Spin dependence of the incident channel distorted wave in the theory of the $A(d,p)B$ reaction

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The adiabatic distorted wave approximation (ADWA) model of $A(d,p)B$ reactions is generalized to include the effects of nucleon-nucleus spin-orbit potentials and the neutron-proton tensor force and associated D-state terms in the interaction between the neutron and proton in the incident deuteron. The ADWA predicts that the incident channel distorting potential must include momentum-dependent rank-2 spin tensor force terms with novel properties not normally considered in the deuteron-nucleus scattering.

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I. INTRODUCTION

In the adiabatic distorted wave approximation (ADWA) model of the $A(d,p)B$ reaction the transition amplitude has a similar structure to the distorted wave Born approximation (DWBA). Reference [1] gives a review of the origins of the ADWA and a discussion of open problems. Both models use a distorted wave to describe the initial and final state interactions. However, in the ADWA the distorted wave in the incident channel does not describe elastic deuteron scattering from A, but is rather a coherent superposition of elastic scattering and continuum deuteron break-up components projected onto the region of configuration space in which the neutron, n, and proton, p, of the incident deuteron are within the range of the n-p interaction V_{np}. For a recent successful application of the ADWA method to the analysis of experimental data see Ref. [2]. This application involves a version of the ADWA that does not include full spin dependence and considers only differential cross sections.

Other methods of calculating (d,p) reactions which go beyond the DWBA have been developed. In particular the continuum discretized coupled-channels (CDCC)x and Faddeev methods have helped to throw some light on the validity of the ADWA. It has been shown [3] using the CDCC method that for typical $A(d,p)B$ reactions at low energies of current experimental interest the transition matrix is completely dominated by the first component of the Weinberg expansion of the three-body scattering wave function. This result suggests that as far as calculating the (d,p) transition matrix is concerned it would be useful to focus on finding viable practical methods of calculating the first Weinberg component of the three-body wave function, rather than following the DWBA and CDCC routes which calculate the three-body wave function everywhere. This is precisely the thrust of the ADWA method.

Unlike Faddeev methods that are based on three-body models of the $d + A$ nucleus system, the ADWA method focuses on the evaluation of a many-body expression for the $A(d,p)B$ and therefore lends itself well to the practical analysis of experiments designed to probe the overlap functions of nuclei A and B. Faddeev calculations of three-body models cannot be used in this way. However, Faddeev calculations can be used to check the ADWA when both the ADWA and the Faddeev calculations are based on the same three-body Hamiltonian.

There is evidence from comparisons of this type [4] that the ADWA is inadequate at very low and very high energies. These calculations are based on a Faddeev model that cannot fully account for the true many-body problem; for example, the treatment of neutron absorption in the Faddeev calculations in Ref. [4] is not consistent. However, the results in Ref. [4] do indicate that over a large energy range the ADWA gives an excellent first approximation to the component of the three-body scattering wave function of most relevance to the (d,p) reaction.

The Faddeev approach has been generalized [5] to take into account explicitly some features arising from the internal degrees of freedom of nucleus A as well as treating the three-body aspects completely, but this method has not yet been developed to the point that it can be used as a tool for analyzing experimental data in a routine way. In any case both the Faddeev and CDCC methods require the input of far more information than is probably needed to calculate the first Weinberg component to useful accuracy.

This paper focuses on the spin dependence of the ADWA distorting potential in the incident channel of the $A(d,p)B$ reaction. The results are probably of most relevance to $A(d,p)B$ experiments that involve polarized nuclei as beams or targets. In fact such an experiment with polarized deuteron beams and proton polarization measurements was the first to show convincingly that failure of DWBA calculations to give a good account of experimental data could be remedied by an ADWA approach [6]. It is a challenge to theory to see how far the results obtained here will be of use in understanding the modern polarization experiments with exotic beams that are beginning to become available [7].

The Faddeev calculations reported in Ref. [4] included both a tensor force in the n-p interaction and spin-orbit forces in the nucleon-A potentials. Therefore, the calculated transition amplitudes used in Ref. [4] must contain spin-dependent effects that would be included in an approximate way in an ADWA calculation using the spin-dependent ADWA potential described here. Experience [6] shows that these effects are most likely to show up in the calculation of nucleon and deuteron polarizations and analyzing powers which were not discussed in Ref. [4]. A virtue of the ADWA approach is that the tensorial nature and origin of any spin dependence

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is relatively transparent. It would certainly be of interest to compare polarization observables, especially deuteron analyzing powers, with the two methods.

Standard CDCC calculations [8] do not fully take into account spin-dependent couplings and therefore cannot reproduce the spin-dependent effects reported here. CDCC calculations of (d,p) observables including both the n-p tensor force and nucleon-A spin-orbit potentials have been reported [9]. It is not clear how far complete spin couplings were taken into account in the breakup channels in this work. It should be noted that difficulties in accounting for polarization data were reported. None of these calculations, or those reported in Refs. [2,4], took into account recent insights [1,10] into the role of nonlocality in the theory of the $A(d,p)B$ reaction.

Section II reviews the definition of the ADWA distorting potential in terms of nucleon optical potentials and properties of the n-p system, including n-p tensor force effects. Sections III and IV explain how the contribution to the ADWA potential from the central and spin-orbit terms in the nucleon optical potentials is calculated. A comparison with earlier work is included where possible. A novel type of tensor force in the ADWA is described in Sec. IV H and a simplifying low-energy approximation is derived and interpreted in Secs. IV I and IV J. Explicit formulas for the various components of the ADWA distorting potential are collected in Sec. V and the paper ends with some comments in Sec. VI. In Appendix C the general symmetry properties of the ADWA potential, including Hermiticity and time-reversal invariance, are discussed. It is explained why the ADWA potential cannot be Hermitian in general even when the underlying two-body forces are Hermitian. Appendix D gives explicit formulas for the matrix elements of the novel tensor force obtained in Sec. IV in a convenient basis for coupled-channel calculations of the ADWA distorting potential.

II. THE ADWA DISTORTING POTENTIAL

It has been shown that in a well defined approximation [3], and under simplifying assumptions about the effective interactions of the three-body model [1,10], the projection of the $d+A$ scattering wave function most relevant to the $A(d,p)B$ transition amplitude should be calculated as a distorted wave in a potential that is defined in terms of local n-A and p-A optical potentials corresponding to a well defined energy related to the incident deuteron kinetic energy and an average kinetic energy associated with the n-p interaction, V_{np} [10–12]. Apart from terms linear in momentum arising from the nucleon-nucleus spin-orbit force, nonlocalities are not treated explicitly. It is assumed here that the effective interaction in a three-body model of the $n+p+A$ system is $V_{nA}(n)+V_{pA}(p)$, the sum of local n-A and p-A optical potentials evaluated at nucleon kinetic energies as described in Ref. [10]. This means that the incident channel distorted wave in the ADWA transition amplitude is calculated using the potential

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{d'},M_d} = \langle \phi^{(1)}_{M_{d'}} | (V_{nA}+V_{pA}) | \phi^{(0)}_{M_d} \rangle. \quad (1)$$

The bra and ket on the right-hand side in Eq. (1) are states in the space of n and p spin variables and the relative n-p coordinate $\boldsymbol{r}$ and are defined below. The notation used for the matrix element on the right implies an inner product in this space to give an operator in the space of $\boldsymbol{R}$, the relative coordinate of the n-p center of mass and the target nucleus A. Current versions of the ADWA assume that there is only coupling via elastic breakup to states in which the total internal angular momentum of the n-p system is $S_d=1$. As a result V^{JT} is an operator in the space of a spin-1 particle with $M_{d'},M_d$ labeling possible projections on the z axis. An estimate of the contribution from singlet n-p, $L=0$, states can be found in Ref. [13]. Coupling to singlet states without target excitation arises from the isospin dependence of the nucleon-nucleus spin-orbit force. There is great uncertainty in the magnitude of these terms in the nucleon optical potential but the effects found in Ref. [13] are not large.

It is shown below that the spin-orbit components of $V_{nA}(n)$ and $V_{pA}(p)$ generate a dependence on the momentum operator, $\boldsymbol{p}_R(=-\imath\nabla_R)$, associated with $\boldsymbol{R}$ and therefore appear as one of the arguments of V^{JT} in Eq. (1).

The expression for V^{JT} in Eq. (1) first appeared in work by Johnson and Tandy in Ref. [14]. A superscript JT will be used to distinguish it from the zero-range form introduced earlier by Johnson and Soper [15]. Both of these potentials are designed to include the coherent superposition of elastic deuteron and breakup components that contribute to the $A(d,p)B$ transition amplitude. The Johnson-Soper potential is an approximation treatment of the same physics as the Johnson-Tandy potential. Both of them should be distinguished from the Watanabe potential discussed below which also involves a folding of nucleon optical potentials but describes completely different physics. The potential V^{JT} defined in Eq. (1) is referred to as the ADWA distorting potential.

The ket on the right of the V^{JT} matrix element is the deuteron ground-state wave function, $\phi^{(0)}_{M_d}$, where $M_d=0,\pm1$ is the projection of the deuteron ground-state spin on the z axis. The state $\phi^{(0)}_{M_d}$ is assumed to be a combination of components with orbital angular momentum $L=0$ and $L=2$ and radial components $\phi^{(0)}_L(r)$ and therefore has the form

$$\phi^{(0)}_{M_d}(\boldsymbol{r},p,n) = \sum_{L=0,2,\Lambda,\sigma} (L\,\Lambda,1\sigma|S_d\,M_d) \times Y_{L\,\Lambda}(\boldsymbol{r})\chi_{1,\sigma}(p,n)\phi^{(0)}_L(r). \quad (2)$$

Both components have the intrinsic spins of n and p in a spin triplet state $\chi_{1,\sigma}(p,n),\sigma=\pm1,0$, which is coupled to the angular momentum to give a definite total angular momentum. For clarity in some of the subsequent formulas this total angular momentum is written S_d. It has the numerical value 1. The operator $(\boldsymbol{S}_d)^2$ that appears as $\boldsymbol{S}_d$ in the argument of V^{JT} in Eq. (1) has eigenvalue $S_d(S_d+1)=2$.

The state in the bra on the right-hand side of Eq. (1) is defined by

$$|\phi^{(1)}_{M_{d'}}\rangle = \frac{V_{np}|\phi^{(0)}_{M_{d'}}\rangle}{\langle\phi^{(0)}_{M_{d'}}|V_{np}|\phi^{(0)}_{M_{d'}}\rangle}. \quad (3)$$

Note that the denominator in this expression is independent of $M_{d'}$ if V_{np} is a tensor of rank zero in combined $\boldsymbol{r}-$ and spin space.

It is readily seen from the Schrödinger equation satisfied by $\phi^{(0)}_{M_{d'}}$ that $\phi^{(1)}_{M_{d'}}$ has a similar structure and can be written

$$\phi^{(1)}_{M_{d'}}(\boldsymbol{r},p,n) = \sum_{L'=0,2,\Lambda',\sigma'} (L'\,\Lambda,1\sigma'|S_d\,M_{d'}) \times Y_{L'\,\Lambda'}(\boldsymbol{r})\chi_{1,\sigma'}(p,n)\phi^{(1)}_{L'}(r), \tag{4}$$

where

$$\phi^{(1)}_{L'}(r) = C_{np}(-\epsilon_0 - T_{L'})\phi^{(0)}_{L'}(r), \tag{5}$$

where C_{np} is defined as

$$C_{np} = \frac{1}{\langle\phi^{(0)}_{M_{d'}}|V_{np}|\phi^{(0)}_{M_{d'}}\rangle} \tag{6}$$

and ϵ_0 is the deuteron binding energy. The radial operator $T_{L'}$ is the n-p kinetic energy operator in a state with orbital angular momentum L'. It is defined by

$$T_{L'} = -\frac{\hbar^2}{2\mu_{np}}\left[\frac{1}{r}\frac{d^2}{d\,r^2}\,r - \frac{L'(L'+1)}{r^2}\right], \tag{7}$$

where μ_{np} is the n-p reduced mass.

The Watanabe optical potential for elastic deuteron scattering

A formal expression similar to Eq. (1) appears in folding models of the optical potential for elastic deuteron scattering in which deuteron elastic breakup is neglected. This approximation is often referred to as the Watanabe model [16]. It is interesting to compare this model with the ADWA potential.

The Watanabe model leads to an expression like Eq. (1) except that $\phi^{(1)}_{M_{d'}}$ in the bra is replaced by $\phi^{(0)}_{M_{d'}}$. The Watanabe potential is therefore given by

$$V^{\rm Watanabe}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{d'},M_d} = \langle\phi^{(0)}_{M_{d'}}|(V_{nA}+V_{pA})|\phi^{(0)}_{M_d}\rangle. \tag{8}$$

I show that the ADWA potential has qualitative features that are quite different from the Watanabe potential. In the former case the presence of the function $\phi^{(1)}_{M_{d'}}$, which has a much shorter range than weakly bound deuteron ground-state function $\phi^{(1)}_{M_{d'}}$, also has important quantitative effects.

III. CONTRIBUTION TO THE ADWA DISTORTING POTENTIAL FROM THE CENTRAL NUCLEON OPTICAL POTENTIALS

The evaluation of the central part of V^{JT} using a realistic n-p interaction, including D-state effects, has been thoroughly explored quantitatively in Ref. [17]. The purpose of the present paper is discuss new features in the spin dependence of V^{JT} that arise from the n-p tensor force. The main new effects appear in the predicted spin dependence of V^{JT} in which a new type of tensor force appears.

For completeness I first give expressions for the contributions to V^{JT} from the central neutron optical potential. These results are a simple generalization of the results already given in Ref. [18], except that I include terms quadratic in the deuteron D-state components ignored in Ref. [18].

The central parts of $V_{nA}(n)$ and $V_{pA}(p)$ are denoted $V^{\rm Cen}_{nA}(n)$ and $V^{\rm Cen}_{pA}(p)$. Formulas for the ADWA potential that arise from these terms are very similar to the corresponding Watanabe potential, although quantitatively there are important differences. I consider the neutron contribution explicitly. The proton contributions have exactly the same form.

The central neutron optical potential depends only on the magnitude $|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}|$ and therefore can be expanded in the form

$$V^{\rm Cen}_{nA}(|\boldsymbol{R}+\boldsymbol{r}/2|) = \sum_{k,q} Y^*_{k\,q}(\boldsymbol{R})\,Y_{k\,q}(\boldsymbol{r})V^{\rm Cen}_{nA}(k,R,r), \tag{9}$$

where

$$V^{\rm Cen}_{nA}(k,R,r) = \int d\Omega_r\, V^{\rm Cen}_{nA}(|\boldsymbol{R}+\boldsymbol{r}/2|)P_k(\cos\theta(\boldsymbol{R},\boldsymbol{r})) \tag{10}$$

and

$$\cos\theta(\boldsymbol{R},\boldsymbol{r}) = \frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R\,r}. \tag{11}$$

Inserting this expansion into Eq. (1) and separating the integration over the direction and magnitude of $\boldsymbol{r}$ gives

$$\begin{aligned}(V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,{\rm Cen}})_{M_{d'},M_d} &= \sum_{k,q} Y^*_{k\,q}(\boldsymbol{R})\sum_{L,L'}\langle(L',1)S_d|Y_{k\,q}(\boldsymbol{r})|(L,1)S_d\rangle \\ &\times \int r^2 dr\,\phi^{(1)*}_{L'}(r)V^{\rm Cen}_{nA}(k,R,r)\phi^{(0)}_{L}(r),\end{aligned} \tag{12}$$

where now the angular brackets denote an integration over the intrinsic n and p spins and the direction of $\boldsymbol{r}$ only. This integration can be carried out using the Wigner-Eckart theorem and the general result given in Eq. (5.9), p. 81, of Ref. [19]. The reduced matrix element of $Y_{k\,q}(\boldsymbol{r})$ is also standard formula given in Eq. (4.17), p. 57, of Ref. [19]. In the notation used here one finds

$$\langle Y_L||Y_k||Y_L\rangle = \frac{\hat{L}\hat{k}}{\sqrt{4\pi}\hat{L}'}(L\,0,k\,0|L'\,0). \tag{13}$$

In this way I obtain

$$\begin{aligned}\langle(L',1)S_d\,M_{d'}|Y_{k\,q}(\boldsymbol{r})|(L,1)S_d\,M_d\rangle &= (S_d\,M_d,k\,q|S_d\,M_{d'})(-1)^{S_d+L-k-1}\frac{\hat{S}_d\hat{L}\hat{k}}{\sqrt{4\pi}}W(L\,L'\,S_d\,S_d;k\,1)(L\,0,k\,0|L'\,0) \\ &= (1\,M_{d'},|\tau_{k,q}(1)|1\,M_{d'})\frac{\sqrt{3}\hat{L}}{\sqrt{4\pi}}W(L\,L'\,1\,1;k\,1)(L\,0,k\,0|L'\,0),\end{aligned} \tag{14}$$

where the value $S_d = 1$ has been inserted. The fact that L and L are always even, and hence k must also be even, is also used. The operator $\tau_{k,q}(1)$ is a tensor of rank k constructed from the components of $\boldsymbol{S}_d$. It has matrix elements

$$\langle 1\, M_{d'}|\tau_{k\,q}(1)|1\, M_d\rangle = \hat{k}(1\, M_d, k\, q|1\, M_{d'}). \tag{15}$$

The contribution to V^{JT} from the central part of the n-A optical potential is

$$V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}} = \sum_{k=0,2,q} Y_k(\boldsymbol{R})\cdot\tau_k(1)\sum_{L,L'}\frac{\sqrt{3}\hat{L}}{\sqrt{4\pi}}W(L\,L'\,1\,1;k\,1)(L\,0,k\,0|L'\,0)F_k^{(1,0)}(L',L,R;\mathrm{Cen},nA), \tag{16}$$

where I have defined the radial integral

$$F_k^{(1,0)}(L',L,R;\mathrm{Cen},nA) = \int r^2 dr \phi_{L'}^{(1)*}(r)V_{nA}^{\mathrm{Cen}}(k,R,r)\phi_L^{(0)}(r). \tag{17}$$

I have used the notation

$$Y_k(\boldsymbol{R})\cdot\tau_k(1) = \sum_q Y_{k,q}^*\tau_{k,q}(1). \tag{18}$$

This expression has a central component $V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}}^{k=0}$ with $k=0$, and a rank-2 spin-tensor component, $V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}}^{k=2}$, with $k=2$. Explicitly

$$V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}}^{k=0} = \frac{1}{4\pi}\left[F_0^{(1,0)}(0,0,R;\mathrm{Cen},nA) + F_0^{(1,0)}(2,2,R;\mathrm{Cen},nA)\right], \tag{19}$$

and

$$V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}}^{k=2} = T_R(\boldsymbol{R},\boldsymbol{S}_d)\sum_{L,L'}\frac{3\sqrt{15}}{\sqrt{2}}\frac{\hat{L}}{4\pi}W(L\,L'\,1\,1;2\,1)(L\,0,2\,0|L'\,0)F_2^{(1,0)}(L',L,R;\mathrm{Cen},nA). \tag{20}$$

The operator $T_R(\boldsymbol{R},\boldsymbol{S}_d)$ is the standard form for one of the three possible rank-2 tensor forces for a spin-1 particle as defined by Satchler [20]:

$$\begin{aligned} T_R(\boldsymbol{R},\boldsymbol{S}_d) &= \frac{(\boldsymbol{S}_d\cdot\boldsymbol{R})^2}{R^2} - \frac{2}{3} \\ &= \frac{1}{3}\sqrt{\frac{8\pi}{5}}\,Y_2(\boldsymbol{R})\cdot\tau_2(\boldsymbol{S}_d), \end{aligned} \tag{21}$$

Note that the T_R term vanishes if $L=2$ terms in the deuteron coming from the n-p tensor force are neglected.

Comparison with the Watanabe optical potential

Watanabe model results equivalent to Eq. (16) can be found in Ref. [18], but with terms with $L = L' = 2$ missing. Unlike the Watanabe case, these terms are not necessarily small in V^{JT} because short range properties of V_{np} are involved through the factor $\phi_{M_d}^{(1)}$ in the definition of V^{JT} in Eq. (1).

Although the formulas given in Eqs. (19) and (20) are perfectly adequate for numerical applications, for ease of comparison with Ref. [18] I give here the explicit form of Eq. (20) that is obtained when the numerical values of the Racah coefficient, etc., are inserted:

$$V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{n,\mathrm{Cen}}^{k=2} = T_R(\boldsymbol{S}_d,\boldsymbol{R})U_{T_R(n,\mathrm{Cen})}^{JT}(R), \tag{22}$$

where the form factor is given by

$$\begin{aligned} U_{T_R(n,\mathrm{Cen})}^{JT}(R) = \frac{3}{\sqrt{2}}\frac{1}{4\pi}\Bigg[&F_2^{(1,0)}(0,2,R;\mathrm{Cen},nA) \\ &+ F_2^{(1,0)}(2,0,R;\mathrm{Cen},nA) \\ &- \frac{1}{\sqrt{2}}F_2^{(1,0)}(2,2,R;\mathrm{Cen},nA)\Bigg]. \end{aligned} \tag{23}$$

In the Watanabe case $F_k^{(0,0)}(0,2,R;\mathrm{Cen},nA) = F_k^{(0,0)}(2,0,R;\mathrm{Cen},nA)$ if phases are chosen so that the radial wave function $\phi_L^{(0)}(r)$ is real. The cross terms in Eq. (23) with $L=0, L'=2$ and $L'=2, L=0$ then agree with the result given in Ref. [18] when a correct identification is made with the notation used in Ref. [18] for $u(r)$, the $L=0$ component, and $w(r)$, the $L=2$ component of the deuteron wave function, i.e.,

$$\begin{aligned} \phi_0^{(0)} &= \frac{u(r)}{r}, \\ \phi_2^{(0)} &= \frac{1}{2}\frac{w(r)}{r}. \end{aligned} \tag{24}$$

In Ref. [18] the nuclear parts of the neutron and proton potentials are assumed to have identical forms, in which case the proton terms can be included simply by multiplying all the expressions (16), (19), (20), and (22) by a factor of 2.

IV. CONTRIBUTION TO THE ADWA DISTORTING POTENTIAL FROM THE NUCLEON-NUCLEUS POTENTIAL SPIN-ORBIT POTENTIAL

A. The nucleon-nucleus spin-orbit interaction

It is assumed that that the neutron-nucleus potential has the Thomas form

$$V^{\rm so}_{nA}(\boldsymbol{r}_n,\boldsymbol{s}_n) = \frac{1}{r_n}\frac{dU^{\rm so}_{nA}(r_n)}{dr_n}\boldsymbol{l}_n\cdot\boldsymbol{s}_n, \tag{25}$$

where $\boldsymbol{l}_n = \boldsymbol{r}_{nA}\wedge\boldsymbol{p}_n$ and p_n is the momentum of the neutron. Throughout this paper the nucleon spin-orbit form factor, $U^{\rm so}_{nA}(r_n)$, is assumed to be real.

For the purpose of the folding model being considered here it is convenient to use the alternative expression

$$V^{\rm so}_{nA}(\boldsymbol{r}_n,\boldsymbol{s}_n) = \frac{1}{\imath}\nabla_{r_n}\wedge U^{\rm so}_{nA}(r_n)\nabla_{r_n}\cdot\boldsymbol{s}_n, \tag{26}$$

where the ∇_{r_n} operators act on *everything* to the right that depends on the coordinate $\boldsymbol{r}_n$, including any state vector. I take the nucleon to be a neutron for definiteness. The equivalence of Eqs. (25) and (26) comes from the assumption that $U^{\rm so}_{nA}(r_n)$ depends only on the magnitude of $\boldsymbol{r}_n$ and that $\nabla_{r_n}\wedge\nabla_{r_n} = 0$.

For the deuteron-nucleus case it is necessary to express the right-hand side of Eq. (26) in terms of the coordinates of the center of mass of n and p relative to the target, $\boldsymbol{R}$, and $\boldsymbol{r}$ the relative coordinate of n and p. They are related to the n-A and p-A separation vectors, $\boldsymbol{r}_{nA}$ and $\boldsymbol{r}_{nA}$, by

$$\begin{aligned}\boldsymbol{R} &= \tfrac{1}{2}(\boldsymbol{r}_{nA}+\boldsymbol{r}_{pA}),\\ \boldsymbol{r} &= (\boldsymbol{r}_{nA}-\boldsymbol{r}_{pA}).\end{aligned} \tag{27}$$

I find

$$\begin{aligned}&\frac{1}{r_n}\frac{dU^{\rm so}_{nA}(r_n)}{dr_N}\boldsymbol{l_n}\cdot\boldsymbol{s}_n\\ &\quad= -2\imath\nabla_r\wedge U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}\right|\right)\nabla_r\cdot\boldsymbol{s}_n\\ &\qquad-\imath\frac{1}{2}\nabla_R\wedge U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}\right|\right)\nabla_R\cdot\boldsymbol{s}_n.\end{aligned} \tag{28}$$

Note that there are no cross terms involving different ∇ operators. This has the consequence that folding matrix elements of the neutron spin-orbit potential, such as in V^{JT}, display the expected symmetries more transparently.

The right-hand side of Eq. (28) does not take into account properly that the momentum $\boldsymbol{p}_n$ that appears in $\boldsymbol{l}_n$ should be the momentum of the neutron in the n-A center-of-mass system. As a result the term involving $\boldsymbol{R}$ operators should be multiplied by

$$C_2(A) = \frac{(A+2)}{(A+1)}, \tag{29}$$

and the term involving $\boldsymbol{r}$ operators should be multiplied by

$$C_1(A) = \frac{A}{(A+1)}. \tag{30}$$

I ignore these factors in the development, only restoring them in the final formulas.

B. Some basic results for the neutron spin-orbit contributions to V^{JT}

In general, V^{JT} is a linear combination of tensors of rank 0, 1, and 2 in the space of operator $\boldsymbol{S}_d$ with coefficients that transform like tensors of the same rank in $\boldsymbol{R}$ space constructed from the operators $\boldsymbol{R},\boldsymbol{p}_R$. I already show in Sec. III that the central parts of the neutron optical potential contribute a central term and rank-2 $T_R(\boldsymbol{R},\boldsymbol{S}_d)$ term arising from the deuteron D state. Before showing detailed new expressions I discuss the results obtained from the neutron spin-orbit contribution when the D state is neglected.

If D-state components are ignored the states $\phi^{(i)}_{M_{s'}}$, $i = 0,1$, are both $L = 0$ states and have the form

$$\phi^{(i)}_{M_{s'}}(\boldsymbol{r},n,p) = u^{(i)}_0(r)\chi_{1,M_{s'}}(n,p), \tag{31}$$

where, in the notation of Eqs. (2) and (4), $u^{(i)}_0(r) = \frac{1}{\sqrt{4\pi}}\phi^{(i)}_0(r)$.

I show that in this case the contribution to V^{JT} from the first term on the right-hand side of Eq. (28) vanishes.

This term contributes

$$\begin{aligned}&\left\langle u^{(1)}_0\chi_{1,M_s}\middle|\left(-2\imath\nabla_r\wedge U^{\rm so}_{nA}\nabla_r\cdot\boldsymbol{s_n}\right)\middle|u^{(0)}_0(r)\chi_{1,M_{s'}}\right\rangle\\ &\quad= \left\langle u^{(1)}_0\middle|\left(-2\imath\nabla_r\wedge U^{\rm so}_{nA}\right)\middle|u^{(0)}_0\right\rangle\cdot\langle\chi_{1,M_s}|\boldsymbol{s_n}|\chi_{1,M_{s'}}\rangle.\end{aligned} \tag{32}$$

The ∇_r operator acting on $u^{(0)}_0(r)$ gives a state proportional to $\boldsymbol{r}$. The integrand vanishes for $r\to\infty$ and hence integrating by parts once allows the other ∇_r to act on $u^{(1)}_0(r)$ and also produce a state proportional to $\boldsymbol{r}$. The result is an integrand proportional to $\boldsymbol{r}\wedge\boldsymbol{r} = 0$.

More generally, it is shown in Sec. IV C that when the D-state terms are included the $(-2\imath\nabla_r\wedge U^{\rm so}_{nA}\nabla_r\cdot\boldsymbol{s_n})$ term gives a central and $T_R(\boldsymbol{R},\boldsymbol{S}_d)$ term only. If the deuteron has definite parity there is no vector spin-orbit term, $\boldsymbol{L}_R\cdot\boldsymbol{S}_d$, from the first term on the right-hand side of Eq. (28). The entire vector spin-orbit term contribution to V^{JT} comes from the second term on the right-hand side of Eq. (28), $-\imath\frac{1}{2}\nabla_R\wedge U^{\rm so}_{nA}\nabla_R\cdot\boldsymbol{s}_n$ as shown in detail in Sec. IV D.

When D-state terms are neglected the second term in Eq. (28) gives the contribution

$$\frac{1}{4}\frac{1}{R}\frac{dV_{\rm so}(R)}{dR}\boldsymbol{L}_R\cdot\boldsymbol{S}_d, \tag{33}$$

where

$$V_{\rm so}(R) = \int d\boldsymbol{r}u^{(1)*}_0(r)U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}\right|\right)u^{(0)}_0(r). \tag{34}$$

This is a well known result in the Watanabe case.

When the D state is included the term $(-2\imath\nabla_r\wedge U^{\rm so}_{nA}\nabla_r\cdot\boldsymbol{s_n})$ gives central and second rank tensor forces of a conventional type, whereas the term $-\imath\frac{1}{2}\nabla_R\wedge U^{\rm so}_{nA}\nabla_R\cdot\boldsymbol{s}_n$ also gives some novel non-Hermitian tensor terms that do not appear in the Watanabe model. In the spin-dependent terms the non-Hermitian nature of the ADWA potential is manifest for the first time.

In the following sections I give explicit formulas for the complete contributions to V^{JT} from the neutron spin-orbit potential when $\phi^{(1)}_{M_{s'}}$ and $\phi^{(0)}_{M_s}$ have a general structure.

The analysis makes heavy use of spherical tensor concepts. Key definitions and notations are collected in Appendix A.

I use methods that can readily be generalized to the case when the $\phi^{(i)}_{M_{d'}}$ describe two-body cluster states of a very general structure.

C. Contribution from the term $-2\imath\nabla_r \wedge U^{\rm so}_{nA}(|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}|)\nabla_r \cdot \boldsymbol{s}_n$ in Eq. (28)

First of all it is shown quite generally, as already mentioned in the last section, that the first term on the right-hand side of Eq. (28) gives a vanishing contribution to the rank-1 spin dependence of V^{JT} if $|\phi^{(0)}_{M_s}\rangle$ and $|\phi^{(1)}_{M_{s'}}\rangle$ have the same parity (both positive parity in the case of the deuteron).

The term

$$\left(V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{M_{s'},M_s}\right)_{(n,1)} = \langle\phi^{(1)}_{M_{s'}}|\left(-2\imath\nabla_r \wedge U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}\right|\right)\nabla_r\cdot\boldsymbol{s}_n\right)|\phi^{(0)}_{M_s}\rangle, \tag{35}$$

involves an integration over $\boldsymbol{r}$ in which, if $|\phi^{(0)}_{M_s}\rangle$ and $|\phi^{(1)}_{M_{s'}}\rangle$ have the same parity, all terms except $U^{\rm so}_{nA}(|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}|)$ have positive parity in the variable $\boldsymbol{r}$. Note that the product of the two operators, ∇_r, cannot change parity. Because $U^{\rm so}_{nA}(|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}|)$ depends on the magnitude $|\boldsymbol{R}+\boldsymbol{r}/2|$ only, the integration over $\boldsymbol{r}$ will produce an even parity function that depends on the single spatial vector $\boldsymbol{R}$ only. Therefore, there can be no rank-1 component proportional to $\boldsymbol{R}$. Since the only possible rank-1 spin term in $(V^{JT})_{(n,1)}$ that depends only on $\boldsymbol{S}_d$ and $\boldsymbol{R}$ is $\boldsymbol{S}_d\cdot\boldsymbol{R}$, this term must vanish. There will also be terms of rank 0 and 2 in $(V^{JT})_{(n,1)}$.

The details of the algebra involved in the explicit evaluation are given in Appendix B. The final result is

$$\begin{aligned}\left(V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{M_{s'},M_s}\right)_{(n,1)} &= \sum_{K,Q}\frac{6\sqrt{3}}{\sqrt{4\pi}}Y^*_{K\,Q}(\boldsymbol{R})(1,M_s|\tau_{K\,Q}(1)|1,M_{s'})\\ &\times\sum_{L,L',L'',L'''}\delta_{L''=L\pm1}\delta_{L'''=L'\pm1}C_K(L'''\,L',L''\,L)\,F^{(1,0)}_K(L'''\,L',L''\,L,\,R;{\rm so},nA),\end{aligned} \tag{36}$$

where

$$C_K(L'''\,L',L''\,L) = \sum_{k_1}\hat{L''}\hat{L'''}\hat{k}_1^2\begin{Bmatrix}L & 1 & 1\\ L' & 1 & 1\\ k_1 & 1 & K\end{Bmatrix}\begin{Bmatrix}L & 1 & L''\\ L' & 1 & L'''\\ k_1 & 1 & K\end{Bmatrix}. \tag{37}$$

The quantum numbers appropriate to the deuteron case are inserted into the result (B8) given in Appendix B, namely $S=S'=S_d=S'_d=1$. Note that the pair L and L' always has the same parity as does the pair L'' and L''', but the latter pair has opposite parity to the pair L and L'.

Note that, as anticipated, there is no $K=1$ spin-orbit contribution here because the form factors F_K vanish for odd K.

The spin-independent term corresponds to $K=0$ and Eq. (36) gives

$$\left[\left(V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{M_{s'},M_s}\right)_{(n,1)}\right]_{K=0} = \delta_{M_s,M'_s}\frac{3}{2\pi}\sum_{L,L''}\hat{L''}\,(-1)^L W(L\,1L'\,1;L''\,1)\,W(L\,1L'\,1;1\,1)\,F^{(1,0)}_0(L''\,L',L''\,L,\,R;{\rm so},nA). \tag{38}$$

It can be checked that this agrees with a direct evaluation from Eq. (B1).

The $K=2$ tensor force term is

$$\begin{aligned}\left[\left(V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)\right)_{(n,1)}\right]_{K=2} &= \sum_Q\frac{6\sqrt{3}}{\sqrt{4\pi}}Y^*_{2\,Q}(\boldsymbol{R})\tau_{2\,Q}(1)\sum_{L,L',L'',L'''}\delta_{L''=L\pm1}\delta_{L'''=L'\pm1}C_2(L'''\,L',L''\,L)\,F^{(1,0)}_2(L'''\,L',L''\,L,\,R;{\rm so},nA)\\ &= T_R(\boldsymbol{R},\boldsymbol{S}_d)U^{JT}_{T_R,(n,2,so)}(R),\end{aligned} \tag{39}$$

with a form factor

$$U^{JT}_{T_R,(n,1,so)}(R) = \frac{9\sqrt{30}}{4\pi}\sum_{L,L',L'',L'''}\delta_{L''=L\pm1}\delta_{L'''=L'\pm1}C_2(L'''\,L',L''\,L)\,F^{(1,0)}_2(L'''\,L',L''\,L,\,R;{\rm so},nA), \tag{40}$$

where $T_R(\boldsymbol{R},\boldsymbol{S}_d)$ is defined in Eq. (21).

As noted in the discussion following Eq. (32), both the $K=0$ and $K=2$ terms vanish if components in $\phi^{(0)}$ and $\phi^{(1)}$ with $L>0$ are ignored.

D. Contribution from the term $-\imath\frac{1}{2}\nabla_R \wedge U^{\rm so}_{nA}\nabla_R \cdot \boldsymbol{s}_n$ in Eq. (28)

This neutron contribution to V^{JT} from this term, labeled with the suffix $(n,2)$, is

$$\left(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s}\right)_{(n,2)} = \left\langle\phi^{(1)}_{M_{s'}}\right|\left(-\imath\tfrac{1}{2}\nabla_R \wedge U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{r}\right|\right)\nabla_R\cdot\boldsymbol{s}_n\right)\left|\phi^{(0)}_{M_s}\right\rangle. \quad (41)$$

I use the identity $\boldsymbol{A}\wedge\boldsymbol{B}\cdot\boldsymbol{C} = -\boldsymbol{A}\wedge\boldsymbol{C}\cdot\boldsymbol{B}$ to write this expression so that all terms involving integration over the variable $\boldsymbol{r}$ and the nucleon spin variables n,p are together; thus

$$\left(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s}\right)_{(n,2)} = \imath\nabla_R \wedge \left[\left\langle\phi^{(1)}_{M_{s'}}\right|\left(\tfrac{1}{2}U^{\rm so}_{nA}\boldsymbol{s}_n\left|\phi^{(0)}_{M_s}\right\rangle\right)\right]\cdot\nabla_R. \quad (42)$$

It is shown below that the rank-1 component of the right-hand side of this expression has the form $\imath F(R)\boldsymbol{R}\wedge\langle M_{s'}|\boldsymbol{S}_d|M_s\rangle$ and so the result is a spin-orbit term $F(R)\boldsymbol{R}\wedge\boldsymbol{p}_R\cdot\langle M_{s'}|\boldsymbol{S}_d|M_s\rangle$.

The quantity in square brackets is a matrix element in the basis $|\boldsymbol{R};S_d,M_s\rangle$ of a vector operator $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$, constructed from the components of $\boldsymbol{R}$ and the spin-1 operator $\boldsymbol{S}_d$. Assuming that $U^{\rm so}_{nA}(|\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}|)$ is local gives

$$\langle\boldsymbol{R}';S_d,M_{s'}|\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)|\boldsymbol{R};S_d,M_s\rangle = \delta(\boldsymbol{R}-\boldsymbol{R}')\left\langle\phi^{(1)}_{M_{s'}}\right|\left(\frac{1}{2}U^{\rm so}_{nA}\left(\left|\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{r}\right|\right)\boldsymbol{s}_n\right)\left|\phi^{(0)}_{M_s}\right\rangle. \quad (43)$$

In terms of $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$, Eq. (42) can be written

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,2)} = \imath\nabla_R\wedge\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\cdot\nabla_R = -\imath\,\boldsymbol{p}_R\wedge\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\cdot\boldsymbol{p}_R. \quad (44)$$

I proceed to evaluate the spherical component, μ_2, of the vector $\langle\phi^{(1)}_{M_{s'}}|\frac{1}{2}U^{\rm so}_{nA}\boldsymbol{s}_n|\phi^{(0)}_{M_s}\rangle$ appearing in $\langle\boldsymbol{R}';S_d,M_{s'}|\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)|\boldsymbol{R};S_d,M_s\rangle$ of Eq. (43). Using the expressions for the $\phi^{(i)}$ from Eqs. (2) and (4) the result is

$$\left\langle\phi^{(1)}_{M_{s'}}\right|\frac{1}{2}U^{\rm so}_{nA}(\boldsymbol{s}_n)_{\mu_2})\left|\phi^{(0)}_{M_s}\right\rangle = \frac{1}{2}\sum_{k\,q}Y^*_{k\,q}(\boldsymbol{R})\sum_{L,\Lambda,L'\Lambda',\sigma,\sigma'}(L'\,\Lambda',1\,\sigma'|S_d\,M_{s'})(L\,\Lambda,1\sigma|S_d\,M_s)\frac{\hat{L}\hat{k}}{\sqrt{8\pi}\hat{L}'}(L\,\Lambda,k\,q|L'\,\Lambda')$$
$$\times(L\,0,k\,0|L'\,0)(1\,\sigma,1\,\mu_2|1\,\sigma')\int_0^\infty r^2\,dr\phi^{(1)*}_{L'}(r)U^{\rm so}_{nA}(k,R,r)\phi^{(0)}_L(r), \quad (45)$$

where $U^{\rm so}_{nA}(k,R,r)$ is introduced in Eq. (B5) and I have used the results (13) and (B7). Note that because L and L' always have the same parity the coefficient $(L\,0,k\,0|L'\,0)$ forces k to be even in Eq. (45).

I use the same technique that was applied in Sec. IV C to represent the summation over magnetic quantum numbers in terms of a $9j$ symbol. Inserting the result (B8) in Eq. (45) gives

$$\langle\phi^{(1)}_{M_{s'}}|\frac{1}{2}U^{\rm so}_{nA}(\boldsymbol{s_n})_{\mu_2})|\phi^{(0)}_{M_s}\rangle = \frac{1}{2}\sqrt{\frac{3}{8\pi}}\sum_{K,k\,L,L'}\hat{K}\hat{k}\begin{Bmatrix}L & 1 & 1\\ L' & 1 & 1\\ k & 1 & K\end{Bmatrix}\hat{L}(L\,0,k\,0|L'\,0)$$
$$\times\sum_{q,Q}(k\,q,K\;Q|1\,\mu_2)Y_{k\,q}(\boldsymbol{R})F^{1,0}_k(L',L,R;{\rm so},nA)\langle1\,M_{s'}|\tau_{K\,Q}(1)|1\,M_s\rangle. \quad (46)$$

I define the form factor

$$F^{(1,0)}_k(L',L,R;{\rm so},nA) = \int_0^\infty r^2\,dr\phi^{(1)*}_{L'}(r)U^{\rm so}_{nA}(k,R,r)\phi^{(0)}_L(r). \quad (47)$$

The only difference between this form factor and the one defined in Eq. (17) is the presence of the form factor of the nucleon spin-orbit potential $U^{\rm so}_{nA}(k,R,r)$ instead of $V^{\rm Cen}_{nA}(k,R,r)$.

The structure of the right-hand side of Eq. (46) is clearer if it is rewritten as

$$\left\langle\phi^{(1)}_{M_{s'}}\right|\frac{1}{2}U^{\rm so}_{nA}(\boldsymbol{s_n})_{\mu_2})\left|\phi^{(0)}_{M_s}\right\rangle = \sum_{k,q,K,Q}(k\,q,K\;Q|1\,\mu_2)Y_{k,q}(\boldsymbol{R})G^{(1,0)}_k(K,R,{\rm so},nA)\langle1\,M_{s'}|\tau_{K\,Q}(1)|1\,M_s\rangle, \quad (48)$$

where I define the form factors $G^{(1,0)}_k(K,R,{\rm so},nA)$ by

$$G^{(1,0)}_k(K,R,{\rm so},nA) = \frac{1}{2}\sqrt{\frac{3}{8\pi}}\hat{K}\hat{k}\sum_{L,L'}\begin{Bmatrix}L & 1 & 1\\ L' & 1 & 1\\ k & 1 & K\end{Bmatrix}\hat{L}(L\,0,k\,0|L'\,0)F^{(1,0)}_k(L',L,R;{\rm so},nA). \quad (49)$$

The correct covariant structure displayed in Eq. (48) should be noted. It shows $Y_{k\,q}(\boldsymbol{R})$, a tensor of rank k in $\boldsymbol{R}$ space, coupled to $\tau_{K\,Q}(1)$, a rank-K tensor in deuteron spin space, to form a tensor of rank 1, component μ_2, in the combined space. In Eq. (44) this vector is coupled with two vector operators $\boldsymbol{p}_R$ to make a tensor of rank 0 in the combined space of $\boldsymbol{R}$ and deuteron spin.

Using Eq. (48) it is possible to show the contributions from the n-A spin-orbit force [second term in Eq. (28)] to the ADWA potential as a linear combination of spin tensors of rank $K = 0$ (scalar), vector $K = 1$ (this turns out to be an $\boldsymbol{L}_R \cdot \boldsymbol{S}_d$ interaction), and $K = 2$ tensor terms. It can be seen immediately from the $9j$ symbol in Eq. (49) or the Clebsch-Gordan coefficient in Eq. (48) that the $K = 0$ term always vanishes because $K = 0$ forces k to be 1 which violates the parity condition mentioned above that k must be even.

A $K = 0$ and additional $K = 2$ contributions coming from the first term in Eq. (28) are discussed in Sec. IV C.

E. Comparison with the Watanabe optical potential (2)

A further consequence of the structure of Eq. (48) is to produce a significant difference between the spin dependence of the ADWA potential and the Watanabe model. The latter is a simple model for the deuteron elastic scattering optical potential. In the Watanabe case $\phi_L^{(1)}(r)$ is replaced by the real function $\phi_L^{(0)}(r)$ everywhere. From the definition (47) it is found that for even k

$$F_k^{(0,0)}(L',L,R;\mathrm{so},nA) = F_k^{(0,0)}(L,L',R;\mathrm{so},nA). \quad (50)$$

On the other hand, the $9j$ symbol in Eq. (49) is multiplied by $(-1)^{K+1}$ if L and L' are interchanged. This symmetry appears because a $9j$ symbol is multiplied by $(-1)^p$ when two rows are interchanged, where p is the sum of all nine entries in the symbol. In the current case $(-1)^p = (-1)^{(L+L'+k+3+2+K)} = (-1)^{K+1}$ because $L + L' + k$ must be even by parity conservation. Also,

$$\hat{L}(L\,0,k\,0|L'\,0) = \hat{L}'('L\,0,k\,0|L\,0). \quad (51)$$

Hence, interchanging the dummy summation variables L and L', it follows that if $K = 2$ the form factor $G_k^{(0,0)}(2,R,\mathrm{so},nA)$ must vanish. Thus, in the Watanabe model the first term in Eq. (28) generates only a rank-1 spin-orbit force and no rank-2 forces.

F. Explicit expressions for the ADWA spin dependence generated by the term $-\imath\frac{1}{2}\nabla_R \wedge \boldsymbol{U}_{nA}^{\mathrm{so}} \nabla_R \cdot \boldsymbol{s}_n$

According to Eqs. (44) and Appendix A, Eq. (A2), it is necessary to calculate

$$\begin{aligned} &V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,2)} \\ &= \imath(-\imath\sqrt{2}) \sum_{\mu_1,\mu_2,\mu} (1\,\mu_1,1\,\mu_2|1\,\mu) \\ &\quad \times \left[(\nabla_R)_{\mu_1} (\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d))_{\mu_2}\right](-1)^{\mu}(\nabla_R)_{-\mu}. \end{aligned} \quad (52)$$

The results (B2) and (B3) can again be used to calculate the effect of the gradient operator $(\nabla_R)_{\mu_1}$ in Eqs. (52). Using the expansion (48) for $(\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d))_{\mu_2}$ gives

$$\left(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)\right)_{(n,1)} = \sum_{\mu} \boldsymbol{A}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu}(-1)^{\mu}(\nabla_R)_{-\mu}, \quad (53)$$

where I define the vector operator $\boldsymbol{A}(\boldsymbol{R},\boldsymbol{S}_d)$ by

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu} \\ &= \sum_{K,k,k'=k-1,k+1} \sqrt{6}(-1)^{K+1}\hat{k}' W(1\,1\,k'\,k\,;1\,K) \\ &\quad \times \{Y_{k'}(\boldsymbol{R}) \times \tau_K(1)\}_{1,\mu}\left[\hat{O}_{k'\,k} G_k^{(1,0)}(K,R,\mathrm{so},nA)\right] \\ &= \sum_{K,k,k'=k-1,k+1} \{Y_{k'}(\boldsymbol{R}) \times \tau_K(1)\}_{1,\mu} A_{k',k}^{(1,0)}(K,R,\mathrm{so},nA). \end{aligned} \quad (54)$$

For clarity I introduce a new form factor

$$\begin{aligned} A_{k',k}^{(1,0)}(K,R,\mathrm{so},nA) &= \sqrt{6}(-1)^{K+1}\hat{k}' W(1\,1\,k'\,k\,;1\,K) \\ &\quad \times \left[\hat{O}_{k'\,k} G_k^{(1,0)}(K,R,\mathrm{so},nA)\right], \end{aligned} \quad (55)$$

where $G_k^{(1,0)}(K,R,\mathrm{so},nA)$ is defined in Eq. (49). In deducing Eq. (53) the summation over μ_1, μ_2, and q has been carried out using the result

$$\begin{aligned} &\sum_{\mu_1,\mu_2,q} (k\,q,K\;Q|1\,\mu_2)(k\,q,1\,\mu_1|k'\,q')(1\,\mu_1,1\,\mu_2|1\,\mu) \\ &\quad = \sqrt{3}\hat{k}'(-1)^{k+K+1} W(1\,1\,k'\,k;1\,K)(k'\,q',K\;Q|1\,\mu), \end{aligned} \quad (56)$$

which follows from the definition of a Racah coefficient in terms of Clebsch-Gordan coefficients.

Equations (53), (54), and (55) give an explicit formula for all the spin-tensor components of the ADWA potential generated by the neutron-nucleus spin-orbit interaction [first term in Eq. (28)]. In the next sections formulas for the $K = 1$ and $K = 2$ are given separately.

G. Rank-1 spin dependence: $K = 1$

For $K = 1$, an important result that follows from properties of the Racah coefficient in Eq. (55) is that for the allowed values of k, namely $k = 0$ and $k = 2$, the only allowed value of k' is $k' = 1$. Use can then be made of

$$\begin{aligned} &\sum_{q',Q} (1\,q',1\;Q|1\,\mu) Y_{1\,q'}(\boldsymbol{R})\tau(1)_{1\,Q} \\ &\quad = \sum_{q',Q} (1\,q',1\;Q|1\,\mu)\sqrt{\frac{3}{4\pi}}\frac{R_{q'}}{R}\sqrt{\frac{3}{2}}(S_d)_Q \\ &\quad = \frac{3}{\sqrt{8\pi}} \times \frac{\imath}{\sqrt{2}}\frac{(\boldsymbol{R}\wedge\boldsymbol{S}_d)_{\mu}}{R}, \end{aligned} \quad (57)$$

where the last of Eqs. (A2) and the identities

$$\begin{aligned} \tau_{1\,Q}(1) &= \sqrt{\frac{3}{2}}(\boldsymbol{S}_d)_Q, \\ R\,Y_{1\,q'}(\boldsymbol{R}) &= \sqrt{\frac{3}{4\pi}}R_{q'}, \end{aligned} \quad (58)$$

are used. Putting together all these results into Eq. (53) gives

$$\begin{aligned}(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)&)^{K=1}_{(n,2)}\\ &=\sum_k A^{(1,0)}_{1,k}(1,R,\mathrm{so},nA)\frac{3\imath}{\sqrt{16\pi}}\frac{1}{R}(\boldsymbol{R}\wedge\boldsymbol{S}_d\cdot\nabla_R)\\ &=\sum_k \frac{3}{\sqrt{16\pi}}\frac{1}{R}A^{(1,0)}_{1,k}(1,R,\mathrm{so},nA)(\boldsymbol{L}_R\cdot\boldsymbol{S}_d).\end{aligned}\tag{59}$$

The result is that the ADWA spin-orbit potential arising from the neutron-nucleus spin-orbit potential is

$$V^{JTn}_{\mathrm{so}}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)=U^{JT}_{\mathrm{so}}(R,nA)\,\boldsymbol{L}_R\cdot\boldsymbol{S}_d,\tag{60}$$

with a form factor

$$\begin{aligned}U^{JT}_{\mathrm{so}}&(R,nA)\\ &=\frac{27}{16\pi}\sum_{k=0,2}\hat{k}W(1\,1\,1\,k\,;1\,1)\sum_{L,L'}\begin{Bmatrix}L & 1 & 1\\ L' & 1 & 1\\ k & 1 & 1\end{Bmatrix}\\ &\quad\times\hat{L}(L\,0,k\,0|L'\,0)\frac{1}{R}\big[\hat{O}_{1,k}F^{(1,0)}_k(L',L,R;\mathrm{so},nA)\big],\end{aligned}\tag{61}$$

where

$$\hat{O}_{1\,0}=\frac{1}{\sqrt{3}}\frac{d}{dR},$$

$$\hat{O}_{1\,2}=-\sqrt{\frac{2}{3}}\left(\frac{d}{dR}+\frac{3}{R}\right).\tag{62}$$

Note that the $k=0$ term, which includes both $L=0$ and $L=2$ terms, has the Thomas form. Putting in the explicit values for the Racah and $9j$ symbols gives

$$\begin{aligned}U^{JT}_{\mathrm{so}}(R,nA)^{(k=0)}=\frac{1}{4}\frac{1}{R}\frac{d}{dR}\Bigg[&\frac{F^{(1,0)}_k(0,0,R;\mathrm{so},nA)}{4\pi}\\ &-\frac{1}{2}\frac{F^{(1,0)}_k(2,2,R;\mathrm{so},nA)}{4\pi}\Bigg].\end{aligned}\tag{63}$$

The $L=0$ contribution agrees with that given in Eqs. (33) and (34), but note that the quadratic $L=2$ contribution enters with a different weighting than the $L=2$ contribution to the central component of the ADWA potential from the central n-A potential displayed in Eq. (19).

The $k=2$ to $U^{JT}_{\mathrm{so}}(R,nA)$ in Eq. (61), which is only present when the n-p tensor force is taken into account, does not have the Thomas form. As it includes terms linear in $L=2$ radial functions ($L=0,L'=2$ and $L=2,L'=0$) it may carry the dominant D-state contribution to $U^{JT}_{\mathrm{so}}(R,nA)$.

H. Rank-2 spin dependence: $K=2$

For $K=2$ the only allowed values for k and k' in Eqs. (55) and (54) are $k=2$ and $k'=1,3$, and hence

$$\begin{aligned}(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)}&=\sum_{k'=1,3}A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)\sum_\mu\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_{1,\mu}(-1)^\mu(\nabla_R)_{-\mu}\\ &=\imath\sum_{k'=1,3}A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_1\cdot\boldsymbol{p}_R,\end{aligned}\tag{64}$$

where

$$A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)=-\sqrt{6}\hat{k}'W(1\,1\,k'\,2\,;1\,2)\big[\hat{O}_{k'\,2}G^{(1,0)}_2(2,R,\mathrm{so},nA)\big].\tag{65}$$

Using the explicit formulas for $G^{(1,0)}_2(2,R,\mathrm{so},nA)$ given in Eq. (49) gives

$$G^{(1,0)}_2(2,R,\mathrm{so},nA)=\frac{1}{20}\frac{1}{\sqrt{8\pi}}\big(F^{(1,0)}_2(2,0,R;\mathrm{so},nA)-F^{(1,0)}_2(0,2,R;\mathrm{so},nA)\big),\tag{66}$$

Inserting explicit values for the Racah coefficient in Eq. (65) yields

$$A^{(1,0)}_{1,2}(2,R,\mathrm{so},nA)=\frac{\sqrt{6}}{40\sqrt{5}}\left(\frac{d}{dR}+\frac{3}{R}\right)\frac{\big(F^{(1,0)}_2(2,0,R;\mathrm{so},nA)-F^{(1,0)}_2(0,2,R;\mathrm{so},nA)\big)}{\sqrt{4\pi}},\tag{67}$$

and

$$A^{(1,0)}_{3,2}(2,R,\mathrm{so},nA)=\frac{1}{20\sqrt{5}}\left(\frac{d}{dR}-\frac{2}{R}\right)\frac{\big(F^{(1,0)}_2(2,0,R;\mathrm{so},nA)-F^{(1,0)}_2(0,2,R;\mathrm{so},nA)\big)}{\sqrt{4\pi}}.\tag{68}$$

An alternative form that is more useful in applications is obtained by using the identity

$$\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_1\cdot\boldsymbol{p}_R=(-1)^{k'}\sqrt{\frac{3}{5}}\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2\cdot\tau_2(1),\tag{69}$$

and the corresponding expression for $(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)}$ is

$$(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)}=-\imath\sqrt{\frac{3}{5}}\sum_{k'=1,3}A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2\cdot\tau_2(1).\tag{70}$$

Appendix D gives the matrix elements of the potential (70) in a basis suitable for applications to the calculation of the distorted wave needed in the evaluation of $A(d,p)B$ transition amplitudes in the ADWA.

Symmetry properties of the new tensor potential

It is not obvious from expressions (64) and (70) that they define an anti-Hermitian interaction, although it is shown quite generally in Appendix C that this is in fact the case. By inverting the algebra that led to Eq. (70) this issue and the connection with the work of Satchler [20] can be clarified.

The inverse of Eq. (B2) is

$$\sum_{\mu_1,q}(k\,q,1\,\mu_1|k'\,q')(\nabla_r)_{\mu_1} f(r)Y_{k\,q}(\boldsymbol{r}) = \delta_{k'=k\pm1}Y_{k'\,q'}(\boldsymbol{r})(\hat{O}_{k'\,k}f(r)). \tag{71}$$

Using this and the Racah coefficient identity (56) I find that Eq. (64) is equivalent to

$$(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)} = -\imath\,\boldsymbol{p}_R\cdot\left(\left\{G_2^{(1,0)}(\boldsymbol{R})\times\tau_2(1)\right\}_1\wedge\boldsymbol{p}_R\right), \tag{72}$$

where $G^{(1,0)}_{2q}(\boldsymbol{R})$ is a second-rank tensor defined by

$$\begin{aligned} G^{(1,0)}_{2q}(\boldsymbol{R}) &= Y_{2q}(\boldsymbol{R})G_2^{(1,0)}(2,R;\mathrm{so},n) \\ &= Y_{2q}(\boldsymbol{R})\frac{1}{20}\frac{1}{\sqrt{8\pi}}\left(F_2^{(1,0)}(2,0,R;\mathrm{so},nA)\right. \\ &\quad \left. - F_2^{(1,0)}(0,2,R;\mathrm{so},nA)\right). \end{aligned} \tag{73}$$

Note that both $\boldsymbol{p}_R$ operators in Eq. (72) can act on everything to the right because $\boldsymbol{p}_R\wedge\boldsymbol{p}_R=0$.

It is useful to compare the structure of expression (72) with the form given in Eqs. (43) and (44) for the complete contribution to the adiabatic potential from the second term in Eq. (28). The anti-Hermitian nature of the second-rank spin-tensor component of expression (44) is proved in Appendix C. The proof of the anti-Hermitian property of expression (72) follows similar lines. The crucial point is that the vector $\{G_2^{(1,0)}(\boldsymbol{R})\times\tau_2(1)\}_{1\mu}$ that appears in Eq. (72) is anti-Hermitian.

In Ref. [20] Satchler does not consider a spin-1 tensor force of the type of Eq. (72), although it certainly satisfies all the symmetry properties considered to be essential in Ref. [20], i.e., to be a tensor of rank 0 in combined space and spin space, reflection invariant, and to generate a symmetric S matrix (reciprocity) in the angular momentum basis.

It does not seem possible to rewrite any of the equivalent expressions (64), (70), or (72) in any of the forms considered in Ref. [20]. Note that if the second-rank tensor $G^{(1,0)}_{2q}(\boldsymbol{R})$ is replaced by a first-rank tensor in $\boldsymbol{R}$ then the potential (72) becomes Hermitian and violates the condition for a symmetric S matrix used in Ref. [20].

In the case of the Watanabe potential, which is a model of the elastic deuteron-nucleus optical potential in which open deuteron elastic breakup channels are neglected, this new tensor force would be expected not to contribute because it gives a non-flux-conserving anti-Hermitian potential even when the neutron optical potential is real. It was shown already in Sec. IV E that it does indeed vanish automatically in the Watanabe case. That this does not occur for the ADWA distorting potential is because this potential is designed to describe a coherent superposition of elastic deuteron and elastic breakup channels projected onto a volume of space within the range of V_{np}. There is no reason why the flux associated with this projection should be conserved even when all other channels are closed and the nucleon optical potentials are real. Therefore, the effective interaction that drives this projection can be complex to reflect flux lost into regions outside the range of V_{np}.

I. Low-energy approximation for the new rank-2 spin-tensor contribution to the ADWA potential

In this section I show that in an approximation that is expected to be adequate for low-energy deuterons (center-of-mass energy less than 10 MeV, say) I can replace the momentum-dependent potential (64) by a local second-rank potential of the T_R type.

A low-energy deuteron incident with energy E_d on a medium or heavy nucleus is moving in an attractive potential well with real part $V_d(R)$ of depth approximately 100 MeV. It was pointed out by Goddard [21] that a result of this is that the deuteron's momentum direction in the region of the nucleus is approximately radial; i.e.,

$$\boldsymbol{p}_R \approx -\frac{\boldsymbol{R}}{R}p(R), \tag{74}$$

where

$$p(R) = \sqrt{\frac{2M_d}{\hbar^2}(E_d - V_d(R))}. \tag{75}$$

Goddard showed that using this insight a momentum-dependent tensor potential of the T_p type [20] could be replaced by an approximately equivalent T_R potential. He showed that this equivalence was well satisfied for 10-MeV deuterons and deduced that at this energy it was very difficult to distinguish the effects of T_p and T_R potentials in the phenomenological deuteron optical potential. Here I use the same idea to study the rank-2 momentum-dependent contribution to the ADWA potential derived in Sec. IV H.

Using approximation (74) in the $K=2$ contribution to Eq. (44) gives

$$\begin{aligned} V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{(K=2)}_{(n,2)} &= -\imath\,\boldsymbol{p}_R\wedge\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)\cdot\boldsymbol{p}_R \\ &= -\imath[\boldsymbol{p}_R\wedge\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)]\cdot\boldsymbol{p}_R \\ &\approx +\imath[\boldsymbol{p}_R\wedge\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)]\cdot\boldsymbol{R}\frac{p(R)}{R}, \end{aligned} \tag{76}$$

where, in the last two lines, the square brackets mean that the operator $\boldsymbol{p}_R$ acts only on the factor $\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)$.

Because $\boldsymbol{R}$ commutes with $[\boldsymbol{p}_R\wedge\boldsymbol{M}^{(K=2)}]$ the identity $\boldsymbol{A}\cdot\boldsymbol{B}\wedge\boldsymbol{C}=\boldsymbol{A}\wedge\boldsymbol{B}\cdot\boldsymbol{C}$ can be used to rewrite that last line

of Eq. (76) as

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{(K=2),\text{approx}}_{(n,2)} = +\imath[\boldsymbol{R}\wedge\boldsymbol{p}_R\cdot\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)]\frac{p(R)}{R}$$
$$= \imath[\boldsymbol{L}_R\cdot\boldsymbol{M}^{(K=2)}(\boldsymbol{R},\boldsymbol{S}_d)]\frac{p(R)}{R}. \quad (77)$$

The effect of the angular momentum operator $\boldsymbol{L}_R$ on $\boldsymbol{M}^{(K=2)}$ can be calculated by using the expressions for $\boldsymbol{M}^{(K=2)}$ as a linear combination of spherical harmonics in Eqs. (43), (48), and (49). The result is

$$[\boldsymbol{L_R}\cdot\boldsymbol{M}^{(K=2)}] = \sum_{\mu}(-1)^{\mu}\left[(\boldsymbol{L}_R)_{\mu}\boldsymbol{M}^{(K=2)}_{-\mu}\right]$$
$$= \sum_{\mu}(-1)^{\mu}\sum_{k,q,Q}(k\,q,2\,Q|1\;-\mu)[L_{\mu}Y_{k,q}(\boldsymbol{R})]G^{(1,0)}_k(2,R,\text{so},nA)\tau_{2Q}(1)$$
$$= \sum_{\mu}(-1)^{\mu}\sum_{k,q,Q}(k\,q,2\,Q|1\;-\mu)\left(\sum_{q'}\sqrt{k(k+1)}(k\,q,1\,\mu|k\,q')Y_{k,q'}(\boldsymbol{R})\right)G^{(1,0)}_k(2,R,\text{so},nA)\tau_{2Q}(1), \quad (78)$$

where I used (Ref. [19], p. 150)

$$Ł_{\mu}Y_{k,q}(\boldsymbol{R}) = \sqrt{k(k+1)}\sum_{q'}(k\,q,1\,\mu|k\,q')Y_{k,q'}(\boldsymbol{R}). \quad (79)$$

This just expresses the fact that the spherical harmonics are eigenfunctions of $\boldsymbol{L}^2$ and L_z. Note that $\boldsymbol{L}_R$ commutes with any function of the magnitude R.

The summation over q and μ is carried out using properties of the Clebsch-Gordan coefficients to give

$$\sum_{q,\mu}(-1)^{\mu}(k\,q,2\,Q|1\;-\mu)(k\,q,1\,\mu|k\,q') = \frac{\sqrt{3}}{\sqrt{5}}(-1)^{Q}\,\delta_{k,2}\delta_{q',-Q}, \quad (80)$$

so that

$$[\boldsymbol{L_R}\cdot\boldsymbol{M}^{(K=2)}] = \frac{\sqrt{3}}{\sqrt{5}}\sqrt{6}G^{(1,0)}_2(2,R,\text{so},nA)\sum_{Q}(-1)^{Q}Y_{2,-Q}(\boldsymbol{R})\tau_{2Q}(1)$$
$$= \frac{\sqrt{3}}{\sqrt{5}}\sqrt{6}G^{(1,0)}_2(2,R,\text{so},nA)Y_2(\boldsymbol{R})\cdot\tau_2(1). \quad (81)$$

Finally, Eq. (76) reduces to

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{(K=2),\text{approx}}_{(n,2)} = \imath 3\sqrt{\frac{2}{5}}\frac{p(R)}{R}G^{(1,0)}_2(2,R,\text{so},nA)Y_2(\boldsymbol{R})\cdot\tau_2(1)$$
$$= \imath\frac{9\sqrt{2}}{\sqrt{8\pi}}\frac{p(R)}{R}G^{(1,0)}_2(2,R,\text{so},nA)T_R(\boldsymbol{R},\boldsymbol{S}_d), \quad (82)$$

where T_R is the standard rank-2 tensor force defined in Eq. (21).

Using the explicit formulas for $G^{(1,0)}_2(2,R,\text{so},nA)$ given in Eq. (66) gives

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{(K=2),\text{approx}}_{(n,2)}$$
$$= \imath\frac{9\sqrt{2}}{40}\frac{p(R)}{R}\frac{1}{4\pi}\left(F^{(1,0)}_2(2,0,R;\text{so},nA)\right.$$
$$\left.-F^{(1,0)}_2(0,2,R;\text{so},nA)\right)T_R(\boldsymbol{R},\boldsymbol{S}_d). \quad (83)$$

The factor $\imath$ in this formula shows clearly the anti-Hermitian nature of this tensor potential contribution for a real nucleon spin-orbit potential and how it vanishes in the case of the Watanabe potential when the factor involving $F^{(1,0)}_2$ becomes $(F^{(0,0)}_2(2,0,R;\text{so},nA) - F^{(0,0)}_2(0,2,R;\text{so},nA)) = 0$.

Note that when approximation (74) is applied to the $K=1$ terms in $V^{JT}_{(n,1)}$ it gives a vanishing result and therefore does not give a useful approximation to the vector spin-orbit force.

J. Origin of the new tensor force

Although the definition of the ADWA distorting potential given in Eq. (1) is certainly not Hermitian in general even if the nucleon optical potentials are, this property has not shown up in the usual approximate evaluations that have been published so far. The reason for this is that the contribution from states of the n-p system that couple different orbital angular momenta have

not been included before together with the nucleon-nucleus spin-orbit force. The non-Hermitian tensor force given by Eq. (64) or the approximate form given by Eq. (83) both give a vanishing result if there are no terms with $L \neq L'$.

The physical origin of the difference of form factors appearing in Eq. (83), and hence the source of this interaction, can be made more transparent by making use of the coupled eigenvalue equations satisfied by the radial functions $\phi_L^{(0)}(r)$ and $\phi_L^{(1)}(r)$.

I assume a nucleon-nucleon (NN) potential in the even-parity, isospin zero state appropriate to the deuteron has central (C), spin-orbit (SO), and tensor (T) components and has the form

$$v_{\rm NN}(\boldsymbol{r},\boldsymbol{p}_r,\boldsymbol{S}_d) = v_{\rm NN}^{\rm C}(r) + v_{\rm NN}^{\rm SO}(\boldsymbol{r},\boldsymbol{p}_r,\boldsymbol{S}_d) + v_{\rm NN}^{\rm T}(\boldsymbol{r},\boldsymbol{S}_d). \tag{84}$$

The eigenvalue equations for the radial parts of the deuteron ground state are

$$\begin{aligned}(-\epsilon_d - T_0)\phi_0^{(0)}(r) &= v_{\rm NN}^{\rm C}(r)\phi_0^{(0)}(r) + + v_{\rm NN}^{\rm T}(r)_{0,2}\phi_2^{(0)}(r)\\ (-\epsilon_d - T_2)\phi_2^{(0)}(r) &= \left(v_{\rm NN}^{\rm C}(r) + v_{\rm NN}^{\rm SO}(r,2) + v_{\rm NN}^{\rm T}(r)_{2,2}\right)\phi_2^{(0)}(r)\\ &\quad + v_{\rm NN}^{\rm T}(\boldsymbol{r})_{2,0}\phi_0^{(0)}(r),\end{aligned} \tag{85}$$

where T_L is the kinetic energy operator for a state with orbital angular momentum L, ϵ_d is the deuteron binding energy, and $v_{\rm NN}^{\rm SO}(r,2)$ is the NN spin-orbit interaction in the state $L = 2, J = 1$. The quantities $v_{\rm NN}^{\rm T}(\boldsymbol{r})_{L',L}$ are the matrix elements of the NN tensor potential in the $|L, J = 1, M\rangle$ basis.

Equation (3) gives

$$\phi_L^{(1)}(r) = C_{np}(-\epsilon_d - T_L)\phi_L^{(0)}(r), \tag{86}$$

where C_{np} is the constant

$$C_{np} = \frac{1}{\langle\phi_{M_s}^{(0)}|V_{np}|\phi_{M_s}^{(0)}\rangle}. \tag{87}$$

C_{np} is independent of M_s if V_{np} is a tensor of rank 0 in the combined spin and coordinate space of n and p.

Putting together Eqs. (83), (85), and (87) and assuming that all the radial functions have been chosen to be real gives

$$\begin{aligned}&\left(F_2^{(1,0)}(2,0,R;{\rm so},nA) - F_2^{(1,0)}(0,2,R;{\rm so},nA)\right)\\ &= C_{np}\left[\langle\phi_0^{(0)}v_{\rm NN}^{\rm T}(r)_{20}\phi_0^{(0)}\rangle + \langle\phi_2^{(0)}(v_{\rm NN}^{\rm T}(r)_{22}\right.\\ &\quad \left. + v_{\rm NN}^{\rm SO}(r,2))\phi_0^{(0)}\rangle - \langle\phi_2^{(0)}v_{\rm NN}^{\rm T}(r)_{02}\phi_2^{(0)}\rangle\right].\end{aligned} \tag{88}$$

Here the notation used temporarily is

$$\langle(\cdots)\rangle \equiv \int r^2 dr\, U_{nA}^{\rm s0}(2,R,r)(\cdots). \tag{89}$$

Note that if terms involving $\phi_2^{(0)}$, $v_{\rm NN}^{\rm T}$, and $v_{\rm NN}^{\rm SO}$ are regarded as all being of first order in small quantities then to first order

$$\begin{aligned}&\left(F_2^{(1,0)}(2,0,R,{\rm so},nA) - F_2^{(1,0)}(0,2,R,{\rm so},nA)\right)\\ &\approx C_{np}\langle\phi_0^{(0)}v_{\rm NN}^{\rm T}(r)_{20}\phi_0^{(0)}\rangle,\end{aligned} \tag{90}$$

which to the same approximation is the same as $F_2^{(1,0)}(2,0,R;{\rm so},nA)$. Hence quantity (88) is expected to be of the same order of magnitude as the terms appearing in the T_R potential displayed in Eqs. (22) and (23), modified by the ratio of the strength of the neutron spin-orbit and central potentials.

V. SUMMARY

In this section all formulas for the central, spin-orbit, and rank-2 spin-tensor components of the ADWA distorting potential generated by the n-A optical are brought together with the appropriate recoil correction factors C_1 and C_2 defined in Eqs. (29) and (30) inserted.

A. The central ADWA potential

From Eqs. (19) and (38),

$$\begin{aligned}V^{JT}(\boldsymbol{R})^{n,{\rm Central}} &= \frac{1}{4\pi}\left[F_0^{(1,0)}(0,0,R;{\rm Cen},nA) + F_0^{(1,0)}(2,2,R;{\rm Cen},nA)\right]\\ &\quad + \frac{3}{2\pi}\frac{A}{(A+1)}\sum_{L,L''}\hat{L''}(-1)^L W(L\,1L\,1;L''\,1)\,W(L\,1L\,1;1\,1)\,F_0^{(1,0)}(L''\,L',L''\,L,\,R;{\rm so},nA).\end{aligned} \tag{91}$$

The form factors $F_0^{(1,0)}(L,L,R;{\rm Cen},nA)$ and $F_0^{(1,0)}(L''\,L',L''\,L,\,R;{\rm so},nA)$ are defined in Eqs. (17) and (B5), respectively.

B. The ADWA spin-orbit potential

From Eqs. (60) and (61),

$$V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{n,{\rm so}} = U_{\rm so}^{JT}(R,nA)\,\boldsymbol{L}_R\cdot\boldsymbol{S}_d), \tag{92}$$

with a form factor

$$U_{\rm so}^{JT}(R,nA) = \frac{27(A+2)}{16\pi(A+1)}\sum_{k=0,2}\hat{k}\,W(1\,1\,1\,k\,;1\,1)\sum_{L,L'}\begin{Bmatrix}L & 1 & 1\\ L' & 1 & 1\\ k & 1 & 1\end{Bmatrix}\hat{L}(L\,0,k\,0|L'\,0)\frac{1}{R}\left[\hat{O}_{1,k}F_k^{(1,0)}(L',L,R;{\rm so},nA)\right], \tag{93}$$

where $\hat{O}_{1\,0}$ and $\hat{O}_{1\,2}$ are defined in Eq. (62). The form factor $F_k^{(1,0)}(L',L,R;{\rm so},nA)$ is defined in Eq. (47).

C. The ADWA rank-2 spin tensor of the $T_R(\boldsymbol{R},\boldsymbol{S}_d)$ type

From Eqs. (23) and (40),

$$V^{JT}(\boldsymbol{R},\boldsymbol{S}_d)_{T_R,nA} = T_R(\boldsymbol{S}_d,\boldsymbol{R})U^{JT}_{T_R,nA}(R), \tag{94}$$

where the form factor is given by

$$U^{JT}_{T_R,nA}(R) = \frac{3}{\sqrt{2}}\frac{1}{4\pi}\left(F_2^{(1,0)}(0,2,R;\mathrm{Cen},nA) + F_2^{(1,0)}(2,0,R;\mathrm{Cen},nA) - \frac{1}{\sqrt{2}}F_2^{(1,0)}(2,2,R;\mathrm{Cen},nA)\right)$$
$$+ \frac{9\sqrt{30}A}{4\pi(A+1)}\sum_{L,L',L'',L'''}\delta_{L''=L\pm1}\delta_{L'''=L'\pm1}C_2(L'''\,L',L''\,L)\,F_2^{(1,0)}(L'''\,L',L''\,L,\,R;\mathrm{so},nA), \tag{95}$$

where $C_2(L'''\,L',L''\,L)$ is defined in Eq. (37) and $F_2^{(1,0)}(L'''\,L',L''\,L,\,R;\mathrm{so},nA)$ is defined in Eq. (B5).

D. The new ADWA rank-2 spin-tensor potential

The new rank-2 potential can be expressed in several different forms. The expression most useful for applications to coupled channels codes is probably Eq. (70), viz.,

$$\left(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)\right)^{K=2}_{(n,2)} = -\imath\sqrt{\frac{3}{5}}\sum_{k'=1,3}A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2\cdots\tau_2(1). \tag{96}$$

Explicit forms for the functions $A^{1,0}_{k',2}(2,R)$,$k' = 1,3$ can be found in Eqs. (67) and (68).

An alternative expression that makes the anti-Hermitian property of this potential more evident and could be more useful in a perturbative treatment can be found in Eq. (72), which I rewrite here as

$$(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)} = \frac{1}{\imath}\frac{1}{20\sqrt{2}}\boldsymbol{p}_R\cdot\left(\{Y_2(\boldsymbol{R})\frac{\left(F_2^{(1,0)}(2,0,R;\mathrm{so},nA) - F_2^{(1,0)}(0,2,R;\mathrm{so},nA)\right)}{\sqrt{4\pi}}\tau_2(1)\}_1\wedge\boldsymbol{p}_R\right). \tag{97}$$

VI. COMMENTS

This paper forms part of an investigation into the implications of Ref. [3] for the calculation of $A(d,p)B$ transition amplitudes. It was shown there that for incident deuteron energies of current experimental interest these amplitudes are dominated by the first Weinberg component of the full three-body scattering wave function. In the ADWA this component is approximated by a distorted wave in the Johnson-Tandy potential defined in Eq. (1). The systematics of calculations using this potential with a finite range n-p interaction are reported in Ref. [17] and methods for dealing with nonlocal and energy-dependent nucleon-A optical potentials needed in Eq. (1) are described in Refs. [10–12]. One result of these studies is that when equivalent local potentials are used in the ADWA potential they should be evaluated at an energy shifted by about 40 MeV from the prescription that follows from the Watanabe model [22]. This energy shift is determined by short range properties of the n-p interaction. It will be interesting to see how far this shift is modified by the D-state effects included in the results presented here.

The emphasis here has been on the spin dependence of the ADWA potential. Novel features not seen or expected from studies of the Watanabe optical model are obtained. The quantitative implications for these observations and their relevance to deuteron stripping and pickup experiments, including polarization experiments, will be the subject of further work.

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APPENDIX A: SPHERICAL TENSOR NOTATION

This appendix gathers together the main definitions and notations for spherical tensors and general tensor products used in the main text.

Spherical components of a vector are

$$A_0 = A_z,$$
$$A_{\pm1} = \frac{\mp1}{\sqrt{2}}(A_x \pm \imath A_y). \tag{A1}$$

Expressions for vector products of vectors in terms of spherical components are

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{\mu=0,\pm1}(-1)^\mu A_\mu B_{-\mu},$$
$$(\boldsymbol{A}\wedge\boldsymbol{B})_\mu = -\imath\sqrt{2}\sum_{\mu_1\mu_2}(1\,\mu_1,1\,\mu_2|1\,\mu)A_{\mu_1}B_{\mu_2}. \tag{A2}$$

I use the following notation for general tensor products:

$$\{A_{k_1} \times B_{k_2}\}_{k_3\, q_3} = \sum_{q_1,q_2} (k_1\, q_1, k_2\, q_2 | k_3\, q_3) A_{k_1\, q_1} B_{k_2\, q_2}. \tag{A3}$$

Note the special case

$$\begin{aligned}\{A_{k_1} \times B_{k_1}\}_{0,0} &= \sum_{q_1,q_1'} (k_1\, q_1, k_1\, q_2 | 0\, 0) A_{k_1\, q_1} B_{k_1\, q_1'} \\ &= \sum_{q_1} \frac{(-1)^{k_1 - q_1}}{\hat{k}_1} A_{k_1\, q_1} B_{k_1\, -q_1} \\ &= \frac{(-1)^{k_1}}{\hat{k}_1} A_{k_1} \cdot B_{k_1}. \end{aligned} \tag{A4}$$

APPENDIX B: DERIVATION OF EQ. (36)

To proceed with the evaluation of Eq. (35), an integration by parts once with respect to $\boldsymbol{r}$ is carried out. Since the deuteron wave function vanishes exponentially at infinite r all the surface terms vanish and the result is a change of sign and a factor ∇_r acting to the left on $\phi^{(1)}$. Thus, remembering that $((\nabla_r)_\mu)^* = (-1)^\mu (\nabla_r)_{-\mu}$, I obtain

$$\begin{aligned}&\left(V^{JT}(\boldsymbol{R}, \boldsymbol{S}_d)_{M_{s'}, M_s}\right)_{(n,1)} \\ &= \left\langle \left(\nabla_r \phi^{(1)}_{M_{s'}}\right) \middle| (2\imath \wedge U^{\rm so}_{nA} \nabla_r \cdot \boldsymbol{s_n}) \middle| \phi^{(0)}_{M_s} \right\rangle \\ &= \sum_{\mu_1,\mu_2,\mu} (-\imath\sqrt{2})(1\,\mu_1, 1\,\mu_2 | 1\,\mu)(-1)^{\mu_1} \\ &\quad \times \left\langle (\nabla_r)_{-\mu_1} \phi^{(1)}_{M_{s'}}) \middle| 2\imath U^{\rm so}_{nA} (\nabla_r)_{\mu_2} (-1)^\mu (\boldsymbol{s_n})_{-\mu} \middle| \phi^{(0)}_{M_s} \right\rangle. \end{aligned} \tag{B1}$$

I evaluate the effects of the ∇_r by using Eqs. (2) and (4) and the formula

$$\begin{aligned}&(\nabla_r)_{\mu_1} f(r) Y_{k\,q}(\boldsymbol{r}) \\ &= \sum_{k'=k-1,k+1,\, q'} (k\, q, 1\, \mu_1 | k'\, q') Y_{k'\, q'}(\boldsymbol{r}) (\hat{O}_{k'\, k} f(r)), \end{aligned} \tag{B2}$$

where the radial operators $\hat{O}_{k'\, k}$ are defined by

$$\begin{aligned}\hat{O}_{k+1\, k} &= \sqrt{\frac{(k+1)}{(2k+3)}} \left(\frac{d}{dr} - \frac{k}{r}\right), \\ \hat{O}_{k-1\, k} &= -\sqrt{\frac{k}{(2k-1)}} \left(\frac{d}{dr} + \frac{(k+1)}{r}\right). \end{aligned} \tag{B3}$$

This result is equivalent to the formula given on p. 150 of Ref. [19]. Note the restriction of the sum over k' to terms of opposite parity to k.

Using Eq. (B2) twice in Eq. (B1) together with formula (13) gives

$$\begin{aligned}\left(V^{JT}(\boldsymbol{R}, \boldsymbol{S}_d)_{M_{s'}, M_s}\right)_{(n,1)} &= 2\sqrt{2} \sum_{\mu_1,\mu_2,\mu} (1\mu_1, 1\mu_2 | 1\mu) \sum_{kq} Y^*_{kq}(\boldsymbol{R}) \sum_{L,\Lambda,L'\Lambda',\sigma,\sigma'} (L'\Lambda', 1\sigma' | S_d M_{s'}) (L\Lambda, 1\sigma | S_d M_s) \\ &\quad \times (-1)^\mu (1\sigma, 1-\mu | 1\sigma') \langle 1 || \boldsymbol{s_n} || 1 \rangle \sum_{L'''=L'\pm 1, \Lambda'''} (-1)^{\mu_1} (L'\Lambda', 1-\mu_1 | L'''\Lambda''') \\ &\quad \times \sum_{L''=L\pm 1, \Lambda'} (L\Lambda, 1\mu_2 | L''\Lambda'') \frac{\hat{k}}{\sqrt{4\pi}\hat{L}'''} (L''\Lambda'', kq | L'''\Lambda''') F^{(1,0)}_k (L'''L', L''L, R; {\rm so}, nA). \end{aligned} \tag{B4}$$

I have introduced the radial integral

$$F^{(1,0)}_k (L'''\, L', L''\, L,\, R; {\rm so}, nA) = \hat{L}''(L''\, 0, k\, 0 | L'''\, 0) \int_0^\infty r^2\, dr \left(\hat{O}_{L'''\, L'} \phi^{(1)}_{L'}(r)\right)^* U^{\rm so}_{nA}(k, R, r) \left(\hat{O}_{L''\, L} \phi^{(0)}_L(r)\right), \tag{B5}$$

where $U^{\rm so}_{nA}(k,R,r)$ is defined by an expansion of $U^{\rm so}_{nA}(|\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}|)$ analogous to Eqs. (9) and (10).

The integration over the spin variables of the neutron and proton needed in Eq. (B1) gives

$$\begin{aligned}&\langle \chi_{1,\sigma'}(p,n) | (\boldsymbol{s_n})_{\mu_2} | \chi_{1,\sigma}(p,n) \rangle \\ &= \tfrac{1}{2} \langle \chi_{1,\sigma'}(p,n) | (\boldsymbol{S_d})_{\mu_2} | \chi_{1,\sigma}(p,n) \rangle \\ &= \tfrac{1}{2} (1\sigma, 1\mu_2 | 1\sigma') \langle 1 || \boldsymbol{S_d} || 1 \rangle \\ &= \tfrac{1}{2} (1\sigma, 1\mu_2 | 1\sigma') \sqrt{2}, \end{aligned} \tag{B6}$$

so that the reduced matrix element needed in Eq. (B4) is

$$\langle 1 || \boldsymbol{s_n} || 1 \rangle = \frac{1}{\sqrt{2}}. \tag{B7}$$

In the third line of Eq. (B6) I used the general result, $\langle J || \boldsymbol{J} || J \rangle = \sqrt{J(J+1)}$ (Eq. (4.18) of Ref. [19]).

Note that because L and L' always have the same parity the coefficient $(L0, k0 | L'0)$ forces k to be even in Eq. (B4).

Using the relation between the 9-j symbol and Clebsch-Gordan coefficients given in Ref. [19] (p. 144), the summation over magnetic quantum numbers in Eq. (B4) can be reexpressed in a form that displays its dependence on spin tensors

in the deuteron spin space. In particular the summation over σ and σ' can be reexpressed using

$$\sum_{\sigma,\sigma'}(L'\Lambda',S'\sigma'|S'_dM_{s'})(L\Lambda,S\sigma|S_dM_s)(S\sigma,k_2q_2|S'\sigma') = \sum_{KQ,k_1q_1}(-1)^p\hat{S}'\hat{S}_d\frac{\hat{K}^2}{\hat{k}_2}\frac{\hat{k}_1^2}{\hat{L}'}\begin{Bmatrix} L & S & S_d \\ L' & S' & S'_d \\ k_1 & k_2 & K\end{Bmatrix} \times(L\Lambda,k_1q_1|L'\Lambda')(-1)^{k_1-q_1}(k_1-q_1,KQ|k_2q_2) \times(S_dM_s,KQ|S'_dM_{s'}), \tag{B8}$$

where $p=(L-L'+S-S'+S_d-S'_d)$. Using this, the summations over μ_1, μ_2, μ, Λ, Λ', Λ'', and Λ''' can then also be expressed in terms of a $9j$ symbol. The result is given in Eq. (36) in the main text.

APPENDIX C: PROPERTIES OF THE SPIN-DEPENDENT INTERACTIONS UNDER HERMITIAN CONJUGATION AND TIME-REVERSAL TRANSFORMATIONS

In this section I discuss certain key symmetry properties of the vector operator, $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$, defined in Eqs. (43) and (48) that lead to important features of the ADWA spin dependence. It is assumed that the neutron spin-orbit form factor, $U^{\rm so}_{nA}(r_n)$, is real and that phase conventions are chosen so that the radial functions $\phi^{(i)}_L(r)$, and hence the form factors, $F^{(1,0)}_k(L',L,R;\mathrm{Cen},n)$, in which they appear, are real.

1. Parity conservation

Parity conservation [see the Clebsch-Gordan coefficient in Eq. (47)] means that k must be even and hence $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$ is unchanged under the transformation $\boldsymbol{R}\to-\boldsymbol{R}$.

2. Hermiticity

I examine the Hermiticity of $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$. I have, from Eqs. (48) and (43),

$$\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu_2} = \sum_{k,q,K,Q}(kq,KQ|1\mu_2)Y_{kq}(\boldsymbol{R})G^{(1,0)}_k \times(K,R,\mathrm{so},nA)\tau_{K,Q}(S_d), \tag{C1}$$

and hence

$$(\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu_2})^\dagger = \sum_{k,q,K,Q}(kq,KQ|1\mu_2)(Y_{kq}(\boldsymbol{R}))^*\left(G^{(1,0)}_k(K,R,\mathrm{so},nA)\right)^*\tau^\dagger_{K,Q} = \sum_{k,q,K,Q}(kq,KQ|1\mu_2)(-1)^qY_{k-q}(\boldsymbol{R})G^{(1,0)}_k(K,R,\mathrm{so},nA)(-1)^Q\tau_{K,-Q}(S_d), \tag{C2}$$

where I used

$$\tau^\dagger_{K,Q}(S_d)=(-1)^Q\tau_{K,-Q}(S_d). \tag{C3}$$

Hence

$$\begin{aligned}(\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu_2})^\dagger &= \sum_{k,q,K,Q}(kq,KQ|1\mu_2)(-1)^qY_{k,-q}(\vec{R})(-1)^Q\tau_{K,-Q}(1)G^{(1,0)}_k(K,R,\mathrm{so},nA)\\ &= \sum_{k,q,K,Q}(-1)^{k+K+1}(k\ -q,K\ -Q|1\ -\mu_2)(-1)^{\mu_2}Y_{k,-q}(\vec{R})\tau_{K,-Q}(1)G^{(1,0)}_k(K,R,\mathrm{so},nA)\\ &= (-1)^{\mu_2}\sum_{k,q,K,Q}(-1)^{k+K+1}(k\,q,K\ Q|1\ -\mu_2)Y_{k,q}(\vec{R})\tau_{K,Q}(1)G^{(1,0)}_k(K,R,\mathrm{so},nA)\\ &= (-1)^{\mu_2}(\boldsymbol{M}^{K=1}(\boldsymbol{R},\boldsymbol{S}_d))_{-\mu_2}-(-1)^{\mu_2}(\boldsymbol{M}^{K=2}(\boldsymbol{R},\boldsymbol{S}_d))_{-\mu_2},\end{aligned} \tag{C4}$$

where the fact that k is even has been used.

Referring to the connection between the spherical and Cartesian components given in Appendix A, Eq. (A1), the result (C4) implies that the Cartesian components of the vector $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$ corresponding to a particular K are Hermitian if K is odd, but anti-Hermitian if K is even. In the notation of Eq. (44) this means that the vector operator $\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)$ satisfies

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d) &= \boldsymbol{M}^{K=1}(\boldsymbol{R},\boldsymbol{S}_d)+\boldsymbol{M}^{K=2}(\boldsymbol{R},\boldsymbol{S}_d),\\ (\boldsymbol{M}^{K=1}(\boldsymbol{R},\boldsymbol{S}_d))^\dagger &= \boldsymbol{M}^{K=1}(\boldsymbol{R},\boldsymbol{S}_d),\\ (\boldsymbol{M}^{K=2}(\boldsymbol{R},\boldsymbol{S}_d))^\dagger &= -\boldsymbol{M}^{K=2}(\boldsymbol{R},\boldsymbol{S}_d).\end{aligned} \tag{C5}$$

The n-A spin-orbit form factor $U^{\rm so}_{nA}(r_n)$ has a finite range and the functions $\phi^{(i)}$ vanish at large r so it can be assumed that the momentum operators $\boldsymbol{p}_R$ in Eq. (43) are Hermitian. Therefore, using $(\boldsymbol{A}\wedge\boldsymbol{B}\cdot\boldsymbol{C})^\dagger=-\boldsymbol{C}^\dagger\wedge\boldsymbol{B}^\dagger\cdot\boldsymbol{A}^\dagger$ gives

$$\begin{aligned}(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})_{(n,1)} &= (V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=1}_{(n,1)}\\ &\quad+(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)^{K=2}_{M_{s'},M_s})_{(n,1)},\\ ((V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=1}_{(n,1)})^\dagger &= (V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=1}_{(n,1)},\\ ((V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=2}_{(n,1)})^\dagger &= -(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=2}_{(n,1)}.\end{aligned} \tag{C6}$$

Hence, if the neutron spin-orbit potential is real, the rank-1 components of these contributions to the ADWA potential are Hermitian, but the rank-2 components are anti-Hermitian.

3. Reciprocity and time reversal

It is also desirable that the distorted waves generated by the ADWA potential have appropriate properties under time reversal. In practice this requires

$$\mathcal{K}V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,1)}\mathcal{K}^{-1} = (V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,1)})^\dagger, \quad \text{(C7)}$$

where $\mathcal{K}$ is the antilinear time-reversal operator. When this reciprocity relation is satisfied, phases of scattering wave functions can be chosen so that the corresponding S matrix in the angular momentum basis is symmetric.

To see that this condition is satisfied note that the spin tensors $\tau_{K,Q}$ satisfy

$$\mathcal{K}\tau_{K,Q}\mathcal{K}^{-1} = (-1)^{K+Q}\tau_{K,-Q}. \quad \text{(C8)}$$

The factor $(-1)^K$ arises because the components of $\boldsymbol{S}_d$ satisfy $\mathcal{K}(\boldsymbol{S}_d)_i\mathcal{K}^{-1} = -(\boldsymbol{S}_d)_i, i = x,y,z$, and the $\tau_{K,Q}$ are homogeneous polynomials of degree K in those components.

The $Y_{k,q}(\vec{R})$ satisfy

$$\mathcal{K}Y_{k,q}(\vec{R})\mathcal{K}^{-1} = Y_{k,q}(\vec{R})^* = (-1)^q Y_{k,-q}(\vec{R}). \quad \text{(C9)}$$

Following lines similar to those used in Eq. (C2) gives

$$\begin{aligned}\mathcal{K}\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)_{\mu_2}\mathcal{K}^{-1} &= \sum_{k,q,K,Q}(k\,q,K\;Q|1\;\mu_2)(-1)^q Y_{k,-q}(\vec{R})(-1)^{K+Q}\tau_{K,-Q}(1)G_k^{(1,0)}(K,R,\mathrm{so},nA)^* \\ &= \sum_{k,q,K,Q}(-1)^1(k\;-q,K\;-Q|1\;-\mu_2)(-1)^{\mu_2}Y_{k,-q}(\vec{R})\tau_{K,-Q}(1)G_k^{(1,0)}(K,R,\mathrm{so},nA) \\ &= (-1)^{\mu_2}\sum_{k,q,K,Q}(-1)^1(k\,q,K\;Q|1\;-\mu_2)Y_{k,q}(\vec{R})\tau_{K,Q}(1)G_k^{(1,0)}(K,R,\mathrm{so},nA). \quad \text{(C10)}\end{aligned}$$

A comparison with Eq. (C2) gives

$$\begin{aligned}\mathcal{K}\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\mathcal{K}^{-1} &= -\left(\boldsymbol{M}^{K=1}(\boldsymbol{R},\boldsymbol{S}_d)\right)^\dagger - \left(\boldsymbol{M}^{K=2}(\boldsymbol{R},\boldsymbol{S}_d)\right)^\dagger \\ &= -\left(\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\right)^\dagger. \quad \text{(C11)}\end{aligned}$$

Putting this result into Eq. (44) and using $\mathcal{K}\boldsymbol{p}_R\mathcal{K}^{-1} = -\boldsymbol{p}_R$ and

$$(\boldsymbol{A}\wedge\boldsymbol{B}\cdot\boldsymbol{C})^\dagger = -\boldsymbol{C}^\dagger\wedge\boldsymbol{B}^\dagger\cdot\boldsymbol{A}^\dagger \quad \text{(C12)}$$

gives

$$\begin{aligned}\mathcal{K}V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,1)}\mathcal{K}^{-1} &= +\imath\,\boldsymbol{p}_R\wedge\mathcal{K}\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\mathcal{K}^{-1}\cdot\boldsymbol{p}_R \\ &= -\imath\,\boldsymbol{p}_R\wedge\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)^\dagger\cdot\boldsymbol{p}_R \\ &= (-\imath\,\boldsymbol{p}_R\wedge\boldsymbol{M}(\boldsymbol{R},\boldsymbol{S}_d)\cdot\boldsymbol{p}_R)^\dagger \\ &= \left(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{(n,1)}\right)^\dagger. \quad \text{(C13)}\end{aligned}$$

It can be shown that the last equality is also valid when the neutron spin-orbit form factor, $U_{nA}^{\mathrm{so}}(r_n)$, in Eq. (25) is complex. Thus, although the $K = 2$ terms are anti-Hermitian, they and the $K = 1$ terms do satisfy the required time-reversal condition (C7).

Note that for real nucleon optical potentials the Watanabe potential discussed in Sec. IV E is always Hermitian because it involves no coupling to deuteron breakup channels. The ADWA potential, on the other hand, takes into account physical effects not included in the Watanabe model. It is designed to describe a linear superposition of elastic and breakup channel wave functions projected onto a restricted region of configuration space within the range of V_{np}. There is no reason why probability flux within this region should be conserved.

APPENDIX D: MATRIX ELEMENTS OF THE RANK-2 TENSOR FORCE $(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R),S_d)_{M_{s'},M_s})^{K=2}_{(n,1)}$

In the calculation of distorted waves associated with spin-dependent forces it is natural to use the basis $|(L,S_d)\boldsymbol{J},M\rangle$ where L is the orbital angular momentum corresponding to $\boldsymbol{L}_R$, $S_d = 1$, $\boldsymbol{J} = \boldsymbol{L}_R + \boldsymbol{S}_d$, and M is an eigenvalue of J_z. The matrix elements of the rank-0 potential $(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d)_{M_{s'},M_s})^{K=2}_{(n,1)}$ in this basis will be diagonal in J and M but not necessarily in L.

Equations (70) give

$$(V^{JT}(\boldsymbol{R},\boldsymbol{p}_R,\boldsymbol{S}_d))^{K=2}_{(n,2)} = \imath\sum_{k'=1,3}3\sqrt{\frac{2}{5}}\hat{k}'W(1\,1\,k'\,2;1\,2)[\hat{O}_{k'2}G_2^{(1,0)}(2,R,\mathrm{so},nA)]\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2\cdot\tau_2(1), \quad \text{(D1)}$$

where $G_2^{(1,0)}(2,R,\mathrm{so},nA)$ is given in Eq. (66).

Using the standard result given in Ref. [19], Eq. (5.13) of Ref. [19] gives

$$\begin{aligned}\langle((L',1)J',M'|\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_1\cdot\boldsymbol{p}_R|(L,1)J,M\rangle &= (-1)^{k'}\sqrt{\frac{3}{5}}\langle((L',1)J',M'|\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2\cdot\tau_2(1)|(L,1)J,M\rangle\\ &= (-1)^{k'}\sqrt{\frac{3}{5}}\delta_{J,J'}\delta_{M,M'}\hat{L}'\sqrt{3}(-1)^{J-L'-1}W(L'\,L\,1\,1;2\,J)\\ &\quad\times\langle L'||\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2||L\rangle\langle1||\tau_2(1)||1\rangle.\end{aligned}\tag{D2}$$

Definition (15) results in the reduced matrix element

$$\langle1||\tau_2(1)||1\rangle=\sqrt{5}.\tag{D3}$$

The other reduced matrix element in Eq. (D2) can be evaluated using the standard formulas in Appendix A, Eq. (A2), and Eqs. (13), (B2), and (B3). These give

$$\begin{aligned}\langle L',\Lambda'|\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_{2,Q}|L,\Lambda\rangle &= \sum_{q',\mu}(k'\,q',1\,\mu|2,Q)\langle L',\Lambda'|Y_{k',q'}(\boldsymbol{R})(\boldsymbol{p}_R)_\mu|L,\Lambda\rangle\\ &= \imath^{-1}\sum_{q',\mu}(k'\,q',1\,\mu|2,Q)\langle L',\Lambda'|Y_{k',q'}(\boldsymbol{R})\sum_{L''=L\pm1,\Lambda''}(L\,\Lambda\,1\mu|L''\,\Lambda'')|L'',\Lambda''\rangle\hat{O}_{L'',L}\\ &= \imath^{-1}\sum_{q',\mu,L''=L\pm1,\Lambda''}(k'\,q',1\,\mu|2\,Q)(L\,\Lambda\,1\mu|L''\,\Lambda'')\langle L',\Lambda'|Y_{k',q'}(\boldsymbol{R}|L'',\Lambda''\rangle\hat{O}_{L'',L}\\ &= \imath^{-1}\sum_{q',\mu,L''=L\pm1,\Lambda''}(k'\,q',1\,\mu|2\,Q)\frac{\hat{L}''\hat{k}'}{\sqrt{4\pi}\hat{L}'}(L\,\Lambda\,1\mu|L''\,\Lambda'')(L''\,\Lambda'',k'\,q'|L'\,\Lambda')\\ &\quad\times(L''\,0,k'\,0|L'\,0)\hat{O}_{L'',L}.\end{aligned}\tag{D4}$$

The summation over q',μ,Λ'' can be reduced using the definition of a Racah coefficient to give

$$\begin{aligned}\sum_{q',\mu,\Lambda''}(k'\,q',1\,\mu|2\,Q)(L\,\Lambda\,1\mu|L''\,\Lambda'')(L''\,\Lambda'',k'\,q'|L'\,\Lambda') &= \sum_{q',\mu,\Lambda''}(-1)^{k'+1+2}(1\,\mu,k'\,q'|2\,Q)(L\,\Lambda\,1\mu|L''\,\Lambda'')(L''\,\Lambda'',k'\,q'|L'\,\Lambda')\\ &= (-1)^{k'+1}\hat{L}''\sqrt{5}W(L\,1\,L'\,k';L''\,2)(L\,\Lambda,2\,Q|L'\,\Lambda').\end{aligned}\tag{D5}$$

Putting together these results gives

$$\langle L',||\{Y_{k'}(\boldsymbol{R})\times\boldsymbol{p}_R\}_2||L,\Lambda\rangle=\frac{1}{\imath}\sum_{L''=L\pm1}\frac{\hat{L}''\hat{k}'}{\sqrt{4\pi}\hat{L}'}(L''\,0,k'\,0|L'\,0)(-1)^{k'+1}\hat{L}''\sqrt{5}W(L\,1\,L'\,k';L''\,2)\hat{O}_{L'',L},\tag{D6}$$

and hence

$$\begin{aligned}&\langle(L',1)J'||\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_1\cdot\boldsymbol{p}_R||(L,1)J\rangle\\ &= \frac{\delta_{J,J'}}{\imath}\sqrt{\frac{3}{5}}\hat{L}'\sqrt{3}\sqrt{5}(-1)^{J-L'}W(L'\,L\,1\,1;2\,J)\sum_{L''=L\pm1}\frac{(\hat{L}'')^2\hat{k}'}{\sqrt{4\pi}}\sqrt{5}W(L\,1\,L'\,k';L''\,2)(L''\,0,k'\,0|L'\,0)\hat{O}_{L'',L}\\ &= \frac{\delta_{J,J'}}{\imath}3\sqrt{5}\hat{k}'(-1)^{J-L'}W(L'\,L\,1\,1;2\,J)\sum_{L''=L\pm1}\frac{(\hat{L}'')^2}{\sqrt{4\pi}}W(L\,1\,L'\,k';L''\,2)(L''\,0,k'\,0|L'\,0)\hat{O}_{L'',L}.\end{aligned}\tag{D7}$$

In the special case $L=0$, $L'=2$, $k'=1$, when $L''=1$ is the only nonvanishing contribution, this reduces to

$$\langle(2,1)1||\{Y_{k'}(\boldsymbol{R})\times\tau_2(1)\}_1\cdot\boldsymbol{p}_R||(0,1)J\rangle=-\frac{\delta_{J,1}}{\imath}\sqrt{\frac{6}{5}}\frac{1}{\sqrt{4\pi}}\frac{d}{dR}.\tag{D8}$$

It is readily checked by direct evaluation that this is the correct result.

Using Eq. (D1) the final result for the reduced matrix element of $(V^{JT})^{K=2}_{(n,1)}$ is

$$\begin{aligned}\langle(L',1)J'||(V^{JT})^{K=2}_{(n,1)}||(L,1)J\rangle &= \delta_{J,J'}3\sqrt{\frac{5}{4\pi}}(-1)^{J-L'}W(L'\,L\,1\,1;2\,J)\sum_{k'=1,\,3}\hat{k}'A^{(1,0)}_{k',2}(2,R,\mathrm{so},nA)\\ &\quad\times\sum_{L''=L\pm1}(\hat{L}'')^2W(L\,1\,L'\,k';L''\,2)(L''\,0,k'\,0|L'\,0)\hat{O}_{L'',L},\end{aligned}\tag{D9}$$

where the functions $A^{1,0}_{k',2}(2,R)$, $k'=1,3$, are given explicitly in Eqs. (67) and (68).

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