Spectral-statistics properties of the experimental and theoretical light baryon and meson spectra

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We compare the statistical fluctuation properties of the baryon and meson experimental mass spectra with those obtained from theoretical models (quark models and lattice QCD). We find that for the experimental spectra the statistical properties are close to those predicted by random matrix theory for chaotic systems, while for the theoretical ones they are in general closer to those predicted for integrable systems and safely incompatible with those of chaotic systems. We stress the importance of the agreement of the fluctuation properties between experiment and theoretical models, as they determine the dynamical regime and the complexity of the real interactions. We emphasize the new statistical method we use, adapted to properly analyze the fluctuation properties for very short spectral sequences.

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I. INTRODUCTION

Hadron spectroscopy has played a central role in the study of the strong interaction—aiding in its understanding—and in the development of quantum chromodynamics (QCD). Hadrons constitute bound states for quarks and gluons, and their accurate description is one of the principal aims of QCD. However, so far a quantitative and predictive theory of confined states has not been achieved; hence, in order to study the properties of hadrons we have to rely on models which have to be consistent with the underlying QCD. Constituent quark models [1] are examples of this kind of modeling.

Baryons. It is well known that the number of baryons predicted by quark models [2,3] is substantially larger than what is observed in meson scattering and production experiments [4]. This fact raises the problem of *missing resonances*, which has opened the door to a huge experimental effort in recent years to observe and identify these missing states [5]. These experiments have to achieve high precision due to the important background (which can veil resonances) and the overlap of baryons, as well as the need to survey different meson production channels and observables. The procedure to assess the existence of these elusive baryons consists of analyses of partial waves [6] and polarization observables [7] of the reactions, comparing experimental data from different sources to what is obtained after including or removing the hypothetical resonance. If data are better reproduced, the existence of a resonance is possible but sometimes debatable. The Particle Data Group (PDG) rates the possible existence of the resonances based on the quantity and quality of experimental data. Only after several independent experiments and analyses, a baryon is awarded a well-established status, rated with three or four stars.

Mesons. In the last decade an enormous experimental effort has been made in meson spectroscopy, with several facilities conducting research programs [8] whose main goal has been to find exotic mesons [9] which do not fit within the quark-antiquark picture of quark models. This search has been fruitless so far but has put meson physics at the forefront of scientific research, becoming a thriving research area with experimental collaborations in several facilities, i.e., BES (China) [10], CLAS at JLab (USA) [11], COMPASS at CERN (Switzerland) [12], J-PARC (Japan) [13], and Hall D under construction at JLab (USA) [14].

Theoretical research has not been oblivious to this experimental interest and several quark models of mesons have made their appearance in the literature [15–17], trying to match the low-lying experimental spectrum and complementing the classic calculation by Godfrey and Isgur [18]. Among the theoretical developments, a noteworthy one is the lattice QCD calculation of the meson spectrum by the Hadron Spectrum Collaboration (HSC) at JLab [19,20], although it has the drawback of being computed at a high pion mass of 396 MeV.

Statistics. As hadrons can be considered as aggregates of quarks and gluons, the mass spectrum of low-lying baryons or mesons can be understood as the energy spectrum of an interacting quantum system composed of such quarks and gluons. Hence, the properties of the masses can be characterized in the same terms as the energy spectrum of a similar interacting quantum system, like the atomic nucleus. Since Wigner discovered that the statistical properties of complex nuclear spectra are well described by the Gaussian orthogonal ensemble (GOE) of random matrix theory (RMT) [21], statistical methods have become a powerful tool to study the energy spectra of quantum systems [22,23].

Random matrix theory allows us to establish a connection between statistical properties of energy spectra and quantum chaos. The work of Berry and Tabor [24], which shows that integrable systems lead to energy-level fluctuations that are well described by the Poisson distribution, and the work of Bohigas, Giannoni, and Schmit [25], which conjectured that spectral fluctuation properties of chaotic systems are well described by random matrix theory (known as the BGS conjecture, later proved by Heusler *et al.* [26]), can be considered as a definition of quantum chaos in terms

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of spectral fluctuation properties. That is, the energy-level fluctuations determine if a system is chaotic, integrable, or intermediate. While for integrable and chaotic systems these properties are universal, for intermediate systems different types of transitions from order to chaos have been investigated from different approaches [27–30] but there is not a universal characterization at present.

Most of the initial impulse for the development of random matrix theory came from nuclear physics. Wigner was the first to think of nuclear interactions from a statistical point of view, renouncing the exact knowledge of the system and trying to analyze generic spectral properties instead [31]. The main difference with ordinary statistical mechanics is that one renounces not the exact knowledge of the state of the system but the nature of the system itself, that is, the nature of the interaction, and thus averages are calculated not with an ensemble of states but with an ensemble of Hamiltonians; these are the random matrices. The first experimental verification was carried out on the so-called Nuclear Data Ensemble (NDE), a set of about 1700 data on proton and neutron resonances above the one-nucleon emission threshold, the agreement with RMT being excellent [32]. As nuclei are invariant under time reversal, the matrix representation of the Hamiltonian can accordingly be chosen real and symmetric and thus the Gaussian orthogonal ensemble (GOE) is the one to be used in this case. Hence, one can say that in the high-energy region the picture is clear and one can safely state that nuclei are chaotic systems. In the low-energy domain the situation is less clear because of poorer statistics and uncertainties in the experimental spectra. However, much effort has been dedicated to analyze the experimental data and shed light on this issue. There is not a general result but the type of energy level fluctuations depends on the nuclear mass region and several factors. For example, for light and spherical nuclei they are close to GOE, but for collective states in deformed nuclei they are closer to Poisson, and in other cases the situation is intermediate. For a complete review on the issue see [22].

This kind of statistical analysis was applied to the hadron mass spectrum in [33], obtaining a chaotic-like behavior. In [34] the spectral-statistic techniques have been used to compare the experimental baryon spectrum with theoretical ones, focusing on the problem of missing resonances. The main result of this work is that the spectral fluctuation properties of theoretical quark-model spectra are incompatible with those of the experimental spectrum, with the experimental results being closer to GOE while the theoretical results are incompatible with GOE and closer to Poisson. Given that the lack of levels in a spectrum produces a loss of correlations among levels and thus a displacement towards the Poisson distribution [35], it is the experimental spectrum that should be more uncorrelated—that is, closer to Poisson, because of the lack of the missing resonances with respect to the theoretical models—but in fact the situation is just the opposite. Hence, quark models, as they are presently built, lack a very relevant property of the experimental spectrum: its chaotic behavior. Thus, they may not be suitable to reproduce the low-lying baryon spectrum, and, therefore, to predict the existence of missing resonances. In [36] this work has been extended to the meson spectrum, employing an improved version of the approach used in [34]. Also, for mesons, the fluctuation properties of the experimental spectrum are closer to GOE predictions, safely incompatible with Poisson distribution, with an estimation of 78% of chaos. Moreover, it is also tested that the analysis is robust against the inclusion of the error bars associated with the experimental data. For the theoretical models in this case, five of the six which have been analyzed, including the lattice QCD calculation by the Hadron Spectrum Collaboration (HSC) at JLab, are incompatible with chaos and closer to Poisson, as for baryons. Only one of the quark models predicts an intermediate spectrum with an estimation of 63% of chaos. Thus, all the theoretical models but one predict spectra with fluctuation properties incompatible with the experimental one. This is especially shocking for the lattice QCD spectrum, as lattice QCD is currently the only tool available to compute low-energy observables employing QCD directly. Thus, the current state-of-the-art calculation does not describe properly the statistical properties of the meson spectrum.

With this paper, we aim to fill some gaps coming from the aforementioned works. First, we give a complete description of all the statistical tools to deal with the kind of spectra present in the low-lying regions of few-particle interacting quantum systems, with special emphasis on the new method to perform a proper analysis, taking into account the shortness of the spectral sequences. Thus, besides the meson and baryon mass sequences, it can be also applied to other quantum spectra that present this problem, like, for example, the atomic nucleus. Second, we perform the new analysis used for mesons in [36] on the baryon spectrum, refining and updating the conclusion obtained in [34]. For both baryons and mesons, we perform the analysis on the last updated experimental data from the Review of Particle Physics [37], and compare to theoretical models, giving major support to the conclusions previously obtained.

The article is organized as follows: In Sec. II the techniques used in the analysis of the spectra (experimental and theoretical) are described. In Sec. III we present the results for the experimental baryon and meson spectra with comparison to theoretical models. In Sec. IV we summarize the results and state the main conclusions of the analysis.

II. STATISTICAL ANALYSIS

Prior to any statistical analysis of the spectral fluctuations one has to accomplish some preliminary tasks. First of all, it is necessary to take into account all the symmetries that properly characterize the system. It is well known that mixing different symmetries deflects the statistical properties towards the Poisson statistics [38]. Hence, it is necessary to separate the whole spectrum into sequences of energy levels involving the same symmetries, that is, values of the good quantum numbers. The usual symmetries associated to baryons are spin (J), isospin (I), parity (P), and strangeness; and for mesons the same ones plus C-parity (C). Strangeness can be dropped due to the assumption of flavor $SU(3)$ invariance. Therefore, the baryon spectrum is split into sequences with fixed values of J, I, and P, and the meson spectrum with fixed values of J, I, P, and C.

The energy spectrum of a quantum system is fully characterized by its level density $g(E)$. It can be split into a smooth part $\overline{g}(E)$, giving the secular behavior with the energy, and a fluctuating part $\widetilde{g}(E)$, which is responsible for the statistical properties of the spectrum to be analyzed [39]. Thus, the fluctuation amplitudes of the latter are modulated by $\overline{g}(E)$ and, therefore, in order to compare the statistical properties of different systems or different parts of the same spectrum the main trend defined by $\overline{g}(E)$ must be removed. The standard procedure by which it is removed is called the *unfolding*. It consists of locally mapping the real spectrum $\{E_i\}_{i=1,\ldots,N}$ into another one $\{\varepsilon_i\}_{i=1,\ldots,N}$ with constant mean level density. This can be done by means of the following transformation:

$$\varepsilon_i = \overline{m}(E_i), \quad i = 1,\ldots,N, \tag{1}$$

where $\overline{m}(E)$ is the smooth part of the cumulated level density $m(E)$, which counts the number of levels whose energy is equal to or less than E, and, like the level density $g(E)$, can also be separated into a smooth part and a fluctuating part, and N is the dimension of the spectrum. The transformed level density $\rho(\varepsilon)$ in the new energy variable ε is such that $\overline{\rho}(\varepsilon) = 1$, as required. This general method of unfolding is called *global unfolding* to distinguish it from the *local unfolding*, which we describe in the next paragraph. In practical cases, the unfolding procedure can be a difficult task for systems where there is no analytical expression for the mean level density $\overline{g}(E)$, and it must be stressed that a correct choice of $\overline{g}(E)$ is very important, because if it is not accurate enough it will introduce errors in the fluctuation measures spoiling the statistical analysis [40].

When there is no natural choice for $\overline{g}(E)$ one can resort to simple methods such as *local unfolding*, in which this function is assumed to be approximately constant in a window of v levels on each side of a given energy level E_k, and is given by

$$\overline{g}(E_k) = \frac{2v}{E_{k-v} - E_{k+v}}. \tag{2}$$

It must be noted that this procedure can only be used to study short-range correlations, and it fails to account for the long-range correlations of the spectrum spoiling the relationship between the spectral fluctuations and the regular or chaotic regime of the system [40].

Since the experimental baryon and meson spectra have been divided into very short sequences of levels, we will use the local unfolding procedure. First of all, we can consider that the variation of $\overline{g}(E)$ along these sequences is negligible, since they are short enough. Second, as it is not possible to study long-range correlations for these sequences, the main disadvantage of the local unfolding procedure does not apply to this case. It is important to remark that the usual measures for short-range correlations, like the nearest-neighbor spacing distribution which we use in this work, are not usually spoiled by local unfolding techniques.

The procedure we use in this paper is as follows. Let $\{E_i,\, i = 1,2,\ldots,l_x\}_X$ be an energy-level sequence characterized by the set X of good quantum numbers, and let the distances between consecutive levels be $S_i = E_{i+1} - E_i$. Thus, assuming that the mean level density is constant along the sequence, we can calculate the average value of the spacing between consecutive levels $\langle S\rangle = 1/\overline{g}(E) = (l_x - 1)^{-1}\sum_{i=1}^{l_x-1} S_i$ and use it to rescale the level spacings to obtain the quantities $s_i = S_i/\langle S\rangle$, called generically nearest-neighbor spacings (NNS). For the rescaled spectrum the mean level density $\overline{\rho}(E) = 1$, and $\langle s\rangle = 1$, thus the unfolding is performed.

In this paper, the statistical properties of the NNS are studied by means of the nearest-neighbor spacing distribution (NNSD) [41], denoted $P(s)$, which gives the number of spacings lying between s and $s + ds$, normalized to 1; that is, the probability that the spacing between two consecutive unfolded levels lies between s and $s + ds$. The NNSD follows the Poisson distribution for generic integrable systems [24]:

$$P_P(s) = \exp(-s) \tag{3}$$

while chaotic systems with time reversal and rotational invariance are well described by the GOE of random matrices, whose NNSD follows the Wigner surmise [25]:

$$P_W(s) = \frac{\pi s}{2}\exp\left(-\frac{\pi s^2}{4}\right). \tag{4}$$

For intermediate cases between integrable and chaotic systems, one of the most frequently used distributions is the Berry-Robnik distribution [28], based on the principle of uniform semiclassical condensation (PUSC) [27], which states that certain spectral characteristics can be understood by accounting for the separate chaotic and integrable regions in phase space. Then, denoting by f the volume fraction of the regular phase space, the Berry-Robnik distribution is written as

$$\begin{aligned} P_{BR}(f,s) = \Bigg[& f^2 \operatorname{erfc}\left(\frac{\sqrt{\pi}(1-f)}{2}s\right) \\ & + \left(2f(1-f) + \frac{\pi}{2}(1-f)^3 s\right) \\ & \times \exp\left(-\frac{\pi}{4}(1-f)^2 s^2\right)\Bigg]\exp(-s). \end{aligned} \tag{5}$$

However, despite local unfolding being the usual way to deal with short sequences, it is important to point out that it may cause a distortion on the actual $P(s)$, preventing a direct comparison with the theoretical predictions (Wigner or Poisson). This is a key point in this work, as we have to analyze spectra which have to be split into very short sequences and, as we will show, the effect is too important to ignore and do just the usual comparison to theoretical predictions. Inasmuch as $\langle s\rangle = 1$ for every spacing sequence, none of the spacings can be greater than $l - 1$, where l is the sequence length, and therefore the $P(s)$ distribution must exhibit a sharp cutoff at $s = l - 1$. When l is large enough this cutoff is irrelevant due to the exponential and Gaussian decays of the Poisson and Wigner distributions. But obviously this is not the case for smaller values of l.

In Fig. 1 we show some examples which illustrate the problem clearly. We have performed the local unfolding on GOE and Poisson spectra which have been divided into sequences of length $l = 3$, 4, and 10 levels. Each panel corresponds to an ensemble of 100 spectra with about 100 levels each; that is, similar to the dimensions of the spectra

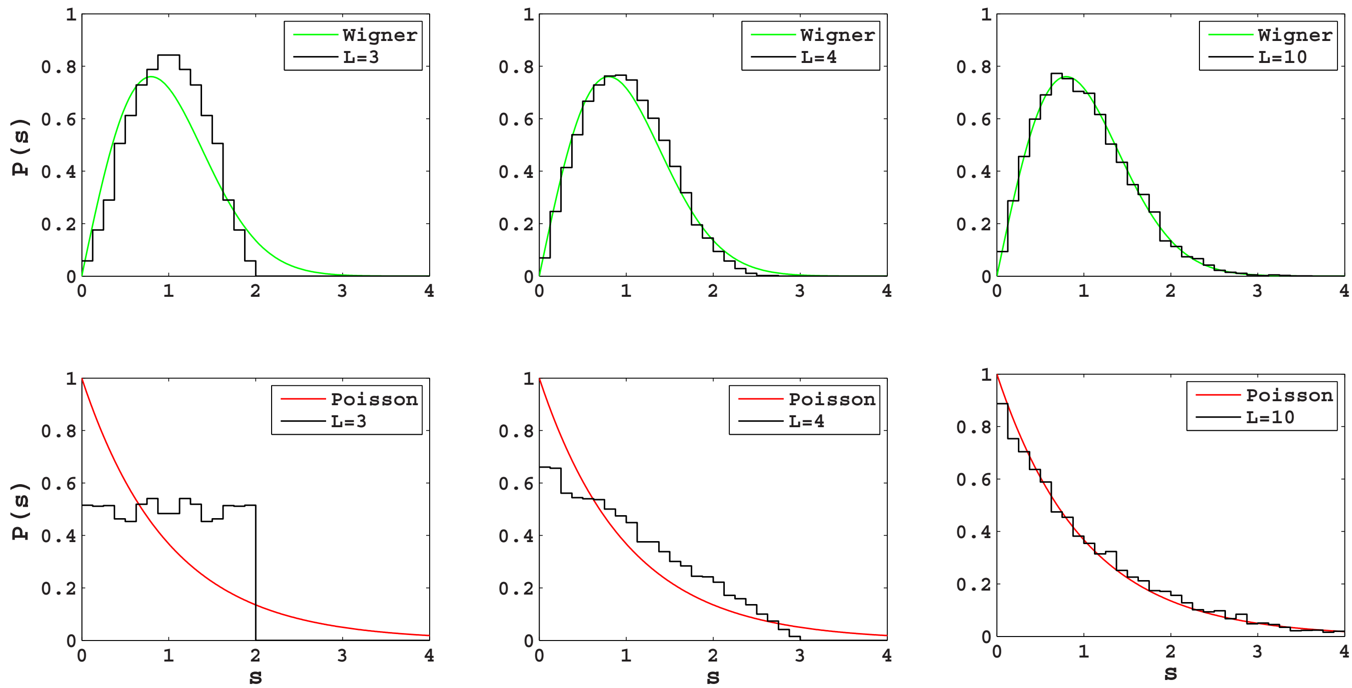

FIG. 1. (Color online) Nearest-neighbor spacing distribution for GOE (up) and Poisson (down) spectra (histograms) divided into sequences of length $l = 3,4,10$ levels in which a local unfolding has been performed, compared to the theoretical predictions, the Wigner surmise, and the Poisson distribution (solid lines).

analyzed in this work. The cutoff at $s = 2$ and 3 can be clearly observed in the first two panels for GOE (up) and Poisson spectra (down), and one can see how the shape of the $P(s)$ distribution changes with respect to the theoretical predictions. This misleading effect is especially relevant for Poisson sequences: GOE-like ones are less affected due to the Gaussian decay of the tail of the distribution. Already for $l = 10$ one can say that the effect is negligible. However, we have to deal with many short sequences when analyzing the low-lying spectra of baryons and mesons, and therefore the correct treatment of this difficulty is a key point to infer well supported conclusions.

This problem was taken into account in [34] by building GOE-like and Poisson-like spectra distorted in the same way by the unfolding procedure as the spectrum which is being studied in each case. That is, by dividing the GOE and Poisson spectra into the same number of sequences with the same lengths as the spectrum under study and performing a local unfolding in the same way. These distributions built *ad hoc* for the spectrum which is being studied are thus more appropriate as reference distributions for comparison than the theoretical predictions.

In this work we take a step further. Instead of building just one GOE-like and one Poisson-like reference distorted spectra to compare, we generate an ensemble of 1000 realizations, and their average will play the role of *theoretical distributions* for comparison with the data in each case. In this way we get a much smoother distribution to compare than with only one sample, which could be too small to be representative of the corresponding theoretical distribution. We will denote these *distorted theoretical predictions* as $P_{\mathrm{DW}}(s)$ for the distorted Wigner distribution, $P_{\mathrm{DP}}(s)$ for the distorted Poisson, and $P_{\mathrm{DBR}}(s)$ for the distorted Berry-Robnik in the cases when it is necessary to build also an intermediate distribution.

In order to have a quantitative comparison of the data with the reference distributions we shall use the Kolmogorov-Smirnov (K-S) test [42], which compares two samples in order to decide if the null hypothesis that both belong to the same distribution can be rejected or not. The statistic D calculated in this test is the largest absolute deviation between the two sample cumulative distribution functions. The obtained p value, which can be used to evaluate the result of the test, corresponds to the tail probability associated with the observed value of D; that is, the probability, under the null hypothesis, of obtaining a value of the test statistic D as extreme as that observed. The usual limit to reject the null hypothesis is $p \lesssim 0.10$. Thus, much larger p values do not allow us to reject the null hypothesis and much smaller p values allow us to safely reject it.

Complementary information can be gained by calculating the moments of the NNS distributions, as they are univocally determined while the distribution itself is sensitive to the bin size. Gathering together all the spacings s_i, the kth moment $M^{(k)}$ is calculated as $M^{(k)} = (d-n)^{-1} \sum_{i=1}^{d-n} s_i^k$, where d stands for the spectrum dimension and n is the number of spacing sequences.

Finally, we perform a test in order to check if our analysis is robust against the inclusion of the error bars associated with the experimental data.

III. RESULTS

In this section we present the results of the statistical analysis, first for baryons and then for mesons. In each case we show first the analysis of the experimental spectrum and then that of the spectra obtained from theoretical models.

A. Baryons

1. Experimental spectrum

We have taken all the resonance states from the Review of Particle Physics (RPP) [37] up to 2.2 GeV. After splitting the spectrum in sequences with the same J, I, and P, we have 53 levels distributed in 14 sequences (only sequences with more than two levels are considered).

Figure 2 shows the $P(s)$ distribution for the experimental spectrum compared to the Wigner surmise $P_W(s)$, the Poisson distribution $P_P(s)$, and the corresponding distorted $P_{\rm DW}(s)$ and $P_{\rm DP}(s)$, which are more appropriate to compare, as explained in the previous section. It can be seen that the distortion is quite noticeable in this case, especially for the Poisson distribution. A cut at $s=2$ can be observed, as expected, because most of the sequences in which the spectrum is divided in this case contain only three levels. To the naked eye, the experimental $P(s)$ seems closer to the Wigner distribution than to the Poisson one. Its most relevant signature is the behavior at small spacings. As it is clearly shown in the figure, $P(s) \xrightarrow[s\to 0]{} 0$. This feature is called "level repulsion" and it is a trademark of chaotic (Wigner-like) spectra, whereas for Poisson sequences $P(0) \neq 0$. Moreover, a quantitative measure is needed before obtaining a conclusion. To do so, we perform the K-S test with the null hypothesis that the experimental distribution coincides with the reference distribution $P_{\rm DW}(s)$ or $P_{\rm DP}(s)$ against the hypothesis that both distributions are different. The results for the p value obtained in each case are $p_{\rm DW}=0.82$ and $p_{\rm DP}=0.26$; that is, though the distribution seems closer to Wigner, none of the null hypotheses can be rejected (the usual limit for the p value is $p \lesssim 0.10$). Thus in this case one could try to compare the $P(s)$ with a Berry-Robnik distribution in order to asses how close the fluctuations are to one limit or the other. But in fact we find that there is no Berry-Robnik distribution $P_{\rm DBR}(f,s)$ which fits the experimental $P(s)$ better than the Wigner distribution $P_{\rm DW}(s)$ itself. This is clearly due to the strong repulsion shown by the experimental $P(s)$, as pointed out above.

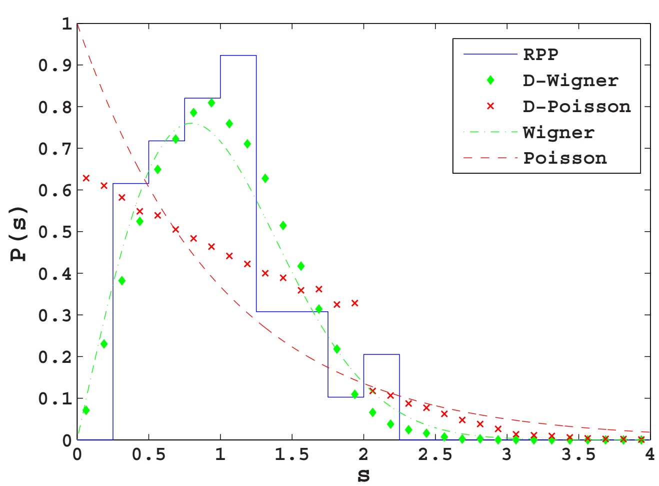

FIG. 2. (Color online) NNSD for the experimental baryon spectrum from the RPP (histogram) compared to the distorted reference distributions [Wigner, $P_{\rm DW}(s)$ (diamonds) and Poisson, $P_{\rm DP}(s)$ (crosses)] and to the theoretical distributions [Wigner surmise (dash-dotted) and Poisson distribution (dashed)].

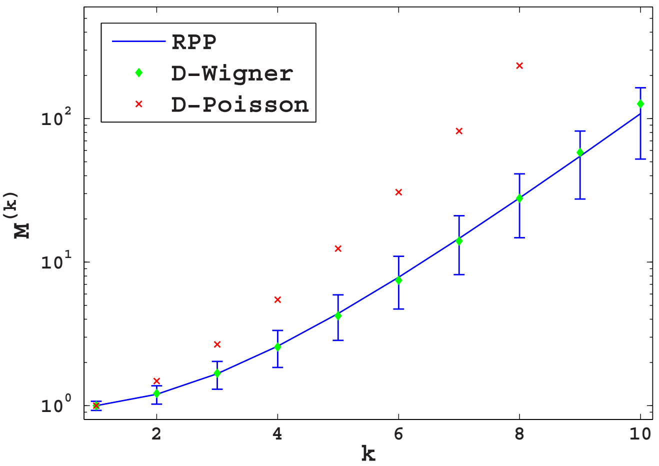

FIG. 3. (Color online) Moments of the NNSD, $M^{(k)}$, for the experimental baryon spectrum (solid line with error bars), compared to those of the distorted distributions: Wigner (diamonds) and Poisson (crosses).

Figure 3 displays the moments $M^{(k)}$, $1 \leqslant k \leqslant 10$ for the experimental spacing distribution (the error bars correspond to the standard deviation), as well as the $M^{(k)}$ corresponding to the distorted distributions $P_{\rm DW}(s)$ and $P_{\rm DP}(s)$. It is shown that the moments of the distorted Poisson distribution are outside of and far away from the error bars. Although the moments of $P_{\rm DW}(s)$ are compatible with the experimental data.

Finally, it remains for us to test if our analysis is robust against the inclusion of the error bars associated with the experimental data. In order to see what happens if we do not consider the experimental masses as exact but as randomly variable inside the interval given by the error bars, we will consider the experimental energies as Gaussian random variables with mean equal to the RPP estimation and variance equal to the corresponding error bar; we generate 1000 "realizations" of the experimental spectrum and analyze them in the same way as the original one. First, we build the NNSD and perform the K-S test for each of them. Figure 4 shows the histograms of the resulting p values for the comparison to $P_{\rm DW}(s)$ and $P_{\rm DP}(s)$. The distribution of $p_{\rm DW}$ values is narrowly concentrated in the region of high values with $\langle p_{\rm DW}\rangle = 0.81 \pm 0.08$; that is, it remains practically unchanged with respect to the $p_{\rm DW}$ obtained for the original experimental spectrum, thus indicating that the result is robust. The distribution of $p_{\rm DP}$ values is more widely spread and the mean value $\langle p_{\rm DP}\rangle = 0.53 \pm 0.13$, which is higher than the one obtained for the original spectrum but still reasonable because if the energy levels are allowed to fluctuate independently (in this case the fluctuation is induced by the error bars) the correlations are usually weakened and thus the statistics can be displaced towards Poisson.

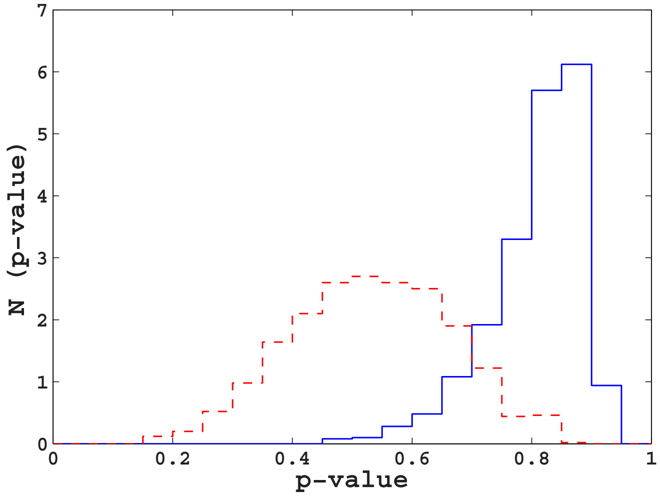

FIG. 4. (Color online) Distributions of the p values of the K-S test for the 1000 "realizations" of the experimental baryon spectrum within the error bars, for the null hypothesis that the distribution coincides with Wigner (solid histogram) and with Poisson (dashed histogram).

To summarize, our analysis is fairly robust against experimental errors and allows us to conclude that the statistical properties of the experimental spectrum are much closer to the Wigner than to the Poisson limit that is, closer to chaos than to integrability, though none of the hypotheses can be safely rejected. At this point one could think that the Poisson limit cannot be completely discarded because of the missing resonances: if we suppose that the statistical properties follow the Wigner prediction but there are missing levels in the spectrum, then it is displaced towards the Poisson prediction [35,38]. If this were the case, then the spectra from theoretical models, if they are complete, would be much closer to Wigner and more clearly incompatible with Poisson. This point is analyzed in the next section.

2. *Theoretical spectra*

Here we analyze the three theoretical spectra from quark models which were analyzed in [34], but now with comparison to these new *theoretical predictions*, the distorted distributions. In this case we do not expect the effect of distortion to be so noticeable as for the experimental spectrum, as the dimensions of the sequences are not so small. Table I displays relevant information on the experimental and the three theoretical spectra: their dimension d, the number n of pure sequences included in the analysis, and the total number of spacings, which is equal to $d - n$. We call CI the spectrum from the model by Capstick and Isgur [2], which is a relativized quark model where the interaction is built employing a one-gluon exchange potential and confinement is achieved through a spin-independent linear potential. It is the immediate and essentially unique generalization of the model by Godfrey and Isgur for mesons [18], that is, from $\overline{q}q$ to qqq. L1 and L2 are the spectra from the model by Löring *et al.* [3], which is a relativistically covariant quark model based on

TABLE I. p values of the K-S test for the experimental and the theoretical baryon spectra. The null hypotheses are that the NNSD coincides with the distorted Wigner surmise ($p_{\rm DW}$) and the distorted Poisson distribution ($p_{\rm DP}$). d stands for the dimension of the spectrum and n for the number of pure sequences.

Set	d	n	$d - n$	$p_{\rm DW}$	$p_{\rm DP}$
RPP (data)	53	14	39	0.82	0.26
CI	145	19	126	5×10^{-4}	0.24
L1	164	24	140	10^{-4}	0.22
L2	124	21	103	2×10^{-4}	0.20

the three-fermion Bethe-Salpeter equation with instantaneous two- and three-body forces (already used in [15] for mesons).

Figure 5 shows the NNSD for the three theoretical spectra, compared to the distributions $P_{\rm DW}(s)$ and $P_{\rm DP}(s)$. As expected, the distortion due to the unfolding is not so appreciable in this case, and thus the result of the analysis remains the same as in [34]. The result is also confirmed by the K-S test. In Table I the p values of the K-S test for the experimental and the theoretical spectra are shown. It is seen that all the theoretical spectra are incompatible with the Wigner distribution ($p_{\rm DW} \sim 10^{-4}$), whereas the experimental one seems to be closer to the Wigner than to the Poisson distribution.

All these results confirm those obtained in [34], giving major support to the conclusions stated there. First, theory and experiment are statistically incompatible. Second, the usual statement of missing resonances cannot account for the discrepancies. As is well known, the existence of missing levels in a spectrum deflects the statistical properties towards Poisson [35,38]. Thus, if the experimental spectrum is not complete due to missing states, it should be closer to the Poisson distribution than the theoretical ones. The situation is just the opposite. Hence, quark models, as they are presently

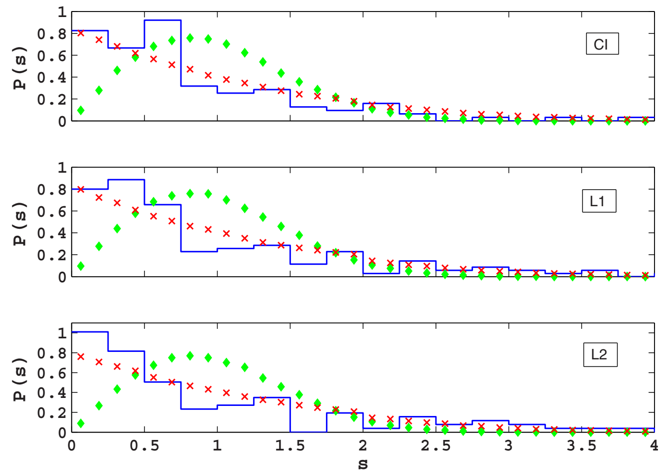

FIG. 5. (Color online) NNSD for the theoretical baryon spectra: (i) top, set CI by Capstick and Isgur [2]; (ii) middle, set L1 by Löring *et al.* [3]; (iii) bottom, set L2 by Löring *et al.* [3]. They are compared to the distorted distributions: Wigner, $P_{\rm DW}(s)$ (diamonds) and Poisson, $P_{\rm DP}(s)$ (crosses).

built, may not be suitable to reproduce the low-lying baryon spectrum, and, therefore, to predict the existence of missing resonances.

B. Mesons

1. Experimental spectrum

We have taken all the resonance states from RPP [37] up to 2.5 GeV. After splitting the spectrum into sequences with the same J, I, P and C, we have 129 levels distributed in 23 sequences.

Figure 6 shows the $P(s)$ distribution for the experimental spectrum together with the distributions $P_{\rm DW}(s)$ and $P_{\rm DP}(s)$. It seems that the statistical properties of the experimental distribution are intermediate between the Poisson and Wigner predictions, though closer to the latter. Then, we fit the experimental $P(s)$ to a Berry-Robnik distribution $P_{\rm DBR}(f,s)$ and in this case, unlike for baryons, we do obtain a best fit which is intermediate between Wigner ($f = 1$) and Poisson ($f = 0$); that is, $f = 0.78 \pm 0.03$. Figure 6 also displays $P_{\rm DBR}(0.78,s)$.

The results for the p value from the K-S test for the comparison with the three reference distributions are the following: $p_{\rm DP} = 0.13$, $p_{\rm DW} = 0.38$, and $p_{\rm DBR} = 0.65$. That is, confirming what can be seen in the figure, the statistical properties of the experimental meson spectrum are intermediate between the Poisson and Wigner limits, though they are closer to the latter since a Berry-Robnik distribution with $f = 0.78$ fits well the experimental NNSD. It is also worth noting that $p_{\rm DP} = 0.13$ is close to the usual limit for the null hypothesis to be rejected ($p \lesssim 0.10$).

Figure 7 displays the moments $M^{(k)}$ for the experimental spacing distribution (the error bars correspond to the standard deviation), and those corresponding to the distributions $P_{\rm DP}(s)$, $P_{\rm DW}(s)$, and $P_{\rm DBR}(s)$. It is shown that the moments of the distorted Poisson distribution are outside of and far away from the error bars. Those of $P_{\rm DW}(s)$ are nearer (note the logarithmic scale). Only the moments of $P_{\rm DBR}(f,s)$ with $f = 0.78$ match the experimental result supporting our choice of f.

Finally, we test if the analysis is robust against the inclusion of the error bars associated with the experimental data. As for baryons, we generate 1000 "realizations" of the experimental spectrum considering the experimental energies as Gaussian random variables with mean equal to the RPP estimation and variance equal to the corresponding error bar. In this case, we compare the random realizations of the experimental spectrum with $P_{\rm DP}(s)$ and with $P_{\rm DBR}(0.78,s)$, as we have seen that it is the distribution which better fits the experimental one. Figure 8 shows that the histograms of the resulting p values are separated with almost no overlap. The distribution of $p_{\rm DBR}$ values

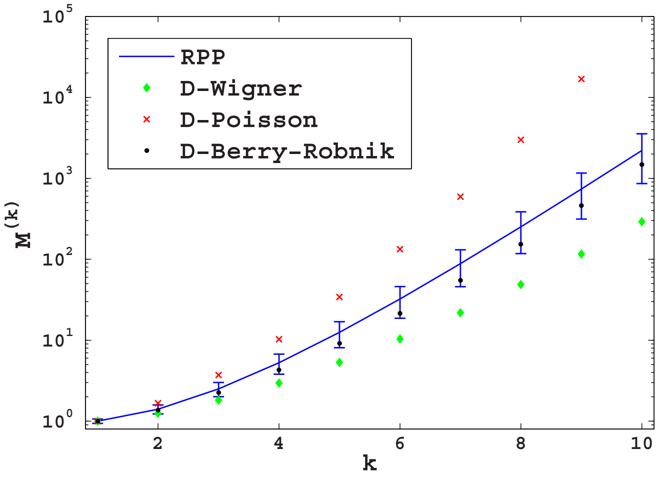

FIG. 7. (Color online) Moments of the NNSD, $M^{(k)}$, for the experimental meson spectrum (solid line with error bars), compared to those of the distorted distributions: Wigner (diamonds), Poisson (crosses), and Berry-Robnik with $f = 0.78$ (dots).

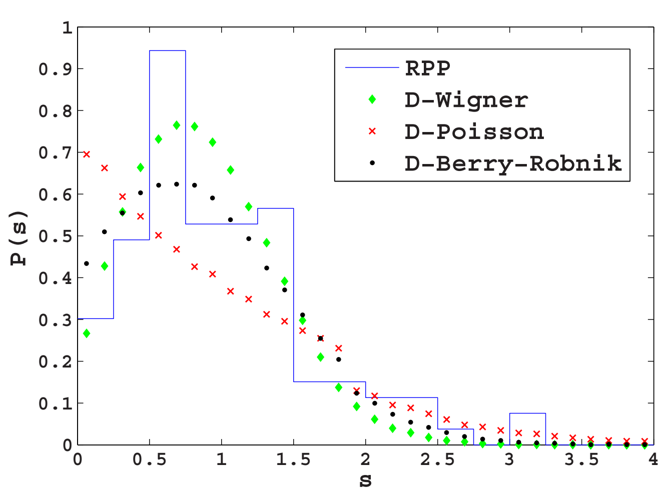

FIG. 6. (Color online) NNSD for the experimental meson spectrum from the RPP (histogram) compared to the distorted distributions: Wigner, $P_{\rm DW}(s)$ (diamonds), Poisson, $P_{\rm DP}(s)$ (crosses), and Berry-Robnik with $f = 0.78$, $P_{\rm DBR}(0.78,s)$ (dots).

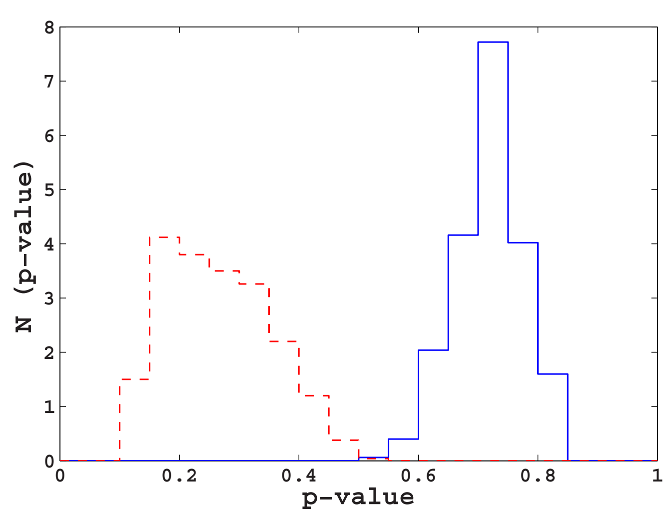

FIG. 8. (Color online) Distributions of the p values of the K-S test for the 1000 "realizations" of the experimental meson spectrum within the error bars, for the null hypothesis that the distribution coincides with Poisson (dashed histogram) and with Berry-Robnik for $f = 0.78$ (solid histogram).

is concentrated in the upper half with $\langle p_{DBR}\rangle = 0.72 \pm 0.06$, and the histogram of the p_{DP} values lies in the lower half with centroid $\langle p_{DP}\rangle = 0.27 \pm 0.09$. It is important to notice that for almost every "realization" of the experimental spectrum $p_{DBR} > p_{DP}$, sustaining the good agreement of the experiment with the Berry-Robnik distribution for $f = 0.78$. For the sake of completeness we have also used as reference distribution the Wigner surmise, obtaining $\langle p_{DW}\rangle = 0.44 \pm 0.11$.

To summarize, our analysis is fairly robust against experimental errors and allows us to conclude that the statistical properties of the experimental spectrum are intermediate between the Wigner and Poisson limits, closer to the former and safely incompatible with the latter. That is, mesons are much closer to chaotic systems than to integrable ones. Moreover, a Berry-Robnik distribution with 78% of chaos provides the best description of the experimental NNSD.

Moreover, it is noteworthy that a higher percentage of chaos is inferred from the statistical analysis of the baryon spectrum. This result is physically reasonable, as the three-quark system is more complex than the two-quark one.

2. *Theoretical spectra*

Next we analyze six theoretical calculations of the light meson spectrum and compare them to the results from previous section. These are (i) the classic model by Godfrey and Isgur (set GI) [18], which is a relativized quark model where the interaction is built employing a one-gluon exchange potential and confinement is achieved through a spin-independent linear potential; (ii) and (iii) the fully relativistic quark models by Koll *et al.* (sets K1 and K2 which correspond, respectively, to models $\mathcal{A}$ and $\mathcal{B}$ in [15]) based on the Bethe-Salpeter equation in its instantaneous approximation, a flavor dependent two-body interaction, and spin-dependent confinement force, the last being the difference between the two models; (iv) the relativistic quark model by Ebert *et al.* (set E) [17] based on a quasipotential (this calculation has the disadvantage that isoscalar and isovector mesons composed of u and d quarks are degenerate); (v) the effective quark model by Vijande *et al.* (set V) [16], based upon the effective exchange of π, σ, η and K mesons between constituent quarks; and (vi) the lattice QCD calculation by the Hadron Spectrum Collaboration at JLab (set LQCD) [20]. Lattice QCD calculation does not include strange mesons in contrast to the previous models, but it includes exotics such as the isoscalar $J^{PC} = 2^{+-}$ states, and it is computed at a high pion mass of 396 MeV.

Table II displays relevant information on the six theoretical spectra, such as their dimension d, the number n of pure sequences included in the analysis, and the total number of spacings, which is equal to $d - n$. It also provides the p values obtained by applying the K-S test to their NNSDs, taking as null hypotheses that the NNSD coincides either with $P_{DW}(s)$ or with $P_{DP}(s)$. The first relevant outcome is that, according to the K-S test, the NNSDs of sets K1, K2, E, and LQCD are incompatible with the Wigner correlations and are closer to the Poisson statistics. Thus, the dynamics predicted by these models is essentially regular, while the statistical properties of the experimental light meson spectrum show that the dynamical regime should be chaotic. This fact resembles the results obtained for baryons: while the fluctuations of the experimental baryon spectrum are well reproduced by Wigner predictions, the theoretical calculations give rise to spectra with Poisson statistics.

TABLE II. p values of the K-S test for the experimental and the theoretical meson spectra. The null hypotheses are that the NNSD coincides with the distorted Wigner surmise (p_{DW}) and the distorted Poisson distribution (p_{DP}). d stands for the dimension of the spectrum and n for the number of pure sequences.

Set	d	n	$d - n$	p_{DW}	p_{DP}
RPP (data)	129	23	106	0.38	0.13
GI	68	17	51	0.84	0.41
K1	162	38	124	0.038	0.55
K2	162	38	124	0.005	0.43
E	190	34	156	0.083	0.21
V	94	18	76	0.51	0.56
LQCD	60	15	45	0.033	0.44

Figure 9 shows the NNSD of the six theoretical spectra. Sets K1 and E provide flat NNSDs with a cut at $s = 2$. The cut is expected as was explained in Sec. II. When the Poisson distribution is distorted it flattens due to the small amount of levels, so actually the NNSDs that we find for sets K1 and E are the ones we expect from a Poisson distribution, confirming that these sets have less correlations than the experimental data as the K-S test suggests. The comparison between models K1 and K2 by Koll *et al.* is particularly interesting because they only differ on the confinement interaction and show how important that interaction can be for the spectral statistics, hinting that it should be revised to obtain a better agreement with the experiment.

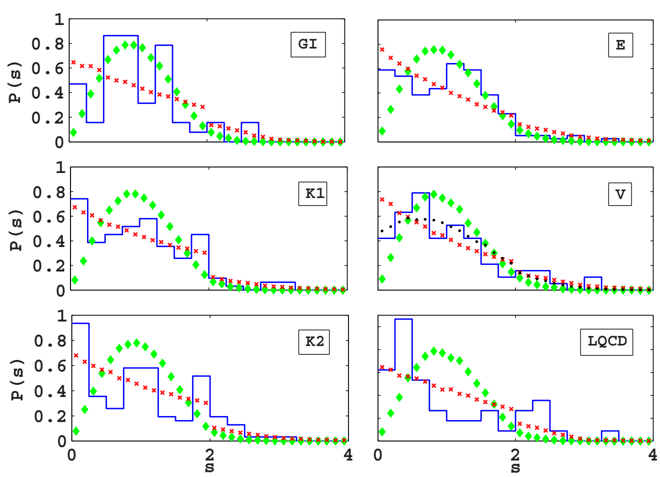

FIG. 9. (Color online) NNSDs for the theoretical meson spectra: (i) Top left, set GI by Godfrey and Isgur [18]; (ii) middle left, set K1 by Koll *et al.* [15]; (iii) bottom left, set K2 by Koll *et al.* [15]; (iv) top right, set E by Ebert *et al.* [17]; (v) middle right, set V by Vijande *et al.* [16] and *ad hoc* distorted Berry-Robnik with $f = 0.63$; (vi) bottom right, set LQCD by Dudek *et al.* [20]. Distorted Wigner, $P_{DW}(s)$, is represented with diamonds; distorted Poisson, $P_{DP}(s)$, with crosses; and distorted Berry-Robnik, $P_{DBR}(f,s)$, with dots.

The result for set LQCD is particularly interesting because lattice QCD is currently the only tool available to compute low-energy observables employing QCD directly. We find that the current state-of-the-art calculation in [20] does not describe properly the statistical properties of the meson spectrum. Lattice QCD NNSD is relatively close to the $P_{\mathrm{DP}}(s)$ as shown in Fig. 9. This is evident at zero spacing where $P_{LQCD}(s=0) \approx 0.6$, thus implying uncorrelated levels, as explained in Sec. III A 1. Our results remain unaltered if the statistical errors of the lattice QCD calculation are taken into account. Thus, the LQCD calculation should be considered a step forward in lattice calculations but still far away from being a description of the data or their structure. This is not unexpected given that the LQCD set has been obtained at a pion mass of 396 MeV, far away from its physical mass, and it is reasonable to expect a drastic change in the statistical properties when calculations get the pion mass closer to its actual value. However, the fact that the lattice QCD calculation has a lot less correlations than the experimental data, being practically uncorrelated, demands, besides the need to bring the calculation closer to the physical pion mass, a careful examination of the approximations employed.

The results of the K-S test for sets GI and V are inconclusive because they suggest that the models are compatible with both chaotic and integrable dynamics. It is thus mandatory to take a close look to the NNSDs (Fig. 9) before obtaining any conclusion. Set GI NNSD has a strange shape with peaks and dips, completely different from any of the usual distributions (Poisson, Wigner, or Berry-Robnik), and therefore is not close at all to experiment (see Fig. 6). It only has some similarity with some very particular integrable systems whose $P(s)$ is equal to a sum of δ functions, constituting an exception to the rule of Poisson distribution [43]. In contrast, set V displays a smooth NNSD, which can be very well fitted to a distorted Berry-Robnik distribution with $f = 0.63 \pm 0.19$ (also displayed in Fig. 9). Then we can conclude that the model by Vijande *et al.* gives a better account of the dynamical regime of the light meson spectrum.

IV. CONCLUSIONS

In this paper we have analyzed the spectral fluctuations of the experimental and theoretical baryon and meson mass spectra in the context of quantum chaos. Comparing the statistical properties of the spectra with random matrix theory (RMT) predictions is a tool to determine the dynamical regime of the system; that is, whether the system is chaotic, regular, or intermediate. We emphasize that, besides the coincidence of the theoretical individual energies with the experimental ones, the agreement in the statistical fluctuation properties is also important, since they determine the dynamical regime and thus can provide insight on the properties of the underlying interactions.

The statistical analysis is described stressing the fact that one has to be very careful when dealing with spectra like those of the baryon and meson masses, which must be divided into very short sequences to perform the analysis, according to symmetry classes (sequences with the same quantum numbers). We show a new method to take into account this problem. It consists of building *distorted theoretical distributions* adapted to compare with the spectrum under study. The distortion is induced by the local unfolding procedure on the actual theoretical predictions exactly in the same way as it is induced on the spectrum under study. For very short sequences, the distorted theoretical predictions are more reliable than comparing directly with the RMT predictions.

Once this analysis is carried out, we obtain that both experimental baryon and meson spectra are closer to a chaotic behavior than to an integrable one. The baryon spectrum seems to be more chaotic than the meson one, a result that is physically reasonable because a three-particle system is more complex than a two-particle one. The best description of the nearest-neighbor spacing distribution (NNSD) for baryons is provided by the Wigner surmise, which is the prediction for chaotic systems, whereas the best description of the NNSD for mesons is given by the intermediate Berry-Robnik distribution with a 78% of chaos.

We have also tested the robustness of the analysis against the inclusion of the experimental errors, by performing the Kolmogorov-Smirnov (K-S) test on an ensemble of spectra generated by considering the experimental masses as Gaussian random variables with a mean given by the Review of Particle Physics (RPP) estimation and variance equal to the corresponding error bar. In both cases, baryon and meson mass spectra, the result shows that our analysis is fairly robust against experimental errors.

As for the theoretical baryon spectra, the three spectra from quark models which have been analyzed are clearly statistically incompatible with the experimental one. From the K-S test, the hypothesis that the NNSD from quark models coincides with the Wigner surmise can be rejected whereas the experimental NNSD is clearly closer to the Wigner surmise than to the Poisson one. Moreover, this discrepancy cannot be accounted for by the existence of the missing resonances. It is well known that the lack of levels in a spectrum deflects the statistical properties towards Poisson. Thus, if this were the origin of the discrepancy, the experimental spectrum should be closer to Poisson statistics than the theoretical ones, but the situation is just the opposite. Hence, having present the importance of the agreement in the statistical spectral properties and its relation with the dynamical regime of the system, one can only state that quark models, as they are presently built, may not be suitable to reproduce the low-lying spectrum, and, therefore, to predict the existence of missing resonances.

In the case of mesons, of the six theoretical spectra which have been analyzed only the one by Vijande *et al.* [16] seems to be compatible with the experimental one, with a NNSD well fitted with the intermediate Berry-Robnik distribution with a 63% of chaos. The other theoretical models, including the lattice QCD (LQCD) calculation, predict a regular or nearly regular dynamics in clear contradiction with the experiment. The disagreement with the LQCD spectrum is especially shocking as LQCD is currently the only tool available to compute low energy observables employing QCD directly. Thus, we find that the current state-of-the-art calculation in [20] does not describe properly the statistical properties of the meson spectrum. The failure could be due to the fact that the calculation is made at an unrealistic pion mass. For the quark models, further work is needed to study the origin of the discrepancy with the experimental spectrum. In particular, it would be interesting to study the differences of the model by Vijande *et al.* with the other quark models, trying to find the signals of chaos.

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