Asymptotic normalization coefficients and radiative widths

A. M. Mukhamedzhanov
Cyclotron Institute, Texas A&M University, College Station, Texas 77843, USA

D. Y. Pang
School of Physics and Nuclear Energy Engineering, Beihang University, Beijing 100191, China and Cyclotron Institute, Texas A&M University, College Station, Texas 77843, USA

(Received 19 January 2015; revised manuscript received 24 May 2015; published 28 July 2015)

The asymptotic normalization coefficient (ANC) is an important quantity in the calculation of radiative width amplitudes, providing limits on the radiative width. Here we present some examples showing the connection between the ANC and radiative width. In particular, the radiative width of the $E1$ transition ^{17}F($1/2^-$, E_x = 3.104 MeV) to ^{17}F($1/2^+$, E_x = 0.495 MeV) reported by Rolfs [Nucl. Phys. A **217**, 29 (1973)] is $(1.2 \pm 0.2) \times 10^{-2}$ eV. Meanwhile the ANC for the first excited state in ^{17}F puts a lower limit on the radiative width, which is $(3.4 \pm 0.50) \times 10^{-2}$ eV. Such a strong disagreement between the measured radiative width and the lower limit imposed by the ANC calls for a new measurement of this radiative width. Other examples are also considered.

DOI: 10.1103/PhysRevC.92.014625 PACS number(s): 21.10.Jx, 25.40.Ny, 23.20.Lv, 27.20.+n

I. INTRODUCTION

The asymptotic normalization coefficient (ANC) plays an important role in nuclear reaction and structure theory and in nuclear astrophysics [1–4]. By definition, the ANC is the amplitude of the tail of the overlap function. The matrix elements determining the amplitudes of the transfer reactions and radiative capture amplitudes contain the overlap functions. For peripheral processes the overlap functions can be replaced by their asymptotic terms. Hence the amplitudes of the peripheral processes are proportional to the corresponding ANCs. That is why the ANC determines the overall normalization of the peripheral process amplitude [2,5]. It is evident that the ANC plays a crucial role in the determination of other nuclear characteristics; to calculate them we need to know the tail of the overlap function. One of these quantities is the radiative width of a resonance decaying to a bound state. In this paper we reveal a role of the ANC in determination of the resonance radiative widths. In particular, we show that the ANC can put a limitation on the value of the radiative width.

For the analysis we use the R-matrix method in which the radiative width amplitude is expressed in terms of the difference of the internal and external (channel) parts. The internal radiative width amplitude is strongly model dependent, and usually in the R-matrix is used as a fitting parameter. In contrast, the channel radiative width amplitude can be expressed in terms of the two observable quantities: the partial resonance width and the ANC of the final bound state to which the resonance decays. It depends only on one model parameter: the R-matrix channel radius. Hence the channel radiative width amplitude can be calculated quite accurately with the well identified uncertainty due to the uncertainties of the partial resonance width, the ANC, and ambiguity in the choice of the channel radius. The calculated channel radiative width amplitude can provide limitations on the total radiative width. It is the goal of this paper to demonstrate this. We provide all the necessary equations and present five examples. Throughout the paper we use the system of units in which $\hbar = c = 1$.

II. RESONANCE RADIATIVE CAPTURE IN THE R-MATRIX APPROACH

To obtain the expression for the radiative width for the resonance decay to a bound state we consider the radiative capture amplitude of the process $a + A \to F^* \to F + \gamma$ proceeding through the intermediate resonance F^* to the final bound state $F = (a\,A)$. The reaction amplitude describing this process is given by

$$M_{\rm rad} = \sqrt{\frac{8\,\pi\,(L+1)}{L}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!}\,\frac{1}{\sqrt{2\,J_f+1}}\,\langle \varphi_F|\hat{A}_L|\Psi_i^{(+)}\rangle. \tag{1}$$

Here $\Psi_i^{(+)}$ is the scattering wave function of the system $a + A$ in the channel where the resonance F^* occurs, φ_F is the bound state wave function of the state $F = (a\,A)$ to which the resonance F^* decays, $\hat{A}_L$ is the electromagnetic transition operator of the multipolarity L, k_γ is the momentum of the emitted photon, and J_f is the spin of the final bound state $F = (a\,A)$. At very low energies we can use the long-wavelength approximation $1/k \gg R$, where R is the size of the resonance system $a + A$, and k is the $a - A$ relative momentum in the initial state. Hence, at this stage we can neglect the internal degrees of freedom of the individual nuclei a and A and can treat them as constituent structureless particles. Then

$$\Psi_i^{(+)} = \varphi_a\,\varphi_A\,\Psi^{(+)}. \tag{2}$$

Here, φ_i is the bound-state wave function of nucleus i, and $\Psi^{(+)}$ is the scattering wave function of the structureless particles a and A. Then the reaction amplitude for the radiative capture to the bound state in the long-wavelength approximation reduces to

$$M_{\rm rad} = \sqrt{\frac{8\,\pi\,(L+1)}{L}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!}\,\frac{1}{\sqrt{2\,J_f+1}}\,\langle I_{aA}^F|\hat{A}_L|\Psi^{(+)}\rangle. \tag{3}$$

Here $I_{aA}^{F} = N\,\langle \varphi_a\, \varphi_A | \varphi_F \rangle$ is the overlap function of the bound-state wave functions of nuclei F, A, and a. Here we recovered the antisymmetrization between the nucleons of a and A what leads to the appearance of the antisymmetrization factor N. In the isospin formalism $N = \frac{(a+A)!}{a!\,A!}$. The integration in the matrix element determining the overlap function is carried over the internal coordinates of nuclei a and A.

The wave function $\Psi^{(+)}$ can be written as the sum of the internal and external components:

$$\Psi^{(+)} = \Psi^{+}_{(\rm int)} + \Psi^{(+)}_{(\rm ext)}. \tag{4}$$

This splitting of the scattering wave function into the internal and external components can be done naturally in the R-matrix approach, which we are going to use in what follows. The internal radial scattering wave function ($r \leqslant r_0$, r is the distance between the particles a and A, r_0 is the channel radius) in the single-channel R-matrix approach in the case of an isolated narrow resonance takes the form [3,6]

$$\Psi^{(+)}_{(\rm int)}(k,r) = i\,\frac{1}{k\,r}\,e^{-i\,\delta^{hs}_{l_i}}\,\frac{\left[\Gamma_{J_i}\right]^{1/2}}{E_R - E - i\,\Gamma_{J_i}/2}\,X_{\rm int}(r)\Big|_{r\leqslant r_0}, \tag{5}$$

where $X_{\rm int}$ is the real eigenstate of the level closest to the resonance, Γ_{J_i} is the partial width of the resonance with the spin J_i for the decay to the channel $a + A$, E_R is the real part of the complex resonance energy $E_R - i\,\Gamma_{J_i}/2$, E is the $a - A$ relative kinetic energy, and $\delta^{hs}_{l_i}$ is the hard-sphere scattering phase shift in the partial wave l_i. At the moment we do not show all the quantum numbers characterizing the resonance and the bound state but we will recover them later. Note that in the case of the resonance scattering the resonance behavior (5) of the internal wave function can be obtained in the potential model from quite general considerations (see Chap. VII of [7]).

In the external region $\Psi^{(+)}$ is

$$\Psi^{(+)}_{(\rm ext)}(k,r) = \sqrt{\frac{1}{v}}\,\frac{1}{k\,r}\left[I_{l_i}(k,r) - S\,O_{l_i}(k,r)\right]\Big|_{r>r_0}. \tag{6}$$

Here, $I_{l_i}(k,r)$ and $O_{l_i}(k,r)$ are the incoming and outgoing spherical waves, $v = k/\mu$ is the $a - A$ relative velocity, and μ is their reduced mass. S is the elastic scattering S-matrix element, which in the vicinity of an isolated resonance takes the form

$$S = e^{-2\,i\,\delta^{hs}_{l_i}}\left(1 + \frac{i\,\Gamma_{J_i}}{E_R - E - i\,\frac{\Gamma_{J_i}}{2}}\right). \tag{7}$$

The nonresonant scattering phase δ^{hs} in the R-matrix approach is the hard-sphere scattering phase shift determined as

$$e^{-i\,2\,\delta^{hs}_{l_i}} = \frac{I_{l_i}(k,r_0)}{O_{l_i}(k,r_0)}. \tag{8}$$

Equating the internal and external wave functions at $r = r_0$ and $E = E_R$ we get

$$X_{\rm int}(r_0) = -\sqrt{2\,\mu\,r_0}\,\gamma_{J_i} \tag{9}$$

with the dimension fm$^{-1/2}$. Here, γ_{J_i} is the reduced width amplitude of the resonance level, which is related to the partial resonance width as

$$\Gamma_{J_i} = 2\,P_{l_i}(k,r_0)\,\gamma^2_{J_i}. \tag{10}$$

$P_{l_i}(k,r_0)$ is the barrier penetrability at the orbital angular momentum l_i at which the resonance occurs:

$$P_{l_i}(k,r_0) = \frac{k\,r_0}{\sqrt{F^2_{l_i}(k,r_0) + G^2_{l_i}(k,r_0)}}. \tag{11}$$

F_{l_i} and G_{l_i} are the regular and singular Coulomb solutions.

Following the split of the initial scattering wave function into the internal and external parts, we can rewrite the radiative capture amplitude as the sum of the internal and external parts:

$$M_{\rm rad} = M_{\rm rad(int)} + M_{\rm rad(ext)}, \tag{12}$$

where the internal and external radiative capture amplitudes are given by

$$\begin{aligned} M_{\rm rad(int)} &= \sqrt{\frac{8\,\pi\,(L+1)}{L}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!} \\ &\quad\times \frac{1}{\sqrt{2\,J_f+1}}\left\langle I^F_{aA}\middle|\hat{A}_L\middle|\Psi^{(+)}_{(\rm int)}\right\rangle\Big|_{r\leqslant r_0} \\ &= i\,e^{-i\,\delta^{hs}_{l_i}}\,\frac{\left[\Gamma_{J_i}\right]^{1/2}\gamma^{J_i}_{\gamma\,J_f}({\rm int})}{E_R - E - i\,\Gamma_{J_i}/2} \end{aligned} \tag{13}$$

and

$$\begin{aligned} M_{\rm rad(ext)} &= \sqrt{\frac{8\,\pi\,(L+1)}{L}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!} \\ &\quad\times \frac{1}{\sqrt{2\,J_f+1}}\left\langle I^F_{aA}\middle|\hat{A}_L\middle|\Psi^{(+)}_{(\rm ext)}\right\rangle\Big|_{r>r_0} \\ &= -i\,e^{-i\,\delta^{hs}_{l_i}}\,\frac{\left[\Gamma_{J_i}\right]^{1/2}\gamma^{J_i}_{\gamma\,J_f}(ch)}{E_R - E - i\,\Gamma_{J_i}/2} + M_{{\rm rad}(nr)}, \end{aligned} \tag{14}$$

correspondingly.

Here, $M_{{\rm rad}(nr)}$ is the external part of the nonresonant (direct) radiative capture amplitude. Note that in the R-matrix approach the internal nonresonant radiative capture amplitude is absorbed into the internal resonant radiative capture amplitude $M_{\rm rad(int)}$. The internal radiative width amplitude for the decay of the resonance with spin J_i to the bound state with spin J_f is

$$\begin{aligned} \gamma^{J_i}_{\gamma\,J_f}({\rm int}) &= \sqrt{\frac{8\,\pi\,(L+1)}{L}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!\,\sqrt{2J_F+1}} \\ &\quad\times \left\langle I^F_{aA}(r)\middle|\hat{A}_L(r)\middle|X_{\rm int}(r)\right\rangle\Big|_{r\leqslant r_0}. \end{aligned} \tag{15}$$

Note that the internal radiative width $\gamma^{J_i}_{\gamma\,J_f}({\rm int})$ is real because $X_{\rm int}$ is a real eigenfunction.

The external (channel) radiative width amplitude is given by [6]

$$\gamma_{\gamma\,J_f}^{J_i}(ch) = \sqrt{\frac{8\,\pi(L+1)}{L\,v}}\,\frac{k_\gamma^{L+1/2}}{(2\,L+1)!!\,\sqrt{2\,J_f+1}} \times e^{-i\,\delta_{l_i}^{hs}}\left[\Gamma_{J_i}\right]^{1/2}\left\langle I_{aA}^{F}(r)\middle|\hat{A}_L(r)\middle|O_{l_i}(r)\right\rangle\Big|_{r>r_0}. \tag{16}$$

The channel radiative width amplitude $\gamma_{\gamma\,J_f}^{J_i}(ch)$, in contrast to the internal radiative width, is complex because it contains the product of the complex functions $e^{-i\,\delta_{l_i}^{hs}}\,O_{l_i}$.

The total radiative width amplitude can be expressed in terms of the internal, $\gamma_{\gamma\,J_f}^{J_i}(\text{int})$, and external (channel), $\gamma_{\gamma\,J_f}^{J_i}(ch)$, radiative width amplitudes:

$$\gamma_{\gamma\,J_f}^{J_i} = \gamma_{\gamma\,J_f}^{J_i}(\text{int}) - \gamma_{\gamma\,J_f}^{J_i}(ch). \tag{17}$$

Then the total radiative width for the resonance decay to the bound state is related to the radiative width amplitude as

$$\Gamma_{\gamma\,J_f}^{J_i} = \left|\gamma_{\gamma\,J_f}^{J_i}\right|^2 = \left|\gamma_{\gamma\,J_f}^{J_i}(\text{int}) - \gamma_{\gamma\,J_f}^{J_i}(ch)\right|^2. \tag{18}$$

In the external region the radial overlap function is given by (we recover here all the quantum numbers characterizing it)

$$I_{aA\,l_f\sigma\,J_f}(r) = C_{l_f\sigma\,J_f}\,W_{-\eta_f^{bs},\,l_f+1/2}(2\,\kappa\,r), \tag{19}$$

where $C_{l_f\sigma\,J_f}$ is the ANC of the virtual decay $F \to a + A$, $W_{-\eta_f^{bs},\,l_f+1/2}(2\,\kappa\,r)$ is the Whittaker function, $\kappa = \sqrt{2\,\mu\,\varepsilon}$, η_f^{bs} and ε are the wave number, Coulomb parameter, and binding energy of the bound state $F = (a\,A)$, correspondingly, l_f is the $a - A$ relative orbital angular momentum in the bound state, and σ is the channel spin. Hence the channel radiative width amplitude is proportional to the ANC of the final bound state.

In the R-matrix approach the channel radiative width amplitude (in $\text{MeV}^{1/2}$) reduces to [8]

$$\gamma_{\gamma J_f}^{J_i}(ch) = \sqrt{\frac{\lambda_N\,m_u}{137\,E}}\,(\mu)^{L+1/2}\left(\frac{Z_a\,e}{m_a^L} + (-1)^L\,\frac{Z_A\,e}{m_A^L}\right)\sqrt{\frac{(L+1)(2L+1)}{L}}\,\frac{1}{(2L+1)!!} \times (k_\gamma\,r_0)^{L+1/2}\,C_{l_f\sigma\,J_f}\,\sqrt{\Gamma_{l_i\sigma\,J_i}}\,\sqrt{P_{l_i}(E,r_0)}\left(\left[F_{l_i}(k,r_0)\right]^2 + \left[G_{l_i}(k,r_0)\right]^2\right) \times W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r_0)\,\langle l_i 0\,L0|l_f 0\rangle\,U(L\,l_f\,J_i\,\sigma;\,l_i\,J_f) \times \left(J_L^{(1)}(l_i,l_f) + i\,J_L^{(2)}(l_i,l_f)\frac{F_{l_i}(k,r_0)\,G_{l_i}(k,r_0)}{\left[F_{l_i}(k,r_0)\right]^2 + \left[G_{l_i}(k,r_0)\right]^2}\right), \tag{20}$$

$$J_L^{(1)}(l_i\,l_f) = \frac{\left[F_{l_i}(k,r_0)\right]^2}{k\,r_0}\,P_{l_i}(E,r_0)\frac{1}{r_0^{L+1}}\int_{r_0}^{\infty} dr\,r^L\,\frac{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r)}{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r_0)}\,\frac{F_{l_i}(k,r)}{F_{l_i}(k,r_0)} + \frac{\left[G_{l_i}(k,r_0)\right]^2}{k\,r_0}\,P_{l_i}(E,r_0)\frac{1}{r_0^{L+1}}\int_{r_0}^{\infty} dr\,r^L\,\frac{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r)}{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r_0)}\,\frac{G_{l_i}(k,r)}{G_{l_i}(k,r_0)}, \tag{21}$$

$$J_L^{(2)}(l_i\,l_f) = \frac{1}{r_0^{L+1}}\int_{r_0}^{\infty} dr\,r^L\,\frac{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r)}{W_{-\eta_f^{bs},l_f+1/2}(2\,\kappa\,r_0)}\left[\frac{F_{l_i}(k,r)}{F_{l_i}(k,r_0)} - \frac{G_{l_i}(k,r)}{G_{l_i}(k,r_0)}\right]. \tag{22}$$

Here, $\langle l_i 0\,L0|l_f 0\rangle$ is the Clebsch-Gordan coefficient, $U(L\,l_f\,J_i\,\sigma;\,l_i\,J_f)$ is the Racah coefficient, $F_{l_i}(k,r)$ and $G_{l_i}(k,r)$ are the regular and singular (at the origin) solutions of the radial Schrödinger equation with pure Coulomb potentials, E and k are the relative kinetic energy (in MeV) and the relative momentum (in fm^{-1}) of the particles a and A in the continuum, and m_i and $Z_i\,e$ are the mass and charge of particle i, $m_u = 931.5$ MeV. From now on we show all the necessary quantum numbers which should be assigned to the partial resonance width.

Note that the dependence of the channel reduced width amplitude on the channel radius is its only model dependence, while to calculate the internal radiative width amplitude a microscopic approach is required. In the R-matrix method the internal radiative width amplitude is a fitting parameter.

We can rewrite the total radiative width amplitude as

$$\begin{aligned}\gamma_{\gamma\,J_f}^{J_i} &= \gamma_{\gamma\,J_f}^{J_i}(\text{int}) - \gamma_{\gamma\,J_f}^{J_i}(ch)\\ &= \left(\gamma_{\gamma\,J_f}^{J_i}(\text{int}) - \text{Re}\left[\gamma_{\gamma\,J_f}^{J_i}(ch)\right]\right) - i\,\text{Im}\left[\gamma_{\gamma\,J_f}^{J_i}(ch)\right].\end{aligned} \tag{23}$$

Then

$$\begin{aligned}\Gamma_{\gamma\,J_f}^{J_i} &= \left|\gamma_{\gamma J_f}^{J_i}(\text{int}) - \gamma_{\gamma J_f}^{J_i}(ch)\right|^2\\ &= \left(\gamma_{\gamma\,J_f}^{J_i}(\text{int}) - \text{Re}\left[\gamma_{\gamma\,J_f}^{J_i}(ch)\right]\right)^2 + \left(\text{Im}\left[\gamma_{\gamma\,J_f}^{J_i}(ch)\right]\right)^2.\end{aligned} \tag{24}$$

Note that the relative phase of $\gamma_{\gamma J_f}^{J_i}(\text{int})$ and the real part of the channel radiative width amplitude $\text{Re}[\gamma_{\gamma J_f}^{J_i}(ch)]$ is, *a priori*, unknown, so these real parts may interfere either constructively or destructively. Hence, $(\text{Im}[\gamma_{\gamma J_f}^{J_i}(ch)])^2$ always provides a lower limit for the radiative width:

$$\Gamma_{\gamma\, J_f}^{J_i} \geqslant \left(\mathrm{Im}\left[\gamma_{\gamma\, J_f}^{J_i}(ch)\right]\right)^2. \tag{25}$$

Additional stronger limits may be obtained if assumptions are made about the interference between the two real contributions. For the constructive interference of the real parts (the signs of $\gamma_{\gamma\, J_f}^{J_i}(\mathrm{int})$ and $\mathrm{Re}[\gamma_{\gamma\, J_f}^{J_i}(ch)]$ are opposite) the channel contribution gives a stronger lower limit because in this case

$$\Gamma_{\gamma\, J_f}^{J_i} \geqslant \Gamma_{\gamma\, J_f}^{J_i}(ch). \tag{26}$$

In the case of the destructive interference of the real parts and $|\mathrm{Re}[\gamma_{\gamma J_f}^{J_i}(ch)]| > |\gamma_{\gamma J_f}^{J_i}(\mathrm{int})|$, the channel contribution gives an upper limit for the radiative width:

$$\Gamma_{\gamma\, J_f}^{J_i} \leqslant \Gamma_{\gamma\, J_f}^{J_i}(ch). \tag{27}$$

Assuming that the experimental radiative width, the ANC of the bound state, and the resonance width are known, we can determine the internal radiative width amplitude

$$\gamma_{\gamma J_f}^{J_i}(\mathrm{int}) = \pm\sqrt{\Gamma_{\gamma\, J_f}^{J_i} - \left(\mathrm{Im}\left[\gamma_{\gamma J_f}^{J_i}(ch)\right]\right)^2} + \mathrm{Re}\left[\gamma_{\gamma J_f}^{J_i}(ch)\right]. \tag{28}$$

Then the internal radiative width takes the form

$$\begin{aligned}\Gamma_{\gamma\, J_f}^{J_i}(\mathrm{int}) &= \left(\gamma_{\gamma\, J_f}^{J_i}(\mathrm{int})\right)^2 \\ &= \left(\pm\sqrt{\Gamma_{\gamma\, J_f}^{J_i} - \left(\mathrm{Im}\left[\gamma_{\gamma J_f}^{J_i}(ch)\right]\right)^2} + \mathrm{Re}\left[\gamma_{\gamma J_f}^{J_i}(ch)\right]\right)^2.\end{aligned} \tag{29}$$

III. EXAMPLES

In this section we consider five different examples. The analysis is done within the framework of the R-matrix method in which the channel radius is a model parameter. If experimental data are available the channel radius can be determined from the fitting of the calculated quantity to the experimental one. Otherwise there are no specific prescriptions how to choose the channel radius, which brings additional uncertainty into quantities calculated using the R-matrix approach. One of the possible recipes is to take the channel radius as the minimal distance between the interacting nuclei at which the polarizing force acting on each nucleus from the other one can be neglected [9]. Often the channel radius is determined as $r_0 = 1.4(a^{1/3} + A^{1/3})$. In what follows we also use this prescription for four cases, except for the last one for which the experimental data are available and the channel radius is determined from the fitting of the calculated astrophysical factor to the experimental one.

1. Transition $^{17}\mathrm{F}(1/2^-, E_x = 3.104\ \mathrm{MeV}) \to {}^{17}\mathrm{F}(1/2^+, E_x = 0.495\ \mathrm{MeV})$. As the first example we consider the decay of the first resonance $1/2^-$ in ^{17}F to the first excited state $1/2^+$. The first example is chosen because it presents the most striking example when the measured radiative width is significantly lower then the lower limit of the radiative width given by Eq. (25).

The proton resonance width is 19 ± 1 keV [10]. The square of the proton ANC of the first excited state measured in [11] is $6490 \pm 680\ \mathrm{fm}^{-1}$. The radiative resonance width for this transition is $\Gamma_{\gamma\,1/2}^{1/2} = (1.2 \pm 0.2) \times 10^{-2}$ eV [12]. Meanwhile, using Eq. (20) we get that at $r_0 = 4.9$ fm $\gamma_{\gamma\,1/2}^{1/2}(ch) = -0.2 - i0.177\ \mathrm{eV}^{1/2}$. Then it appears that the minimum of the radiative width, which is based on the measured proton resonance width and ANC, significantly exceeds the radiative width measured in [12]: $\min\Gamma_{\gamma\,1/2}^{1/2} = (\mathrm{Im}[\gamma_{\gamma\,1/2}^{1/2}(ch)])^2 = (3.1 \pm 0.5) \times 10^{-2}\ \mathrm{eV} > \Gamma_{\gamma\,1/2}^{1/2} = (1.2 \pm 0.2) \times 10^{-2}$ eV. To get the uncertainty of the calculated $(\mathrm{Im}[\gamma_{\gamma\,1/2}^{1/2}(ch)])^2$ we took into account 10% uncertainty of the experimental ANC, 5% uncertainty in the proton resonance width, and 10% uncertainty due to the dependence of $(\mathrm{Im}[\gamma_{\gamma\,1/2}^{1/2}(ch)])^2$ on the channel radius, which gives 15% of the total uncertainty in the determined $(\mathrm{Im}[\gamma_{\gamma\,1/2}^{1/2}(ch)])^2$. Thus we question the experimental result for $\Gamma_{\gamma\,1/2}^{1/2} = (1.2 \pm 0.2) \times 10^{-2}\ \mathrm{eV}^{-1}$ [12] and we think that it should be remeasured.

2. Transition $^{17}\mathrm{F}(5/2^-, E_x = 3.857\ \mathrm{MeV}) \to {}^{17}\mathrm{F}(5/2^+, E_x = 0.0\ \mathrm{MeV})$. As the second example we consider the decay of the second resonance $^{17}\mathrm{F}(5/2^-)$ to the ground state $^{17}\mathrm{F}(5/2^+)$. This example is chosen to demonstrate the case in which the measured radiative width exceeds, as it supposed to be, the lower limit given by Eq. (25). The proton resonance width is <1.5 keV [13]. The square of the proton ANC of the ground ^{17}F state measured in [11] is $1.08 \pm 0.1\ \mathrm{fm}^{-1}$. The radiative resonance width for this transition is $\Gamma_{\gamma\,5/2}^{5/2} = (0.11 \pm 0.02)$ eV [13]. Calculated at $r_0 = 4.9$ fm, our channel radiative width amplitude is $\gamma_{\gamma\,5/2}^{5/2}(ch) = -0.049 - i0.0062\ \mathrm{eV}^{1/2}$. Note that 10% variation of the channel radius leads to only 2% of the $\mathrm{Im}[\gamma_{\gamma\,5/2}^{5/2}(ch)]$. The calculated lower limit of $\Gamma_{\gamma\,5/2}^{5/2}$ is $(\mathrm{Im}[\gamma_{\gamma\,5/2}^{5/2}(ch)])^2 = (3.8 \pm 0.4) \times 10^{-5}$ eV, which is significantly smaller than the measured $\Gamma_{\gamma\,5/2}^{5/2}$. From Eq. (28) we get for the internal radiative width amplitude two values: $\gamma_{\gamma\,5/2}^{5/2}(\mathrm{int}) = -0.38\ \mathrm{eV}^{1/2}$ and $0.28\ \mathrm{eV}^{1/2}$. In the case under consideration the ANC is very small compared to the previous example. That is why the modulus of the channel radiative width is much smaller than that of the internal width and we cannot establish any stronger limits on the radiative width.

3. Transition $^{12}\mathrm{N}(2^-, E_x = 1.191\ \mathrm{MeV}) \to {}^{12}\mathrm{N}(1^+, E_x = 0.0\ \mathrm{MeV})$. As the third example we consider the decay of the second resonance of ^{12}N at 0.576 MeV, $l_i = 0$ and the spin of the resonance $J_i = 2$ to the ground state of ^{12}N with the binding energy 0.6 MeV, $l_f = 1$, and $J_f = 1$. This case is selected because the reaction $^{11}\mathrm{C}(p,\gamma)^{12}\mathrm{N}$ is an important branching point in the alternative path from the slow 3α process to produce CNO seed nuclei [14,15]. This reaction is contributed by the direct and two resonant captures (through the first and second resonances in ^{12}N [15]). The radiative width of the second resonance was a controversial subject theoretically. The proton resonance width of the second resonance is 51 ± 20 keV [16]. The square of the proton ANC of the ground ^{12}N state measured in [15] is $1.73 \pm 0.25\ \mathrm{fm}^{-1}$. The latest measured radiative resonance width for this transition is $\Gamma_{\gamma\,1}^{2} = 13 \pm 0.5$ meV [17]. The previous GANIL measurement $\Gamma_{\gamma\,1}^{2} = 6_{3.5}^{+7}$ meV had a too-large uncertainty [18].

The calculated channel radiative width amplitude at the channel radius $r_0 = 4.5$ fm is $\gamma_{\gamma\,1}^{2}(ch) = 0.176 + i\,0.028\ \mathrm{eV}^{1/2}$. Hence the calculated lower limit of the radiative width $\Gamma_{\gamma\,1}^{2}$ is $(\mathrm{Im}[\gamma_{\gamma\,1}^{2}(ch)])^2 = 0.80 \pm 0.30$ meV, which is significantly smaller than the measured $\Gamma_{\gamma\,1}^{2}$. The calculated channel radiative width is $\Gamma_{\gamma\,1}^{2}(ch) = 31.7 \pm 7.5$ meV, which is significantly higher than the measured radiative width. Here the uncertainty is contributed by the uncertainties of the experimental ANC, partial proton resonance width and ambiguity in the channel radius.

Hence the measured radiative width can be obtained only as the result of the destructive interference between the real part of the channel radiative width and the internal radiative width amplitude. The calculated internal radiative width amplitudes are $\gamma_{\gamma\,1}^{2}(\mathrm{int}) = 0.29\ \mathrm{eV}^{1/2}$ and $0.065\ \mathrm{eV}^{1/2}$. Correspondingly, the internal radiative widths are $\Gamma_{\gamma\,1}^{2}(\mathrm{int}) = 82$ meV and 4.3 meV.

Because the interference between $\gamma_{\gamma\,1}^{2}(\mathrm{int})$ and $\mathrm{Re}[\gamma_{\gamma\,1}^{2}(ch)]$ in both cases is destructive, we cannot impose a stronger lower limit than the one given by Eq. (25). But, if we select the lower value of the internal width, then we can impose an upper limit on the total radiative width: $\Gamma_{\gamma\,1}^{2} < \Gamma_{\gamma\,1}^{2}(ch) = 32.0 \pm 8.0$ meV.

4. Transition $^{13}\mathrm{O}(1/2^+, E_x = 2.69\ \mathrm{MeV}) \to {}^{13}\mathrm{O}(3/2^-, E_x = 0.0\ \mathrm{MeV})$. This is an another example where the lower limit of the radiative width given by Eq. (25) can raise questions about some of the previous radiative width estimates for the decay of the first resonance $^{13}\mathrm{O}(1/2^+, E_x = 2.69\ \mathrm{MeV})$ at the resonance energy $E_R = 1.17$ MeV to the ground state $^{13}\mathrm{O}(3/2^-, E_x = 0.0\ \mathrm{MeV})$. The proton partial resonance width determined in [19] is 0.45 ± 0.10 MeV. The radiative width of that resonance was suggested in [14] to have a value of $\Gamma_{\gamma\,3/2}^{1/2} = 24$ meV with one order of magnitude uncertainty, coming from a Weisskopf estimate of the transition strength. The square of the proton ANC for the ground state of ^{13}O measured in [20] is $2.53 \pm 0.30\ \mathrm{fm}^{-1}$. The lower limit of the resonance radiative width calculated using Eq.(20) at $r_0 = 4.6$ fm, $(\mathrm{Im}[\gamma_{\gamma\,3/2}^{1/2}(ch)])^2 = 36.0$ meV is larger than the value accepted in [14]. Variation of the channel radius by 9% changes the lower limit by 9%. Thus from Eq. (25) we can conclude that the value of the radiative width adopted in [14] is too low and that $\Gamma_{\gamma\,3/2}^{1/2} > 36.0 \pm 5.0$ meV, where the uncertainty is contributed by the uncertainties of the experimental ANC, partial proton resonance width, and ambiguity in the channel radius.

If for the estimation of the internal radiative width amplitude we use the single-particle approach we get $\gamma_{\gamma\,3/2}^{1/2}(\mathrm{int}) = -0.47\ \mathrm{eV}^{1/2}$ and $\Gamma_{\gamma\,3/2}^{1/2}(\mathrm{int}) = 0.22$ eV for $r_0 = 4.6$ fm. For the channel radiative width amplitude we get $\gamma_{\gamma\,3/2}^{1/2}(ch) = 0.57 + i\,0.19\ \mathrm{eV}^{1/2}$ and $\Gamma_{\gamma\,3/2}^{1/2}(ch) = 0.36 \pm 0.07$ eV. The calculated total radiative width $\Gamma_{\gamma\,3/2}^{1/2} = 1.12$ eV is significantly higher than the lower limit $(\mathrm{Im}[\gamma_{\gamma\,3/2}^{1/2}(ch)])^2 = 36.0$ meV given by Eq. (25). Because the interference between the internal radiative width amplitude (calculated in the single-particle model) and the real part of the channel radiative width amplitude in the case under consideration is constructive, the channel radiative width $\Gamma_{\gamma\,3/2}^{1/2}(ch) = 0.36 \pm 0.07$ eV provides a stronger lower limit for the total radiative width than Eq. (25): $\Gamma_{\gamma\,3/2}^{1/2} > 0.36 \pm 0.07$ eV.

5. Transition $^{15}\mathrm{O}(3/2^+, E_x = 6.79\ \mathrm{MeV}) \to {}^{15}\mathrm{O}(1/2^-, E_x = 0.0\ \mathrm{MeV})$. As the last example we consider the transition from the tail of the subthreshold resonance in ^{15}O, which is the bound state with binding energy $\varepsilon = -0.504$ MeV, to the ground state of ^{15}O. This transition plays an important role in the radiative capture $^{14}\mathrm{N} + p \to {}^{15}\mathrm{O} + \gamma$, which is the bottleneck reaction in the CNO cycle [21–23]. A subthreshold resonance is a bound state which is close to the threshold. Then the tail of this bound state extended to the continuum works as a resonance and the radiative capture to the ground state occurs as a capture to the subthreshold resonance at positive energy E with its subsequent decay to the ground state by emitting the photon. The partial width of the subthreshold resonance is expressed in terms of the ANC and is given by [24]

$$\Gamma_{l_i\,\sigma\,J_i} = \frac{1}{\mu}\, P_{l_i}(E, r_0)\, \frac{\left[W_{-\eta_i^{bs}, l_i+1/2}(2\,\kappa_i\, r_0)\right]^2}{r_0}\, (C_{l_i\,\sigma\,J_i})^2. \quad (30)$$

Here $C_{l_i\,\sigma\,J_i}$ is the ANC of the subthreshold state with spin J_i, channel spin σ, and orbital angular momentum l_i, and η_i^{bs} and κ_i are the Coulomb parameter and the bound-state wave number of the subthreshold bound state. The measured squares of the ANCs of the ground and the subthreshold states for the channel spin $\sigma = 3/2$ in ^{15}O are $54 \pm 6.0\ \mathrm{fm}^{-1}$ and $24.0 \pm 5.0\ \mathrm{fm}^{-1}$ [22], correspondingly. The radiative width for the transition of the subthreshold state to the ground state is determined by the product of the squares of the ANCs of these two bound states.

In the case under consideration the experimental astrophysical factor is known [23] and the channel radius $r_0 = 5.5$ fm was determined by fitting the calculated astrophysical factor to the experimental one [23]. For the channel radius $r_0 = 5.5$ fm the channel radiative width is 0.79 ± 0.24 eV. The

TABLE I. Total radiative width and its estimated lower limit.

Resonant state	Bound state	$\Gamma_{\gamma\,J_f}^{J_f}$ (eV)	Low limit of $\Gamma_{\gamma\,J_f}^{J_f}$ (eV)
$^{17}\mathrm{F}(1/2^-, E_x = 3.104\ \mathrm{MeV})$	$^{17}\mathrm{F}(1/2^+, E_x = 0.495\ \mathrm{MeV})$	$(1.2 \pm 0.2) \times 10^{-2}$	$(3.1 \pm 0.5) \times 10^{-2}$
$^{17}\mathrm{F}(5/2^-, E_x = 3.857\ \mathrm{MeV})$	$^{17}\mathrm{F}(5/2^+, E_x = 0.0\ \mathrm{MeV})$	(0.11 ± 0.02)	$(3.8 \pm 0.4) \times 10^{-5}$
$^{12}\mathrm{N}(2^-, E_x = 1.191\ \mathrm{MeV})$	$^{12}\mathrm{N}(1^+, E_x = 0.0\ \mathrm{MeV})$	$(13 \pm 0.5) \times 10^{-3}$	$(0.8 \pm 0.3) \times 10^{-3}$
$^{13}\mathrm{O}(1/2^+, E_x = 2.69\ \mathrm{MeV})$	$^{13}\mathrm{O}(3/2^-, E_x = 0.0\ \mathrm{MeV})$	1.12	0.36 ± 0.07
$^{15}\mathrm{O}(3/2^+, E_x = 6.79\ \mathrm{MeV})$	$^{15}\mathrm{O}(1/2^-, E_x = 0.0\ \mathrm{MeV})$	>0.85	0.79 ± 0.24

uncertainty is contributed by the uncertainties of the ANCs of the subthreshold and ground states and uncertainty of the R-matrix channel radius. The imaginary part of the channel radiative width amplitude is negligible. From the fitting the experimental data the internal radiative width was found to be 1.1 eV [23]. Then for the total radiative width we get two values: 0.026 eV and 3.75 eV. The first value is too low compared to the measurements in [25–27]. The higher value does not contradict to [27] but significantly exceeds the results obtained in [25,26]. Because the higher value of the total radiative width corresponds to the constructive interference of the internal and channel radiative width amplitudes, the channel radiative width 0.79 ± 0.24 eV provides the lower limit of the radiative width. It questions the value of the total radiative width $0.4^{+0.34}_{-0.13}$ eV obtained in [25]. The total radiative width $0.95^{+0.6}_{-0.95}$ eV obtained in [26] has too high uncertainty while [27] gives only the lower limit for the radiative width: $\Gamma^{3/2}_{\gamma\,1/2} > 0.85$ eV. Evidently new, more accurate measurements of the radiative width for this important transition are needed.

In Table I we summarize the results for all five examples.

IV. SUMMARY

Thus we demonstrated the important role of the ANC in calculations of the resonance radiative width. In particular, we showed two examples in which the adopted radiative widths were smaller than the lower limit imposed by the measured ANC and the proton partial resonance width. Especially interesting is the case of the transition $^{17}\mathrm{F}(1/2^-, E_x = 3.104\ \mathrm{MeV}) \to {}^{17}\mathrm{F}(1/2^+, E_x = 0.495\ \mathrm{MeV})$. The experimental radiative width for this transition given in the compilations [13] based on the measurements in [12] should be remeasured because it is significantly lower than the lower limit provided by the ANC and the proton resonance width. Finally, we discussed the role of the ANC in determination of the radiative width for the capture to the bound state through the subthreshold resonance. In this case the radiative width is determined by the product of the squares of the ANCs for the subthreshold bound state and the final bound state. The presented cases require new more accurate measurements of the radiative width.

ACKNOWLEDGMENTS

A.M.M. acknowledges that this material is based upon work supported by the US Department of Energy, Office of Science, Office of Nuclear Science, under Grants No. DE-FG02-93ER40773 and No. DE-SC0004958. It is also supported by the US Department of Energy, National Nuclear Security Administration, under Grant No. DE-FG52-09NA29467 and by the US National Science Foundation under Grant No. PHY-1415656. D.Y.P. gratefully acknowledges the support of the National Natural Science Foundation of China (Grants No. 11275018 and No. 11035001), the Chinese Scholarship Council (Grant No. 201303070253) and US Department of Energy, Office of Science, Office of Nuclear Science, under Grant No. DE-SC0004958.

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