Wave functions of the $Q \cdot Q$ interaction in terms of unitary 9-j coefficients

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We obtain wave functions for two protons and two neutrons in the $g_{9/2}$ shell expressed as column vectors with amplitudes $D(J_p,J_n)$. When we use a quadrupole-quadrupole interaction ($Q \cdot Q$) we get, in many cases, a very strong overlap with wave functions given by a single set of unitary 9-j coefficients—U9j = $\langle (jj)^{2j}(jj^{J_B}|(jj)^{J_p}(jj)^{J_n})^I\rangle$. Here $J_B = 9$ for even $IT = 0$ states. For both even and odd $T = 1$ states we take J_B equal to 8 whilst for odd I, $T = 0$ we take J_B to be 7. We compare the $Q \cdot Q$ results with those of a more realistic interaction.

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I. INTRODUCTION

In previous works, the problem of maximum $JT = 0$ pairing was addressed [1–3] and comparisons were made with $J = 0T = 1$ pairing. In the course of these works it was found that, to an excellent approximation, some wave functions of the maximum-J pairing Hamiltonian were very close to single sets of unitary 9-j coefficients. In this work we wish to disengage this simple result from the complexities of the maximum-J pairing problem. To this end, we consider better Hamiltonians than maximum J and show that the results hold there as well, and we make comparisons with wave functions obtained from the simple $Q \cdot Q$ interaction as well as from a more realistic interaction CCGI [4]. The calculations are for two protons and two neutrons in the $g_{9/2}$ shell, i.e., ^{96}Cd.

II. OVERLAPS OF $Q \cdot Q$ AND CCGI WITH U9j

To facilitate comparisons of the two interactions, we add constants so that the $J = 0$ matrix elements are zero for both interactions. The ten matrix elements from $J = 0^+$ to $J = 9^+$ are then

$Q \cdot Q$: 0, 0.1222, 0.3485, 0.6515, 0.9848, 1.2879, 1.4849, 1.4849, 1.1818, 0.4546;

CCGI: 0, 0.8290, 1.6500, 1.8770, 2.2170, 2.3018, 1.6049, 2.3830, 1.8019, 2.5270, 0.9150.

The $J = 0$ values are -1.0000 and -2.3170 MeV, respectively. Of course the $Q \cdot Q$ interaction can be multiplied by a positive constant without changing the overlaps. We first compare the overlaps of wave functions obtained with the popular $Q \cdot Q$ interactions with wave functions that are basically single sets of U9j coefficients. This interaction has the nice feature of having attractive $J = 0^+$, 1^+, and 9^+ two-body matrix elements. For even I we compare the yrast-state wave functions of $Q \cdot Q$ with those of the U9j's, i.e., $N\langle (jj)^{J_{\max}}(jj)^{J_B}|(jj)^{J_p}(jj)^{J_n}\rangle^I$, where in the $g_{9/2}$ shell $J_{\max} = 2j = 9$. To compare with yrast $T = 0$ even I states of $Q \cdot Q$ we take $J_B = 2j = 9$. The normalization N is close to $\sqrt{2}$. More precisely, $N(9)^{-2} = 1/2 - 1/2\langle (jj)^9(jj)^9|(jj)^9(jj)^9\rangle$ [1].

We show in Table I the following overlaps: (ψ, U9j) for both $Q \cdot Q$ and CCGI [4].

Note the very strong overlaps for $I = 0^+$, 2^+, 4^+, and 6^+ and then the sudden drop to almost zero overlap for $I = 8^+$ and the small overlap of 0.3635 for $I = 10^+$. In selected cases we consider overlaps with the next excited states, e.g., $T = 0$, $I = 8^+$ and 10^+. The results were 0.9505 for $I = 8^+$ and 0.8540 for $I = 10^+$. In other words, for $I = 8^+$ we can, to a good approximation, identify the simple U9j wave function with the first exited state rather than the ground state. For $I = 10^+$ there is fragmentation. For $I = 12^+$ and 14^+ we get poor overlaps of 0.6586 and 0.8374, respectively. However, for $I = 16^+$ we get a perfect overlap. But this case is trivial. There is only one $I = 16^+$ state with $J_p = 8$ and $J_n = 8$.

We next briefly consider the other (J,T) combinations. If the wave function amplitudes are $D(J_p,J_n)^I$ then we have $D(J_n,J_p) = (-1)^{(I+T)}D(J_p,J_n)$ where T is the isospin. Thus, for even $IT = 1$ and for odd $IT = 1$ we take $J_B = 8$ while for odd $IT = 0$ we take $J_B = 7$. Note that there are no $I = 0T = 1$ states in this model space and that all $I = 1$ states have isospin $T = 1$.

Referring to Table I we here note the (J,T) states of these other combinations with overlap greater than 0.9: (2,1), (4,1), (6,1), (3,0), (5,0), (3,1), (5,1).

There is also the special case (14,1); here there is a perfect overlap because there is only one state of this configuration. Except for the last case, we get the best overlaps for the lowest angular momenta. Note that we cannot associate U9js with $T = 2$ states. Those are double analogs of states of four identical nucleons and the wave functions are uniquely constrained by the Pauli principle.

By choosing $J_B = 8$, we get a spectacular overlap of 0.9990 for the lowest $I = 1^+$, $T = 1$ state. Equally impressive for the $I = 3^+$, $T = 1$ state, the overlap is 0.9997. Choosing $J_B = 7$ we get for the $I = 3^+$, $T = 0$ an overlap of 0.9939 and for $I = 5^+$, $T = 0$ we get 0.9125. For values of I beyond $I = 5$, however, things begin to erode just as they do for large even I.

In general, we get better overlaps with $Q \cdot Q$ than we do with the more realistic CCGI. The values for $T = 0$, $I = 0$ are respectively 0.9996 and 0.9451. For $I = 1$, $T = 1$ they are 0.9990 and 0.8369. This may be due in part to the fact that $Q \cdot Q$ has a more attractive $J_{\max}$ matrix element than CCGI does.

But there are some surprises. For $J = 12^+$ and 14^+ we get much better results for CCGI than for $Q \cdot Q$. The ($Q \cdot Q$,CCGI) values are (0.6590, 0.9807) for $I = 12^+$ and (0.8370, 0.9722) for $I = 14^+$. As discussed in Ref. [5] the

TABLE I. Overlaps.

J	T	$[Q\cdot Q\mathrm{U}9j]$ $Q\cdot Qa$	Non-yrast $Q\cdot Qb$	[CCGI, U9j] CCGIa	Non-yrast CCGIb
0	0	0.9996		0.9451	
2	0	0.9999		0.9817	
4	0	0.9986		0.9178	
6	0	0.9871		0.8034	
8	0	0.0481	0.9505	0.2315	0.0488
10	0	0.3635	0.8540	0.6399	0.7591
12	0	0.6590	0.5884	0.9806	
14	0	0.8374		0.9722	
16	0	1		1	
2	1	0.9975		0.8870	
4	1	0.9870		0.8850	
6	1	0.9061		0.6319	
8	1	0.0568		0.2027	
10	1	0.3366		0.2711	
12	1	0.7746		0.3916	
3	0	0.9939		0.9912	
5	0	0.9125		0.3329	
7	0	0.7536		0.7607	
9	0	0.3021		0.3925	
1	1	0.9990		0.8369	0.5357
3	1	0.9974		0.9541	
5	1	0.9634		0.9315	
7	1	0.3120		0.6317	
9	1	0.1055		0.1318	

$J=0^+$, $T=1$ matrix element is not involved in these high-spin states. The more relevant comparison here is between the $J_{\max}=9$ matrix element and the one with $J=2$. With $Q\cdot Q$ the $J=2$ and $J=7$ matrix elements are 0.3482 and 0.4546. Hence $J=2$ is more attractive than $J=9$. In contrast with CCGI [4] the respective numbers are 1.6500 and 0.9140. Hence with CCGI the $J=9$ matrix element is more attractive than $J=2$.

III. COMPARISON WITH $E(J_{\max})$

Previously we had studied the overlaps of U9j with wave functions of the $E(J_{\max})$ interaction [5], i.e., an interaction in which all two-body matrix elements vanish except for the ones with $J=2j$. This interaction can only occur only between a neutron and a proton. Studying this interaction gave us the idea that the above set of U9j coefficients could, in many cases, be excellent approximations to wave functions that result from more realistic interactions. Indeed, the single U9$j\langle(jj)^9(jj)^{J_B}|(jj)^{J_p}(jj)^{J_n}\rangle^I$, with both J_p and J_n even is an exact eigenfunction of $E(9)$ for two protons and two neutrons in the $g_{9/2}$ shell. Indeed, we get perfect overlap in the following cases with the $J_{\max}$ interaction: $I=0$, $T=0$ ($J_B=9$), $I=1$, $T=1$ ($J_B=8$), $I=2$, $T=1$ ($J_B=8$), and $I=3$, $T=0$ ($J_B=7$). Although for $I=2$, $T=0$ a single U9j is not an eigenstate, to an excellent approximation the lowest two $I=2^+$ states have $J_B=9$ and $J_B=7$.

However, the $E(J_{\max})$ interaction taken by itself does not give a reasonable spectrum. One of the sturdiest results in nuclear structure is that all even-even nuclei have $I=0$ ground states. However with an attractive E(9) interaction the lowest state has $I=16^+$.

The $Q\cdot Q$ interaction has a much more reasonable spectrum, with an $I=0^+$ ground state for even-even nuclei. *A priori* it is not clear that single U9js could be reasonable approximations to the eigenfunctions of $Q\cdot Q$. The two-particle matrix elements of this interaction are strongly attractive not only for $J=J_{\max}$ but also for $J=0$ (even more so) and $J=1$. For this reason, it is more significant that the simple sets of U9js noted above have high overlaps with wave functions arising from $Q\cdot Q$, at least for low spins. On the other hand, without the study of the $E(J_{\max})$ interaction we would never have guessed that a set of U9js existed which were close to wave functions of realistic interactions.

IV. CLOSING REMARKS

In closing, we note that we have obtained an interesting result for a system of two protons and two neutrons in a single j shell. We find that, with the $J_{\max}$ and $Q\cdot Q$ interactions, states of low total angular momentum are spectacularly well described by a single unitary 9-j coefficient. With more realistic interactions the overlaps, although not quite as good, are still greater than 0.9 in many cases. It is of course well known that 9-j coefficients are building blocks which could be used to construct wave functions, but the surprise here is that the unitary 9-j coefficients are themselves in many cases very close to the exact wave functions.

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