Vibrational and rotational excited states within a Bohr Hamiltonian with a deformation-dependent mass formalism

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In a recent work [Phys. Rev. C **84**, 044321 (2011)] M. J. Ermamatov and P. R. Fraser have studied rotational and vibrational excited states of axially symmetric nuclei within the Bohr Hamiltonian with different mass parameters. However, the energy formula that the authors have used contains some inaccuracies. So the numerical results they obtained seem to be controversial. In this paper, we revisit all calculations related to this problem and determine the appropriate formula for the energy spectrum. Moreover, in order to improve such calculations, we reconsider this problem within the framework of the deformation-dependent mass formalism. Also, unlike the work of Bonatsos *et al.* [Phys. Rev. C **83**, 044321 (2011)], in which the mass parameter has not been considered, we will show the importance of this parameter and its effect on numerical predictions.

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I. INTRODUCTION

Thanks to its relatively simple structure the Bohr Hamiltonian [1] continues to play an undeniable role in the study of nuclear structure within collective models [2,3] in competition with more sophisticated methods such as the quasiparticle random phase approximation (QRPA) [4,5] and the interacting boson model (IBM) [6]. Moreover, its advantage in respect to these microscopic methods resides in its ability to provide collective-states eigenenergies and corresponding wave functions of nuclei in analytical form. So far, the Bohr Hamiltonian has been widely used with a constant mass parameter [7–13]. Recently, this assumption has been reexamined in the framework of the deformation-dependent mass formalism (DDMF) [14,15], emphasizing that the mass tensor of the collective Hamiltonian cannot be taken as a constant but it has to depend on the collective coordinates. Such a formalism allows us to enhance the precision of numerical calculations of nuclear characteristics. Moreover, Jolos *et al.* [16–19] have shown that this mass parameter should split into a ground-state band and β-band and γ-band coefficients for deformed nuclei. Each coefficient is set to its average value over the wave function of the corresponding band state. Following the latter procedure, M. J. Ermamatov *et al.* have studied rotational and vibrational spectra of axially symmetric nuclei [20]. Their calculations have been based on an analytical energy formula that the authors claimed they obtained in a previous work [21]. However, the used formula in Ref. [20] together with the corresponding wave functions were inaccurate as we will show in this paper. Therefore, the calculated transition rates by the same authors are also questionable. Besides, the Bohr Hamiltonian's dependence on two separable collective coordinates β and γ, where β also represents nuclear shape deformation, enables one to choose nuclear collective potentials as a sum of two separate terms, namely a β potential $V(\beta)$ and a γ term $V(\gamma)$. In the present paper, where we revisit the M. J. Ermamatov *et al.* work [20] with the purpose of improving their calculations within the DDMF, the potential term $V(\beta)$ is chosen to be equal to the Davidson potential [22] as in Ref. [20] and the γ potential $V(\gamma)$ is taken to be equal to the harmonic oscillator. Such a problem has been solved in Ref. [14] but with equal mass coefficients by means of the supersymmetric quantum mechanical method (SUSYQM) [23,24]. Furthermore, we will display the essential role played by the mass parameter in the evaluation of nuclear characteristics unlike the Bonatsos *et al.* work [11–15] in which this parameter has been hidden. Thus, the eigenenrgies formula and the corresponding wave functions are derived by means of the asymptotic iteration method (AIM) [25]. This method has proved to be a useful tool when dealing with physical problems involving Schrödinger-type equations [26–28].

This paper is organized as follows: In Sec. II the position-dependent mass formalism is briefly described. In Sec. III, we propose the Bohr Hamiltonian with three different mass coefficients that we use in Sec. IV in accordance with the deformation-dependent mass formalism. The exact separation of the Bohr Hamiltonian in the case of axially symmetric prolate deformed nuclei and the solutions of angular equation are achieved in Sec. V. The radial equation is given in Sec. VI. Analytical expressions for the energy levels and excited-state wave functions are presented in Secs. VII and VIII, respectively, while the $B(E2)$ transition probabilities are given in Sec. IX. Finally, Sec. X is devoted to the numerical calculations for energy spectra and $B(E2)$ transition probabilities with their comparisons with experimental data and the available IBM ones, while Sec. XI contains the conclusion. An overview of the asymptotic iteration method is given in Appendix A, while in Appendix B, we give the used formulas for the calculations of $B(E2)$.

II. POSITION-DEPENDENT MASS FORMALISM

The general form of the Hamiltonian with effective mass depending on position was originally introduced by Von Roos

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[29],

$$H = -\frac{\hbar^2}{4}[m^{\delta'}(x)\nabla m^{\kappa'}\nabla m^{\lambda'} + m^{\lambda'}(x)\nabla m^{\kappa'}\nabla m^{\delta'}] + V(x), \tag{1}$$

where V is the relevant potential and the parameters δ',κ',λ' are constrained by the condition $\delta'+\kappa'+\lambda' = -1$. Assuming a position-dependent mass of the form [30]

$$m(x) = m_0 M(x), M(x) = \frac{1}{[f(x)]^2}, f(x) = 1 + g(x), \tag{2}$$

where m_0 is a constant mass and $M(x)$ is a dimensionless position-dependent mass, the Hamiltonian (1) becomes [30]

$$H = -\frac{\hbar^2}{4m_0}[f^{\delta}(x)\nabla f^{\kappa}\nabla f^{\lambda} + f^{\lambda}(x)\nabla f^{\kappa}\nabla f^{\delta}] + V(x), \tag{3}$$

with $\delta+\kappa+\lambda = 2$. It is known [30] that this Hamiltonian can be put into the form

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(x)}\nabla f(x)\nabla\sqrt{f(x)} + V_{\text{eff}}(x), \tag{4}$$

with

$$V_{\text{eff}}(x) = V(x) + \frac{\hbar^2}{2m_0}\left\{\frac{1}{2}(1-\delta-\lambda)f(x)\nabla^2 f(x) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)[\nabla f(x)]^2\right\}, \tag{5}$$

where δ and λ are free parameters.

III. BOHR HAMILTONIAN WITH MASS COEFFICIENTS

In the laboratory frame, the Bohr Hamiltonian can be written as [17]

$$H = \frac{1}{4}\left[\sum_{\mu}\pi^{+}_{2\mu}\pi_{2\mu}\frac{1}{B(\alpha_2)} + \frac{1}{B(\alpha_2)}\sum_{\mu}\pi^{+}_{2\mu}\pi_{2\mu}\right] + V(\alpha_2), \tag{6}$$

where $\alpha_{2\mu}$ is a collective variable and $\pi_{2\mu}$ is an operator of the conjugate momentum. In the intrinsic frame we obtain from Eq. (6)

$$H = -\frac{\hbar^2}{4B(\beta,\gamma)}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] - \left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] \times \frac{\hbar^2}{4B(\beta,\gamma)} + V(\beta,\gamma). \tag{7}$$

For small amplitudes of γ vibration around $\gamma = 0$ and β vibration around $\beta = \beta_0 \neq 0$, the collective coordinates could be considered as separable in the axial symmetry nuclei case. Thus, we can consider three separable states of nuclei, namely the ground state and the β- and γ-vibrational states. Each one of these states will have its own mass parameter equal to its average value over the wave function of the state under consideration:

(1) The ground-state mass parameter

$$\langle \text{g.s.}|B(\beta,\gamma)|\text{g.s.}\rangle \equiv B_{\text{rot}}, \tag{8}$$

where we consider the ground-state rotational band;

(2) the γ-mass parameter,

$$\langle\gamma|B(\beta,\gamma)|\gamma\rangle \equiv B_{\gamma}, \tag{9}$$

where we consider the γ-vibrational state; and

(3) the β-mass parameter,

$$\langle\beta|B(\beta,\gamma)|\beta\rangle \equiv B_{\beta}, \tag{10}$$

where we consider the β-vibrational state.

The procedure described above assumes the use of projection operators. Using Eqs. (8)–(10), we obtain from Eq. (7) the following Hamiltonian:

$$H = -\frac{\hbar^2}{2\langle i|B|i\rangle}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \tag{11}$$

where $i = $ g.s., β, or γ band depending on which state is considered. In the case of a small axially symmetric deformation of nuclei, the Bohr Hamiltonian with three different mass coefficients can be written as [17]

$$H = H_{\text{rot}} + H_{\gamma} + H_{\beta}, \tag{12}$$

where

$$H_{\text{rot}} = \frac{\hbar^2}{6B_{\text{rot}}\beta^2}(\hat{\mathcal{Q}}^2 - \hat{Q}_3^2), \tag{13}$$

$$H_{\gamma} = -\frac{\hbar^2}{2B_{\gamma}\beta^2}\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{\hbar^2}{2B_{\gamma}}\frac{\hat{Q}_3^2}{4\beta^2\gamma^2} + \frac{V(\gamma)}{\beta^2}, \tag{14}$$

and

$$H_{\beta} = -\frac{\hbar^2}{2}\left(\frac{1}{B_{\beta}}\frac{\partial^2}{\partial\beta^2} + \frac{2}{B_{\gamma}}\frac{1}{\beta}\frac{\partial}{\partial\beta} + \frac{2}{B_{\beta}}\frac{1}{\beta}\frac{\partial}{\partial\beta}\right) + V(\beta). \tag{15}$$

IV. BOHR HAMILTONIAN WITH DIFFERENT DEFORMATION-DEPENDENT MASS PARAMETERS

To construct a Bohr Hamiltonian with a mass depending on the deformation coordinate β, in accordance with the formalism described in Sec. II,

$$B = \frac{\langle i|B_0|i\rangle}{[f(\beta)]^2}, \tag{16}$$

we have to follow the procedure in Ref. [17]. Since the deformation function f depends only on the radial coordinate β, only the β part of the resulting equation will be affected.

The final result reads [14]

$$\frac{\hbar^2}{2\langle i|B_0|i\rangle}\Big[-\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{f^2}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}\Big]\Psi + V_{\rm eff}\Psi = E\Psi \quad (17)$$

with

$$V_{\rm eff} = V(\beta,\gamma) + \frac{\hbar^2}{2\langle i|B_0|i\rangle}\Big[\frac{1}{2}(1-\delta-\lambda)f\,\nabla^2 f + \Big(\frac{1}{2}-\delta\Big)\Big(\frac{1}{2}-\lambda\Big)(\nabla f)^2\Big]. \quad (18)$$

V. SEPARATION OF THE BOHR HAMILTONIAN OR AXIALLY SYMMETRIC PROLATE DEFORMED NUCLEI

Exact separation of the variables β and γ may be achieved when the potential is chosen as in Refs. [10,31]:

$$V(\beta,\gamma) = U(\beta) + \frac{f^2}{\beta^2}W(\gamma), \quad (19)$$

where the potential $W(\gamma)$ has a minimum around $\gamma = 0$. Then one can write the angular momentum of Eq. (17) in the form [32]

$$\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)} \approx \frac{4}{3}\left(Q_1^2+Q_2^2+Q_3^2\right) + Q_3^2\left(\frac{1}{\sin^2\gamma}-\frac{4}{3}\right). \quad (20)$$

In the same context, we consider a wave function of the form [32]

$$\Psi(\beta,\gamma,\theta_i) = F_{n_\beta L}(\beta)\eta_{n_\gamma,K}(\gamma)\mathcal{D}^L_{M,K}(\theta_i), \quad (21)$$

where $\mathcal{D}(\theta_i)$ are Wigner functions of the Euler angles $\theta_i (i = 1,2,3)$ and L is the total angular momentum, where M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z axis and the body-fixed z' axis, respectively. As a result, Eq. (17) can be approximately separated into three equations:

$$\Big\{\frac{\hbar^2}{2\langle i|B_0|i\rangle}\Big[-\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} + \frac{f^2}{\beta^2}\Lambda + \frac{1}{2}(1-\delta-\lambda)f\,\nabla^2 f + \Big(\frac{1}{2}-\delta\Big)\Big(\frac{1}{2}-\lambda\Big)(\nabla f)^2\Big] + V(\beta)\Big\}F_{n_\beta L}(\beta) = E F_{n_\beta L}(\beta), \quad (22)$$

$$\Big[-\frac{\hbar^2}{2B_\gamma}\Big(\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{K^2}{4}\frac{1}{\sin^2\gamma}\Big) + W(\gamma)\Big]\eta_{n_\gamma,K}(\gamma) = \bar{\Lambda}\eta_{n_\gamma,K}(\gamma), \quad (23)$$

and

$$\frac{\hbar^2}{6B_{\rm rot}}(\hat{\hat{Q}}^2 - Q_3^2)\mathcal{D}^L_{M,K}(\theta_i) = \Lambda'\mathcal{D}^L_{M,K}(\theta_i). \quad (24)$$

The eigenvalues of the rotational part equation (24) are easily obtained since $\hat{\hat{Q}}^2$ is the quadratic casimir operator of $O(3)$ and $\hat{\hat{Q}}_3^2$ is the projection of the angular momentum on the z axis,

$$\Lambda' = \frac{\hbar^2}{6B_{\rm rot}}[L(L+1) - K^2]. \quad (25)$$

Note that Eq. (23) for $\gamma \approx 0$ can be treated as in Ref. [32].

For the γ part, we use a harmonic oscillator potential [20],

$$W(\gamma) = \tfrac{1}{2}(\beta_0^4 C_\gamma)\gamma^2, \quad (26)$$

where β_0 denotes the position of the minimum of the potential in β and C_γ is a free parameter. In this case, Eq. (23) transforms into the usual harmonic oscillator equation

$$\Big[-\frac{\hbar^2}{2B_\gamma}\Big(\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} - \frac{K^2}{4}\frac{1}{\gamma^2}\Big) + \frac{1}{2}(\beta_0^4 C_\gamma)\gamma^2\Big]\eta_{n_\gamma,K}(\gamma) = \bar{\Lambda}\eta_{n_\gamma,K}(\gamma). \quad (27)$$

To solve this equation through AIM, we propose the following ansatz for the γ-part eigenvectors $\eta_{n_\gamma,K}(\gamma)$:

$$\eta_{n_\gamma,K}(\gamma) = \gamma^{|K/2|}e^{-\frac{\gamma^2}{2g}}\Gamma_{n_\gamma,K}(\gamma) \quad (28)$$

with $g = \frac{1}{\beta_0^2}\frac{\hbar}{\sqrt{B_\gamma C_\gamma}}$. For this form of the angular wave function, the γ-part equation (27) reduces to a standard form given in Eq. (A1) in Appendix A. According to the AIM procedure, the eigenvalues are calculated by means of the termination condition Eq. (A5) and the recurrence relations Eq. (A4), hence one can derive the generalized form of the eigenvalues,

$$\bar{\Lambda} = \frac{2}{g}\frac{\hbar^2}{B_\gamma}\left(2\tilde{n}_\gamma + \frac{K}{2} + 1\right), \quad \tilde{n}_\gamma = 0,1,2,\ldots. \quad (29)$$

By inserting $\tilde{n}_\gamma = \frac{n_\gamma - |K/2|}{2}$ in Eq. (28), where n_γ is the quantum number related to γ oscillations, one obtains

$$\bar{\Lambda} = \frac{2}{g}\frac{\hbar^2}{B_\gamma}(n_\gamma + 1), \quad n_\gamma = 0,1,2,\ldots. \quad (30)$$

As a result, we found

$$\frac{B_\beta}{\hbar^2}\Lambda = \left\{\frac{2}{g}\frac{B_\beta}{B_\gamma}(n_\gamma+1) + \frac{1}{3}\frac{B_\beta}{B_{\rm rot}}[L(L+1)-K^2]\right\}. \quad (31)$$

The allowed bands are characterized by

$$\begin{aligned} n_\gamma &= 0, \quad K = 0; \\ n_\gamma &= 1, \quad K = \pm 2; \\ n_\gamma &= 2, \quad K = 0, \pm 4; \ldots. \end{aligned} \quad (32)$$

In the standard case of constant mass where $B_\gamma = B_\beta = B_{\rm rot} = 1$ and $\hbar = 1$, our formula Eq. (31) matches up with Eq. (41) of Ref. [14]. In Ref. [14], the coefficient of γ^2 in $u(\gamma)$ is equal to $(3c)^2$ compared to $(\beta_0^4 C_\gamma)$ Eq. (26) used in this work.

The eigenfunctions corresponding to eigenvalues (30) are obtained in terms of the confluent hypergeometric

function

$$\Gamma_{n_\gamma,K}(\gamma) = N_{n_\gamma,K}\ {}_1F_1\left(-\tilde{n}_\gamma, 1+\frac{|K|}{2}, \frac{\gamma^2}{g}\right), \quad (33)$$

where $N_{n_\gamma,K}$ is a normalization constant. According to the relation between hypergeometric functions and the Laguerre polynomials, the γ angular wave functions for axially symmetric prolate deformed nuclei can be written as:

$$\eta_{n_\gamma,K} = N_{n_\gamma,K}\ \gamma^{|K/2|}\ e^{-\frac{\gamma^2}{2g}} L^{|K/2|}_{\tilde{n}_\gamma}\left(\frac{\gamma^2}{g}\right), \quad (34)$$

where $L^{K/2}_{\tilde{n}_\gamma}$ represents the Laguerre polynomial and N_γ the normalization constant, determined from the normalization condition

$$\int_0^{\pi/3} \eta^2_{n_\gamma,K}(\gamma)|\sin 3\gamma|d\gamma = 1. \quad (35)$$

In the case of small γ vibration, we can write $|\sin 3\gamma| \simeq |3\gamma|$, and then the integral Eq. (35) is easily calculated by using Eq. (8.980) of Ref. [33]. This leads to

$$N_{n_\gamma,K} = \left[\frac{2}{3} g^{-1-|K/2|} \frac{\tilde{n}_\gamma!}{\Gamma(\tilde{n}_\gamma + |K/2| + 1)}\right]^{1/2}. \quad (36)$$

The normalization constants for the $(n_\gamma,K) = (0,0)$ and $(n_\gamma,K) = (1,2)$ states are found to be $N^2_{0,0} = \frac{2}{3g}$, $N^2_{1,2} = \frac{2}{3g^2}$, respectively, and then $\frac{N^2_{0,0}}{N^2_{1,2}} = g$. This result will be used to calculate the $B(E2)$ values in the $\gamma \to$ ground and $\gamma \to \beta$ transitions ($\Delta K = 2$).

VI. THE RADIAL SCHRÖDINGER EQUATION

The β-vibrational states of deformed nuclei with mass parameter are determined by the solution of the radial Schrödinger equation,

$$\frac{\hbar^2}{2}\left\{\frac{1}{B_\beta} f^2 F'' + \left(\frac{1}{B_\beta}+\frac{1}{B_\gamma}\right)\left(ff' + \frac{2f^2}{\beta}\right)F' + \left(\frac{1}{B_\beta}+\frac{1}{B_\gamma}\right)\left[\frac{(f')^2}{8} + \frac{ff''}{4} + \frac{ff'}{\beta}\right]F\right\} - \frac{f^2}{2\beta^2}\Lambda F + EF - V_{\rm eff}F = 0 \quad (37)$$

with

$$V_{\rm eff} = V + \frac{\hbar^2}{2}\left(\frac{1}{B_\beta}+\frac{1}{B_\gamma}\right)\left[\frac{1}{4}(1-\delta-\lambda)ff'' + \frac{1}{2}\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right)(f')^2\right]. \quad (38)$$

Setting a standard transformation of the radial wave function

$$F_{n_\beta L}(\beta) = \beta^{-(1+B_\beta/B_\gamma)} R_{n_\beta L}(\beta), \quad (39)$$

we get

$$-f^2R'' - \left(1+\frac{B_\beta}{B_\gamma}\right)ff'R' - \left(1+\frac{B_\beta}{B_\gamma}\right)\left[\frac{(f')^2}{8}+\frac{ff''}{4}\right]R + 2U_{\rm eff}R = \frac{2B_\beta}{\hbar^2}ER, \quad (40)$$

where

$$U_{\rm eff} = \frac{B_\beta}{\hbar^2}V_{\rm eff} + \frac{1}{2}\frac{B_\beta}{B_\gamma}\left(1+\frac{B_\beta}{B_\gamma}\right)\frac{f^2+\beta ff'}{\beta^2} + \frac{B_\beta}{\hbar^2}\frac{f^2}{2\beta^2}\Lambda. \quad (41)$$

In the frame without mass parameters, we can reduce the first three terms in Eq. (44) by $(\sqrt{f}\frac{d}{d\beta}\sqrt{f})^2 R$.

VII. THE EFFECTIVE POTENTIAL AND ENERGY LEVELS

As in Ermamatov *et al.* [20], we use in our calculations the Davidson potential [22]

$$V(\beta) = V_0\left(\frac{\beta}{\beta_0} - \frac{\beta_0}{\beta}\right)^2, \quad (42)$$

where V_0 represents the depth of the minimum, located at β_0.

According to the specific form of the potential Eq. (42), we are also going to consider for the deformation function the special form

$$f(\beta) = 1 + a\beta^2, \quad a \ll 1. \quad (43)$$

Inserting these forms for the potential and the deformation function in Eq. (41) one gets

$$2U_{\rm eff} = k_2\beta^2 + k_0 + \frac{k_{-2}}{\beta^2} \quad (44)$$

with

$$\begin{aligned} k_2 =& \frac{a^2}{2}\left\{\left(1+\frac{B_\beta}{B_\gamma}\right)\left[6\frac{B_\beta}{B_\gamma} + (1-2\delta)(1-2\lambda) + (1-\delta-\lambda)\right] + 2\frac{B_\beta}{\hbar^2}\Lambda\right\} + 2\frac{g_\beta}{\beta_0^4} \\ k_0 =& \frac{a}{2}\left\{\left(1+\frac{B_\beta}{B_\gamma}\right)\left[8\frac{B_\beta}{B_\gamma} + (1-\delta-\lambda)\right] + 4\frac{B_\beta}{\hbar^2}\Lambda\right\} - 4\frac{g_\beta}{\beta_0^2} \\ k_{-2} =& \frac{B_\beta}{B_\gamma}\left(1+\frac{B_\beta}{B_\gamma}\right) + \frac{B_\beta}{\hbar^2}\Lambda + 2g_\beta, \end{aligned} \quad (45)$$

where $g_\beta = \frac{B_\beta V_0 \beta_0^2}{\hbar^2}$.

To solve the radial equation Eq. (40) through the AIM [25], one needs the following parametrization:

$$R_{n_\beta L}(y) = y^\rho(1+ay)^\nu \chi_{n_\beta L}(y), \quad y = \beta^2, \quad (46)$$

where

$$\begin{aligned} \rho =& \frac{1}{4}(1+\sqrt{1+4k_{-2}}) \\ \nu =& -\frac{1}{2}\left\{\frac{B_\beta}{B_\gamma} + \left[\left(\frac{B_\beta}{B_\gamma}+\frac{1}{2}\right)\left(\frac{B_\beta}{B_\gamma}-1\right) + k_{-2} - \frac{k_0}{a} + \frac{k_2}{a^2} + \frac{2B_\beta}{a\hbar^2}E\right]^{1/2}\right\}. \end{aligned} \quad (47)$$

For this form of the radial wave function, the Eq. (40) reads

$$\chi''_{n_\beta}(y) = -\left[\frac{1+4\rho+ay\left(3+2\frac{B_\beta}{B_\gamma}+4\rho+4\nu\right)}{2y(1+ay)}\right]\chi'_{n_\beta}(y) - a\left[\frac{2(\rho+\nu)\left(1+2\frac{B_\beta}{B_\gamma}+2\nu+2\rho\right)+1+\frac{B_\beta}{B_\gamma}-\frac{k_2}{a^2}}{4y(1+ay)}\right]\chi_{n_\beta}(y). \quad (48)$$

The first and the second terms on the right-hand side of Eq. (48) represent λ_0 and s_0 of Eq. (A1), respectively. After calculating λ_n and s_n, by means of the recurrence relations of Eq. (A4), we get the generalized formula of the radial energy spectrum from the roots of the termination condition of Eq. (A5),

$$E_{n_\beta n_\gamma LK} = \frac{\hbar^2}{2B_\beta}\left[k_0 + \frac{a}{2}\left(2+\frac{B_\beta}{B_\gamma}+2p+2q+pq\right) + 2a(2+p+q)n_\beta + 4an_\beta^2\right], \quad (49)$$

where n_β is the principal quantum number of β vibrations and

$$q \equiv q_{n_\gamma}(L,K) = \sqrt{1+4k_{-2}}$$
$$p \equiv p_{n_\gamma}(L,K) = \sqrt{4\frac{B_\beta}{B_\gamma}-3+4\frac{k_2}{a^2}}. \quad (50)$$

The quantities k_2, k_0, k_{-2} are given by Eq. (45), where Λ is the eigenvalue of the γ-vibrational part of the Hamiltonian for axially symmetric prolate deformed nuclei. In the numerical results part of the paper, the energies are normalized to the first excited state. So, the results depend on six parameters B_β/B_γ, $B_\gamma/B_{\rm rot}$, g, g_β, a, and β_0.

A few interesting low-lying bands are classified by the quantum numbers n_β, n_γ, and K, such as the ground-state band (g.s.) with $n_\beta = 0$, $n_\gamma = 0$, $K = 0$; the β band with $n_\beta = 1$, $n_\gamma = 0$, $K = 0$; and the γ band with $n_\beta = 0$, $n_\gamma = 1$, $K = 2$.

A. Special case 1: Without mass coefficients

If we assume $B_\beta = B_\gamma = B_{\rm rot} = 1$, one gets from Eq. (45)

$$k_2 = a^2[(1-\delta-\lambda)+(1-2\delta)(1-2\lambda)+6+\Lambda]+2\frac{V_0}{\beta_0^2}$$
$$k_0 = a[(1-\delta-\lambda)+8+2\Lambda]-4V_0$$
$$k_{-2} = 2+\Lambda+2V_0\beta_0^2. \quad (51)$$

Thus, the energy spectrum formula Eq. (49) is identical to Eq. (82) of Ref. [14] obtained by means of the SUSYQM [23,24]. The slight difference between our coefficients k_2, k_0, and k_{-2} and those of Ref. [14] comes from the adopted expression of Davidson potential.

B. Special case 2: No dependence of the mass on the deformation

If $a = 0$, the dependence of the mass on the deformation, is canceled, then one obtains from Eq. (45)

$$k_2 = 2\frac{g_\beta}{\beta_0^4}, \quad k_0 = -4\frac{g_\beta}{\beta_0^2}$$
$$k_{-2} = \frac{B_\beta}{B_\gamma}\left(1+\frac{B_\beta}{B_\gamma}\right)+\frac{B_\beta}{\hbar^2}\Lambda+2g_\beta. \quad (52)$$

In this case, the energy spectrum becomes

$$E_{n_\beta n_\gamma LK} = \frac{\hbar^2}{2B_\beta}\left\{k_0+\sqrt{4k_2}\left[1+2n_\beta+\frac{1}{2}q_{n_\gamma}(L,K)\right]\right\}. \quad (53)$$

For axially symmetric prolate deformed nuclei, the energy formula reads

$$E_{n_\beta n_\gamma LK} = \sqrt{2\frac{V_0^2}{g_\beta}}\left[1+2n_\beta+\frac{1}{2}q_{n_\gamma}(L,K)-\sqrt{2g_\beta}\right] \quad (54)$$

with

$$\frac{1}{2}q_{n_\gamma}(L,K) = \sqrt{\frac{1}{4}+\frac{B_\beta}{B_\gamma}\left(1+\frac{B_\beta}{B_\gamma}\right)+\frac{B_\beta}{\hbar^2}\Lambda+2g_\beta} \quad (55)$$

and

$$\frac{B_\beta}{\hbar^2}\Lambda = \frac{2}{g}\frac{B_\beta}{B_\gamma}(n_\gamma+1)+\frac{1}{3}\frac{B_\beta}{B_{\rm rot}}[L(L+1)-K^2]. \quad (56)$$

Note that Eq. (54) represents the correct formula of the energy spectrum, compared to Eq. (11) given in Ref. [20], where the mass parameter term is missed in the analog formula of Eq. (55).

It is also worth noting that, in this case, Eq. (37) reduces to a standard confluent hypergeometric equation which can be converted to a Laguerre differential equation. The resolution of such a problem is carried out in Sec. VIII B.

C. Special case 3: Standard case

For γ-unstable nuclei, in the limit case of $a = 0$ and $B_\beta = B_\gamma = B_{\rm rot}$, our Eq. (49) reduces to

$$E_{n_\beta L} = \sqrt{2\frac{V_0^2}{g_\beta}}\left(1+2n_\beta+\sqrt{\frac{9}{4}+\Lambda+2g_\beta}\right)-2V_0 \quad (57)$$

with

$$\Lambda = \tau(\tau+3) \quad (58)$$

and $\tau = L/2$ is the seniority quantum number. This formula is similar to the energy spectrum Eq. (80) in Ref. [34].

VIII. EXCITED-STATE WAVE FUNCTIONS

The used wave functions in our calculations are given by

$$\Psi(\beta,\gamma,\theta_i) = \beta^{-1-\frac{B_\beta}{B_\gamma}}R_{n_\beta,L}(\beta)\eta_{n_\gamma,K}(\gamma)\mathcal{D}^L_{M,K}(\theta_i). \quad (59)$$

The radial function $R_{n_\beta,L}(\beta)$ corresponds to the $n^{\rm th}$ eigenstate of Eq. (40), $\eta_{n_\gamma,K}(\gamma)$ is given by Eq. (34), and the symmetries

TABLE I. The values of free parameters used in the calculations.

Nucleus	g	g_β	B_β/B_γ	$B_\beta/B_{\rm rot}$	$g(B_\beta = B_\gamma = B_{\rm rot})$	$g_\beta(B_\beta = B_\gamma = B_{\rm rot})$
^{154}Sm	0.0187	281.66	1.36	3.99	0.0489	0.357
^{156}Gd	0.0252	308.84	1.53	4.64	0.0673	0.884
^{172}Yb	0.0064	2469.46	1.32	11.14	0.0453	− 1.909
^{182}W	0.0249	619.74	2.01	6.62	0.0714	0.512

TABLE II. The comparison of the theoretical predictions of energy levels Eq. (49) of the ground-state band and the β and γ bands normalized to the energy of the first excited state $E(2^+_{\rm g.s.})$ using the parameters given in Table I for ^{154}Sm for this work with those from Ref. [18] and experimental values taken from Ref. [38]. β_0 and a indicate the position of the minimum of Davidson potential Eq. (42) and the deformation dependence of the mass Eq. (43), respectively, while σ is the quality measure Eq. (77).

L	Expt.	$B_\beta \neq B_\gamma \neq B_{\rm rot}$		$B_\beta = B_\gamma = B_{\rm rot}$		Ref. [18]
		$a = 0$	DDM	$a = 0$	DDM	
g.s.						
4	3.26	3.31	3.31	3.25	3.27	3.28
6	6.63	6.89	6.89	6.59	6.68	6.76
8	11.01	11.65	11.65	10.82	11.07	11.28
10	16.26	17.52	17.52	15.75	16.29	16.65
12	22.27	24.41	24.41	21.22	22.24	22.68
σ		1.289	1.289	0.592	0.043	0.320
a			0.0000		0.0483	
β_0			22.41		0.54	
β_1						
0	13.40	13.40	13.03	13.40	13.13	13.40
2	14.37	14.40	14.03	14.40	14.14	14.40
4	16.32	16.71	16.35	16.65	16.41	16.68
6	19.23	20.29	19.94	19.99	19.78	20.16
8		25.05	24.73	24.22	24.08	24.68
10		30.93	30.64	29.15	29.11	30.05
12		37.81	37.58	34.62	34.73	36.08
σ		0.651	0.501	0.479	0.384	0.576
a			0.0335		0.0039	
β_0			0.88		0.88	
γ_1						
2	17.56	17.56	18.01	17.56	18.67	17.56
3	18.77	18.47	18.97	18.28	19.49	16.56
4	20.30	19.68	20.26	19.23	20.57	19.87
5	22.01	21.18	21.86	20.39	21.91	21.48
6	23.73	22.96	23.78	21.77	23.50	23.38
7	26.27	25.03	26.01	23.33	25.33	25.53
8		27.36	28.56	25.08	27.40	27.93
9		29.95	31.41	26.99	29.70	30.55
10		32.79	34.58	29.05	32.21	33.36
11		35.88	38.06	31.26	34.94	36.33
12		39.19	41.84	33.60	37.87	39.44
13		42.73	45.93	36.06	40.99	42.65
σ		0.812	0.260	1.817	0.743	1.097
a			0.0199		0.0086	
β_0			11.53		1.67	
$\sigma_{\rm total}$		0.895	0.874	1.153	0.754	0.728
a			0.0219		0.0054	
β_0			1.07		1.60	

TABLE III. The comparison of the theoretical predictions of energy levels Eq. (49) of the ground-state band and the β and γ bands normalized to the energy of the first excited state $E(2^+_{\rm g.s.})$ using the parameters given in Table I for ^{156}Gd for this work with those from Ref. [18] and experimental values taken from Ref. [38]. β_0 and a indicate the position of the minimum of Davidson potential Eq. (42) and the deformation dependence of the mass Eq. (43), respectively, while σ is the quality measure Eq. (77).

L	Expt.	$B_\beta \neq B_\gamma \neq B_{\rm rot}$		$B_\beta = B_\gamma = B_{\rm rot}$		Ref. [18]
		$a = 0$	DDM	$a = 0$	DDM	
g.s.						
4	3.24	3.31	3.30	3.23	3.25	3.29
6	6.57	6.87	6.82	6.49	6.60	6.76
8	10.84	11.62	11.50	10.55	10.87	11.22
10	15.91	17.44	17.30	15.22	15.92	16.51
12	21.63	24.25	24.18	20.34	21.62	22.38
σ		1.575	1.492	0.747	0.025	0.470
a			0.0600		0.0217	
β_0			60.00		1.11	
β_1						
0	11.79	11.79	9.93	11.79	11.79	11.79
2	12.69	12.79	10.94	12.79	12.79	12.79
4	14.68	15.10	13.29	15.02	15.02	15.08
6	17.30	18.66	16.93	18.28	18.28	18.54
8	20.76	23.41	21.82	22.34	22.34	23.01
10	24.94	29.23	27.89	27.01	27.01	28.30
12	30.43	36.04	35.09	32.13	32.13	34.17
σ		3.135	2.585	1.339	1.339	2.311
a			0.0230		0.0000	
β_0			2.77		43.81	
γ_1						
2	12.97	12.97	12.97	12.97	13.75	12.97
3	14.02	13.90	13.90	13.70	14.58	13.96
4	15.22	15.12	15.12	14.66	15.67	15.27
5	16.93	16.64	16.64	15.83	17.02	16.88
6	18.47	18.45	18.45	17.20	18.61	18.76
7	20.79	20.54	20.54	18.75	20.44	20.90
8	22.60	22.90	22.90	20.47	22.50	23.27
9	25.28	25.51	25.51	22.34	24.78	25.85
10	27.44	28.38	28.38	24.35	27.28	28.60
11	30.19	31.48	31.48	26.49	29.97	31.50
12	32.84	34.80	34.80	28.74	32.86	34.52
13	35.67	38.34	38.34	31.08	35.94	37.63
σ		1.122	1.122	2.725	0.390	0.982
a			0.0000		0.0219	
β_0			32.35		1.42	
$\sigma_{\rm total}$		1.897	1.866	2.029	1.008	1.379
a			0.0499		0.0582	
β_0			0.98		0.80	

eigenfunctions of the angular momentum are

$$\mathcal{D}^{L}_{M,K}(\theta_i) = \sqrt{\frac{2L+1}{16\pi^2(1+\delta_{K0})}}\left[D^{L*}_{MK} + (-1)^L D^{L*}_{M-K}\right]. \quad (60)$$

To get the radial eigenvectors $R_{n_\beta,L}(\beta)$ of Eq. (40), we insert the expression of the energy spectrum Eq. (49) into Eq. (47). Then we get from Eq. (48) and Eq. (46):

$$R_{n_\beta,L}(y) = y^{\frac{1}{4}(1+q)}(1+ay)^{-n_\beta-\frac{1}{2}(1+\frac{B_\beta}{B_\gamma})-\frac{1}{4}(q+p)}\chi_{n_\beta,L}(y), \quad (61)$$

where q and p are given in Eq. (50).

TABLE IV. The comparison of the theoretical predictions of energy levels Eq. (49) of the ground-state band and the β and γ bands normalized to the energy of the first excited state $E(2^+_{\text{g.s.}})$ using the parameters given in Table I for ^{172}Yb for this work with those from Ref. [18] and experimental values taken from Ref. [38]. β_0 and a indicate the position of the minimum of Davidson potential Eq. (42) and the deformation dependence of the mass Eq. (43), respectively, while σ is the quality measure Eq. (77).

L	Expt.	$B_\beta \neq B_\gamma \neq B_{\text{rot}}$		$B_\beta = B_\gamma = B_{\text{rot}}$		Ref. [18]
		$a=0$	DDM	$a=0$	DDM	
g.s.						
4	3.29	3.33	3.30	3.25	3.28	3.32
6	6.84	6.96	6.84	6.58	6.78	6.91
8	11.54	11.87	11.54	10.79	11.45	11.71
10	17.34	18.02	17.33	15.69	17.26	17.65
12	24.14	25.36	24.15	21.12	24.21	24.64
σ		0.719	0.087	1.766	0.076	0.309
a			0.0036		0.0800	
β_0			40.09		1.98	
β_1						
0	13.20	13.20	11.68	13.20	13.20	13.20
2	14.15	14.20	12.69	14.20	14.20	14.20
4	16.34	16.53	15.02	16.45	16.45	16.52
6	19.53	20.16	18.67	19.78	19.78	20.11
8	23.54	25.07	23.62	23.99	23.99	24.93
10	28.10	31.22	29.83	28.89	28.89	30.89
12	33.11	38.56	37.27	34.32	34.32	37.91
σ		4.593	2.135	0.628	0.628	2.311
a			0.0418		0.0000	
β_0			1.74		1.60	
γ_1						
2	18.63	18.63	18.71	18.63	19.08	18.63
3	19.68	19.59	19.69	19.33	19.89	19.63
4	21.06	20.87	20.98	20.26	20.95	20.95
5	22.60	22.47	22.60	21.39	22.28	22.60
6		24.38	24.54	22.73	23.87	24.56
7		26.60	26.80	24.26	25.73	26.83
8		29.12	29.37	25.96	27.84	29.39
9		31.95	32.26	27.83	30.20	32.24
10		35.07	35.46	29.85	32.83	35.37
11		38.48	38.97	32.01	35.70	38.76
12		42.17	42.78	34.29	38.83	42.20
13		46.15	47.91	36.70	42.22	46.28
σ		0.121	0.065	0.862	0.347	0.070
a			0.0075		0.0600	
β_0			17.10		2.19	
σ_{total}		1.719	1.413	1.067	3.458	1.495
a			0.0010		0.0100	
β_0			11.12		90.01	

TABLE V. The comparison of the theoretical predictions of energy levels Eq. (49) of the ground-state band and the β and γ bands normalized to the energy of the first excited state $E(2^+_{\text{g.s.}})$ using the parameters given in Table I for ^{182}W for this work with those from Ref. [18] and experimental values taken from Ref. [38]. β_0 and a indicate the position of the minimum of Davidson potential Eq. (42) and the deformation dependence of the mass Eq. (43), respectively, while σ is the quality measure Eq. (77).

L	Expt.	$B_\beta \neq B_\gamma \neq B_{\text{rot}}$		$B_\beta = B_\gamma = B_{\text{rot}}$		Ref. [18]
		$a=0$	DDM	$a=0$	DDM	
g.s.						
4	3.29	3.32	3.29	3.22	3.29	3.30
6	6.80	6.91	6.78	6.45	6.78	6.81
8	11.44	11.71	11.40	10.47	11.40	11.41
10	17.12	17.65	17.07	15.05	17.07	16.94
12	23.72	24.64	23.77	20.07	23.76	23.2
σ		0.548	0.042	2.161	0.037	0.276
a			0.0335		0.0470	
β_0			18.49		1.25	
β_1						
0	11.36	11.36	11.36	11.36	11.36	11.36
2	12.57	12.36	12.36	12.36	12.36	12.36
4	15.10	14.68	14.68	14.58	14.58	14.66
6		18.26	18.26	17.81	17.81	18.17
8		23.07	23.07	21.83	21.83	22.77
10		29.01	29.01	26.41	26.41	28.30
12		36.00	36.00	31.43	31.43	34.57
σ		0.335	0.335	0.395	0.395	0.345
a			0.0000		0.0000	
β_0			53.35		52.55	
γ_1						
2	12.21	12.21	12.42	12.21	12.76	12.21
3	13.31	13.16	13.19	12.94	13.55	13.21
4	14.43	14.41	14.46	13.89	14.60	14.52
5	16.24	15.97	16.03	15.05	15.90	16.14
6	17.70	17.83	17.90	16.41	17.42	18.05
7		19.71	20.07	17.95	19.17	20.24
8	22.61	22.41	22.52	19.65	21.13	22.68
9		25.11	25.26	21.50	23.28	25.35
10		28.08	28.26	23.48	25.62	28.24
11		31.31	31.53	25.58	28.14	31.31
12		34.78	35.06	27.79	30.83	34.55
13		38.49	38.83	30.10	33.68	37.92
σ		0.168	0.158	0.935	0.380	0.194
a			0.0538		0.0215	
β_0			0.95		1.07	
σ_{total}		0.358	0.357	1.369	1.019	0.240
a			0.0000		0.0054	
β_0			50.35		2.16	

After inserting Eq. (61) into Eq. (40), we obtain

$$\chi''_{n_\beta,L}(y) = -\left[\frac{1+\frac{q}{2}+a\left(1-2n_\beta-\frac{p}{2}\right)y}{y(1+ay)}\right]\chi'_{n_\beta,L}(y) - \left[\frac{an_\beta\left(n_\beta+\frac{p}{2}\right)}{y(1+ay)}\right]\chi_{n_\beta,L}(y). \quad (62)$$

The excited-state wave functions of this equation are obtained through Eq. (A2),

$$\chi(y) = N_{n_\beta,L}\,{}_2F_1\left[-n_\beta,-n_\beta-\frac{p}{2};-2n_\beta-\frac{(q+p)}{2};1+ay\right], \quad (63)$$

where $N_{n_\beta,L}$ is a normalization constant and ${}_2F_1$ are hypergeometrical functions. Therefore, according to the relation between hypergeometrical functions and the generalized Jacobi polynomials, Eq. (4.22.1) of Ref. [35], the radial wave function can be written as

$$R_{n_\beta,L}(t) = N_{n_\beta,L}\,2^{-(1+\frac{B_\beta}{B_\gamma})/2-(q+p)/4}a^{-(1+q)/4}(1-t)^{(1+2\frac{B_\beta}{B_\gamma}+p)/4}(1+t)^{(q+1)/4}P_{n_\beta}^{(q/2,p/2)}(t), \quad t=\frac{-1+ay}{1+ay}. \quad (64)$$

To determine $N_{n_\beta,L}$, we use the usual orthogonality relation of Jacobi polynomials Eq. (7.391.7) of Ref. [33]. This leads to

$$N_{n_\beta,L} = \left(2a^{q/2+1}n_\beta!\right)^{\frac{1}{2}}\left[\frac{\Gamma\left(n_\beta+\frac{q+p}{2}+1\right)\Gamma\left(2n_\beta+\frac{q+p}{2}+1+\frac{B_\beta}{B_\gamma}\right)}{\Gamma\left(n_\beta+\frac{q}{2}+1\right)\Gamma\left(n_\beta+\frac{B_\beta}{B_\gamma}+\frac{p}{2}\right)\Gamma\left(2n_\beta+\frac{q+p}{2}+1\right)}\right]^{\frac{1}{2}}. \quad (65)$$

TABLE VI. The comparison of the theoretical predictions of $B(E2)$ in units of $B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})$ using the parameters given in Table I for ^{154}Sm in this work with those from Ref. [18] and experimental values.

	Expt.	$B_\beta \neq B_\gamma \neq B_{\text{rot}}$		$B_\beta = B_\gamma = B_{\text{rot}}$		Ref. [18]
		$a=0$	DDM	$a=0$	DDM	
			$\frac{B(E2;L_{\text{g.s.}}+2\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}$			
$4^+_{\text{g.s.}}\to 2^+_{\text{g.s.}}$	1.40(5)	1.44	1.44	1.47	1.47	1.44
$6^+_{\text{g.s.}}\to 4^+_{\text{g.s.}}$	1.67(7)	1.61	1.61	1.70	1.70	1.61
$8^+_{\text{g.s.}}\to 6^+_{\text{g.s.}}$	1.83(11)	1.72	1.72	1.90	1.90	1.72
$10^+_{\text{g.s.}}\to 8^+_{\text{g.s.}}$	1.81(11)	1.81	1.81	2.10	2.10	1.82
$12^+_{\text{g.s.}}\to 10^+_{\text{g.s.}}$		1.90	1.90	2.30	2.30	1.91
			$\frac{B(E2;L_\beta\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}\times 10^3$			
$2^+_\beta\to 0^+_{\text{g.s.}}$	5.4(13)	6.5	6.4	23.3	24.3	6.7
$4^+_\beta\to 2^+_{\text{g.s.}}$		5.5	5.5	20.0	21.3	5.6
$6^+_\beta\to 4^+_{\text{g.s.}}$		3.1	3.1	12.4	13.7	2.9
$2^+_\beta\to 2^+_{\text{g.s.}}$		12.9	11.9	46.4	47.9	13.3
$4^+_\beta\to 4^+_{\text{g.s.}}$		11.7	11.6	42.3	43.6	12.1
$6^+_\beta\to 6^+_{\text{g.s.}}$		11.5	11.4	41.5	42.9	11.9
$0^+_\beta\to 2^+_{\text{g.s.}}$		59.7	59.0	216.5	221.5	61.7
$2^+_\beta\to 4^+_{\text{g.s.}}$	25(6)	42.2	35.6	152.0	154.4	43.8
$4^+_\beta\to 6^+_{\text{g.s.}}$		48.5	47.7	169.9	171.8	51.3
$6^+_\beta\to 8^+_{\text{g.s.}}$		57.4	56.2	191.6	193.1	62.8
			$\frac{B(E2;L_\beta\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}\times 10^3$			
$2^+_\gamma\to 0^+_{\text{g.s.}}$	18.4(29)	18.4	14.8	46.5	48.7	18.4
$2^+_\gamma\to 2^+_{\text{g.s.}}$		26.5	21.3	68.6	71.4	26.2
$2^+_\gamma\to 4^+_{\text{g.s.}}$	3.9(6)	1.4	1.1	3.7	3.8	1.3
$3^+_\gamma\to 2^+_{\text{g.s.}}$		33.1	29.3	85.0	88.8	32.8
$3^+_\gamma\to 4^+_{\text{g.s.}}$		13.5	11.8	36.5	37.6	13.0
$4^+_\gamma\to 2^+_{\text{g.s.}}$		11.0	9.7	28.0	29.4	11.0
$4^+_\gamma\to 4^+_{\text{g.s.}}$		33.0	28.7	88.6	91.7	32.1
$4^+_\gamma\to 6^+_{\text{g.s.}}$		2.9	2.4	8.4	7.8	2.7
$5^+_\gamma\to 4^+_{\text{g.s.}}$		29.8	25.8	79.4	82.65	29.3
$5^+_\gamma\to 6^+_{\text{g.s.}}$		17.6	14.8	49.9	51.0	16.5

(i) In the case where $B_\beta = B_\gamma = B_{\rm rot} = 1$ and $\hbar = 1$, the wave function Eq. (64) and the normalization constant Eq. (65) match up with Eq. (108) and Eq. (112) of Ref. [14], respectively.

(ii) In the limit case $a \to 0$, no dependence of the mass on the deformation, the second-order differential equation Eq. (40) must have a solution of the form

$$R_{n_\beta,L}(\beta) = \beta^{\frac{1}{2}[1+q_{n_\gamma}(L,k)]} e^{-b\beta^2} G_{n_\beta,L}(\beta), \tag{66}$$

where $b = \sqrt{\frac{g_\beta}{2\beta_0^4}}$. By using this radial function in Eq. (40) and introducing a new variable $y = \beta^2$, one can get

$$G''_{n_\beta,L}(y) = -\left(\frac{1+\frac{q}{2}-2by}{y}\right) G'_{n_\beta,L}(y) - \frac{2bn_\beta}{y} G_{n_\beta,L}(y). \tag{67}$$

From Eq. (A1) of the AIM, one can define $\lambda_0(0)$ and $s_0(y)$. Then $\lambda_n(y)$ and $s_n(y)$ are calculated by use of the recurrence relations given in Eq. (A4) and the solution of this equation is found through Eq. (A2) to be

$$G_{n_\beta,L}(y) = N_{n_\beta,L} L_{n_\beta}^{\frac{1}{2}q_{n_\gamma}(L,k)}(2by), \tag{68}$$

where L denotes the Laguerre polynomials and $N_{n_\beta,L}$ is a normalization coefficient determined from the normalization

TABLE VII. The comparison of the theoretical predictions of $B(E2)$ in units of $B(E2; 2^+_{\rm g.s.} \to 0^+_{\rm g.s.})$ using the parameters given in Table I for ^{156}Gd in this work with those from Ref. [18] and experimental values.

	Expt.	$B_\beta \neq B_\gamma \neq B_{\rm rot}$		$B_\beta = B_\gamma = B_{\rm rot}$		Ref. [18]
		$a = 0$	DDM	$a = 0$	DDM	
			$\frac{B(E2;L_{\rm g.s.}+2 \to L_{\rm g.s.})}{B(E2;2^+_{\rm g.s.} \to 0^+_{\rm g.s.})}$			
$4^+_{\rm g.s.} \to 2^+_{\rm g.s.}$	1.41(5)	1.44	1.37	1.48	1.48	1.44
$6^+_{\rm g.s.} \to 4^+_{\rm g.s.}$	1.58(6)	1.61	1.42	1.73	1.73	1.61
$8^+_{\rm g.s.} \to 6^+_{\rm g.s.}$	1.71(10)	1.72	1.38	1.96	1.95	1.73
$10^+_{\rm g.s.} \to 8^+_{\rm g.s.}$	1.68(9)	1.82	1.32	2.19	2.18	1.83
$12^+_{\rm g.s.} \to 10^+_{\rm g.s.}$	1.60(16)	1.91	1.26	2.43	2.42	1.93
			$\frac{B(E2;L_\beta \to L_{\rm g.s.})}{B(E2;2^+_{\rm g.s.} \to 0^+_{\rm g.s.})} \times 10^3$			
$2^+_\beta \to 0^+_{\rm g.s.}$	3.4(3)	6.1	4.4	24.7	24.7	6.3
$4^+_\beta \to 2^+_{\rm g.s.}$		4.6	3.9	19.5	19.5	4.7
$6^+_\beta \to 4^+_{\rm g.s.}$		2.1	1.9	11.0	11.0	0.9
$2^+_\beta \to 2^+_{\rm g.s.}$	18(2)	12.6	4.2	51.9	51.9	13.0
$4^+_\beta \to 4^+_{\rm g.s.}$		11.5	9.4	47.3	47.3	11.8
$6^+_\beta \to 6^+_{\rm g.s.}$		11.2	3.6	46.4	46.4	11.6
$0^+_\beta \to 2^+_{\rm g.s.}$		60.5	48.9	251.1	251.1	62.5
$2^+_\beta \to 4^+_{\rm g.s.}$	22(2)	44.2	1.9	181.4	181.4	46.0
$4^+_\beta \to 6^+_{\rm g.s.}$		52.0	41.2	204.9	204.9	53.3
$6^+_\beta \to 8^+_{\rm g.s.}$		62.4	48.9	230.6	230.6	68.9
			$\frac{B(E2;L_\beta \to L_{\rm g.s.})}{B(E2;2^+_{\rm g.s.} \to 0^+_{\rm g.s.})} \times 10^3$			
$2^+_\gamma \to 0^+_{\rm g.s.}$	25.0(8)	25.0	25.0	65.9	68.3	25.0
$2^+_\gamma \to 2^+_{\rm g.s.}$	38.7(13)	36.1	36.1	97.7	100.6	35.5
$2^+_\gamma \to 4^+_{\rm g.s.}$	4.1(2)	1.8	1.8	5.3	5.3	1.8
$3^+_\gamma \to 2^+_{\rm g.s.}$	39.0(75)	44.9	44.9	121.1	125.2	44.6
$3^+_\gamma \to 4^+_{\rm g.s.}$	27.2(35)	18.3	18.3	52.3	53.3	17.7
$4^+_\gamma \to 2^+_{\rm g.s.}$	9.6(27)	14.9	14.9	39.9	41.5	14.9
$4^+_\gamma \to 4^+_{\rm g.s.}$	53.16(16)	44.9	44.9	127.2	130.3	43.6
$4^+_\gamma \to 6^+_{\rm g.s.}$		4.0	4.0	12.1	11.1	3.7
$5^+_\gamma \to 4^+_{\rm g.s.}$	43(43)	40.5	40.5	114.1	117.6	39.8
$5^+_\gamma \to 6^+_{\rm g.s.}$	59(59)	23.9	23.9	72.1	72.8	22.4

condition

$$\int_0^\infty \beta^{2(1+\frac{B_\beta}{B_\gamma})} F^2(\beta) d\beta = 1, \tag{69}$$

leading to

$$N_{n_\beta,L} = \left[2(2b)^{\frac{1}{2}q_{n_\gamma}(L,k)+1} \frac{n_\beta!}{\Gamma\left(n_\beta + \frac{1}{2}q_{n_\gamma}(L,k)+1\right)} \right]^{1/2}. \tag{70}$$

IX. $B(E2)$ TRANSITION RATES

The electric quadrupole operator for axially deformed nuclei around $\gamma = 0$ is given by [32]

$$T^{(E2)}_M = t\beta \left[\mathcal{D}^{(2)}_{M,0} \cos\gamma + \frac{1}{\sqrt{2}} \left(\mathcal{D}^{(2)}_{M,2} + \mathcal{D}^{(2)}_{\mu,-2} \right) \sin\gamma \right], \tag{71}$$

where t is a scaling factor. The first term describes $\Delta K = 0$ transitions and the second is for $\Delta K = 2$ transitions.

The $B(E2)$ transition rates from an initial to a final state are given by [36]

$$B(E2; L_i K_i \to L_f K_f) = \frac{5}{16\pi} \frac{|\langle L_f K_f \| T^{(E2)} \| L_i K_i \rangle|^2}{2L_i + 1}, \tag{72}$$

and the reduced matrix element can be obtained by using the Wigner-Eckrat theorem [36],

$$\langle L_f M_f K_f | T^{(E2)}_M | L_i M_i K_i \rangle = \frac{(L_i 2 L_f | M_i M M_f)}{\sqrt{2L_f + 1}} \langle L_f K_f \| T^{(E2)} \| L_i K_i \rangle. \tag{73}$$

TABLE VIII. The comparison of the theoretical predictions of $B(E2)$ in units of $B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ using the parameters given in Table I for ^{172}Yb in this work with those from Ref. [18] and experimental values.

	Expt.	$B_\beta \neq B_\gamma \neq B_{\text{rot}}$		$B_\beta = B_\gamma = B_{\text{rot}}$		Ref. [18]
		$a = 0$	DDM	$a = 0$	DDM	
		$\frac{B(E2;L_{\text{g.s.}}+2 \to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})}$				
$4^+_{\text{g.s.}} \to 2^+_{\text{g.s.}}$	1.42(10)	1.43	1.37	1.47	1.34	1.43
$6^+_{\text{g.s.}} \to 4^+_{\text{g.s.}}$	1.51(7)	1.59	1.41	1.70	1.36	1.59
$8^+_{\text{g.s.}} \to 6^+_{\text{g.s.}}$	1.89(19)	1.67	1.36	1.90	1.31	1.67
$10^+_{\text{g.s.}} \to 8^+_{\text{g.s.}}$	1.77(11)	1.74	1.29	2.11	1.26	1.74
$12^+_{\text{g.s.}} \to 10^+_{\text{g.s.}}$		1.79	1.31	2.32	1.22	1.79
		$\frac{B(E2;L_\beta \to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} \times 10^3$				
$2^+_\beta \to 0^+_{\text{g.s.}}$	1.1(1)	2.4	3.6	23.5	23.5	2.4
$4^+_\beta \to 2^+_{\text{g.s.}}$		2.0	2.1	19.9	19.9	2.0
$6^+_\beta \to 4^+_{\text{g.s.}}$		1.1	1.2	12.3	12.3	1.0
$2^+_\beta \to 2^+_{\text{g.s.}}$		4.8	4.6	47.1	47.1	4.17
$4^+_\beta \to 4^+_{\text{g.s.}}$		4.4	4.3	42.8	42.8	4.4
$6^+_\beta \to 6^+_{\text{g.s.}}$		4.3	4.2	42.1	42.1	4.3
$0^+_\beta \to 2^+_{\text{g.s.}}$		22.3	21.4	220.3	220.3	22.5
$2^+_\beta \to 4^+_{\text{g.s.}}$	12(1)	15.9	6.3	155.1	155.1	16.0
$4^+_\beta \to 6^+_{\text{g.s.}}$		18.4	17.1	173.7	173.7	18.8
$6^+_\beta \to 8^+_{\text{g.s.}}$		22.1	20.3	195.8	195.8	22.8
		$\frac{B(E2;L_\gamma \to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})} \times 10^3$				
$2^+_\gamma \to 0^+_{\text{g.s.}}$	6.3(5)	6.3	3.7	42.3	23.4	6.3
$2^+_\gamma \to 2^+_{\text{g.s.}}$		9.0	5.5	62.6	34.9	9.0
$2^+_\gamma \to 4^+_{\text{g.s.}}$	0.60(5)	0.5	0.3	3.4	1.8	0.4
$3^+_\gamma \to 2^+_{\text{g.s.}}$		11.3	10.3	77.5	43.0	11.2
$3^+_\gamma \to 4^+_{\text{g.s.}}$		4.6	4.2	33.4	18.1	4.4
$4^+_\gamma \to 2^+_{\text{g.s.}}$	33(24)	3.8	3.4	25.5	14.1	3.8
$4^+_\gamma \to 4^+_{\text{g.s.}}$		11.1	10.1	81.1	43.8	11.0
$4^+_\gamma \to 6^+_{\text{g.s.}}$		1.0	0.9	7.7	3.6	0.9
$5^+_\gamma \to 4^+_{\text{g.s.}}$		10.1	9.1	72.5	39.0	10.0
$5^+_\gamma \to 6^+_{\text{g.s.}}$		5.8	5.2	45.9	23.0	5.7

The final result [37] reads

$$B(E2;n_\beta L n_\gamma K \to n'_\beta L' n'_\gamma K') = \frac{5}{16\pi}\langle L,K,2,K'-K|L',K'\rangle^2 I^2_{n_\beta L,n'_\beta L'} C^2_{n_\gamma K,n'_\gamma K'} \quad (74)$$

with

$$I_{n_\beta,L,n'_\beta,L'} = \int \beta F_{L,n_\beta}(\beta)F_{L',n'_\beta}(\beta)\beta^{-2-2\frac{B_\beta}{B_\gamma}}d\beta = \int \beta R_{L,n_\beta}(\beta)R_{L',n'_\beta}(\beta)d\beta, \quad (75)$$

where $C_{n_\gamma K,n'_\gamma K'}$ contains the integral over γ. For $\Delta K = 0$ corresponding to transitions (g.s. $\to$ g.s., $\gamma \to \gamma, \beta \to \beta$ and $\beta \to$ g.s.), the γ-integral part reduces to the orthonormality condition of the γ-wave functions: $C_{n_\gamma K,n'_\gamma K'} = \delta_{n_\gamma,n'_\gamma}\delta_{K,K'}$. While for $\Delta K = 2$ corresponding to transitions ($\gamma \to$ g.s., $\gamma \to \beta$), this integral takes the form

$$C_{n_\gamma K,n'_\gamma K'} = \int \sin\gamma\,\eta_{n_\gamma K}\eta_{n'_\gamma K'}|\sin 3\gamma|d\gamma. \quad (76)$$

In the next sections, all values of $B(E2)$ are calculated in units of $B(E2;2^+_1 \to 0^+_1)$.

X. NUMERICAL RESULTS AND DISCUSSION

Before starting any calculations of the energy spectra and transition rates for the axially symmetric prolate deformed nuclei ^{154}Sm, ^{156}Gd, ^{172}Yb, and ^{182}W, which have been the object of Ermamatov *et al.* study [20], and before trying to improve them within the DDMF, we have to reevaluate the parameters of the problem through the corrected formulas of these nuclear characteristics Eqs. (54) and (55). For this purpose, we determine the free parameters $B_\gamma/B_\beta, g$, and g_β

TABLE IX. The comparison of the theoretical predictions of $B(E2)$ in units of $B(E2;2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ using the parameters given in Table I for ^{182}W in this work with those from Ref. [18] and experimental values.

	Expt.	$B_\beta \neq B_\gamma \neq B_{\text{rot}}$		$B_\beta = B_\gamma = B_{\text{rot}}$		Ref. [18]
		$a=0$	DDM	$a=0$	DDM	
		$\frac{B(E2;L_{\text{g.s.}}+2\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}$				
$4^+_{\text{g.s.}} \to 2^+_{\text{g.s.}}$	1.43(8)	1.44	1.36	1.49	1.48	1.44
$6^+_{\text{g.s.}} \to 4^+_{\text{g.s.}}$	1.46(6)	1.60	1.40	1.74	1.72	1.60
$8^+_{\text{g.s.}} \to 6^+_{\text{g.s.}}$	1.53(10)	1.71	1.35	1.98	1.92	1.71
$10^+_{\text{g.s.}} \to 8^+_{\text{g.s.}}$	1.48(9)	1.79	1.28	2.22	2.11	1.80
$12^+_{\text{g.s.}} \to 10^+_{\text{g.s.}}$		1.87	1.23	2.48	2.30	1.88
		$\frac{B(E2;L_\beta\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}\times 10^3$				
$2^+_\beta \to 0^+_{\text{g.s.}}$	6.6(10)	4.4	4.4	25.1	25.1	5.2
$4^+_\beta \to 2^+_{\text{g.s.}}$		3.2	3.2	19.2	19.2	2.1
$6^+_\beta \to 4^+_{\text{g.s.}}$		1.4	1.4	10.5	10.5	0.1
$2^+_\beta \to 2^+_{\text{g.s.}}$	4.6(6)	9.3	9.3	53.6	53.6	13.7
$4^+_\beta \to 4^+_{\text{g.s.}}$		8.4	8.4	48.8	48.8	12.5
$6^+_\beta \to 6^+_{\text{g.s.}}$		8.3	8.3	47.9	47.9	12.2
$0^+_\beta \to 2^+_{\text{g.s.}}$		44.9	44.9	262.3	262.3	77.0
$2^+_\beta \to 4^+_{\text{g.s.}}$	13(1)	33.1	33.1	191.1	191.1	64.1
$4^+_\beta \to 6^+_{\text{g.s.}}$		39.4	39.4	216.3	216.3	81.3
$6^+_\beta \to 8^+_{\text{g.s.}}$		47.81	47.81	243.2	243.2	101.6
		$\frac{B(E2;L_\gamma\to L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\to 0^+_{\text{g.s.}})}\times 10^3$				
$2^+_\gamma \to 0^+_{\text{g.s.}}$	24.8(8)	24.8	24.8	70.2	72.1	24.8
$2^+_\gamma \to 2^+_{\text{g.s.}}$	49.2(13)	35.7	35.7	104.2	106.5	35.3
$2^+_\gamma \to 4^+_{\text{g.s.}}$	0.2(2)	1.8	1.8	5.6	5.7	1.8
$3^+_\gamma \to 2^+_{\text{g.s.}}$		44.5	44.6	129.2	132.5	44.2
$3^+_\gamma \to 4^+_{\text{g.s.}}$		18.1	18.1	56.0	56.8	17.6
$4^+_\gamma \to 2^+_{\text{g.s.}}$	17.2(17)	14.8	14.8	42.6	43.8	14.8
$4^+_\gamma \to 4^+_{\text{g.s.}}$	75.9(73)	44.3	44.3	136.3	138.7	43.3
$4^+_\gamma \to 6^+_{\text{g.s.}}$		3.9	3.8	13.0	11.8	3.7
$5^+_\gamma \to 4^+_{\text{g.s.}}$		40.0	40.1	122.2	125.0	39.5
$5^+_\gamma \to 6^+_{\text{g.s.}}$		23.4	23.4	77.5	78.1	22.3

from the experimental data of $E(2^+_\gamma)/E(2^+_1)$, $E(0^+_\beta)/E(2^+_1)$, and $B(E2;2^+_\gamma \to 0^+_1/B(E2;2^+_1 \to 0^+_1)$ by solving a system of three nonlinear algebraic equations (Appendix B), while $B_\beta/B_{\rm rot}$ is fixed to the value given in Ref. [18]. With the new parameters (Table I) we have calculated the correct values that Ermamatov *et al.* [20] should obtain the ratios $E(L^+_{\rm g.s.})/E(2^+_{\rm g.s.})$ for the ground-state band, $E(L^+{}_\beta)/E(2^+_{\rm g.s})$ for the β band, and $E(L^+_\gamma)/E(2^+_{\rm g.s.})$ for the γ band. Here $E(L^+_i)$ $(i = {\rm g.s.},\beta,\gamma)$ is the energy of the level characterized by the angular momentum L^+ in the band i and $E(2^+{}_{\rm g.s.})$ is the energy of the first excited level of the ground-state band. As a qualitative test of agreement between the theoretical results and the experimental data, we evaluated the rms differences given by

$$\sigma = \sqrt{\frac{\Sigma^n_{i=1}[E_i({\rm exp}) - E_i({\rm th})]^2}{(n-1)E(2^+_1)^2}}, \qquad (77)$$

where $E_i({\rm exp})$ is the experimental energy of the $i^{\rm th}$ level, $E_i({\rm th})$ the corresponding theoretical value, n the maximum number of considered levels, and $E(2^+_1)$ the head energy of the band under consideration.

In Table II, we compare our results for ^{154}Sm in both cases $B_\beta \neq B_\gamma \neq B_{\rm rot}$ (the third column with $a = 0$) and $B_\beta = B_\gamma = B_{\rm rot}$ (the fifth column with $a = 0$) with experimental data [38] and the data from Ref. [18]. One can see that our results for $B_\beta \neq B_\gamma \neq B_{\rm rot}$ agree with experimental data, particularly in the β and γ bands ($\sigma < 1$), but slightly differ from the data of Ref. [18]. This slight discrepancy could be reduced within the framework of the DDMF. While in the ground-state band the precision of our results ($\sigma > 1$) is obviously affected by the energy value of the level $L = 12$, which is nearly 10% higher than the experimental one. From the same table, we can also see that the obtained values in the case $B_\beta \neq B_\gamma \neq B_{\rm rot}$ are more precise ($\sigma_{\rm total} < 1$) than those for which $B_\beta = B_\gamma = B_{\rm rot}$ ($\sigma_{\rm total} > 1$). For ^{156}Gd (Table III) our results are relatively better in the γ band for $B_\beta \neq B_\gamma \neq B_{\rm rot}$ but are globally more precise than for $B_\beta = B_\gamma = B_{\rm rot}$. Moreover, our energy spectrum for ^{172}Yb given in Table IV well reproduce the standard ones, particularly in the ground-state and γ bands with $B_\beta \neq B_\gamma \neq B_{\rm rot}$ unlike those of the case where these mass parameters are taken to be equal to 1. On the other hand, our results for the nucleus ^{182}W (Table IV) are more accurate ($\sigma < 1$) in the three bands with $B_\beta \neq B_\gamma \neq B_{\rm rot}$ than in the case of $B_\beta = B_\gamma = B_{\rm rot}$.

In order to improve the obtained numerical results, we recalculated the energy ratios in the framework of the DDMF with the more elaborated formula given in Eq. (49). Such a formula contains two supplementary parameters, namely a and β_0. The optimal values of both parameters are evaluated through rms fits of energy levels by making use of Eq. (77) for each band of each nucleus.

From Tables II–V one can see that a fair enhancement of numerical results has been achieved within the DDMF in both cases $B_\beta \neq B_\gamma \neq B_{\rm rot}$ and $B_\beta = B_\gamma = B_{\rm rot}$. Indeed, from the numerical calculations for nuclei ^{154}Sm, ^{156}Gd, ^{172}Yb, and ^{182}W, we note that the precision in the case of $B_\beta \neq B_\gamma \neq B_{\rm rot}$ increases with the mass number.

TABLE X. The comparison of the theoretical predictions of $E(L^+_i)$ ($i = {\rm g.s.},\beta,\gamma$ bands) normalized to the energy of the first excited state $E(2^+_{\rm g.s.})$ using the parameters given in Table I for ^{154}Sm and ^{182}W in this work with those from IBM-1 Refs. [39,40] and experimental values Ref. [38].

	^{154}Sm				^{182}W			
L	Expt.	$a = 0$	DDM	IBM-1 [39]	Expt.	$a = 0$	DDM	IBM-1 [40]
g.s.								
4	3.26	3.31	3.31	3.19	3.29	3.32	3.29	3.33
6	6.63	6.89	6.89	7.33	6.80	6.91	6.78	6.95
8	11.01	11.65	11.65	12.44	11.44	11.71	11.40	12.00
10		17.12	17.65	17.07	18.33			
σ		0.490	0.490	1.127		0.350	0.039	0.775
β_1								
0	13.40	13.40	13.03	14.04	11.36	11.36	11.36	11.41
2	14.37	14.40	14.03	14.78	12.57	12.36	12.36	11.46
4	16.32	16.71	16.35	17.31	15.10	14.68	14.68	13.81
6	19.23	20.29	19.94	17.66		18.26	18.26	17.50
σ		0.651	0.501	1.158		0.332	0.332	1.204
γ_1								
2	17.56	17.56	18.01	18.53	12.21	12.21	12.24	12.41
3	18.77	18.47	18.97	18.97	13.31	13.16	13.19	12.47
4	20.30	19.68	20.26	21.72	14.43	14.41	14.46	14.94
5	22.01	21.18	21.86	24.12	16.24	15.97	16.03	15.48
6	23.73	22.96	23.78		17.70	17.83	17.90	18.69
σ		0.623	0.298	1.576		0.168	0.158	0.801

TABLE XI. The comparison of the theoretical predictions of $B(E2)$ in units of $B(E2; 2^+_{\text{g.s.}} \longrightarrow 0^+_{\text{g.s.}})$ using the parameters given in Table I for ^{154}Sm and ^{182}W in this work with those from IBM-1 Refs. [39,40] and experimental values in Ref. [38].

	^{154}Sm				^{182}W			
	Expt.	$a = 0$	DDM	IBM-1 [39]	Expt.	$a = 0$	DDM	IBM-1 [40]
				$\frac{B(E2;L_{\text{g.s.}}+2\longrightarrow L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\longrightarrow 0^+_{\text{g.s.}})}$				
$4^+_{\text{g.s.}} \longrightarrow 2^+_{\text{g.s.}}$	1.40(5)	1.44	1.44	1.35	1.43(8)	1.44	1.36	1.33
$6^+_{\text{g.s.}} \longrightarrow 4^+_{\text{g.s.}}$	1.67(7)	1.61	1.61	1.53				
				$\frac{B(E2;L_\beta\longrightarrow L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\longrightarrow 0^+_{\text{g.s.}})}\times 10^3$				
$2^+_\beta \longrightarrow 0^+_{\text{g.s.}}$	5.4(13)	6.5	6.4	7.78	6.6(10)	4.4	4.4	113.0
$4^+_\beta \longrightarrow 2^+_{\text{g.s.}}$		5.5	5.5	29.57		3.2	3.2	56.3
$2^+_\beta \longrightarrow 2^+_{\text{g.s.}}$		12.9	12.8	15.33	4.6(6)	9.3	9.3	48.4
$2^+_\beta \longrightarrow 4^+_{\text{g.s.}}$	25(6)	42.2	41.4	30.67	13(1)	33.1	33.1	3.7
				$\frac{B(E2;L_\beta\longrightarrow L_{\text{g.s.}})}{B(E2;2^+_{\text{g.s.}}\longrightarrow 0^+_{\text{g.s.}})}\times 10^3$				
$2^+_\gamma \longrightarrow 0^+_{\text{g.s.}}$	18.4(29)	18.4	16.5	16.43				
$2^+_\gamma \longrightarrow 2^+_{\text{g.s.}}$		26.5	23.6	24.10				
$2^+_\gamma \longrightarrow 4^+_{\text{g.s.}}$	3.9(6)	1.4	1.2	0.99	0.2(2)	1.8	1.8	153.6

Similarly, we have also calculated transition rates $B(E2; L^+_{\text{g.s.}}+2 \to L^+_{\text{g.s.}})$, $B(E2; L^+_\beta \to L^+_{\text{g.s.}})$, and $B(E2; L^+_\gamma \to L^+_{\text{g.s.}})$ in units of $B(E2; 2^+_{\text{g.s.}} \to 0^+_{\text{g.s.}})$ for the same nuclei in both cases $B_\beta \neq B_\gamma \neq B_{\text{rot}}$ and $B_\beta = B_\gamma = B_{\text{rot}}$ within and without the DDMF. Within the DDMF, we have used the same optimal values of the two parameters a and β_0 previously obtained for the energy ratios.

Then, in Tables VI–IX, it is clearly shown that our results in the case of $B_\beta \neq B_\gamma \neq B_{\text{rot}}$ are better than those with $B_\beta = B_\gamma = B_{\text{rot}}$. Indeed, in the β band and in the case of different mass coefficients with $a = 0$, the mean difference between the theoretical value of transition rate and the experimental one corresponding to transition $2^+_\beta \to 0^+_{\text{g.s.}}$ is about 1.8, while in the case of equal mass parameters, it is equal to 20. Likewise, in the transition $2^+_\beta \to 4^+_{\text{g.s.}}$, the mean difference between the theory and the experiment in the first case is about 15.9 and in the second one it is equal to 152. For $a \neq 0$, in the case of different mass coefficients, for the same transitions $2^+_\beta \to 0^+_{\text{g.s.}}$ and $2^+_\beta \to 0^+_{\text{g.s.}}$, the mean difference value is about 1.7 and 14, respectively, while in the case of equal mass coefficients it is equal to 20.3 and 152.5, respectively. Such a fact can also be seen in the γ band. We underline here that, in the case of equal mass parameters, the obtained results reproduce those of Bonatsos *et al.* [14]. The slight difference between them came from the fact that the Bonatsos *et al.* fitting calculations have been carried on a given number of levels which differ from the number we considered in our calculations. This is further proof that our formulas, given in Eq. (49) and Eq. (64), respectively, for the energy and the wave functions, are more accurate than those erroneously derived by Ermamatov *et al.* [20]. Moreover, this comparison corroborates the fact that the mass parameter should be taken into account in such calculations. As mentioned in the Introduction, the Bohr Hamiltonian is a quite competitive method in respect to other methods like IBM-1 [6]. To make a simple comparison between them, we give in Tables X–XI our obtained results compared with the available IBM-1 data.

XI. CONCLUSION

In this paper we have revisited all calculations performed in a recent work [20] based on inaccurate formulas for the energy spectrum and transition rates for axially symmetric prolate nuclei. With the asymptotic iteration method we have derived the correct formulas for these nuclear observables. Morover, we have extended our calculations into a deformation-dependent effective masses formalism in order to improve the numerical results. Moreover, we have shown the importance of the mass parameter to be introduced in numerical calculations, unlike what has been done by other authors who have neglected the important role played by this parameter in such calculations. Through a comparison with IBM-1, the Bohr Hamiltonian with mass parameters has proved to be more accurate.

APPENDIX A: ASYMPTOTIC ITERATION METHOD (AIM)

The asymptotic iteration method [25] is proposed to solve the second-order homogeneous differential equation of the form

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x), \tag{A1}$$

where the variables λ_0 and s_0 are sufficiently differentiable.

The differential equation (A1) has a general solution [25],

$$y(x) = \exp\left[-\int^{x} \alpha(x_1)dx_1\right]\left(C_2 + C_1\int^{x} \exp\left\{\int^{x_1}[\lambda_0(x_2) + 2\alpha(x_2)]dx_2\right\}dx_1\right). \tag{A2}$$

If we have $n > 1$, then for sufficiently large n $\alpha(x)$ values can be obtained,

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x), \tag{A3}$$

with the sequences

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \tag{A4a}$$

$$s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x), \quad n = 1,2,3,\ldots, \tag{A4b}$$

and the energy eigenvalues are then computed by means of the following termination condition [25]:

$$\delta = s_n\lambda_{n-1} - \lambda_n s_{n-1} = 0. \tag{A5}$$

APPENDIX B: FORMULAS USED FOR THE CALCULATIONS OF THE $B(E2)$

In this Appendix we present the expressions used for calculations of the transition probabilities $B(E2)$:

$$\frac{B(E2; L'^{+}_{\text{g.s.}} \to L^{+}_{\text{g.s.}})}{B(E2; 2^{+}_{\text{g.s.}} \to 0^{+}_{\text{g.s.}})} = 5\left(C^{L0}_{L'020}\right)^2\left(\frac{\Gamma\{0.5[q_0(L',0) + q_0(L,0)] + 1.5\}}{\Gamma[0.5[q_0(2,0) + q_0(0,0)] + 1.5]}\right)^2 \frac{\Gamma[q_0(2,0)+1]\Gamma[q_0(0,0)+1]}{\Gamma[q_0(L',0)+1]\Gamma[q_0(L,0)+1]} \tag{B1}$$

$$\frac{B(E2; L'^{+}_{\beta} \to L^{+}_{\text{g.s.}})}{B(E2; 2^{+}_{\text{g.s.}} \to 0^{+}_{\text{g.s.}})} = \frac{5}{4}\left(C^{L0}_{L'020}\right)^2\left(\frac{\Gamma\{0.5[q_0(L',0) + q_0(L,0)] + 1.5\}}{\Gamma\{0.5[q_0(2,0) + q_0(0,0)] + 1.5\}}\right)^2 \frac{\Gamma[q_0(2,0)+1]\Gamma[q_0(0,0)+1]}{\Gamma[q_0(L',0)+1]\Gamma[q_0(L,0)+1]} \times \frac{[q_0(L',0) - q_0(L,0) - 1]^2}{q_0(L',0)+1} \tag{B2}$$

$$\frac{B(E2; L'^{+}_{\gamma} \to L^{+}_{\text{g.s.}})}{B(E2; 2^{+}_{\text{g.s.}} \to 0^{+}_{\text{g.s.}})} = 5g\left(C^{L0}_{L'020}\right)^2\left(\frac{\Gamma\{0.5[q_0(L',0) + q_0(L,0)] + 1.5\}}{\Gamma\{0.5[q_0(2,0) + q_0(0,0)] + 1.5\}}\right)^2 \frac{\Gamma[q_0(2,0)+1]\Gamma[q_0(0,0)+1]}{\Gamma[q_0(L',0)+1]\Gamma[q_0(L,0)+1]}, \tag{B3}$$

where $C^{L0}_{L'020}$ is the Clebsch-Gordan coefficient.

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