Hadron resonance gas equation of state from lattice QCD

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The Monte Carlo results in lattice QCD for the pressure and energy density at small temperature $T < 155$ MeV and zero baryonic chemical potential are analyzed within the hadron resonance gas model. Two extensions of the ideal hadron resonance gas are considered: the excluded volume model, which describes a repulsion of hadrons at short distances, and the Hagedorn model with an exponential mass spectrum. Considering both of these models we do not find conclusive evidence in favor of either of them. The controversial results appear because of rather different sensitivities of the pressure and energy density to both excluded volume and Hagedorn mass spectrum effects. On the other hand, we have found clear evidence for a *simultaneous* presence of both of them. They lead to rather essential contributions: suppression effects for thermodynamical functions of the hadron resonance gas due to the excluded volume effects and enhancement due to the Hagedorn mass spectrum.

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I. INTRODUCTION

Monte Carlo calculations in lattice QCD at finite temperature T (see, e.g., Refs. [1–4], and references therein) reveal two physical phases of strongly interacting matter: a hadron phase at small T and a deconfined quark-gluon phase at high T. A nontrivial analysis within lattice QCD indicates that these phases at zero chemical potential are not separated by a true phase transition but that they are rather connected by an analytic crossover [5]. In Fig. 1 the lattice results of Ref. [3] for the pressure and energy density ($3p/T^4$ and ε/T^4) obtained at zero baryonic chemical potential in QCD with 2 + 1 quark flavors and extrapolated to the thermodynamical and continuum limits are shown as functions of temperature T.

From Fig. 1 one observes a steep increase of thermodynamical quantities near the crossover temperature T_c. This temperature is estimated in the range of 150–160 MeV. The values of $3p/T^4$ and ε/T^4 in the deconfined quark-gluon phase approach slowly from below the Stefan-Boltzmann limit $3p_{\rm SB}/T^4 = \varepsilon_{\rm SB}/T^4 = \sigma_{\rm SB}$, a value which equals $19\pi^2/12 \cong 15.6$ in the three-flavor QCD. At $T < T_c$ the confined hadron phase emerges. In the present paper the lattice data [3] will be used to constrain the equation of state of the hadronic matter.

A description of hadron multiplicities in high-energy nucleus-nucleus collisions shows a surprisingly good agreement between the results of the hadron resonance gas (HRG) model (see, e.g., Refs. [6–11]) and the experimental data. In most statistical model formulations the ideal HRG (Id-HRG) is used. It is argued that the presence of all known resonance states in the thermal system takes into account attractive interactions between hadrons [12].

Two extensions of the Id-HRG model have been widely discussed. The first one is the excluded volume HRG (EV-HRG) model in which the effects of hadron repulsions at short distances are introduced. One usually uses the van der Waals procedure [13,14] and substitutes a system volume V by the available volume $V - \sum_i v_i N_i$, where v_i is the volume parameter for the ith hadron species, N_i is the number of particles of the ith type, and the sum is taken over all types i of hadrons and resonances. Another example of attractive and repulsive interactions between hadrons is given within relativistic mean-field theory [15] (see also the recent paper [16], and references therein). Note that the EV-HRG model can be equivalently formulated in terms of the mean field (see Refs. [17–19]). This makes it possible to incorporate other hadron interactions within a unified mean-field approach.

The second extension of the HRG model is the inclusion of the exponentially increasing mass spectrum $\rho(m)$ proposed by Hagedorn about 50 years ago [20,21]. These excited colorless states (named fireballs or strings) are considered as a continuation of the resonance spectrum at masses m higher than 2 GeV.

In the present paper we use the lattice data [3] at small temperature $T < 155$ MeV to confirm the presence of the excluded volume effects and effects of the Hagedorn mass spectrum. The HRG model had been used for comparison with the lattice data in the hadronic sector [22–24]. There were as well several publications in which the EV-HRG model (see, e.g., Refs. [25,26]) or the Hagedorn mass spectrum (see, e.g., Refs. [27–29]) were confronted with the lattice data. Our analysis extends these previous attempts to the case when both these physical effects are treated simultaneously. Included together they essentially improve the agreement of the HRG model with the lattice results, while treated separately neither of them can be clearly established.

The paper is organized as follows. In Sec. II the grand canonical ensemble formulation of the Id-HRG and EV-HRG models are considered. In Sec. III the EV-HRG model is extended by inclusion of the Hagedorn mass spectrum. A summary in Sec. IV closes the article.

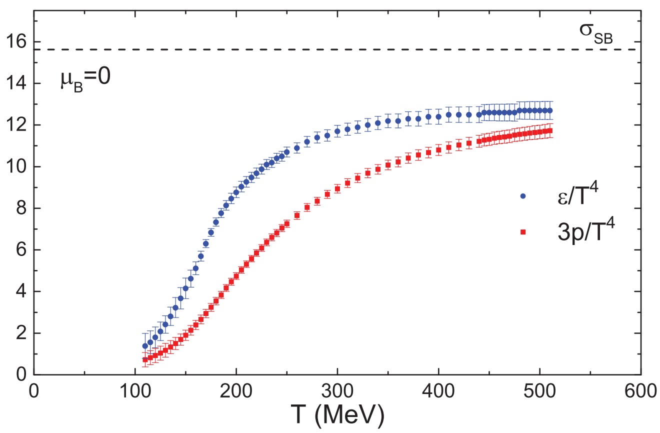

FIG. 1. (Color online) The lattice results from Ref. [3] for $3p/T^4$ (circles) and ε/T^4 (squares) at zero baryonic chemical potential.

II. HADRON RESONANCE GAS

A. Ideal hadron resonance gas

In the grand canonical ensemble the pressure and energy density of the Id-HRG are given by

$$p^{\rm id}(T,\mu) = \sum_i p_i^{\rm id}(T,\mu_i)$$
$$= \sum_i \frac{d_i}{6\pi^2}\int dm\, f_i(m)\int_0^\infty \frac{k^4 dk}{\sqrt{k^2+m^2}} \times\left[\exp\left(\frac{\sqrt{k^2+m^2}-\mu_i}{T}\right)+\eta_i\right]^{-1}, \quad (1)$$

$$\varepsilon^{\rm id}(T,\mu) = \sum_i \varepsilon_i^{\rm id}(T,\mu_i)$$
$$= \sum_i \frac{d_i}{2\pi^2}\int dm\, f_i(m)\int_0^\infty k^2 dk\,\sqrt{k^2+m^2} \times\left[\exp\left(\frac{\sqrt{k^2+m^2}-\mu_i}{T}\right)+\eta_i\right]^{-1}, \quad (2)$$

where d_i is the spin degeneracy of the ith particle and the normalized function $f_i(m)$ takes into account the Breit-Wigner shape of the resonance with finite width Γ_i around their average mass m_i for the stable hadrons, $f_i(m) = \delta(m - m_i)$. The sum over i in Eqs. (1) and (2) is taken over all nonstrange and strange hadrons that are listed in Particle Data Tables [30]. These include mesons up to $f_2(2340)$ and (anti)baryons up to $N(2600)$. We also note that in these equations $\eta_i = -1$ for bosons and $\eta_i = 1$ for fermions, while $\eta = 0$ corresponds to the Boltzmann approximation. The chemical potential for the ith hadron is given by

$$\mu_i \;=\; b_i\,\mu_B \,+\, s_i\,\mu_S \,+\, q_i\,\mu_Q \quad (3)$$

with $b_i = 0,\pm1$, $s_i = 0,\pm1,\pm2,\pm3$, and $q_i = 0,\pm1,\pm2$ being the corresponding baryonic number, strangeness, and electric charge of ith hadron. The notation μ will be used to denote all chemical potentials: $\mu \equiv (\mu_B,\mu_S,\mu_Q)$.

B. Excluded volume hadron resonance gas

In this section the role of the repulsive interactions is considered within the EV-HRG model. The van der Waals excluded volume procedure corresponds to a substitution of the system volume V by the available volume V_{av},

$$V \to V_{\rm av} \;=\; V - \sum_i v_i N_i, \quad (4)$$

where N_i is the particle number, $v_i = 4(4\pi r_i^3/3)$ is the excluded volume parameter with r_i being the corresponding hard-core radius of particle i, and the sum is taken over all hadrons and resonances. This result, in particular, the presence of the factor of 4 in the expression for v_i, can be rigorously obtained for a low-density gas of particles of a single type (see, e.g., Ref. [31]). In the grand canonical ensemble, the substitution (4) leads to a transcendental equation for the EV-HRG pressure [14]:

$$p^{\rm ev}(T,\mu) = \sum_i p_i^{\rm id}(T,\tilde{\mu}_i); \quad \tilde{\mu}_i = \mu_i - v_i\, p^{\rm ev}, \quad (5)$$

and the energy density is calculated as

$$\varepsilon^{\rm ev}(T,\mu) = \frac{\sum_i \varepsilon_i^{\rm id}(T,\tilde{\mu}_i)}{1+\sum_j v_j n_j^{\rm id}(T,\tilde{\mu}_j)}, \quad (6)$$

where $n_i^{\rm id}$ is the ideal-gas particle number density of the ith hadron species,

$$n_i^{\rm id}(T,\mu_i) = \frac{d_i}{2\pi^2}\int dm f_i(m)\int_0^\infty k^2 dk \times\left[\exp\left(\frac{\sqrt{k^2+m^2}-\mu_i}{T}\right)+\eta_i\right]^{-1}. \quad (7)$$

In what follows we restrict our consideration to the case of equal volume parameters v_i for all hadrons and resonances, $v_i = v \equiv 16\pi r^3/3$. The Boltzmann approximation $\eta_i = 0$ in Eqs. (1), (2), and (7) simplifies Eqs. (5) and (6) to

$$p^{\rm ev}(T,\mu) = \kappa^{\rm ev}\, p^{\rm id}(T,\mu) = \kappa^{\rm ev}\, T\, n^{\rm id}(T,\mu), \quad (8)$$

$$\varepsilon^{\rm ev}(T,\mu) = \frac{\kappa^{\rm ev}\,\varepsilon^{\rm id}(T,\mu)}{1+v\,\kappa^{\rm ev}\,n^{\rm id}(T,\mu)}, \quad (9)$$

where the excluded volume suppression factor $\kappa_{\rm ev}$ and the total particle number density $n^{\rm id}$ in the Id-HRG are introduced as

$$\kappa^{\rm ev} \equiv \exp\left(-\frac{v\,p^{\rm ev}}{T}\right), \quad n^{\rm id}(T,\mu) \equiv \sum_i n_i^{\rm id}(T,\mu_i). \quad (10)$$

Expressions (8)–(10) can be also obtained in the framework of thermodynamically self-consistent mean-field theory (see Sec. V in Ref. [19]). This approach gives a sequential treatment of the problem when one can examine various different mean fields that mimic the repulsive and attractive interactions (for details see Refs. [17–19]).

The EV-HRG model was used to fit the data on hadron multiplicities in Ref. [32] with values of r in the region of 0.2–0.8 fm. A numerical value of the hard-core radius was estimated as $r = 0.3$ fm in Ref. [33]. Note that, if radii of all hadrons are assumed to be the same, the chemical freeze-out parameters, temperature and baryon chemical potential,

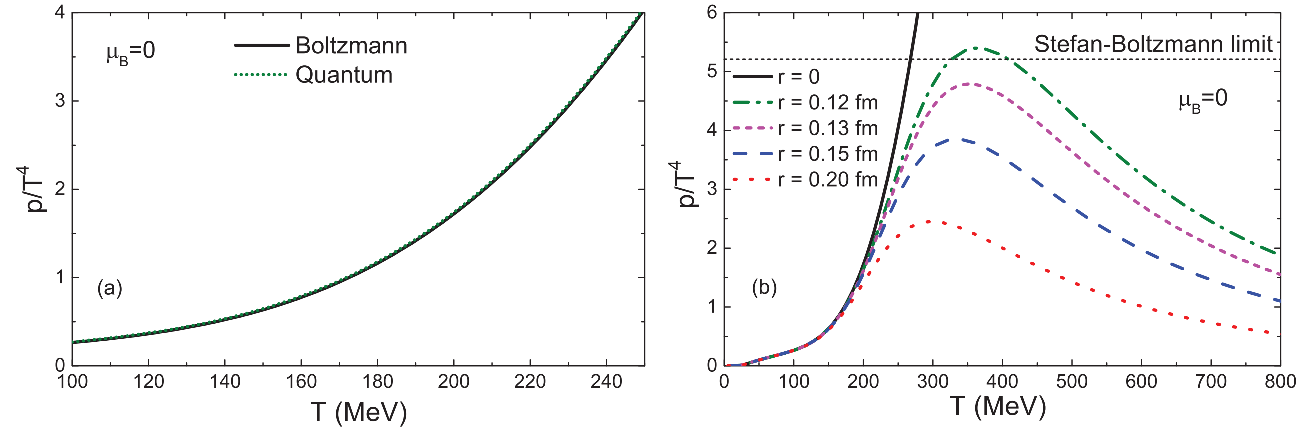

FIG. 2. (Color online) (a) The Id-HRG pressure, $p^{\mathrm{id}}(T)/T^4$, as a function of temperature at $\mu = 0$ (dotted line) and the Boltzmann approximation $\eta_i = 0$ (solid line). (b) The Id-HRG pressure and EV-HRG pressure functions for several different values of hard-core radius r. The Stefan-Boltzmann limit for the deconfined quark-gluon phase, $p_{\mathrm{SB}}/T^4 = \sigma_{\mathrm{SB}}/3 \cong 5.2$, is indicated by the horizontal dotted line.

fitted to the data on hadron multiplicities, are identical to those obtained within the Id-HRG model. Indeed, the particle number ratios are not sensitive to the numerical value of r. Hence, in order to establish the presence of nonzero hard-core hadron radii, independent measurements of the total system volume are needed. On the other hand, it was shown that the particle number fluctuations depend straightforwardly on the hard-core hadron radius [34,35]. Thus, an interpretation of these data within the EV-HRG model opens the way to estimate the value of r from the data. This is, however, not an easy task as there are many other effects which influence the particle number fluctuations.

C. Id-HRG and EV-HRG versus the lattice data

In Fig. 2(a) the Id-HRG pressure (1) divided by T^4 is shown as a function of temperature for the case of zero chemical

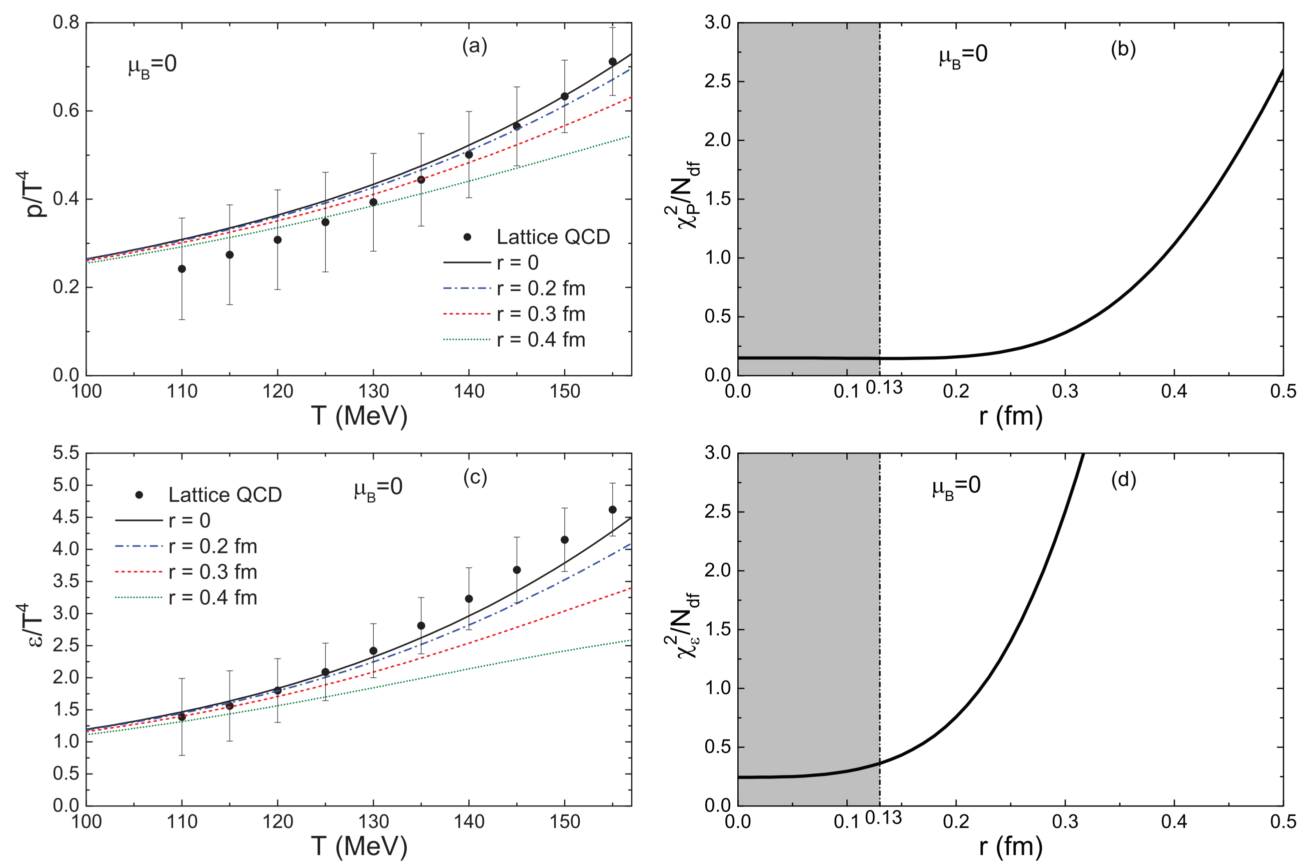

FIG. 3. (Color online) The results of the EV-HRG model for different values of r compared to the lattice data for p/T^4 (a) and ε/T^4 (c). The values of χ_p^2/N_{df} (b) and $\chi_\varepsilon^2/N_{\mathrm{df}}$ (d) shown as functions of r; the shaded gray area corresponds to $r \leqslant 0.13$ fm.

potentials,

$$\mu_B = \mu_S = \mu_Q = 0. \tag{11}$$

Equation (11) corresponds to zero values of all conserved charges, baryonic number, strangeness, and electric charge of the strongly interacting matter. This is approximately valid for matter created at the Large Hadron Collider (LHC) of the European Organization for Nuclear Research (CERN). As seen from Fig. 2(a) the Boltzmann approximation $\eta_i = 0$ (solid line) gives a very accurate evaluation of $p^{\rm id}(T)/T^4$ calculated with the quantum statistics (dashed line). In fact, the difference between the solid and dashed lines is hardly seen in Fig. 2(a). The Boltzmann approximation will be thus adopted for our further analysis. Note that a shift of the chemical potential according to Eq. (5) makes the Boltzmann approximation in the EV-HRG model even more accurate than in the case of the Id-HRG model.

In Fig. 2(b) the Id-HRG pressure (1) is compared with EV-HRG pressures (8) at several values of the hard-core radius r. Note that the Id-HRG model shows a strong increase of $p^{id}(T)/T^4$ at high T which exceeds the Stefan-Boltzmann pressure of the deconfined quarks and gluons, $p_{\rm SB}(T)/T^4 = \sigma_{\rm SB}/3 \cong 5.2$. Therefore, according to the Gibbs criterion pointlike hadrons would always be the dominant phase at high temperatures [36] due to the large number of different types of mesons and baryons. This feature of the Id-HRG equation of state contradicts the lattice QCD results; hence it shows a shortcoming of the model based on the concept of pointlike particles. Just the excluded volume effects ensure a transition from a gas of hadrons and resonances to the quark-gluon plasma. One needs therefore the EV-HRG equation of state for the hydrodynamic model calculations of nucleus-nucleus collisions (see, e.g., Refs. [37–39]). For any finite particle volume, i.e. $r > 0$, the behavior of pressure is found as $p^{\rm ev}(T)/T^4 \sim (vT^3)^{-1} \to 0$ at $T \to \infty$. However, as seen from Fig. 2(b), a more rigid restriction, $r \geqslant 0.13$ fm, is needed to guarantee that $p^{\rm ev}(T)/T^4 < \sigma_{\rm SB}/3$ at all $T > T_c$.

In Fig. 3(a), the EV-HRG results for $p^{\rm ev}/T^4$ are compared to the lattice data $p^{\rm lat}/T^4$ [3] at $T < 155$ MeV for several different values of hard-core radius r. In Fig. 3(b), the value of $\chi^2_p/N_{\rm df}$ at different r values is shown. This quantity is calculated as

$$\chi^2_p/N_{\rm df} = \frac{1}{N_{\rm df}} \sum_{i=1}^{N} \frac{[(p^{\rm ev}/T^4)_i - (p^{\rm lat}/T^4)_i]^2}{[\Delta(p^{\rm lat}/T^4)_i]^2}, \tag{12}$$

where $N_{\rm df}$ is the number of points, N (equal to 10 in our case), minus the number of fitting parameters (one parameter r in our fit). The most essential part of uncertainties in the lattice data is not statistical. The systematic uncertainties dominate, and these uncertainties are significantly correlated. In this case, the use of the $\chi^2/N_{\rm df}$ criterion is not perfectly reasonable. However, we still use this quantity as a way to quantify the deviations of HRG calculations from the lattice data.

From Fig. 3(b) one observes that the lattice data for $p^{\rm lat}/T^4$ are fitted well in a rather wide range of hard-core radius of $r \lesssim 0.4$ fm. Therefore, the lattice data for the hadron pressure are consistent with the presence of rather significant excluded volume effects and suggest reasonable numerical values for the hard-core radius r. However, a comparison of the EV-HRG model with the lattice results for $\varepsilon^{\rm lat}/T^4$ shown in Fig. 3(c) does not indicate the presence of excluded volume effects. This is clear from Fig. 3(d), where we show the value of $\chi^2_\varepsilon/N_{\rm df}$ calculated as in Eq. (12) but with ε/T^4 instead of p/T^4. The value of $r = 0$ corresponds to the best fit of $\varepsilon^{\rm lat}/T^4$. Let us, however, recall that Fig. 2(b) shows that too small values of the hard-core radius, $r \leqslant 0.13$ fm, look doubtful. For these small values of r the EV HRG pressure becomes larger than the Stefan-Boltzmann limit for quarks and gluons. The "forbidden" region of the hard-core radius, $r < 0.13$ fm, is shown as the gray area in Figs. 3(b), 3(d), 5(a), and 5(b).

Therefore, while a reasonable r value, e.g., $r = 0.3$ fm, gives a good agreement, $\chi^2_p/N_{\rm df} \cong 0.4$, with the $p^{\rm lat}/T^4$ lattice data, it also leads to a rather large value of $\chi^2_\varepsilon/N_{\rm df} \cong 2.5$ and, thus, looks unreasonable for the fit of $\varepsilon^{\rm lat}/T^4$.

III. EXCLUDED VOLUME HRG WITH HAGEDORN MASS SPECTRUM

The analysis presented in the previous section gives no conclusive answer about the presence of the excluded volume effects. The value of $r = 0.3$ fm in the EV-HRG model leads to sizable suppression effects of the Id-HRG pressure and to a good agreement with the lattice data $p^{\rm lat}/T^4$, whereas the $\varepsilon^{\rm lat}/T^4$ data prefer the value of $r \cong 0$ and are thus consistent with the Id-HRG model. From our point of view, this observation may indicate the presence of additional contributions to $p^{\rm ev}$ and $\varepsilon^{\rm ev}$ in the EV-HRG model. These contributions should be small enough for the pressure and much larger for the energy density. We argue that massive Hagedorn states are the ideal candidates for this role. Indeed, each heavy particle with $m \gg T$ gives its contribution, T, to the pressure, and a much larger contribution, $m + 3T/2$, to the energy density.

For a further analysis we use the following parametrization for the Hagedorn mass spectrum [21]:

$$\rho(m) = C \frac{\theta(m - M_0)}{\left(m^2 + m_0^2\right)^a} \exp\left(\frac{m}{T_H}\right). \tag{13}$$

The spectrum (13) with the parameters $M_0 = 2$ GeV, $T_H = 160$ MeV, $m_0 = 0.5$ GeV, and $a = 5/4$ will be used. The parameter C in (13) will be the only free parameter in the following analysis. We have checked that another set of parameters, e.g., the same set with $M_0 = 2.5$ GeV, and also a set with $m_0 = 0$, $a = 3/2$, and $M_0 = 3$ GeV, lead to very similar results.

Our final assumption concerns the proper volume for the Hagedorn states. To avoid additional free parameters we adopt the same value of $v = 16\pi r^3/3$ for all known hadrons and resonances as well as for the Hagedorn states. The EV-HRG model with the Hagedorn mass spectrum will be denoted as EV-HRG-H. The pressure $p^{\rm H}(T)$ in the EV-HRG-H model is given by the following equation:

$$p^{\rm H} = \exp\left(-\frac{vp^{\rm H}}{T}\right) T \int dm \int_0^\infty \frac{k^2 dk}{2\pi^2} \times \exp\left(-\frac{\sqrt{m^2 + k^2}}{T}\right) \left[\sum_i d_i f_i(m) + \rho(m)\right]. \tag{14}$$

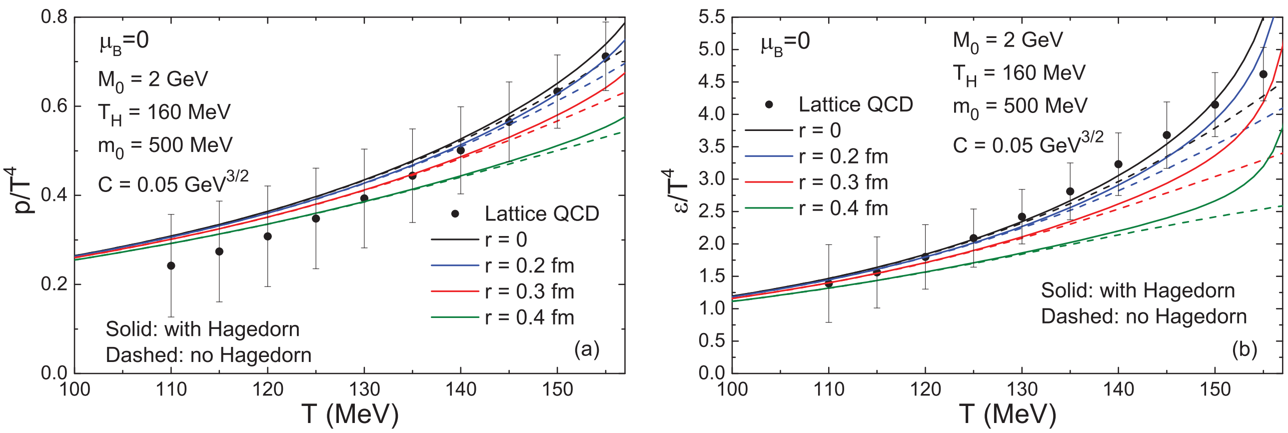

FIG. 4. (Color online) The results of the EV-HRG-H model for different values of r compared to the lattice data for p/T^4 and ε/T^4 in (a) and (b), respectively. The value of C is fixed as $C = 0.05\ \text{GeV}^{3/2}$.

The expression (14) can be equivalently rewritten similarly to Eq. (10) as

$$p^{\text{H}}(T) = \kappa^{\text{H}}\, p^{\text{id}}_{\text{H}}(T) = \kappa^{\text{H}}\, T\, n^{\text{id}}_{\text{H}}(T), \qquad (15)$$

where $\kappa^{\text{H}} \equiv \exp(-v p^{\text{H}}/T)$, and p^{H}_{id} and n^{H}_{id} are, respectively, the expressions for the pressure and total number density of all particles (hadrons, resonances, and Hagedorn excited states) in the ideal HRG with the Hagedorn mass spectrum (Id-HRG-H), i.e., with the Hagedorn mass spectrum but without excluded volume effects. One can easily calculate the energy density as

$$\varepsilon^{\text{H}}(T) = T\,\frac{dp^{\text{H}}}{dT} - p^{\text{H}} = \frac{\kappa^{\text{H}}\,\varepsilon^{\text{H}}_{\text{id}}}{1 + v\,\kappa^{\text{H}}\,n^{\text{H}}_{\text{id}}}, \qquad (16)$$

where

$$\varepsilon^{\text{H}}_{\text{id}}(T) = \int dm \int_0^\infty \frac{k^2 dk}{2\pi^2}\sqrt{m^2+k^2}\,\exp\left(-\frac{\sqrt{m^2+k^2}}{T}\right) \times \left[\sum_i d_i\, f_i(m) + \rho(m)\right] \qquad (17)$$

denotes the energy density of the Id-HRG-H model.

In Fig. 4, a comparison of the EV-HRG-H model with the lattice data [3] is presented. The results for p^{H}/T^4 and $\varepsilon^{\text{H}}/T^4$ are presented by the solid lines in Figs. 4(a) and 4(b), respectively. These lines correspond to different values of r but fixed $C = 0.05\ \text{GeV}^{3/2}$. The dashed lines show the EV-HRG results without the Hagedorn mass spectrum, i.e., at $C = 0$. The inclusion of the Hagedorn mass spectrum become clearly visible at $T > 130$ MeV, and its contribution to the energy density is essentially larger than that to the pressure.

A simultaneous fit of the p^{lat}/T^4 and $\varepsilon^{\text{lat}}/T^4$ lattice data is done. The quality of the fit is now controlled by χ^2/N_{df} with $20 = 10 + 10$ number of points and two fitting parameters (r and C). At each value of r one can find the C parameter which minimizes χ^2/N_{df} at fixed r. This introduces the correlation between parameters C and r, which is shown in Fig. 5(a). The data are well fitted for a rather wide range of values for hard-core radius: $\chi^2/N_{\text{df}} \lesssim 1$ for $r \lesssim 0.4$ fm. In Fig. 5(b) the dependence of χ^2/N_{df} on r is shown for $C = 0$ and $C = C(r)$, where $C(r)$ is depicted in Fig 5(a). A simultaneous fit of p^{lat}/T^4 and $\varepsilon^{\text{lat}}/T^4$ within the EV-HRG model (i.e., for $C = 0$) does not show the necessity of $r > 0$. Similarly, the Id-HRG-H model (i.e., at $r = 0$) admits only very small contributions from Hagedorn states to the thermodynamical functions [a small value of C at $r = 0$ seen in Fig. 5(a)]. Therefore, no clear evidence for $r > 0$ or $C > 0$ can be found if

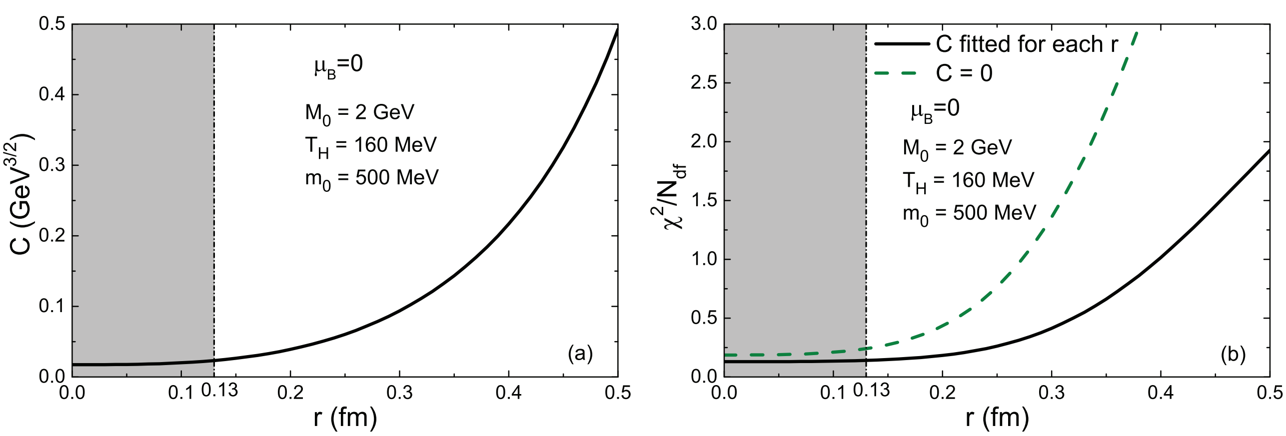

FIG. 5. (Color online) (a) Parameter C which minimizes χ^2/N_{df} at each value of r shown as a function of r. (b) The quantity χ^2/N_{df} as a function of r. For each value of r, parameter C is fitted in order to minimize χ^2/N_{df}. The shaded gray area corresponds to $r \leqslant 0.13$.

these two effects are considered separately, i.e., within the EV-HRG or the Id-HRG-H model. On the other hand, taking them simultaneously within the EV-HRG-H model indicates the presence of these two effects and improvement of the Id-HRG model.

Let us also note that the inclusion of the Hagedorn mass spectrum led to a successful description of the lattice data for the confined glueball phase in the pure SU(3) case (without quarks) in Ref. [40]. This analysis is based on the assumption that glueball decay widths and interactions (i.e., excluded volume effects too) are rather small and can be neglected. In the full SU(3) theory (with quarks) this assumption may not be valid.

IV. SUMMARY

In summary, the lattice data of Ref. [3] for p^{lat}/T^4 and $\varepsilon^{\mathrm{lat}}/T^4$ are considered with the HRG model. Two extensions of this model are analyzed: the excluded volume model (with the same hard-core radius r for all particles) and the exponential the Hagedorn mass spectrum model. A condition that the pressure of the HRG should not exceed the Stefan-Boltzmann limit for quarks and gluons indicates that hadrons should have a nonzero hard-core radius of at least 0.13 fm. However, a comparison of the excluded volume HRG model with the lattice data at $T < 155$ MeV yields no conclusive evidence in favor of the presence of excluded volume effects. Namely, the fit of p^{lat}/T^4 prefers values of $r \lesssim 0.4$ fm, while the best fit of $\varepsilon^{\mathrm{lat}}/T^4$ corresponds to $r \cong 0$. If $r = 0$, there is also not much room for the contribution from the Hagedorn states, with the best fit in this case corresponding to $C \cong 0$; i.e., it suggests the absence of contributions from the Hagedorn states.

These results mean that neither the excluded volume HRG model nor the ideal HRG model with additional Hagedorn states being considered separately demonstrates any advantages for fitting the lattice data in a comparison to the ideal HRG model (with no excluded volume effects and no Hagedron states). On the other hand, if both these physical effects are considered simultaneously the situation is changed: the data are well fitted for $r \lesssim 0.4$ fm and $C \lesssim 0.2$ GeV$^{3/2}$ with $\chi^2/N_{\mathrm{df}} \lesssim 1$; i.e., there is a clear indication that both the hard-core repulsion and the Hagedorn mass spectrum should be taken into account simultaneously in the framework of the hadron resonance gas model. Accounting for these leads to rather essential contributions: suppression effects for p^{H}/T^4 and $\varepsilon^{\mathrm{H}}/T^4$ due to the excluded volume effects and enhancement due to the Hagedorn mass spectrum. These *simultaneous* contributions ensure a better agreement with the lattice data and lead, therefore, to improvement of the ideal HRG model.

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