Resonance-state properties from a phase shift analysis with the *S*-matrix pole method and the effective-range method

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Asymptotic normalization coefficients (ANCs) are fundamental nuclear constants playing an important role in nuclear physics and astrophysics. We derive a new useful relationship between ANCs of the Gamow radial wave function and the renormalized (due to the Coulomb interaction) Coulomb-nuclear partial scattering amplitude. We use an analytical approximation in the form of a series for the nonresonant part of the phase shift which can be analytically continued to the point of an isolated resonance pole in the complex plane of the momentum. Earlier, this method which we call the *S*-matrix pole method was used by us to find the resonance pole energy. We find the corresponding fitting parameters for the ^{5}He, ^{5}Li, and ^{16}O concrete resonance states. Additionally, based on the theory of the effective range, we calculate the parameters of the $p_{3/2}$ and $p_{1/2}$ resonance states of the nuclei ^{5}He and ^{5}Li and compare them with the results obtained by the *S*-matrix pole method. ANC values are found which can be used to calculate the reaction rate through the ^{16}O resonances which lie slightly above the threshold for the α^{12}C channel.

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I. INTRODUCTION

It is known that many reactions important for nuclear astrophysics proceed through subthreshold bound states and lower-lying resonance states above the threshold, and the single-channel approach can be applied to describe these states. To calculate the rate of such reactions, we need to find the asymptotic normalization coefficient (ANC) of the radial wave function for bound and resonance states. The ANC method has been explored as an indirect experimental method for the determination of the cross sections of peripheral reactions at low energy [1]. There are several methods to determine the bound state ANC from experimental data (see [2,3] and references therein). Recently the effective-range expansion method has been developed to find the ANC for bound and resonant states from an elastic scattering phase shift analysis (see [4,5] and references therein). We note that a sufficiently precise measurement of elastic scattering can give crucial information concerning the ANC. However, finding the ANC for a resonance is more difficult than for a bound state. It was shown earlier that for narrow resonances the ANC is proportional to the square root of the width Γ of the resonance considered [6]. It is known that the normalization procedure for the Gamow wave function of a resonance, particularly in the case of a broad resonance when one cannot apply the Zel'dovich formula [7], is difficult because the outgoing wave increases exponentially due to the complex momentum. However, having the ANC, we know the asymptotic part of the wave function which allows us to normalize it correctly if we choose a nuclear potential of the interaction between the two nuclei considered, thus describing the resonant state.

The problem of the exponential increase of the Gamow resonance wave function in the asymptotic region can be solved by using a complex scaling method based on the so-called ABC theorem [8]. This method can be applied to charged particles as well (see, for example, [9]) because the Coulomb potential satisfies the scaling condition of the ABC theorem. The complex scaling method using the Zel'dovich formula appears quite widely in the literature (see [10] and references therein). However, the application of this method to a numerical normalization of the Gamow wave function is rather difficult. In [10] the problem of calculating the resonance pole was solved using a similar *S*-matrix pole approach but for a potential model, unlike in our present work.

Usually *R*-matrix theory is applied to define the parameters of low-lying resonances and to describe nuclear resonance reactions. One of the shortcomings of this theory is the need to fix a value of the channel radius, which is impossible to measure experimentally. Therefore, it is important to develop a theory based on the general properties of the scattering or reaction amplitudes, which can be used for an analytical continuation to the nonphysical Riemann energy surface. We would like to point out that knowing the parameters of low-lying isolated resonances (in particular the ANC values) allows us to predict accurately the crucial reaction rates for nuclear astrophysics.

II. THE ANC FROM THE ELASTIC SCATTERING AMPLITUDE BASED ON THE ANALYTIC PROPERTIES OF THE *S* MATRIX

As we mentioned above, the application of the analytic properties of the *S* matrix makes it easy to link the ANC to the width Γ of an isolated narrow resonance [6]. However, this relationship is not valid for a wide resonance. In this section,

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we show how to obtain this relationship for a resonance with a broad width as well.

The partial amplitude of the nuclear scattering modified by the Coulomb interaction is[1]

$$f_l(k) = \frac{e^{i2\sigma_l}(e^{i2\delta_l} - 1)}{2ik}, \tag{1}$$

where k is the relative momentum of the colliding nuclei, and δ_l is the nuclear scattering phase shift for the orbital momentum l modified by the Coulomb interaction. (This also depends on the total angular momentum J, which we omit because Coulomb effects do not depend on the spin.) The σ_l is the pure Coulomb scattering phase shift

$$\sigma_l = \arg\Gamma(l + 1 + i\eta), \tag{2}$$

or

$$e^{i2\sigma_l} = \frac{\Gamma(l + 1 + i\eta)}{\Gamma(l + 1 - i\eta)}, \tag{3}$$

where $\Gamma(x)$ is the gamma function, $\eta = z_1 z_2 \mu\alpha/k$ is the Sommerfeld parameter, α is the fine-structure constant, and μ is the reduced mass of the colliding nuclei with the charge numbers z_1 and z_2.

In the single-channel elastic scattering case the partial S-matrix element, without the pure Coulomb part, is

$$S_l(k) = e^{i2\delta_l}. \tag{4}$$

Near an isolated resonance it can be represented as [11]

$$S_l(k) = e^{2i\nu_l(k)}\frac{(k + k_r)(k - k_r^\star)}{(k - k_r)(k + k_r^\star)}, \tag{5}$$

where $k_r = k_0 - ik_i$ is the complex wave number of a resonance [$k_0 > k_i > 0$, and the symbol (*) means the complex conjugate operation]. Energy E_r of this resonance and its width Γ are

$$E_r = \frac{k_0^2 - k_i^2}{2\mu}, \quad \Gamma = \frac{2k_0 k_i}{\mu}. \tag{6}$$

The partial scattering nonresonant phase shift $\nu_l(k)$ is a smooth function near the pole of the S-matrix element, corresponding to the resonance. The S-matrix element defined by Eq. (5) fulfills the conditions of analyticity, unitarity, and symmetry. Using Eq. (5), one can rewrite Eq. (4) in the form

$$S_l(k) = e^{2i(\nu_l + \delta_r + \delta_a)}, \tag{7}$$

where

$$\delta_r = -\arctan\frac{k_i}{k - k_0}$$

represents the resonance phase shift, while

$$\delta_a = -\arctan\frac{k_i}{k + k_0}$$

is the additional phase shift which contributes to the whole scattering phase shift. Thus the total phase shift is

$$\delta_l = \nu_l + \delta_r + \delta_a. \tag{8}$$

[1]Here and below we use the unit system $\hbar = c = 1$.

The amplitude (1) has a complicated analytical property in the complex momentum plane due to the Coulomb factor. According to Refs. [4,12,13], we renormalize the partial amplitude of the elastic scattering multiplying it by the function

$$h_l(k) = \frac{(l!)^2 e^{\pi\eta}}{[\Gamma(l + 1 + i\eta)]^2}. \tag{9}$$

Applying Eq. (3), we can write the renormalized amplitude as

$$\tilde{f}_l(k) = \frac{(e^{i2\delta_l} - 1)}{2ik}\frac{\Gamma(l + 1 + i\eta)}{\Gamma(l + 1 - i\eta)}\frac{(l!)^2 e^{\pi\eta}}{[\Gamma(l + 1 + i\eta)]^2}. \tag{10}$$

After simplification and replacing $e^{i2\delta_l}$ by $S_l(k)$ we get

$$\tilde{f}_l(k) = \frac{S_l(k) - 1}{2ik\rho_l(k)}, \tag{11}$$

where ρ_l is equal to

$$\rho_l(k) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}\prod_{n=1}^{l}\left(1 + \frac{\eta^2}{n^2}\right). \tag{12}$$

This renormalized amplitude $\tilde{f}_l(k)$ can be analytically continued like the partial scattering amplitude, corresponding to the short-range interaction, and has its pole at point k_r according to Eq. (5). But we should note that the Coulomb interaction leads to an essential singularity at zero energy and also (see [14]) to an infinite number of poles of $\tilde{f}_l(k)$ in addition to the poles of a purely nuclear nature.

In the vicinity of the pole k_r, the partial scattering amplitude (11) can be represented as

$$\tilde{f}_l(k) = \frac{W}{k - k_r} + \tilde{f}_{\text{nonres}}(k), \tag{13}$$

where the function $\tilde{f}_{\text{nonres}}(k)$ is regular at the point k_r.

The simple derivation of the residue W leads to the expression

$$W = \operatorname{res}\tilde{f}_l = \lim_{k\to k_r}[(k - k_r)\tilde{f}_l(k)] = -\frac{k_i e^{i2\nu_l(k_r)}}{k_0\rho_l(k_r)}. \tag{14}$$

According to the definition of the nuclear vertex constant $\tilde{G}_l$ (NVC), [15] the relationship between NVC and the residue W can be written as

$$W = -\frac{\mu^2}{2\pi k_r}\tilde{G}_l^2. \tag{15}$$

So we get

$$\tilde{G}_l^2 = \frac{2\pi}{\mu^2}\frac{k_r k_i e^{i2\nu_l(k_r)}}{k_0\rho_l(k_r)} = \frac{\pi\Gamma}{\mu k_0}\frac{(1 - ik_i/k_0)e^{i2\nu_l(k_r)}}{\rho_l(k_r)}. \tag{16}$$

Using the relationship between NVC $\tilde{G}_l$ and ANC C_l [15], we obtain

$$\begin{aligned} C_l &= \frac{i^{-l}\mu}{\sqrt{\pi}}\frac{\Gamma(l + 1 + i\eta_r)}{l!}e^{-\frac{\pi\eta_r}{2}}\tilde{G}_l \\ &= i^{-l}\sqrt{\frac{\mu\Gamma}{k_0}}e^{-\frac{\pi\eta_r}{2}}\frac{\Gamma(l + 1 + i\eta_r)}{l!} \\ &\quad \times e^{i\nu_l(k_r)}\sqrt{(1 - ik_i/k_0)/\rho_l(k_r)}. \end{aligned} \tag{17}$$

The derived equations are valid for both narrow and broad resonances. For narrow resonances, when $\Gamma \ll E_r$ ($k_i \ll k_0$), one can simplify Eq. (17) for the ANC replacing k_r by k_0 and using the equality

$$e^{-\frac{\pi\eta}{2}}\frac{\Gamma(l+1+i\eta)}{l!\sqrt{\rho_l(k_0)}} = e^{i\sigma_l} \tag{18}$$

to obtain

$$C_l^a = \sqrt{\frac{\mu\Gamma}{k_0}}e^{i(\nu_l(k_0)+\sigma_l(k_0)-\pi l/2)}, \tag{19}$$

which coincides with the result obtained in Ref. [6].

The nonresonant phase shift $\nu_l(k)$ is the analytical function excluding the origin. In Ref. [16], the authors presented the behavior of $\nu_l(k)$ near origin as

$$\nu_l(k) = -\frac{2\pi}{(l!)^2}k^{2l+1}\eta^{2l+1}a_l\, e^{-2\pi\eta}, \tag{20}$$

where a_l is the scattering length for colliding nuclei. We see that $k = 0$ is an essential singularity of the scattering phase shift. However, as a function of the momentum k, it has normal analytical properties near the point corresponding to the resonance. Therefore we can expand $\nu_l(k)$ to a series

$$\nu_l(k) = \sum_{n=0}^{\infty} c_n(k-k_s)^n \tag{21}$$

in the vicinity of the pole corresponding to the resonance. The point k_s denotes a centered point, and the radius of convergence should be shorter than the distance from the centered point to the closest singular point. The last can be due to an exchange Feynman diagram for the elastic scattering, leading to the logarithmic singularity which is absent in our model.

If we wish to determine the value of the phase shift $\nu_l(k)$ by applying Eq. (21) at a point on the complex plane close to the centered point k_s, then only the first few items of the convergent series for calculating $\nu_l(k)$ can be taken into account with certain precision. The expansion coefficients c_n of Eq. (21) as well as k_0 and k_i are determined by fitting the experimental values of the elastic scattering phase shifts δ_l given by Eq. (8).

III. EFFECTIVE-RANGE METHOD

The effective-range theory is also based on the analytical property of the elastic scattering amplitude when an ingoing particle collides with another nuclei at low energy. This is a very good method to find the NVC and ANC of the bound states from phase shift analyses (see Refs. [4,5] and references therein).

Substituting the expression Eq. (4) of the partial S matrix into Eq. (11) we easily obtain the renormalized amplitude in the following form:

$$\tilde{f}_l(k) = \frac{1}{k(\cot\delta_l - i)\rho_l(k)}, \tag{22}$$

where the function $\rho(k)$ is defined by Eq. (12) and δ_l is the nuclear phase shift modified by the Coulomb interaction. From Eq. (22) it follows that the position of the pole corresponds to the condition

$$\cot\delta_l - i = 0. \tag{23}$$

Exactly the same condition (23) is fulfilled for the pole of the elastic scattering amplitude of the uncharged particles. Following Ref. [17] we write the effective-range function, which is an analytical function, except for possible poles (zeros of the scattering amplitude), and relates to the phase shift δ_l as

$$K_l(k^2) = k^{2l+1}D_l(\eta)\big[C_0^2(\eta)(\cot\delta_l - i) + 2\eta h(\eta)\big], \tag{24}$$

where

$$C_0^2(\eta) = \frac{2\pi\eta}{\exp(2\pi\eta)-1}, \tag{25}$$

$$h(\eta) = \psi(i\eta) + (2i\eta)^{-1} - \ln(i\eta), \tag{26}$$

$$D_l(\eta) = \prod_{n=1}^{l}(1+\eta^2/n^2), \quad D_0(\eta) = 1, \tag{27}$$

and $\psi(x)$ is the di-gamma function. We note that the effective-range function $K_l(k^2)$ is real in the positive energy region.

If the interaction of colliding particles is purely nuclear, i.e., without the Coulomb tail, the effective-range function (24) is simplified and expressed through the partial scattering phase shift by the well-known equation

$$K_l(k^2) = k^{2l+1}\cot\delta_l. \tag{28}$$

Since the effective-range function is an analytic function (except for possible poles), it can be expanded in a power series over k^2 in the low-energy region, where only the elastic scattering channel is open. Typically, the following expansion is used

$$K_l(k^2) = -\frac{1}{a_l} + \frac{1}{2}r_l^2k^2 - P_lr_l^3k^4 + \cdots, \tag{29}$$

where a_l, r_l, and P_l are real and called the scattering length, effective range, and shape parameter, respectively. An alternative form to Eq. (29) is the Padé approximation used in Ref. [18].

The expansion coefficients of Eq. (29) are defined by fitting the effective-range function expressed through experimental phase shifts for the positive energy in the form of Eq. (24) or Eq. (28), depending on whether a charged or uncharged particle is scattered by the target nucleus. The effective-range function Eq. (29) with the fitted parameters is used to find the pole of the elastic scattering amplitude, corresponding to the condition of (23) which leads to the equation

$$K_l(k^2) - 2\eta k^{2l+1}D_l(\eta)h(\eta) = 0. \tag{30}$$

Actually, Eq. (30) can be taken as a condition for parameter fitting when a resonance pole energy and a width are included as an input like the phase shift data.

For the pole of the elastic scattering amplitude in the case of an uncharged particle, the pole condition is simplified to

$$K_l(k^2) - ik^{2l+1} = 0. \tag{31}$$

Solving Eq. (30) or (31), we find the pole momentum value of the elastic scattering amplitude and the energy which has complex value for a resonance, respectively. Then we calculate

the residue W of the renormalized scattering amplitude of a charged particle (22) at this pole point. The equation for W is

$$W = \left.\frac{k^{2l}}{\frac{d}{dk}[K_l(k^2) - 2\eta k^{2l+1} D_l(\eta) h(\eta)]}\right|_{k=k_r} \quad (32)$$

for a charged particle, and

$$W = \left.\frac{k^{2l}}{\frac{d}{dk}[K_l(k^2) - ik^{2l+1}]}\right|_{k=k_r}, \quad (33)$$

in the case of an uncharged particle scattering. The expressions for the NVC and ANC are defined through the residue W by Eqs. (15) and (17), which are given in the previous section.

IV. RESULTS FOR THE ^{5}He AND ^{5}Li GROUND AND FIRST EXITED STATES

The ^{5}Li and ^{5}He nuclei are interesting in that the ground and first excited states are resonance states which can be treated as single-channel systems. The phase shift of the elastic $N\alpha$ scattering with total angular momentum and parity equal to $J^\pi = 3/2^-$ passes rapidly through $\pi/2$ and therefore leads to a narrow resonance. However, the phase shift of the elastic $N\alpha$ scattering with $J^\pi = 1/2^-$ does not pass through $\pi/2$ and therefore the corresponding resonance is wide enough. This fact leads to certain difficulties, not only in determining the position of the resonance and its width, but also in finding such characteristics as the NVC and ANC.

The coefficient values of the effective-range expansion obtained from a phase shift analysis of the elastic scattering data in the region up to 3 MeV for neutron and 5 MeV for proton were found by the authors of Ref. [19]. Using these parameters, the authors of Ref. [20] determined the values of the energy and width of the resonances. The article [20] was cited in Ref. [21] where a separable potential fits the resonance parameters for the $n\alpha$ scattering in the $p_{1/2}$ and $p_{3/2}$ states. Agreement of the phase shifts calculated in [21] with the experimental ones is good for the narrow $p_{3/2}$ resonance but is poor for the broad $p_{1/2}$ resonance. The N/D method was applied in [12,22] for calculating the values of the parameters of these resonances. Additionally, the residues W of the renormalized scattering amplitude were calculated at the resonance poles in the complex k plane using the effective-range method in [4].

We applied the $N\alpha$ phase shifts data presented in Ref. [23] to calculate W, NVC, and ANC. According to the authors of Ref. [23] the $N\alpha$ phase shifts are obtained by an accurate R-matrix analysis of the elastic scattering data. In Fig. 1 we show the results of fitting the phase shifts for the n-^{4}He and p-^{4}He elastic scattering, using the S-matrix pole method. A good agreement is achieved in the wide energy region, including the resonances considered.

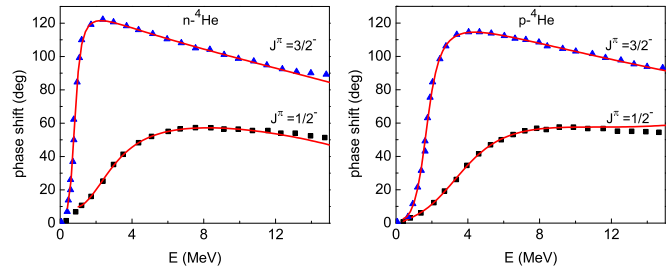

FIG. 1. (Color online) Comparison of the fitted phase shifts for the $n(p)$-^{4}He elastic scattering obtained by the S-matrix pole method with the experimental values. The experimental data are taken from Ref. [23]. The energy is given in the laboratory frame.

In Table I we present the parameter values related to the ^{5}He and ^{5}Li nuclei, which are calculated using the analytic properties of the S matrix outlined in Sec. II. In Fig. 2 we compare the fitted effective-range function with the corresponding values calculated by the effective-range method, using the experimental phase shift data taken from [23]. The obtained agreement is quite good. Table II shows the calculation results of the same parameters for the same nuclei and states, but found using the effective-range method described in Sec. III. A comparison of the results presented in Tables I and II shows that both methods lead to quite consistent results. The essential difference between some of the results for the two methods considered may be explained by the fact that these results are more sensitive to the applied approach in the case of broad resonances. The same conclusion was noted in Ref. [24], where the authors also analyzed the parameters of the $N\alpha$ states given in Refs. [25,26]. We would like to point out that the difference between the energies of states $1/2^-$ and $3/2^-$ for ^{5}He received by both methods applied is $\sim$0.9 MeV, which is comparable to the difference $\sim$1.1 $-$ 1.3 MeV between results given by other authors. (See the tables in [4,24]). The same differences for the states ^{5}Li are 1.17 and 0.71 MeV, which are obtained using the presentation of the S-matrix [Eq. (5)] and the effective-range method, respectively. The results found by the other authors lead to values where the limits are relatively wide. As to the widths of the corresponding levels, the range of differences of the values obtained by the different authors is similar to

TABLE I. Nucleus, channel, state, energy, and width, corresponding values of the residue ($|W|$), NVC ($\tilde{G}_l^2$), and ANC (C_l) obtained by fitting the elastic $N\alpha$ scattering phase shifts presented in Ref. [23]. Results are found using the analytical properties of the S matrix outlined in Sec. II. Four terms of Eq. (21) are used for fitting. The last column shows the ANC (C_l^a) calculated by Eq. (19). The energy of the resonance is given in the center-of-mass system of $N\alpha$.

Nucleus	J^π	E_r (MeV)	Γ(MeV)	$\lvert W\rvert$	$\tilde{G}_l^2$(fm)	C_l (fm$^{-1/2}$)	C_l^a (fm$^{-1/2}$)
^{5}He; nα	$3/2^-$	0.629	0.448	0.147	0.005$-$ i0.009	$-$0.105 $-i$0.190	$-$0.095 $-i$0.214
	$1/2^-$	1.476	3.520	0.194	$-$0.019$-i$0.016	$-$0.320 $-i$0.116	$-$0.391 $-i$0.314
^{5}Li; pα	$3/2^-$	1.328	0.994	0.320	0.018$-i$0.027	$-$0.115 $-i$0.231	$-$0.103 $-i$0.269
	$1/2^-$	2.504	4.667	0.261	$-$0.011$-i$0.040	$-$0.276 $-i$0.196	$-$0.355 $-i$0.374

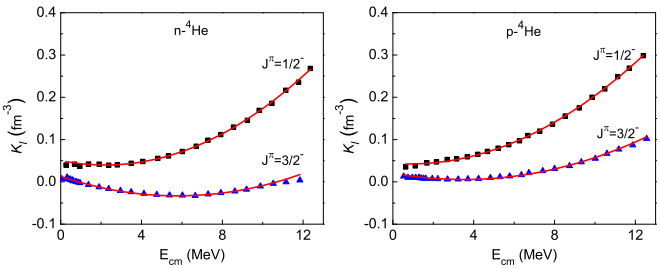

FIG. 2. (Color online) Comparison of the fitted effective-range functions for the $n(p)$-^{4}He elastic scattering with the experimental values calculated by using the experimental data taken from Ref. [23]. The energy is given in the c.m. frame.

the range of differences for the real parts of the resonance energies [4,24].

According to our results, the difference in the level energies calculated by the two methods described above are 6–7% for ^{5}He and 11% for ^{5}Li, while the width differences are 7% at state $J^\pi = 3/2$ and 18% for $J^\pi = 1/2$ of ^{5}He. For the levels of ^{5}Li, differences in the widths calculated by the two methods are very small. Comparing the results of Tables I and II we can see that most of the calculated data have similar values with a maximum difference of ∼20%. From this comparison it can be concluded that it is difficult to decide which method of calculation is preferable. Comparing the values of the ANC of the penultimate and last columns, we see a difference in ∼60%, which gives us grounds to say that the asymptotic formula defined by Eq. (19) leads to incorrect values of the ANC for broad resonances. The values of the residue [Eqs. (14) and (32)] calculated by both methods are similar in absolute values to the corresponding values presented in Refs. [4,22].

V. RESULTS FOR THE ^{16}O LOW-LYING RESONANCES SITUATED ABOVE THE α-^{12}C THRESHOLD

In our previous work [27], we determined the position and the width of the resonance in ^{16}O, using Eqs. (8) and (21) by fitting the phase shift for the elastic scattering of the α particles on the nucleus ^{12}C given in Ref. [28]. It was found that the dependence of the results on the location of k_s is insignificant if it is within the area of the maximum increase of the full scattering phase shift. To verify the almost linear behavior of the phase shift $\nu_l(k)$, we checked its dependence on the momentum k by subtracting the sum of the phase shifts $\delta_r(k)$ and $\delta_a(k)$ from the experimental phase shift within the resonance region. In Fig. 3 we demonstrate a good description of the energy dependence of the experimental α-^{12}C elastic scattering phase shifts which is obtained using the S-matrix

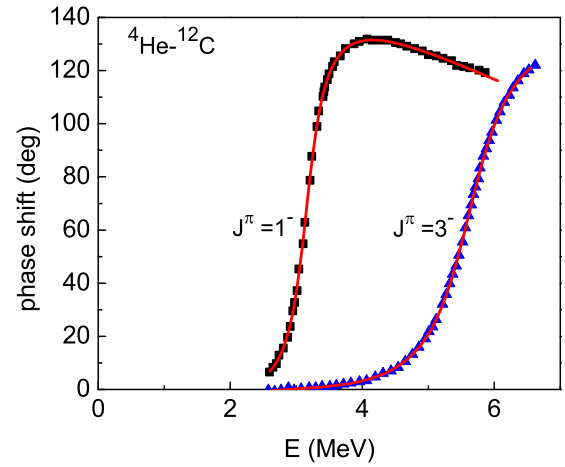

FIG. 3. (Color online) Comparison of the fitted phase shifts for the α-^{12}C elastic scattering obtained by the S-matrix pole method with the experimental values. The experimental data are taken from Ref. [28]. The energy is given in the laboratory frame.

pole method. As examples, we take the $J^\pi = 1^-$ and $J^\pi = 3^-$ states, when the resonances are broad enough.

We note that all known methods of fitting the elastic scattering phase shift lead to the same values of the energy and width for narrow resonances. However, the results diverge for broad resonances. Therefore, we can expect a difference in the results of the ANC evaluations for broad resonances compared with calculations by Eq. (19). Table III shows our calculation results for the energy and width of the resonances for the nucleus ^{16}O, and the corresponding NVC and ANC values. In the second and third columns of Table III we show the results obtained by a R-matrix analysis [28] while our results received by a S-matrix analysis are displayed in the fourth and fifth columns. Readers can see that these results for the energy and width coincide when the resonance is narrow, but there are essential differences for broad resonances (in particular for states 1^- and 3^-). The values of the renormalized NVCs ($\tilde{G}_l^2$) and ANCs (C_l), which were found by using our calculated values of the energies, widths, and the nonresonant phase shifts are shown in the next two columns. In the last column the values of ANCs (C_l^a) which were calculated by using Eq. (19) are presented. We note that these values are found at real momentum values.

As the experimental phase shifts are determined with some uncertainties, it is reasonable to assess the change of NVC and ANC as functions of the resonance energy and width. Therefore, we calculated the value of the nonresonant phase shift ν_l and found the values of NVC and ANC at the resonance

TABLE II. Same as in Table I, but in the frame of the effective-range method outlined in Sec. III.

Nucleus	J^π	E_r (MeV)	Γ (MeV)	$\lvert W\rvert$	$\tilde{G}_l^2$ (fm)	C_l (fm$^{-1/2}$)	C_l^a (fm$^{-1/2}$)
^{5}He; nα	$3/2^-$	0.675	0.560	0.171	$0.007 - i0.010$	$-0.111 - i0.212$	$-0.076 - i0.245$
	$1/2^-$	1.563	4.155	0.220	$-0.015 - i0.026$	$-0.323 - i0.187$	$-0.384 - i0.367$
^{5}Li; pα	$3/2^-$	1.481	1.041	0.295	$0.019 - i0.025$	$-0.109 - i0.236$	$-0.062 - i0.281$
	$1/2^-$	2.213	4.640	0.305	$-0.016 - i0.043$	$-0.300 - i0.193$	$-0.375 - i0.369$

TABLE III. States, energies, and widths of ^{16}O nucleus levels above the α^{12}C threshold from our fit, as well as the corresponding values of the calculated NVC and ANC from the elastic α^{12}C scattering phase shifts [28]. Four terms of Eq. (21) are used for fitting. The energies of the resonances are given in the center-of-mass system of α^{12}C.

J^π	E_r (MeV) [28]	Γ (keV) [28]	E_r (MeV)	Γ (keV)	$\tilde{G}_l^2$ (fm)	C_l (fm$^{-1/2}$)	C_l^a (fm$^{-1/2}$)
0^+	4.887	3.0	4.887	3.0	$0.0023 - i0.0042$	$0.0122 - i\,0.0104$	$0.0122 - i\,0.0104$
1^-	2.416	388.0	2.364	356.2	$4.9703 - i1.7969$	$0.1530 - i\,0.1032$	$0.1759 - i\,0.1135$
2^+	2.683	0.76	2.683	0.76	$0.0031 - i0.0002$	$0.0038 - i\,0.0086$	$0.0038 - i\,0.0086$
2^+	4.339	83.0	4.350	79.1	$0.0383 - i0.0079$	$-0.0125 - i\,0.0831$	$-0.0124 - i\,0.0838$
3^-	4.320	864.0	4.214	811.7	$0.2762 - i0.1420$	$-0.2332 - i\,0.0201$	$-0.2718 - i\,0.0311$
4^+	3.196	25.6	3.199	26.5	$0.0284 - i0.0014$	$-0.0491 + i\,0.0190$	$-0.0494 + i\,0.0190$

point for the state $J^\pi = 3^-$, fixing the resonance energy and width fitted by R-matrix method [28]. It was found that the differences in energy and resonance were 2.5% and 6.4%, respectively, while the renormalized NVC and ANC differ by 2.9% and 5.3%, respectively. It should be noted that the percentage difference of the NVC and ANC values is a consequence of the calculation of ANC through the value NVC, because it is multiplied by the $\Gamma(x)$ function at the different values of the Coulomb factor. One can see that the uncertainties of NVC and ANC are roughly the same as those of the resonance energy value. For narrow resonances, it is quite reasonable to evaluate ANC using Eq. (19), taking the value of the nonresonant phase shift for the real values of energy or momentum from the experimental data. It is obvious that for broad resonances the width of which is greater than their energy, the uncertainty of the ANC value should be related to the uncertainty of the width which is determined by fitting the experimental scattering phase shifts. The effective-range method is not able to reproduce the widths of the ^{16}O resonances. This may be due to the single-channel approximation which we use in this work.

VI. CONCLUSION

The S-matrix pole prescription [Eq. (5)] and expansion of the nonresonant phase to series [Eq. (21)] give consistent resonance parameters for the ground and first excited states of ^{5}He and ^{5}Li as well as for the low-lying states of ^{16}O situated above the α^{12}C threshold in spite of their resonance widths.

The standard expansion of the effective-range function $K_l(k^2)$ to find the NVC $\tilde{G}_l$ and other parameters of the two first resonance states of ^{5}He and ^{5}Li are used successfully. We have found results which are a little different from those obtained by other methods used. In our opinion, these differences can be explained by the fact that in the first method, a centered point of the expansion of the nonresonant phase shift to a series is the point which is closest to the position of the resonance, while in the method of the effective range, we use a centered point of the expansion at zero momentum, which is far from the resonance pole.

In the case of a bound state, the binding energy can be considered as an additional parameter unlike in the S-matrix method with the phase shift fitting. Therefore, we expect that the method using the S-matrix pole prescription [Eq. (5)] can lead to quite different results, and so we recommend using the S-pole prescription to specify resonance parameters. At the same time, the effective-range expansion method in the convergence energy region is applicable in the case of a bound state when the the S-matrix pole prescription does not work. The results of this paper can be used for solving nuclear astrophysical problems and may be applied in the theory of nuclear reactions using Feynman diagrams to describe the reaction mechanisms.

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