\xi_{0,q} = -0.5$, $\xi_{0,g} = -0.25$) initial conditions, respectively. The coding of the lines is the same as in Fig. 1.
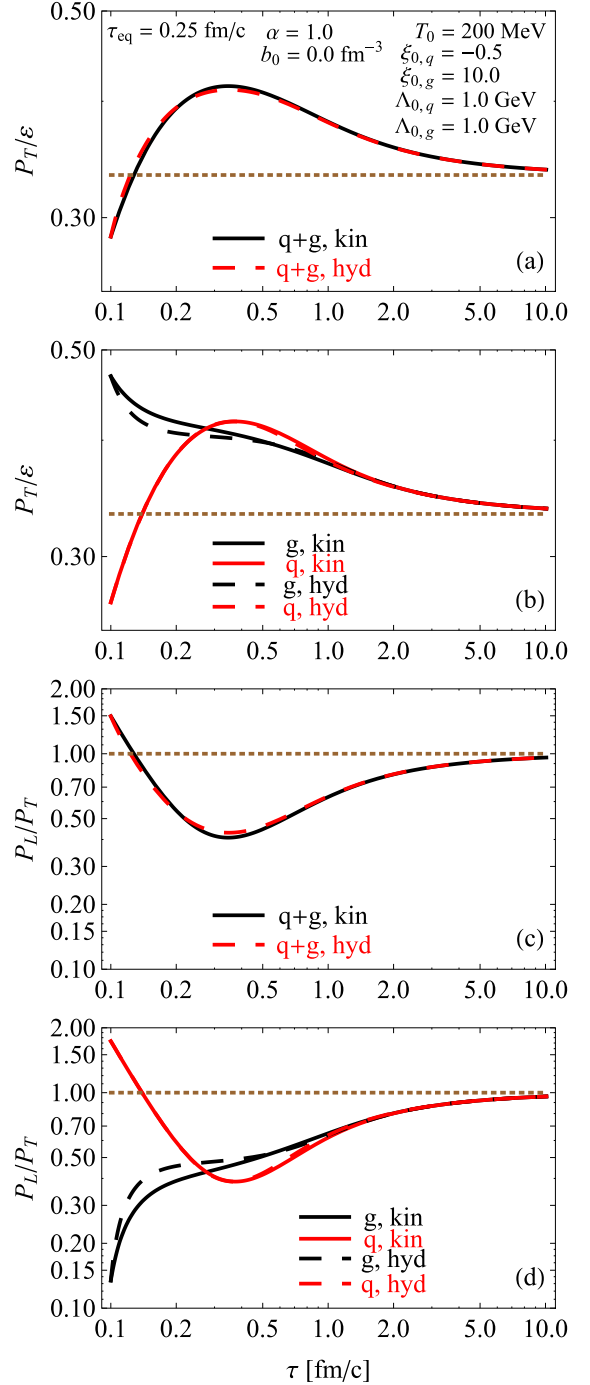
The results presented in Fig. 1 show that the hydrodynamic description agrees very well with the kinetic results. One may note, however, that the best agreement is achieved for total quantities that include both quarks and gluons. On the other hand, the worst agreement is obtained for gluons alone. This situation changes if we set $\alpha = 0$ instead of $\alpha = 1$, however, the overall results for the case $\alpha = 1$ are the best. In Figs. 2 and 3, again, a good agreement between the hydrodynamic and the kinetic-theory results is shown, especially for the
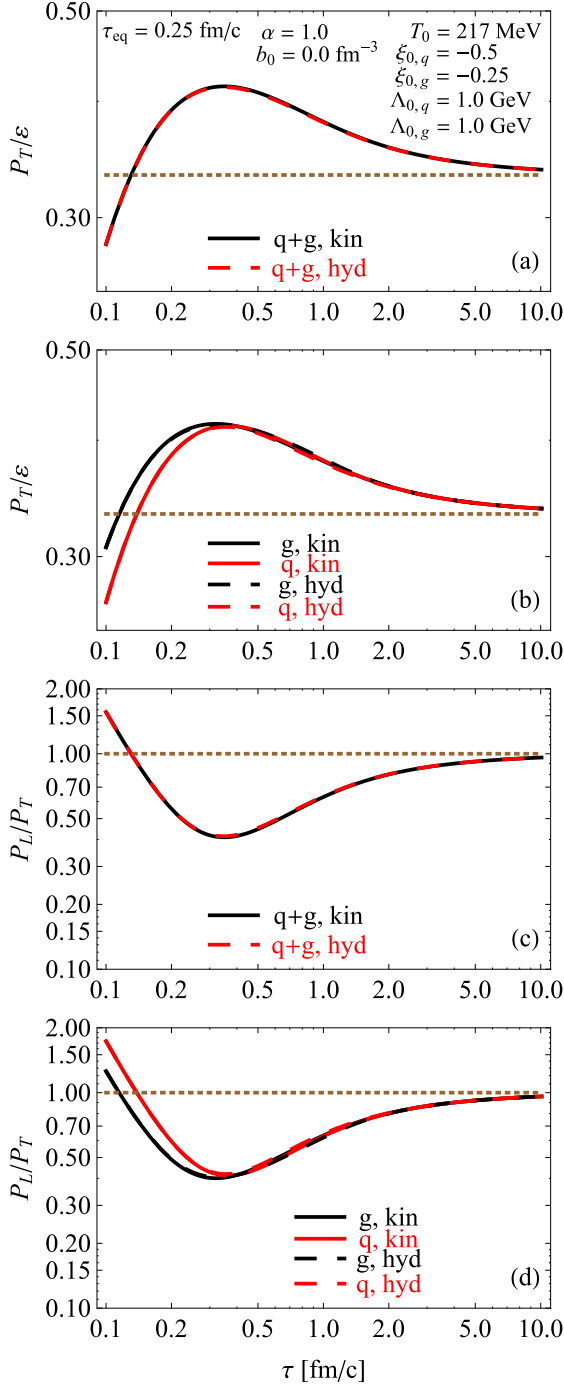
FIG. 3. (Color online) The same as Fig. 1, but for the prolate-prolate configuration.

FIG. 4. (Color online) Comparison of the results obtained with a zero vs a finite initial baryon density for the oblate-oblate configuration.

prolate-prolate configurations. For the prolate-oblate case, the agreement is a bit worse, but we have to keep in mind that such an initial configuration is extremely out of equilibrium: not only are the two distributions highly anisotropic in the momentum space, but also their individual types of anisotropy are different. The correct description of such a nonequilibrium case within a hydrodynamic approach is challenging.

Finally, in Fig. 4 we show the effect of a finite baryon density. The calculations are done for the two cases $b_0 = 0$
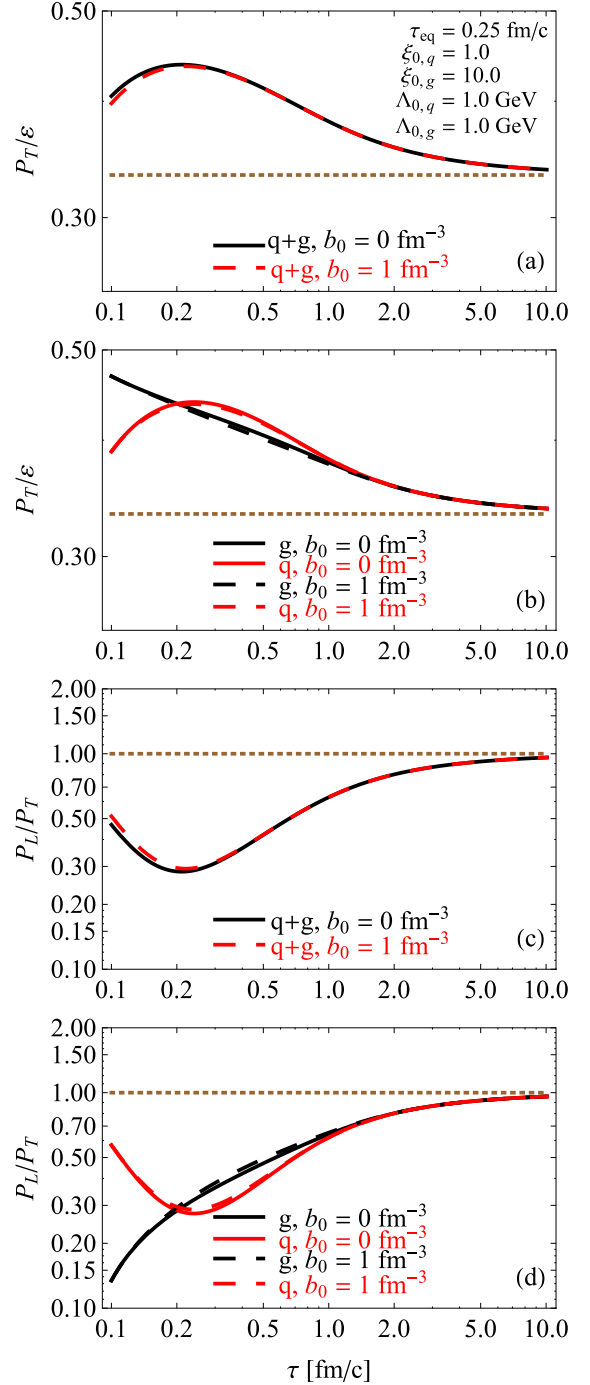
and $b_0 = 1$ fm^{-3}. In these two cases the initial oblate-oblate configuration is assumed. In agreement with earlier studies we find that the effect of a finite baryon density is very small, unless the initial baryon number density is extremely high ($b_0 \gg 1$ fm^{-3}).

We emphasize that the results shown in Figs. 1–4 were obtained with $\alpha = 1$ in Eq. (16), which has turned out to be

the best choice. A similar agreement was also obtained for the case $\alpha = 0$. On the other hand, the case $\alpha = 0.5$ gives a much worse agreement.

VII. CONCLUSIONS

We have constructed a new set of equations for anisotropic hydrodynamics describing a mixture of anisotropic quark and gluon fluids. The consistent treatment of the zeroth, first, and second moments of the kinetic equations allows us to construct our approach with more general forms of the anisotropic phase-space distribution functions than used in similar earlier studies [19–21]. In this way, the main problems of the previous formulations of anisotropic hydrodynamics for mixtures have been overcome and a good agreement with the exact kinetic-theory results is obtained.

Our considerations have been based on the RTA form of the collision term. However, we expect that the hydrodynamic model formulated in this work should also describe well kinetic systems with more general collision terms, as long as the cross sections appearing in those collision terms determine properly the relaxation time used in the RTA.

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