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Higher partial-wave potentials from supersymmetry-inspired factorization and nucleon-nucleus elastic scattering

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A simple potential model of the Hulthen type without spin-orbit coupling is considered as the ground-state interaction, and in conjunction with supersymmetric quantum mechanics, higher partial-wave interactions are developed to study the scattering of nucleons from light nuclei. The phase function method is adopted to deal with scattering phase shifts. Applying certain energy-dependent correction factors to our interactions, a close agreement with experimental data is obtained for the elastic scattering of nucleons from alpha particles up to 12 MeV.

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I. INTRODUCTION

The theory of nuclear forces is one of the oldest branches in nuclear physics and has a long history. Although a vast amount of works has already been devoted to the nucleon-nucleon (NN) problem still the NN interaction is the most fundamental problem in nuclear physics yet. These systems have been studied extensively and they provide a large number of reliable experimental data. The first field theoretic approach to find the fundamental theory of nuclear forces was started by Yukawa [1]. Since then a vast theoretical effort was made to derive 2π exchange contribution to the nuclear force to develop high precession parametrized potentials such as Paris potential [2], Nijm 93, Nijm-I, Nijm-II [3], CD-Bonn [4], and many others [5–7]. The data base regarding phase shift analysis was augmented considerably by a number of groups [8–15]. The phase shift data presented by these groups do not differ much except the methods employed were refined in one way or another. Thus, one can safely rely on these NN phase shift values.

But the situation is not so reliable for heavier systems. The elastic scattering between light nuclei is generally treated within the framework of the generator coordinate method (GCM) or the resonating group method. Within this model good agreement with experimental data regarding elastic scattering phase shifts have been achieved [16,17] by using phenomenological two-body interactions. Satchler *et al.* [18] with an optical potential model and Dohet-Eraly and Baye [19] within the formalism of unitary correlation operator method have studied $(\alpha - n)$ and $(\alpha - p)$ elastic scattering below 20 MeV and found good agreement with experimental data [20]. In the recent past we have exploited the supersymmetry-inspired factorization method to develop higher partial-wave potentials for nucleon-nucleon systems from their ground-state interactions and compute the related phase shifts [21–25]. Inspired by this we propose here a simple potential model (s wave) for the nuclear part of the $(\alpha - n)$ and $(\alpha - p)$ systems and generate their higher partial-wave interactions through the application of the well-known formalism of supersymmetric quantum mechanics (SQM) [21–29]. For the charged hadrons system one has to add an electromagnetic interaction with the nuclear part. Here the atomic Hulthen potential takes care of the electromagnetic interaction. We compute the scattering phase shifts for the associated interactions by judicious use of the phase function method (PFM) [30]. The present article is an effort in this direction. In Sec. II we propose ground-state interactions along with a brief outline of the development of the next higher partial-wave interactions through SQM formalism. Section III is devoted to computation and discussion on phase shifts. Finally, in Sec. IV some concluding remarks are presented.

II. SUPERSYMMETRY AND HIGHER PARTIAL-WAVE INTERACTIONS

Arnold and MacKellar [31] have parametrized the Hulthen potential to fit the deuteron binding energy and s-wave scattering length. Hereafter we shall designate it as nuclear Hulthen potential. It is also well known that the Hulthen potential is applicable for the s wave only. In SQM [26–29] one often deals with the hierarchy problems, for example, within the framework of the SQM one can generate a Hamiltonian hierarchy, the adjacent members of which are the supersymmetric partners in that they share the same eigenvalue spectrum except the missing ground state. Therefore, it may be of considerable interest to generate the supersymmetric partners of the nuclear Hulthen potential and study their properties. We designate these generated supersymmetric partners as p- and d-wave potentials.

The s-wave nuclear Hulthen potential [31] for the N-N system reads

$$V_N^0(r) = -(\beta^2 - \alpha^2)\frac{e^{-\beta r}}{(e^{-\alpha r} - e^{-\beta r})}, \tag{1}$$

where β stands for the inverse range parameter and the wave number α is related to the strength of the interaction. Following the SQM formalism, in our recent few papers [21–25], we have constructed the higher partial-wave NN potentials. Those are

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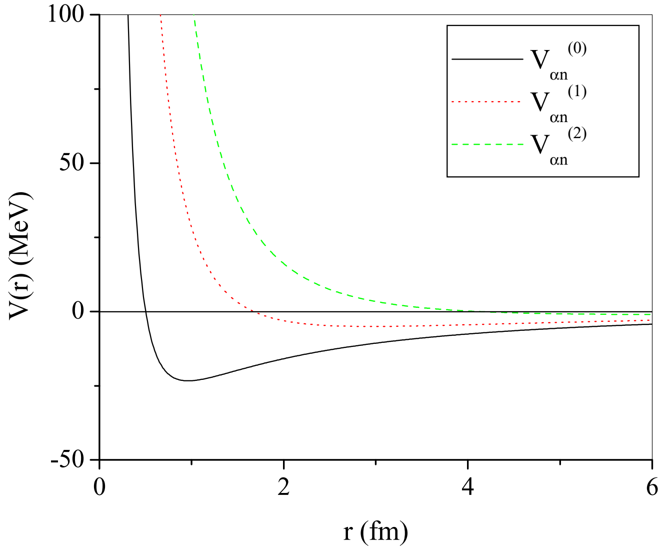

FIG. 1. (Color online) Potentials (s, p, and d states) for the ($\alpha - n$) system.

expressed as

$$V_N^{\ell}(r) = V_N^0(r) + \frac{\ell(\ell+1)\,(\beta-\alpha)^2 e^{-(\alpha+\beta)r}}{2\,(e^{-\alpha r} - e^{-\beta r})^2}; \quad \ell = 1,2,\ldots \tag{2}$$

For the ($\alpha - n$) and ($\alpha - p$) systems, unlike the NN system, the s-wave nuclear interaction is proposed by the following two-term potential:

$$V_{\alpha n}^{(0)}(r) = V_N^{(0)}(r) + (\beta-\alpha)^2 \frac{e^{-(\alpha+\beta)r}}{(e^{-\alpha r} - e^{-\beta r})^2}. \tag{3}$$

Applying the basic formalism of SQM the higher partial-wave (p- and d-wave) nuclear potentials are obtained as

$$V_{\alpha n}^{(1)}(r) = V_N^{(0)}(r) + 3(\beta-\alpha)^2 \frac{e^{-(\alpha+\beta)r}}{(e^{-\alpha r} - e^{-\beta r})^2} \tag{4}$$

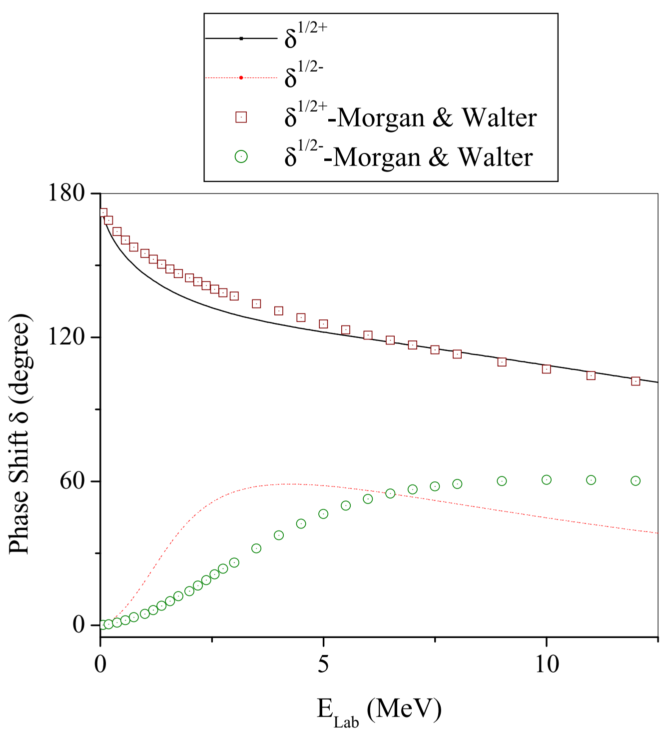

FIG. 2. (Color online) Phase shifts (s and p states) for the ($\alpha - n$) system.

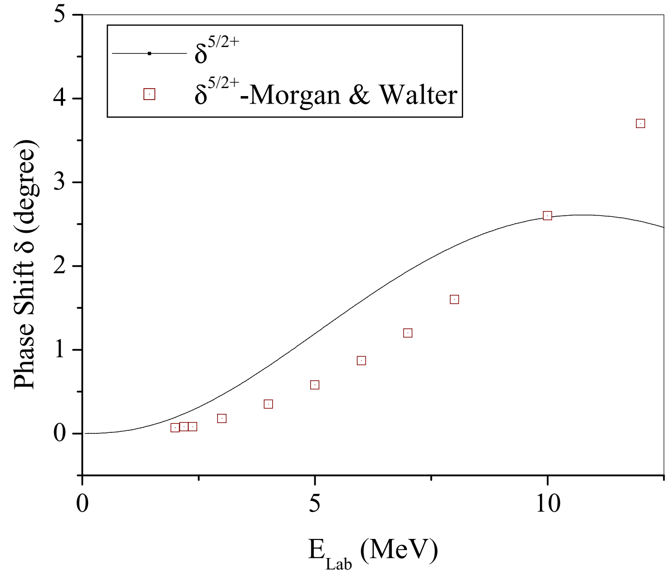

FIG. 3. (Color online) Phase shifts (d state) for the ($\alpha - n$) system.

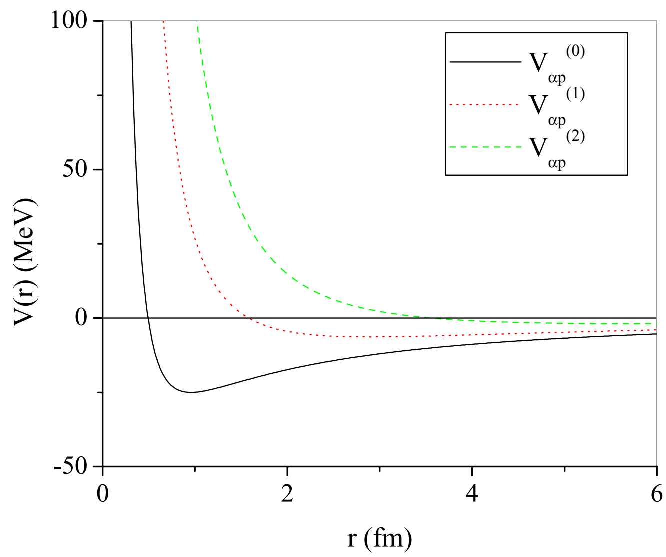

FIG. 4. (Color online) Potentials (s, p, and d states) for the ($\alpha - p$) system.

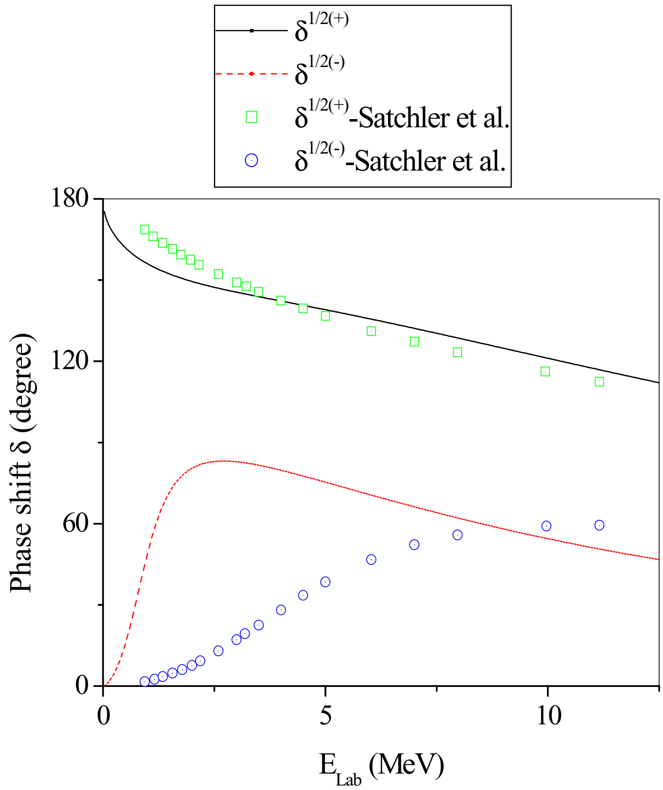

FIG. 5. (Color online) Phase shifts (s and p states) for the $(\alpha - p)$ system.

and

$$V_{\alpha n}^{(2)}(r) = V_N^{(0)}(r) + 6(\beta - \alpha)^2 \frac{e^{-(\alpha+\beta)r}}{(e^{-\alpha r} - e^{-\beta r})^2}. \quad (5)$$

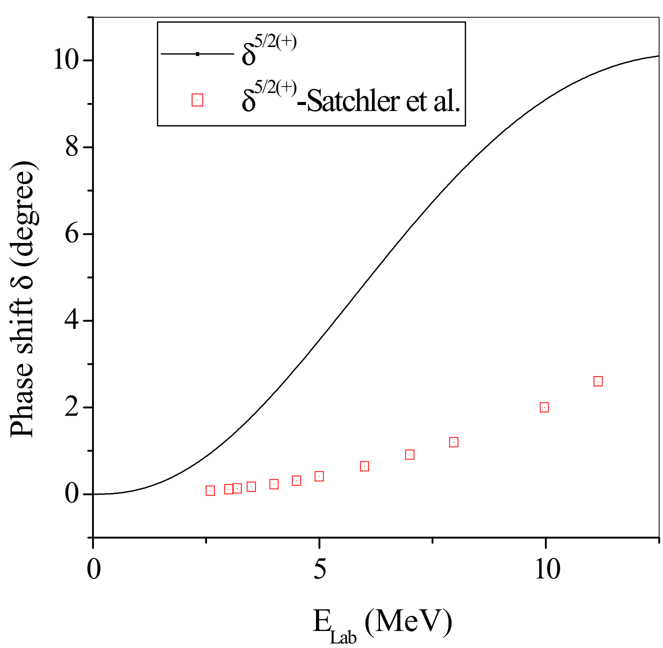

FIG. 6. (Color online) Phase shifts (d state) for the $(\alpha - p)$ system.

For the $(\alpha - p)$ system the s-, p-, and d-wave interactions read

$$V_{\alpha p}^{(0)}(r) = V_H(r) + V_{\alpha n}^{(0)}(r), \quad (6)$$

$$V_{\alpha p}^{(1)}(r) = V_H(r) + V_{\alpha n}^{(1)}(r), \quad (7)$$

and

$$V_{\alpha p}^{(2)}(r) = V_H(r) + V_{\alpha n}^{(2)}(r). \quad (8)$$

The quantity $V_H(r)$ stands for the two-parameter (V_0 and a) atomic Hulthen potential [32] written as

$$V_H(r) = V_0 \frac{e^{-r/a}}{(1 - e^{-r/a})}. \quad (9)$$

III. COMPUTATION OF PHASE SHIFTS AND DISCUSSIONS

The phase function method (PFM) is an efficient tool for evaluating scattering phase shifts for quantum mechanical problems involving local and nonlocal interactions [30,33,34]. For a local potential the phase function $\delta_\ell(k,r)$ satisfies a first-order nonlinear differential equation [30] that reads

$$\delta'_\ell(k,r) = -k^{-1}\, V(r)[\hat{j}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\, \sin\, \delta_\ell(k,r)]^2, \quad (10)$$

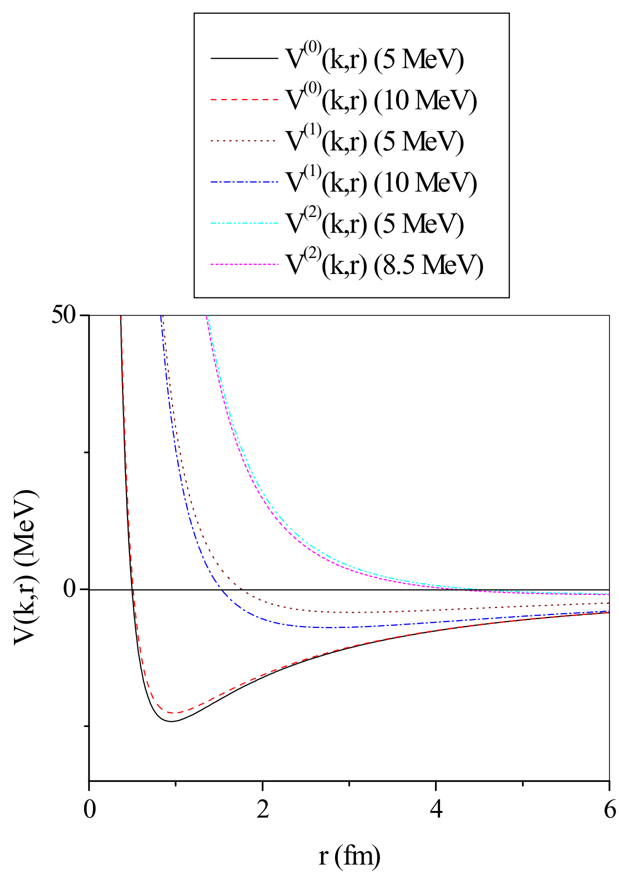

FIG. 7. (Color online) Potentials (s, p, and d states) for the $(\alpha - n)$ system with correction.

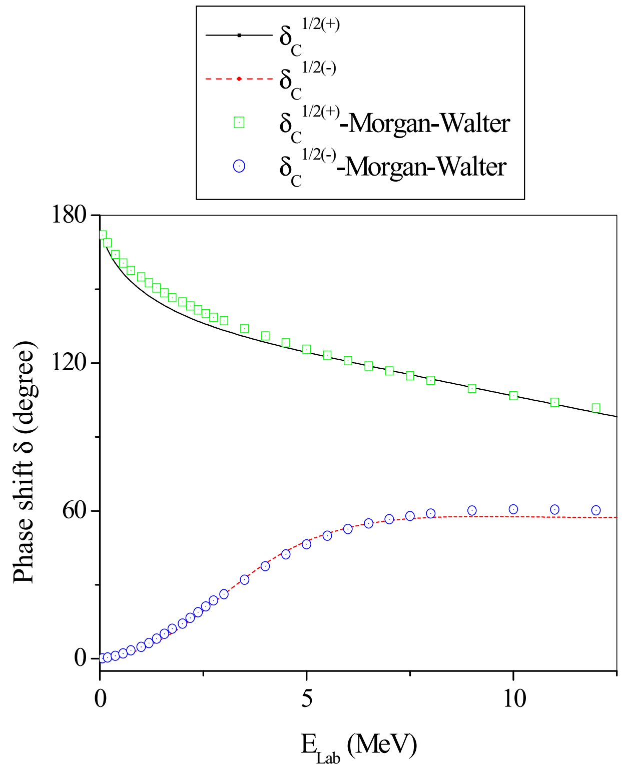

FIG. 8. (Color online) Phase shifts (s and p states) for the $(\alpha - n)$ system with correction.

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati Bessel functions with $\hat{h}_\ell^{(1)}(x) = -\hat{\eta}_\ell(x) + i\,\hat{j}_\ell(x)$. The scattering phase shift $\delta_\ell(k)$ is obtained by solving the equation from the origin to the asymptotic region with the initial condition $\delta_\ell(k,0) = 0$.

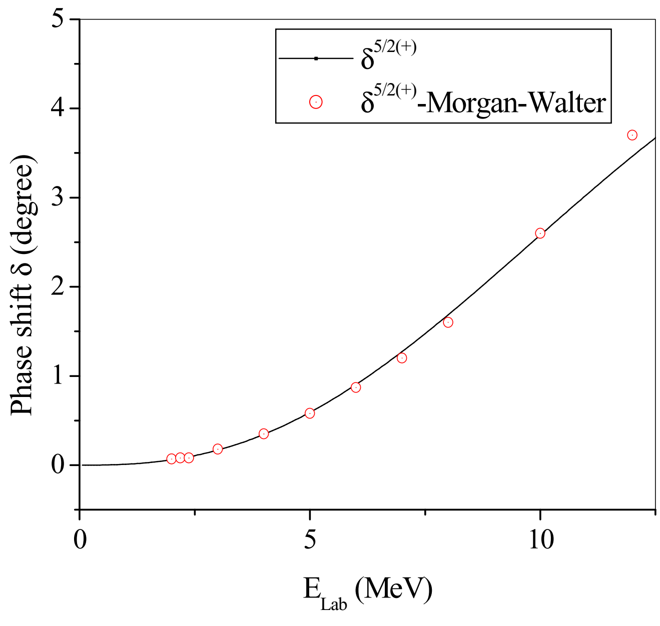

FIG. 9. (Color online) Phase shifts (d state) for the $(\alpha - n)$ system with correction.

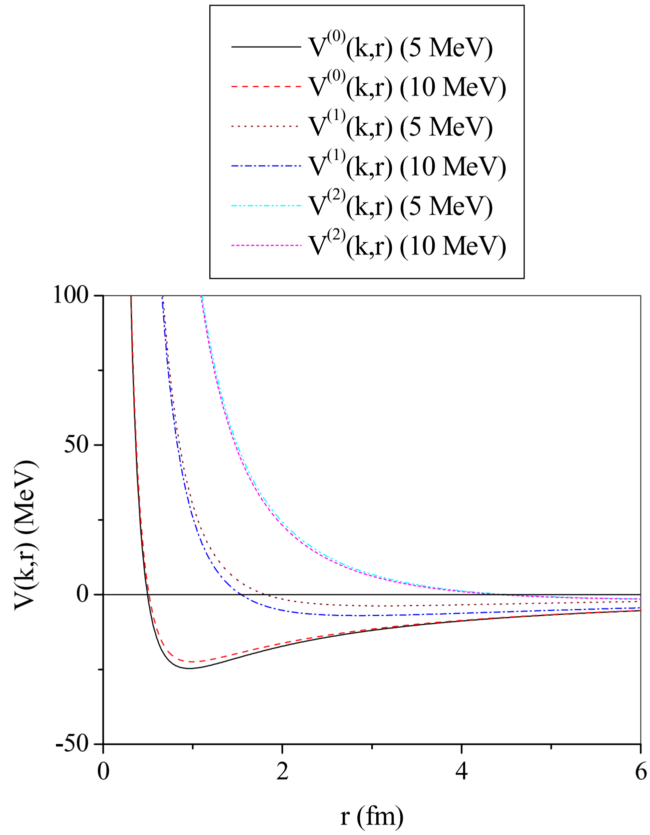

FIG. 10. (Color online) Potentials (s, p, and d states) for the $(\alpha - p)$ system with correction.

We have computed and plotted the s- and p-wave potentials for the systems under consideration. With the parameters $\beta = 1.135\ \text{fm}^{-1}$ and $\alpha = 0.949\ \text{fm}^{-1}$ the s-, p-, and d-wave interactions and the corresponding phase shifts for the $(\alpha - n)$ system are portrayed in Figs. 1–3. For the $(\alpha - p)$ system the same are depicted in Figs. 4–6 with the parameters $\beta = 1.135\ \text{fm}^{-1}$ and $\alpha = 1.011\ \text{fm}^{-1}$. We have chosen to work with $(2k\eta)^{-1} = 18.12908$ fm and $a = 20$ fm. The phase shifts are computed for the potentials in Eqs. (3)–(9) by applying Eq. (10).

In Figs. 1 and 4 it is observed that stronger repulsive cores develop in the SQM generated p- and d-wave interactions than their s-wave counterparts. Looking closely into Figs. 2, 3, 5, and 6 it is observed that the phase shifts for our proposed s wave and SQM generated p- and d-wave potentials produce the correct nature of the phase shifts for $1/2^{(+)}$, $1/2^{(-)}$, and $5/2^{(+)}$ states of the $(\alpha - n)$ and $(\alpha - p)$ systems, respectively. For the $(\alpha - n)$ system we notice that our phase shift values $\delta^{1/2(+)}$, $\delta^{1/2(-)}$, and $\delta^{5/2(+)}$ differ more or less symmetrically on either side of $E_{\text{Lab}} = 7.5$ MeV, $E_{\text{Lab}} = 6.5$ MeV, and $E_{\text{Lab}} = 10.0$ MeV, respectively, from those of experimental results [20]. On the other hand, for the $(\alpha - p)$ system these values are $E_{\text{Lab}} = 4.3$ MeV and $E_{\text{Lab}} = 9.0$ MeV for s and p waves, respectively. However, for the d wave our phase shifts differ significantly and no such point of coincidence is detected within the range of consideration. Therefore, our

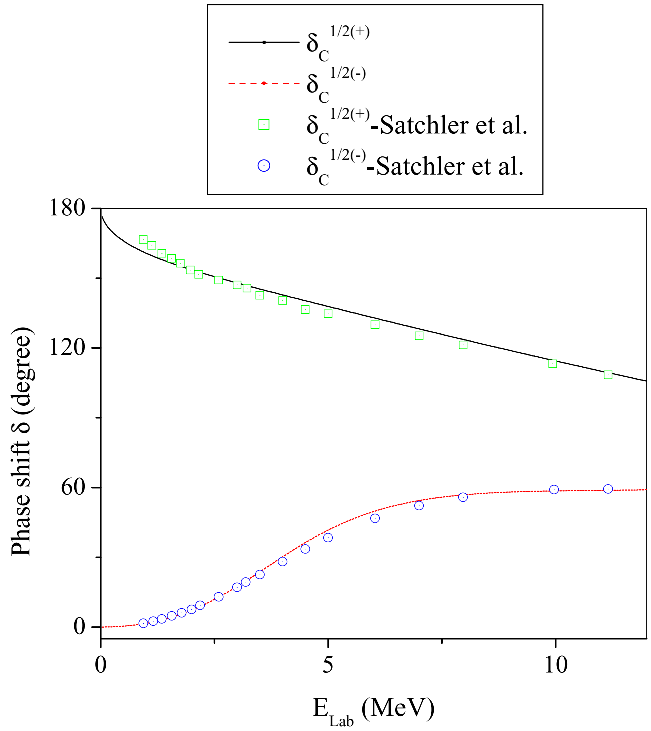

FIG. 11. (Color online) Phase shifts (s and p states) for the $(\alpha - p)$ system with correction.

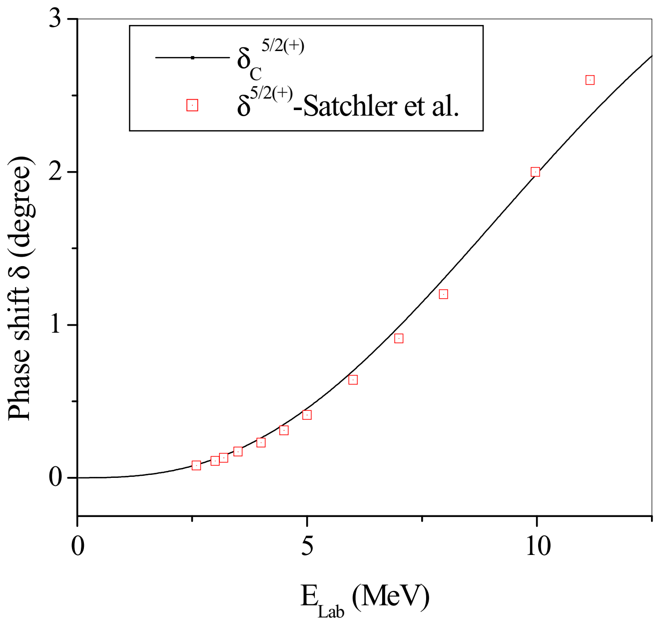

FIG. 12. (Color online) Phase shifts (d state) for the $(\alpha - p)$ system with correction.

data need correction on either side of the point of coincidence to have a better agreement with standard data [18–20]. To simulate the effect of such correction in the phase data we have identified and incorporated an energy-dependent correction factor to the s, p, and d-wave interactions to achieve good agreement with the experiment [20]. These are $V_{\alpha n}^{(0)C}(k,r) = 1.25(k^2 - 0.231)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.58$, $V_{\alpha n}^{(1)C}(k,r) = 1.25(k^2 - 0.2)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.097$, $V_{\alpha n}^{(2)C}(k,r) = 1.25(k^2 - 0.308)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.273$ for the $(\alpha - n)$ system and $V_{\alpha p}^{(0)C}(k,r) = 1.25(k^2 - 0.133)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.38$, $V_{\alpha p}^{(1)C}(k,r) = 1.25(k^2 - 0.28)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.067$ for the $(\alpha - p)$ system. Although no point of coincidence is noticed in our d-wave data with the standard result [18–20] still we have applied this correction factor $V_{\alpha p}^{(2)C}(k,r) = 1.25(k^2 - 0.772)e^{-\gamma(\alpha+\beta)r}$; $\gamma = 0.25$ to our generated $(\alpha - p)$ potential to achieve a close agreement with the experiment [20]. Here γ is an additional adjustable parameter. These phase shifts and potentials with respective correction factors are included in Figs. 7–12. The phase shifts and the potentials with correction factors to s, p, and d states are denoted by $\delta_C^{1/2(+)}$, $\delta_C^{1/2(-)}$, $\delta_C^{5/2(+)}$, and $V^{(0)}(k,r)$, $V^{(1)}(k,r)$, $V^{(2)}(k,r)$, respectively.

IV. CONCLUSION

In the present text we have proposed a simple potential model without taking recourse to the spin-orbit interaction for the $(\alpha - n)$ and $(\alpha - p)$ systems for the partial wave $\ell = 0$. The higher partial-wave interactions have been generated by exploiting the formalism of supersymmetric quantum mechanics and analyzed its effectiveness through the phase shift calculations. Although they differ in their numerical values our computed phase shifts produce the correct nature of the respective states. With an additional energy-dependent correction factor to the potential we have achieved fairly good agreement with the experiment [20]. In contrast to earlier approaches to the problem [18,19] the present model is much simpler; it involves two parameters (α &β) related to range and depth of the ground interaction, and γ, another adjustable one. From our observation it is reflected that the algebra of SQM also has the qualitative ability to describe higher partial-wave state interactions and physical observables from the knowledge of their ground state in subatomic regions. The quantitative agreement of phase shifts was obtained by applying an additional energy-dependent correction factor to respective states. Thus, one may conclude by noting that the energy-dependent correction factors of the interactions, to some extent, have the ability to produce the effect of spin-orbit coupling factors. Therefore, it our belief that the combined approach of SQM and PFM may be of considerable interest for the nucleon-nucleus scattering and deserves some attention.

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