

Combustion of a hadronic star into a quark star: The turbulent and the diffusive regimes

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We argue that the full conversion of a hadronic star into a quark or a hybrid star occurs within two different regimes separated by a critical value of the density of the hadronic phase \bar{n}_h . The first stage, occurring for $n_h > \bar{n}_h$, is characterized by turbulent combustion and lasts typically a few ms. During this short time scale, neutrino cooling is basically inactive and the star heats up due to the heat released in the conversion. In the second stage, occurring for $n_h < \bar{n}_h$, turbulence is not active anymore, and the conversion proceeds on a much longer time scale (of the order of tens of seconds), with a velocity regulated by the diffusion and the production of strange quarks. At the same time, neutrino cooling is also active. The interplay between the heating of the star due to the slow conversion of its outer layers (with densities smaller than \bar{n}_h) and the neutrino cooling of the forming quark star leads to a quasiplateau in the neutrino luminosity which, if observed, would possibly represent a unique signature for the existence of quark matter inside compact stars. We will discuss the phenomenological implications of this scenario, in particular in connection with the time structure of long gamma-ray bursts.

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I. INTRODUCTION

The Bodmer-Witten hypothesis on the absolute stability of strange quark matter [1,2] stimulated many interesting investigations on the possible existence of compact stars entirely composed of this kind of matter [3,4] or small nuggets of strange quark matter which would propagate in the Universe as cosmic rays [5]. The exothermic process of conversion of ordinary nuclear matter into strange quark matter was studied for the first time in Ref. [6], where it was modeled as a slow combustion by means of a one-dimensional stationary reaction-diffusion-advection equation for the strange quark concentration. This is a kinetic theory calculation in which the microphysical processes occurring within a finite width combustion zone are taken into account, which allows us to determine the velocity of the conversion as a function of the quark diffusion coefficient and the rate of conversion of down quarks into strange quarks. It turns out that typical values for the burning velocities are within 10^3 – 10^4 cm/s for a quark chemical potential $\mu_q \sim 300$ MeV and a temperature of the quark phase $T \sim 10$ MeV.

The limit of this kinetic theory approach is that it does not allow us to take into account possible macroscopic collective flows and hydrodynamical instabilities driven by pressure and density gradients between the fuel and the ash fluids. Gravity would also play an important role in the dynamics of the conversion. For a complete treatment of the problem one would have to couple the equations of hydrodynamics (i.e., the equations of conservation of baryon number, momentum, and energy) and the equation of conservation of chemical species (which includes the diffusion and the reaction rates within the combustion zone) in multidimensional numerical simulations; see [7,8]. The width of the combustion zone δ can be estimated by the simple relation $\delta \sim \sqrt{D\tau}$ [9] where D is the quark diffusion coefficient ($D \sim 10^{-1}$ cm²/s for $\mu_q \sim 300$ MeV and $T \sim 10$ MeV [8]) and τ is the inverse of the rate of conversion of down quarks into strange quarks ($\tau \sim 10^{-9}$ s for $\mu_q \sim 300$ MeV [10]). One obtains $\delta \sim 10^{-5}$ cm. Clearly, it would be numerically unfeasible to resolve such a small

length scale within a numerical simulation aiming at studying a compact star whose size is of the order of 10 km.

A similar problem exists in the context of numerical simulations of type Ia supernovae where length scales from 10^{-4} to 10^8 cm characterize the physical system [11] and an alternative scheme has been devised: the combustion zone being much smaller than the size of the system, one can assume that it is actually an infinitely thin layer and it can be treated as a surface of discontinuity which separates the ashes from the fuel. In this scheme, called the *flamelet regime*,¹ one has to impose the Hugoniot jump conditions to relate the thermodynamical variables of the fluid at both sides of the discontinuity. This approach, implemented in 3+1 dimensions (3+1D), is the one most used in the context of type Ia supernovae, and it has demonstrated the crucial role played in such explosive events by the hydrodynamical instabilities, specifically the Rayleigh-Taylor instability and the Landau-Darrieus instabilities [12,13], which turn the laminar combustion into a much faster turbulent combustion. In this sense the flamelet approximation is very sensible: since the burning velocity is strongly enhanced by turbulence, the importance of knowing the exact value of the laminar velocity (governed by the microphysics of the combustion zone) is subordinate.

The same method has been adopted for studying the conversion of nuclear matter into strange quark matter in semi-analytical calculations [14–17]. This process is most likely a deflagration (subsonic combustion) and not a detonation (supersonic combustion) although its velocity is substantially increased, up to two orders of magnitude, by the development of hydrodynamical instabilities. A common finding within this approach is that at some critical density this rapid combustion stops and the neutron star cannot fully convert into a strange quark star. These results have been recently confirmed in 3+1D hydrodynamics numerical simulations: the star is converted on

¹The flamelet regime is also characterized by the fact that the large turbulent eddies do not alter the structure of the flame at the microscopic level.

a time scale of ms thanks to hydrodynamical instabilities but a sizable fraction of the star, of the order of a few $0.1M_{\odot}$, is left unburnt [18,19]. A natural question that we will address in this paper arises: Does the combustion really stop or does it proceed on a longer time scale?

After clarifying the reason for which, in a purely hydrodynamical approach, the turbulent combustion stops (the critical density being given by the condition proposed by Coll in Ref. [20]), we will model the subsequent slow conversion process (which is driven by strangeness diffusion and weak interaction processes). We will demonstrate that, in such a two-step scenario, a first stage of fast turbulent combustion and a second stage of slow laminar combustion, one could obtain a noticeable imprint on the neutrino signal emitted from the formation of a strange quark star.

The paper is organized as follows: in Sec. II we will review the procedure adopted for treating the combustion within a purely hydrodynamical approach and we will explain the meaning of Coll's condition. In Sec. III we will model the slow combustion process and provide an example of the neutrino signals released by a strange star at birth. We draw our conclusions in Sec. IV.

II. TURBULENT HYDRODYNAMICAL COMBUSTION

We review here the way the combustion is treated if the combustion zone is so thin as to be considered as a surface of discontinuity, usually called the *flame front*. The generalization of the classical combustion theory within relativistic (magneto)hydrodynamics has been presented by Coll and Anile in Refs. [20,21]. Denoting with p_i , e_i , n_i and $X_i = (e_i + p_i)/n_i^2$ the pressure, energy density, baryon density, and dynamical volume of fluid i , the so-called “condition for exothermic combustion”—we will name it “Coll's condition”—for the conversion of fluid 1 into fluid 2 reads $e_1(p, X) > e_2(p, X)$. That is, at fixed pressure and dynamical volume, the energy density of fluid 1, the fuel, must be larger than the one of fluid 2, the ash; see also [22] for the case of classical hydrodynamics. Introducing the enthalpy density $w_i = e_i + p_i$, Coll's condition can be equivalently expressed as $w_1(p, X) > w_2(p, X)$. As we will show, this condition is necessary in the case of a detonation, while in the case of a deflagration it determines the appearance of hydrodynamical instabilities dramatically accelerating the combustion process. To our knowledge this issue is not discussed in any standard textbook of hydrodynamics nor in any research paper. Our result also implies that Coll's condition corresponds to the requirement of exothermicity in the case of detonations while its violation does not exclude the possibility of slow processes of combustion.

Let us consider the two fluids to be hadronic matter and quark matter. For the hadronic matter we adopt a generic equation of state at zero temperature $e_h(p_h, X_h)$ (this situation

corresponds to the case of the conversion of a cold hadronic star) while for quark matter we consider the simple case of the equation of state of the MIT bag model with massless quarks as in Ref. [18]. The relation between energy density and pressure in this case reads $e_q = 3p_q + 4B$ where B is the usual bag constant (notice that in the case of massless quarks the energy density is a function of only p and not of X). The more general case of a polytrope is discussed in Appendix A; see also [23].

Similarly to the case of the discontinuity associated with a shock wave, also in the case of the flame front one has to impose the continuity equations for the fluxes of baryon number (or mass flux), momentum, and energy. By indicating with j the number of baryons ignited per unit time and unit area of the flame front, one obtains the following relations between the thermodynamical quantities of the hadronic fluid and of the quark fluid:

$$n_h u_h = n_q u_q = j, \quad (1)$$

$$(p_q - p_h)/(X_h - X_q) = j^2, \quad (2)$$

$$w_h(p_h, X_h)X_h - w_q(p_q, X_q)X_q = (p_h - p_q)(X_h + X_q); \quad (3)$$

the last equation corresponding to the so-called relativistic detonation adiabat. u_h and u_q are the four-velocities of hadronic and quark matter in the flame front rest frame. Starting from hadronic matter in the state A , $p_h = p_A$ and $X_h = X_A$, and, for a given value of j , Eqs. (1)–(3) allow us to determine the final state B of quark matter, $p_q = p_B$ and $X_q = X_B$, which lies on the detonation adiabat. In particular, the second equation represents a line in the (p, X) plane passing through A and with angular coefficient equal to $-j^2$. The intersections of this line with the detonation adiabat allow us to find the state B of quark matter. The baryon number flux j , or equivalently the flame front velocity with respect to one of the two fluids u_i , in general cannot be expressed in terms of the thermodynamical variables of the states A and B . It is instead related to the specific microscopic properties of the chemical reactions of the combustion, the heat transfer, and the diffusion of chemical species across the flame front, and therefore it must be determined within a kinetic theory approach such as the one of Ref. [6]. We remark however that hydrodynamical instabilities can substantially affect the flame front velocities as we will discuss later on.

We will now show that Coll's condition determines the position of the initial state A with respect to the detonation adiabat. To this purpose we will consider a point on the detonation adiabat having the same value of X as the initial state A ($X_A = X_B$) and we will compute the corresponding value of the pressure p_B (this procedure corresponds to the limit $j \rightarrow \infty$); in this way we will find if A lies below or above the detonation adiabat. Coll's condition, written in terms of the enthalpy density, reads [20] $\Delta(p, X) = e_h(p, X) - e_q(p, X) = w_h(p, X) - w_q(p, X) > 0$. One can write Eq. (3), adding and subtracting $e_q(p_A, X_A)$, as

$$\begin{aligned} w_h(p_A, X_A) - w_q(p_B, X_A) &= 2p_A - 2p_B, \\ e_h(p_A, X_A) + p_A - e_q(p_B, X_A) - p_B + e_q(p_A, X_A) - e_q(p_A, X_A) &= 2p_A - 2p_B, \\ \Delta(p_A, X_A) + e_q(p_A, X_A) - e_q(p_B, X_A) &= p_A - p_B, \\ \Delta(p_A, X_A) + 3p_A + 4B - 3p_B - 4B &= p_A - p_B, \\ \Delta(p_A, X_A) &= 2(p_B - p_A). \end{aligned} \quad (4)$$

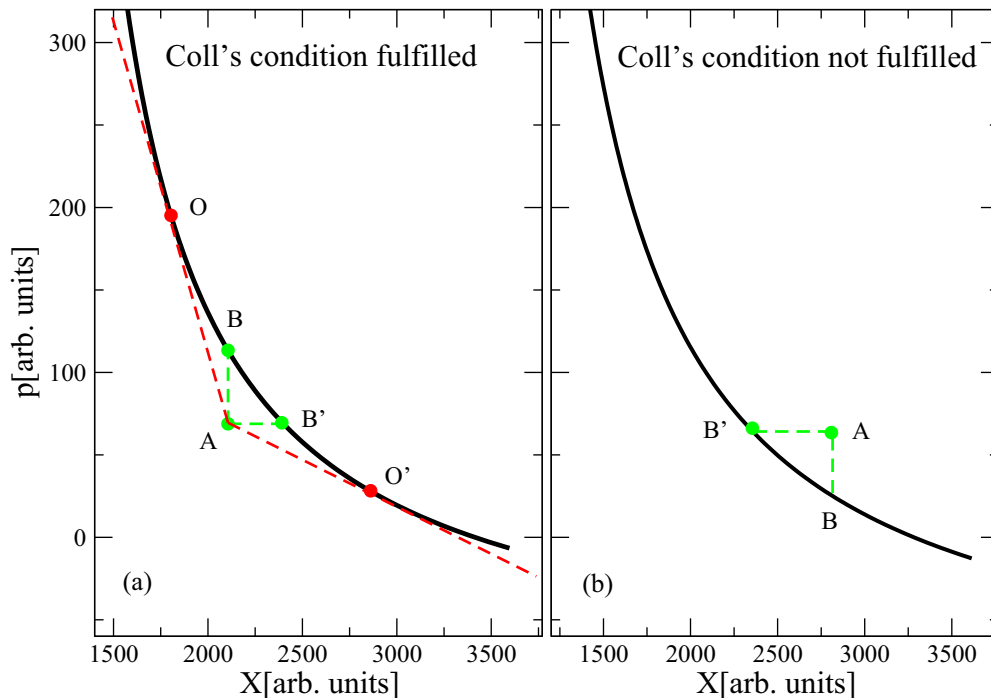


FIG. 1. (Color online) Illustrative plot of the detonation adiabat in the case in which Coll's condition is fulfilled (a) or not (b). A, B, B' indicate respectively the initial hadronic state and two possible final states for the quark phase. O and O' are the Chapman-Jouget points.

Therefore if $\Delta(p_A, X_A) > 0$, i.e., if Coll's condition is fulfilled, then $p_B > p_A$ which means that the initial state of the hadronic phase A lies in the region of the (p, X) plane below the detonation adiabat. In turn, this implies that there are two specific values of j , j_O and $j_{O'}$, for which the lines passing through A are tangent to the detonation adiabat. The two points of tangency are the Chapman-Jouget points; see Fig. 1. Following the standard treatment, one could obtain strong and weak detonations and strong and weak deflagrations depending on the specific values of j and on the boundary conditions of the problem. The point O corresponds to the Chapman-Jouget detonation and it is the only possible realization of detonation in a physical system, such as a compact star, in which no external force is producing the shock wave; see [9]. Moreover a detonation taking place in a compact star can be assimilated into a detonation in a closed pipe, and also in this case it can take place only at the Chapman-Jouget point [9]. Clearly, if Coll's condition is not fulfilled, state A lies in the region of the (p, X) plane above the detonation adiabat (see Fig. 1 right panel) and there are no Chapman-Jouget points in this case. Detonation is therefore excluded when Coll's condition is not fulfilled.

Let us discuss now how Coll's condition is related to the deflagration regimes. We consider the simplest case of a slow combustion, i.e., a process in which the velocities v_h and v_q are much smaller than the sound velocities c_h and c_q of the two fluids (this case is particularly interesting for the combustion of hadronic stars into quark stars for which the laminar velocities, found in [6], are much smaller than the sound velocities). By using Eqs. (1)–(3) one finds in this regime that $p_h = p_A = p_q = p_{B'}$ and $w_A/n_A = w_{B'}/n_{B'}$ or equivalently

$(e_A + p_A)/n_A = (e_{B'} + p_A)/n_{B'}$; i.e., the enthalpy per baryon is conserved during the combustion (see [9] for the case of nonrelativistic hydrodynamics). One sees immediately that in this case Coll's condition implies that $X'_{B'} > X_A$, i.e., $(e_{B'} + p_A)/n_{B'}^2 > (e_A + p_A)/n_A^2$, which together with the conservation of the enthalpy per baryon implies $n_{B'} < n_A$; see Fig. 1 (left panel). Moreover, from $n_A(e_{B'} + p_A) = n_{B'}(e_A + p_A) < n_A(e_A + p_A)$ one obtains $e_{B'} < e_A$. Thus the quark phase is produced with baryon density and energy density smaller than the ones of the hadronic phase. This fact opens the possibility of obtaining the Rayleigh-Taylor instabilities. These instabilities arise, in the presence of gravity, if the gradient of the gravitational potential and the gradient of the energy density point in opposite directions (inverse density stratification). Indeed, as shown in Refs. [17,18], the Rayleigh-Taylor instabilities do occur during the conversion of a hadronic star and they substantially increase the efficiency of burning, leading to time scales of the order of ms for the conversion of a big portion of the star. The relevance of the Rayleigh-Taylor instabilities in the context of the conversion of hadronic matter into quark matter was pointed out already in Refs. [24,25]. In particular, through a linear stability analysis of the propagation of the flame front, it has been shown that deflagrations are always Rayleigh-Taylor unstable in dense matter. Another very interesting result of those studies is the possibility that the deflagration, due to turbulence, turns into a detonation. Let us remind first that in a system with spherical symmetry (like a star), similarly to the case of a closed pipe, the burning is accompanied by a shock wave which propagates ahead of the flame. Due to the hydrodynamical instabilities the flame front is so much accelerated that it can merge with the

precompression shock, a feature that characterizes indeed a detonation wave. Although such a deflagration to detonation transition was not found in the numerical simulations of [18] nor in the semi-analytic study of [17], this scenario cannot be excluded: simulations are performed with a space resolution significantly larger than the Gibson scale. Therefore, one cannot resolve small scale eddies which could eventually increase the burning velocity up to supersonic velocities.

Let us discuss now what happens if Coll's condition is violated. In this case the new phase is produced (again in the case of a slow combustion) with $e_{B'} > e_A$ and therefore the burning can proceed but with velocities which are dominated by the diffusion and the rate of the chemical reactions and which are therefore much smaller than the velocities obtained during the turbulent regime. Moreover one can notice that, since the new phase is produced with an energy density larger than the one of the fuel, one cannot speak in this case of a deflagration (for which the inequality $e_{B'} < e_A$ must hold true).

One can separate the turbulent from the diffusive regime by finding the critical density of the hadronic phase, \bar{n}_h , for which Coll's equality is satisfied: $e_h(p, X) = e_q(p, X)$. For this value of density the state A lies on the detonation adiabat. Moreover one also obtains that $n_q = \bar{n}_h$; thus pressure, energy density and baryon density are continuous across the interface of the flame front. The evolution of the flame for $n_h < \bar{n}_h$ is not turbulent anymore.

A simple way to understand the role played by the instabilities is to use the analytic model proposed in [13] which allows us to estimate the increase in the laminar velocities u_i due to turbulence in terms of the fractal dimension of the flame front. In particular, the fractal excess of the wrinkled front surface ΔD is proportional to the square of $\Gamma = 1 - e_B/e_A$ and it therefore vanishes when Coll's condition is met at $n_A = \bar{n}_h$. The mean velocity v_{mh} of the front (with respect to the hadronic fluid) is significantly larger than the laminar velocity v_{lh} and it reads

$$v_{mh} = v_{lh} (\lambda_{\max}/\lambda_{\min})^{\Delta D}, \quad (5)$$

where λ_{\max} and λ_{\min} are two length scales which regulate the maximal and the minimal size of the wrinkle for which the velocity of the Rayleigh-Taylor growing modes is larger than the laminar velocity; see [17]. As one can indeed notice in the simulations of [18], the velocity of the flame front approaches the laminar velocity when the density is close to \bar{n}_h . Clearly a fraction of the star cannot be converted by means of a fast hydrodynamical deflagration and actually a mass of a few $0.1 M_\odot$ remains unburnt at the end of the turbulent regime [18].

To provide an example of the results obtained by imposing Coll's condition, we display in Fig. 2 the difference between the energy density of hadronic matter and of quark matter (again massless quarks for simplicity) for the following equations of state: the LS180 model from [26], the GM3 model from [27], and the SFHo model from [28] also with the inclusion of delta resonances and hyperons [29]. Pure nucleonic equations of state provide two values of baryon density for which Coll's equation holds (as also found in [14,18]). In general, the stiffer the hadronic equation of state the smaller the density window for which the turbulent hydrodynamical combustion can take place. One the other

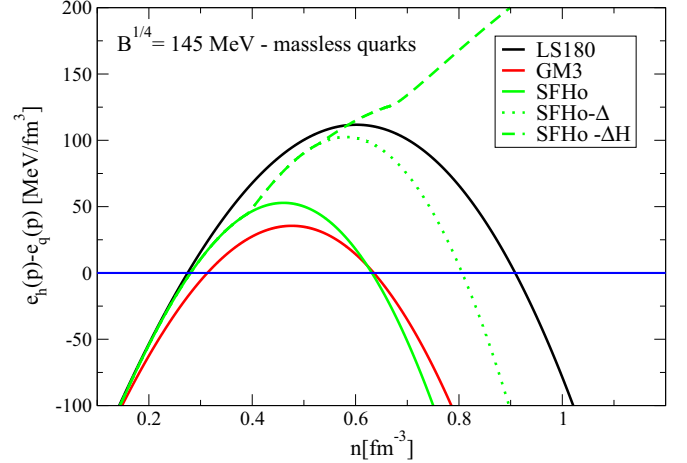


FIG. 2. (Color online) Difference between the energy density of the hadronic phase and the quark phase as a function of the baryon density. The solid black and red lines correspond to the LS180 and GM3 equations of state with only nucleons. The green lines correspond to the SFHo model with only nucleons (solid line), deltas (dotted line), and deltas and hyperons (dashed line). Coll's condition is fulfilled within the density window for which this difference is positive.

hand, when considering deltas and hyperons, within the SFHo model the turbulent hydrodynamical combustion is always possible above the threshold for the formation of deltas (see dashed line). Notice that, once a certain amount of hyperons are present in a hadronic star, the conversion is a necessary process in the scenario of Witten's hypothesis [30].

III. DIFFUSIVE REGIME

A. Diffusion of strangeness and exothermicity

As explained before, when the flame front reaches $n_h = \bar{n}_h$ the hydrodynamical instabilities responsible for the very fast conversion of the inner part of the star are not active anymore. The subsequent evolution of the system can be described by using the laminar burning velocity of the front v_{lh} computed in [6,10]: it corresponds to v_{mh} in the limit $\Delta D \rightarrow 0$ [see Eq. (5)]. Let us first review how the conversion process is described within microscopic kinetic theory approaches; we follow in particular the more recent treatment presented in [10]. The conversion is basically due to two simultaneous processes: diffusion of quarks within the combustion layer and flavor changing weak interactions among quarks (the process $u + d \rightarrow u + s$ being the most relevant for our case). One starts by introducing a position-dependent strangeness imbalance $a(x) = [n_K(x) - n_K^Q]/n_Q$, where the coordinate $x = 0$ defines the position of the conversion boundary, $n_K(x) = [n_d(x) - n_s(x)]/2$, and $n_K^Q = n_K(x)$ for $x \rightarrow +\infty$ (it would vanish for instance in the case of massless strange quarks). In the limit $x \rightarrow +\infty$ one has bulk equilibrated strange quark matter with baryon density n_Q , while for $x < 0$, $a(x) = \text{const} = a(0^-) \equiv a_N$. If the hadronic phase is assumed to be made only of neutrons and if $n_K^Q = 0$ then $a_N = n_N/n_Q$ with n_N being the baryon density of neutron

matter. In the following, for simplicity, we will adopt these conditions. Within the combustion zone, where both diffusion and chemical reactions are active, strange quark matter is out of beta equilibrium. There are two useful reference frames: the one in which the flame front is at rest and the one in which quark matter is at rest. If v_{lh} and v_{lq} are the (laminar) velocities of hadronic matter and quark matter in the front frame, in the quark matter frame the velocity of the front is $v_{lf} = -v_{lq}$ and $v'_{lh} = v_{lh} - v_{lq}$. We remind that v_{lq} and v_{lh} are related by the baryon flux continuity equation: $v_{lq}n_q = v_{lh}n_h$.

By solving the steady state transport equation for a given in [6,10] with the boundary conditions $a(0^+) \equiv a_{Q^*} = n_K^Q(0^+)/n_Q$ and $a(x \rightarrow +\infty) = 0$ [notice that $a(x)$ monotonically decreases from $x = 0$ to $x \rightarrow +\infty$] one obtains for the velocity of the front in the quark matter rest frame (in the suprathreshold regime of Ref. [10])

$$v_{lf} = \sqrt{\frac{D}{\tau} \frac{a_{Q^*}^4}{2a_N(a_N - a_{Q^*})}}. \quad (6)$$

The quark diffusion coefficient D and the inverse rate of chemical reactions (for the process $u + d \rightarrow u + s$) τ are given by [8,10]

$$D = 0.1 \left(\frac{\mu_q}{300 \text{ MeV}} \right)^{2/3} \left(\frac{T}{10 \text{ MeV}} \right)^{-5/3} \text{ cm}^2/\text{s},$$

$$\tau = 1.3 \times 10^{-9} \left(\frac{300 \text{ MeV}}{\mu_q} \right)^5 \text{ s}. \quad (7)$$

Notice that v_{lf} scales as $\sim T^{-5/6}$: the more the star is heated up by the conversion process the slower the burning velocity. This fact will have important consequences for the thermal evolution of the star and the neutrino signal which will be discussed in the following. The value of the parameter a_{Q^*} is crucial: the larger a_{Q^*} , the larger v_{lf} . This quantity is just the boundary condition for the transport equation but its value cannot be taken arbitrarily large because one needs to impose the exothermicity of the process of combustion. In other terms, if the imbalance a_{Q^*} is too large the front cannot move because there are not enough strange quarks to trigger deconfinement. Its maximum value $a_{Q^*}^{\max}$ is estimated in Appendix B.

Let us now explain how we model the conversion during the diffusive regime. Once the initial state A of the hadronic phase is fixed, at a density $n_h \leq \bar{n}_h$, one needs two equations to determine the state of the newly produced quark phase (for instance in terms of μ_q and T). As in the case of a weak deflagration discussed before, since the velocities during the diffusive regime are small with respect to the sound velocities, we can set $p_h = p_A = p_q = p_{B'}$ (as also assumed in [6,10]) and moreover $w_h/n_h = w_A/n_A = w_q/n_q = w_{B'}/n_{B'}$.² Notice that these two equations match continuously at

$n_h = \bar{n}_h$ with the equations for the turbulent hydrodynamical regime.

The process of conversion is exothermic if during the temporal evolution of the system some heat is released to the environment. Let us discuss how the heat per baryon generated by the conversion can be calculated. The equation of conservation of the enthalpy per baryon $w_A/n_A(p_A, T_A) = w_B/n_B(p_A, T_B)$ allows us to find the value of T_B , and if it turns out that $T_B > T_A$, the process is exothermic. By indicating with N the number of baryons composing the system, the total initial enthalpy is given (for uniform matter) by $Nw_A/n_A(p_A, T_A)$. Because of cooling, asymptotically the system will reach again the initial temperature T_A and the total enthalpy of the system after the full conversion and after the cooling process will be $Nw_B/n_B(p_A, T_A)$. Hence, the total heat Q released by the process of conversion and emitted into the environment (via neutrino cooling in the case of a compact star) is given by $Q \equiv N(w_B/n_B(p_A, T_B) - w_B/n_B(p_A, T_A)) = N(w_A/n_A(p_A, T_A) - w_B/n_B(p_A, T_A))$. Thus, the difference between the enthalpy per baryon of the fuel and of the ashes calculated at the same pressure and temperature corresponds to the heat per baryon q released by the conversion: $q = w_A/n_A(p_A, T_A) - w_B/n_B(p_A, T_A)$.

A natural question concerns the point at which the conversion will stop. Let us indicate with $p_h(r)$ and $p_q(r)$ the hadronic matter and quark matter pressure profiles inside the star (r is the radial coordinate). The radius of the star R , is obtained by imposing $p(r) = 0$ regardless of the composition of the surface of the star. When the burning front is close to the surface of the star, the conservation of the enthalpy per baryon (for an initially cold star) implies $e_A/n_A(T_A = 0, p_A = 0) = e_B/n_B(T_B > 0, p_A = 0) > e_B/n_B(T_B = 0, p_A = 0)$. But $e_A/n_A(T_A = 0, p_A = 0) = 930 \text{ MeV}$ (corresponding the energy per baryon of the iron nuclei composing the outer crust) and it is necessarily larger than $e_B/n_B(T_B = 0, p_A = 0)$ under the hypothesis of absolutely stable strange quark matter. This means that the conversion is exothermic, and therefore it will continue, up to the surface of the star. Therefore, just by thermodynamics arguments, one would expect that the whole hadronic star converts into a quark star. On the other hand, the existence of a crystalline structure in the outer crust can significantly hinder the conversion process [6]. It has been noticed in Ref. [31] that the preheating of the crust due to the heat released by the conversion of the inner region leads to the dissociation of nuclei, which greatly facilitates the burning into quark matter. In [31] it has been estimated that the time needed to dissolve and convert the outer crust is of the order of few tens of ms. We will neglect in our calculation the conversion of the outer crust and consider only the slower conversion of the hadronic layer between \bar{n}_h and the neutron drip line density $n_d = 2.6 \times 10^{-4} \text{ fm}^{-3}$. The

²One obtains this relation by using again the continuity equations at the front interface under the hypothesis of small velocities. Alternatively, it can be obtained from the first principle of thermodynamics applied to a system in which a number ΔN_1 of baryons of fluid 1 are converted to ΔN_2 baryons of fluid 2 at a constant pressure p : the first principle of thermodynamics reads in this

case $\Delta Q = p\Delta V_1 + \Delta U_1 + p\Delta V_2 + \Delta U_2$. One can write $\Delta V_i = \Delta N_i/n_i$ and $\Delta U_i = e_i/n_i\Delta N_i$. From the conservation of baryon number $-\Delta N_1 = \Delta N_2 > 0$. Moreover, in a purely hydrodynamical framework no heat is assumed to be transported out of the system and therefore $\Delta Q = 0$. This leads to $w_1/n_1 = w_2/n_2$, i.e., to the conservation of the enthalpy per baryon.

conversion of the outer layer can be important, e.g., when discussing short gamma-ray bursts. We will consider this problem in a future paper.

B. Solving the diffusion equations

We want now to discuss an example of the temporal evolution of the burning of a hadronic star during the diffusive regime. As a first ingredient, we need an initial profile for the star after the first stage of turbulent burning: this configuration is composed of hot quark matter for densities larger than \bar{n}_h and by cold hadronic matter for densities smaller than \bar{n}_h . The equation of state of hot quark matter is computed as explained above by requiring that, at fixed pressure, the enthalpy per baryon of the quark phase is equal to that of the hadronic phase as in the case of a slow combustion. This assumption is not correct for the turbulent regime, during which the conversion velocity is significantly increased by the hydrodynamical instabilities; one can expect that the kinetic energy of the fluid flow is completely dissipated into heat once turbulence is over.

We start from a $1.5M_\odot$ hadronic star obtained with the SFHo model with the inclusion of deltas and hyperons. The central density of this stellar object is larger than the critical density for the formation of hyperons. In Fig. 3, the black lines correspond to the enclosed gravitational mass (upper panel) and the radius (lower panel) as a function of the baryon density for the initial hadronic star configuration. The black dashed line indicates the onset of hyperons which are needed, in our scenario, to seed quark matter. The red lines correspond

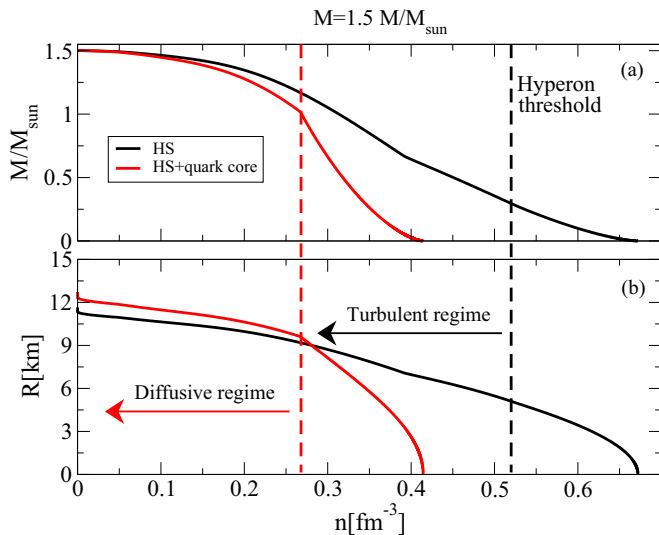


FIG. 3. (Color online) Enclosed gravitational mass (a) and radius (b) as a function of the baryon density for a $1.5M_\odot$ hadronic star before the turbulent conversion (black lines) and after the turbulent conversion (red lines). The black dashed line marks the appearance of hyperons; the seed of strange quark matter is formed at densities larger than this threshold. The red dashed line marks the density below which Coll's condition is no longer fulfilled and the turbulent combustion does not occur anymore. Below this density, the combustion proceeds via the slow diffusive regime.

to the hybrid configuration, hot quark core, and cold hadronic layer, after the turbulent regime. For the quark matter equation of state we have adopted the MIT bag model with massive strange quarks and with the inclusion of the perturbative QCD corrections (parameter set 1 of Ref. [19]). Finally, the red dashed line located at $n = \bar{n}_h$ separates the turbulent from the diffusive regime. During the conversion the total baryonic mass, which turns out to be $M_B = 1.71M_\odot$, is conserved. Notice that the total gravitational mass is also conserved during the turbulent conversion; this is due to the fact that cooling is not active during this stage and can be neglected. After the turbulent regime, a layer of hadronic matter of about $0.5M_\odot$ with a width of about 3 km is left unburnt. This situation is exactly the one obtained also in the numerical simulation of Ref. [19] and depicted in Figs. 2 and 3 of that paper.

Let us now introduce the two differential equations describing the propagation of the flame front and the thermal evolution of the star. We work in the reference frame of quark matter. Concerning the position of the flame front, by labeling with $r_f(t)$ its radial coordinate, one can write

$$\frac{dr_f}{dt} = v_{lf}(\mu_q, T) \quad (8)$$

with the initial condition $r_f(0) = \bar{r}$ where \bar{r} is obtained from the baryon density profile by using the equation $n_h(\bar{r}) = \bar{n}_h$.

For handling the thermal evolution of the star, in principle, one should couple Eq. (8) with a partial differential equation describing the heat transport through neutrinos and the heat source term related to the energy released by the conversion. Moreover, one should also introduce the thermal conductivity which determines how the new release of energy is distributed within the star. The effect of gravity should also be considered to take into account the readjustment of the star during this stage. This is a very complicated task that we will not face in this work. We will adopt instead some simplifying assumptions that allow us to understand qualitatively how the conversion process proceeds and to obtain some order-of-magnitude estimates.

First we can notice that the equations describing heat diffusion (and also the readjustment of the star) have been numerically solved in Ref. [19]. In that paper the diffusion regime was not discussed, but since strangeness diffusion is a rather slow process we can assume that the first few seconds of the thermal evolution of the star are dominated by the diffusion of the heat deposited during the rapid burning of its central region. We therefore assume that the temperature of the surface of the star and the neutrino luminosity evaluated in [19] are the correct ones during the first seconds, even if the slow conversion of the outer layer is not taken into account. From Fig. 3 of Ref. [19] one can also notice that after a few seconds the thermal profile inside the star flattens and that in particular the outer region ($r > \bar{r}$) is almost isothermal after about 7–8 seconds. This first contribution to the neutrino luminosity can be approximated by using the simple formula

$$L(t) = Q/\tau e^{-t/\tau}, \quad (9)$$

where Q is the total energy released during the rapid conversion and τ is the time scale of neutrino diffusion [32].

The results of [19] are well approximated by taking $\tau \sim 3$ s and $Q \sim 8.5 \times 10^{52}$ erg.

To simplify the description of the conversion of the outer layer we assume that (i) the conduction of heat is extremely fast (infinite thermal conductivity) in comparison with the burning velocity and therefore the heat generated by the conversion is distributed throughout the star (the very same assumption has been made in [6]); (ii) we assume that neutrinos are completely trapped and only the blackbody surface emission is considered for the cooling. The corresponding luminosity reads $L = 21/8\sigma(T/K)^4 4\pi r_s^2$ erg/s [32] with r_s the radius of the neutrinosphere (we will assume that it is located at the interface between the inner crust and the outer crust where $n_h = n_d$.) These two assumptions imply that the temperature is uniform within the star and one can write a simple equation that expresses the energy conservation:

$$C(T) \frac{dT}{dt} = -L(T) + 4\pi r_f^2 j(r_f, T) q(r_f, T) \quad (10)$$

where C is the heat capacity of the star, L is the neutrino luminosity, and $j = n_h v'_{lh}$ is the number of baryons ignited per unit time and unit area. The thermodynamical variables n_h , q , and μ_q (which appears in v'_{lh}) are all functions of $r_f(t)$ and $T(t)$. Concerning the heat capacity, we use $C = 2 \times 10^{39} M/M_\odot (T/10^9)$ erg/K obtained in Ref. [32] for a uniform density quark star or a hadronic star.

By solving simultaneously Eqs. (8) and (10) with initial conditions $r(0) = \bar{r}$, $T(0) = T_0$ MeV (which is the temperature of the star for $r > \bar{r}$ after the turbulent regime and it is of the order of 5 MeV as found in [19]) we can calculate the time needed to complete the conversion of the star and the neutrino luminosity due to the conversion of the material left unburnt after the turbulent stage. In Fig. 4, we show three cases corresponding to different values of $a_{Q^*}^{\max}$ (a_N is fixed to one as in [6]). We also display the curve of luminosity corresponding to Eq. (9).

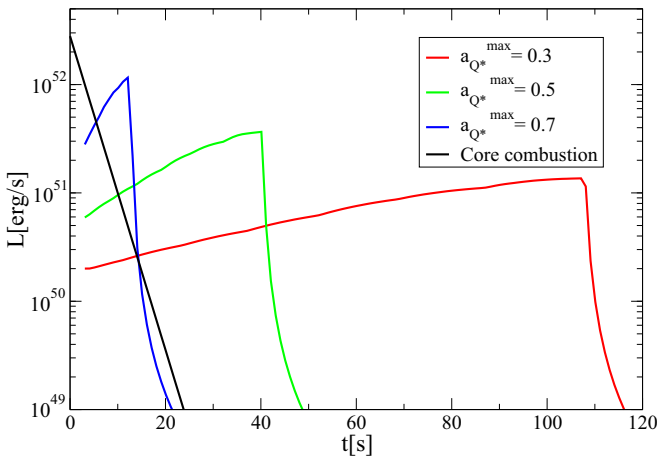


FIG. 4. (Color online) Neutrino luminosity associated with the burning during the diffusive regime of the combustion for three choices of the parameter $a_{Q^*}^{\max}$. The black line represents the luminosity obtained from the rapid combustion of the core as estimated by using Eq. (9).

As discussed above, we assume that the neutrino luminosity estimated in [19] and approximated by Eq. (9) is close to the exact one during the first seconds. After some 10 seconds the luminosity estimated by taking into account the conversion of the outer layer becomes larger than the one obtained in [19] where the outer layer remained unburnt. From that moment forward we assume that the estimate for the neutrino luminosity obtained by solving Eqs. (8) and (10) is close to the exact one. In other words, we have separated the complicated problem of the heat diffusion and of the burning of the outer layer into two stages: the first one lasting a few seconds during which the neutrino luminosity is dominated by the heat deposited during the rapid combustion of the core and is evaluated by solving the problem of heat transport as in [19], and a second stage, starting after a few seconds, during which the process of exothermic conversion of the outer layer provides enough heat to dominate the temperature of the star and the neutrino luminosity. During this second slow stage we have assumed the star to be isothermal.

A quasiplateau in the neutrino luminosity associated with the combustion of the hadronic layer is obtained (particularly evident for the smallest value of $a_{Q^*}^{\max}$). This feature is a necessary consequence of the temperature dependence of the burning velocity: as the conversion proceeds, the temperature increases due to the release of energy and therefore the velocity decreases. It is a self-regulating mechanism which rapidly leads to an almost constant velocity of burning and an almost constant luminosity of neutrinos. The process goes on until the whole star is converted. The kink appearing in the luminosity curves signals the end of the conversion; the following evolution is governed only by the cooling and the standard power-law luminosity is obtained.³

IV. CONCLUSIONS

The conversion of a hadronic star into a strange quark star and the related neutrino emission can be divided into three different stages:

- (1) Turbulent conversion of the inner part of the star, $r < \bar{r}$, lasting a few ms. During this phase cooling is negligible due to neutrino trapping.
- (2) Diffusive conversion which burns the region $r > \bar{r}$:
 - (a) During the first few seconds (at least for stars initially cold and having masses $\lesssim 1.5M_\odot$) the neutrino luminosity is dominated by the cooling of the inner region $r < \bar{r}$.
 - (b) Neutrino luminosity dominated by the cooling of the outer region $r > \bar{r}$. This phase lasts a few tens of seconds and displays a quasiplateau which originates from the burning velocity which is inversely proportional to the temperature.

³The times scales needed for the conversion can be reduced by considering that, due to gravity, the external hadronic matter layer will tend to fall onto the conversion front as the flame propagates. This would lead to an acceleration of the front which reduces the time of the conversion by roughly a factor of 3 to 4 [33].

The quasiplateau produced during the phase (b) has a rather large luminosity $\sim 10^{51-52}$ erg/s for tens of seconds and, if detected, would be a unique feature of the conversion of a hadronic star into a quark star. This very interesting possibility deserves a more refined treatment of the cooling process possibly via a neutrino transport code.

The prolonged conversion of the star is very promising also in connection with the mechanism generating long gamma-ray bursts within the protomagnetar model [34]. This model needs three crucial ingredients: high rotation frequency, high magnetic field, and a significant neutrino emission (provided by the cooling of the protomagnetar) which, through ablation of the external layers of the compact star, allows us to obtain the correct value of the Lorentz factor of the wind within which the gamma-ray burst is generated. The neutrino emission caused by the conversion of an old hadronic star could lead to a long gamma-ray burst if the other two requirements (i.e., high magnetic field and high rotation frequency) are fulfilled. One possibility is given by a merger of a neutron star with a white dwarf. It has been shown that this process produces a spinning Thorne-Zytkow-like object with a low temperature, $T \sim 10^9$ K [35]. If large magnetic fields are generated, for instance via magnetorotational instabilities, the conditions for producing a gamma-ray burst are fulfilled. The accretion of matter onto the hadronic star would trigger the conversion to a quark star and the expected neutrino luminosities are similar to the ones presented in Fig. 4. Such a gamma-ray burst would be similar to a short gamma-ray burst because it is associated with the merger of two compact stars but its duration would be comparable to the one of long gamma-ray bursts. These features are in agreement with the analyses of GRB060614 [36–38] where it has been argued that this burst is not associated with a supernova.

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APPENDIX A

We generalize the discussion presented in Sec. II, where we have adopted the quark matter equation of state with massless quarks, to the case in which the equation of state of fluid 2 is a generic polytrope given by $e_q = \alpha n_q + p_q/(\gamma - 1)$ (Ref. [39]), $p_q = kn_q^\gamma$, where γ is the adiabatic index, and $1 < \gamma \leq 2$ (the second inequality implying that the equation of state is causal at all densities [40]).

One can derive the following expression for the energy density as a function of p_q and X_q :

$$e_q(p_q, X_q) = \frac{\alpha^2(\gamma - 1) + 2p_q X_q + \alpha\sqrt{\gamma - 1}\sqrt{\alpha^2(\gamma - 1) + 4\gamma p_q X_q}}{2X_q(\gamma - 1)}. \quad (\text{A1})$$

Let us fix $X_B = X_A$ and assume $\Delta(p_A, X_A) > 0$. If $p_B > p_A$, the initial point A lies in the region of the (p, X) plane below the detonation adiabat. The detonation adiabat reads [adding and subtracting $e_q(p_A, X_A)$]:

$$\Delta(p_A, X_A) = p_A - p_B + e_q(p_B, X_A) - e_q(p_A, X_A), \quad (\text{A2})$$

which after some manipulation and using Eq. (A1) reads

$$\Delta(p_A, X_A) = \frac{\alpha}{2X_A\sqrt{\gamma - 1}}(\sqrt{\alpha^2(\gamma - 1) + 4\gamma p_B X_A} - \sqrt{\alpha^2(\gamma - 1) + 4\gamma p_A X_A}) + (p_B - p_A)\frac{2 - \gamma}{\gamma - 1}. \quad (\text{A3})$$

Since $1 < \gamma \leq 2$ the sign of $\Delta(p_A, X_A)$ clearly determines the sign of $p_B - p_A$. Thus, if $\Delta(p_A, X_A) > 0$, i.e., if Coll’s condition holds true, the initial point A lies in the region of the (p, X) plane below the detonation adiabat. One sees again that Coll’s condition establishes the position of the initial state of the hadronic phase with respect to the detonation adiabat.⁴

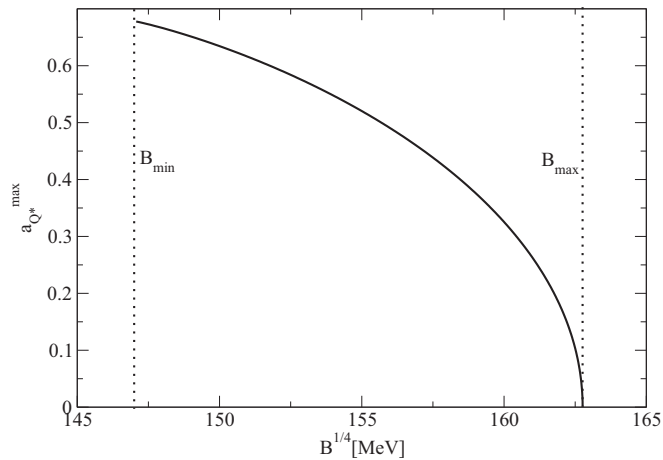
APPENDIX B

In this section we aim at providing an estimate on the allowed values of the parameter $a_{Q^*} \equiv [n_d(0^+) - n_s(0^+)]/(2n_Q)$. Please notice that in this definition there appear quantities computed at the interface and a quantity computed for $x \rightarrow +\infty$. In particular we estimate $a_{Q^*}^{\max}$, which is the maximum value of the strangeness unbalance at the interface between hadronic matter and quark matter in order for the process of conversion to still proceed because of its exothermicity; see Fig. 4 of Ref. [10]. We assume, as in Ref. [10], that the deconfinement always takes place with maximal velocity, i.e., with $a_{Q^*} = a_{Q^*}^{\max}$. We consider the case of a bag-model equation of state with massless quarks and we limit our discussion to the case of vanishing pressure p . This case is particular relevant for the burning of the most external layers of the hadronic star where the pressure approaches zero. Following the scheme of Ref. [41], we introduce the parameter r_s which is the ratio between the baryonic density of strange quarks and the baryon density: $r_s = n_s/3n$. The Fermi momenta of up, down, and strange quarks read $k_{F_{u,d}}(n, r_s) = k_F(n, r_s) = [1.5\pi^2 n(1 - r_s)]^{1/3}$, $k_{F_s}(n, r_s) = [3\pi^2(r_s n)]^{1/3}$, where we have assumed that up and down quarks have the same Fermi momenta (as in beta-stable quark matter for the case of massless quarks). The energy density reads

$$e = \frac{6k_F^4}{4\pi^2} + \frac{3k_{F_s}^4}{4\pi^2} + B. \quad (\text{B1})$$

The bag constant B varies in the range $B_{\min} < B < B_{\max}$, where for $B = B_{\max}$ the energy per baryon of strange quark matter is 930 MeV and for $B = B_{\min}$ the energy per baryon of two-flavor quark matter is 930 MeV.

⁴For values of $\gamma > 2$, one can still obtain the same conclusion, but depending on the specific values of α , p_A , and X_A .


 FIG. 5. Dependence of $a_{Q^*}^{\max}$ on the bag constant.

We first compute n_Q which corresponds to finding the baryon density $n = n_Q$ for which beta-stable quark matter (i.e., $r_s = 1/3$) has vanishing pressure. At this value of density the energy per baryon $(e/n)_\beta$ is strictly smaller than 930 MeV because we are assuming the hypothesis of absolutely stable strange quark matter.

We now calculate the maximum value of $n_d(0^+) - n_s(0^+)$. We define \bar{n} and \bar{r}_s as the values of n and r_s for which the pressure $p = 0$ and the energy per baryon of non-beta-stable quark matter $(e/n)_{\text{non-}\beta} = 930$ MeV. \bar{r}_s represents therefore the minimum amount of strangeness for which non-beta-stable strange quark matter has an energy per baryon

equal to the one of iron. A quark phase with $r_s > \bar{r}_s$ would be favored with respect to nuclear matter and the conversion process would therefore be exothermic.

$a_{Q^*}^{\max}$ is related to \bar{r}_s and \bar{n} by the following equation:

$$a_{Q^*}^{\max} = \frac{k_F^3(\bar{n}, \bar{r}_s) - k_{F_s}^3(\bar{n}, \bar{r}_s)}{2\pi^2 n_Q}. \quad (\text{B2})$$

For each value of B , the values of \bar{r}_s , \bar{n} , and n_Q are completely determined and thus also $a_{Q^*}^{\max}$ is fixed.

We display the results in Fig. 5. In general $a_{Q^*}^{\max} < 1$; its maximum value equals 1 only if one considers for $x \rightarrow +\infty$ non-beta-stable two-flavor quark matter with $n_d = 2n_u$ (see [6]). We consider instead, as in [10], beta-stable strange quark matter for $x \rightarrow +\infty$. One can show analytically that in this case the maximum value of $a_{Q^*}^{\max}$ is $1.5^{3/4}/2 \sim 0.667$ and it is obtained for $B = B_{\min}$.⁵

The minimum value of $a_{Q^*}^{\max}$ is zero and it is obtained at $B = B_{\max}$. Notice that, for this value of B , $\frac{da_{Q^*}^{\max}}{dB} \rightarrow \infty$. This basically implies that the parameter space (i.e., the range of B) has little room for the minimum value of $a_{Q^*}^{\max}$. In other words, it is very unlikely that $a_{Q^*}^{\max} \sim 0$. In turn, this means that while the value of the velocity of conversion is quite uncertain it cannot be vanishingly small.

⁵This value is obtained by using the formulas for the pressure of two-flavor and three-flavor quark matter: $p_{2F} = k_F^4/(2\pi^2) - B$ and $p_{3F} = 3k_F^4/(4\pi^2) - B$ where k_F is the quark Fermi momentum which is the same for all flavors. By imposing $p_{2F} = p_{3F} = 0$ one obtains the quark Fermi momentum for $x = 0$ and for $x \rightarrow +\infty$. Finally one uses (B2) to obtain $a_{Q^*}^{\max}$.

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