$J = 0, T = 1$ pairing-interaction selection rules

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Wave functions arising from a pairing Hamiltonian $E(0)$, i.e., one in which the interaction is only between $J = 0^+$, $T = 1$ pairs, lead to magnetic dipole and Gamow-Teller (GT) transition rates that are much larger than those from an interaction $E(J_{\max})$ in which a proton and a neutron couple to $J = 2j$. With realistic interactions the results are between the two extremes. In the course of this study we found that certain $M1$ and GT matrix elements vanish with $E(0)$. These are connected to seniority and reduced isospin selection rules. We also relate our results to the single j scissors mode.

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I. INTRODUCTION

We have recently performed single j shell studies of both schematic and realistic interactions [1]. They ranged from the $J = 0^+$, $T = 1$ to the $J_{\max}$, $T = 0$ interactions. In this work we focus more on the experimental consequences of choosing a given interaction. In particular we study Gamow-Teller (GT) and isovector $M1$ matrix elements for transitions in Sc and Ti isotopes. Some of the problems have been addressed numerically in a previous publication [2], but here we present analytical proofs.

A. The interactions

For two particles in a single j shell the states of even angular momentum J have isospin $T = 1$ and those of odd J have $T = 0$. For convenience, we define $E(J)$ as a two-body interaction that is zero except when the two particles couple to J. Hence, we have the $J = 0^+$, $T = 1$ pairing interaction designated as $E(0)$ and the other extreme $E(J_{\max})$, which acts only in the $T = 0$ state with $J_{\max} = 2j$. The $T = 0$ odd-J interaction acts only between a neutron and a proton. We only consider charge-independent interactions in this work. For a "realistic" interaction in the $f_{7/2}$ shell we use the MBZE interaction [3] in which the two body matrix elements are obtained from the experimental spectrum of ^{42}Sc. This is based on the works of Bayman *et al.* [4] and McCullen *et al.* [5] but with improved $T = 0$ two-body matrix elements [4]. From $J = 0$ to $J_{\max} = 7$, the matrix elements that were obtained from experiment are

$$0.0000, 0.6111, 1.5863, 1.4904, 2.8153, 1.5101, 3.2420, \quad \text{and} \quad 0.6163.$$

Although the $J = 0^+$ matrix element is the most attractive in MBZE one also has low-lying $T = 0$ levels with $J = 1^+$ and $J = J_{\max} = 7^+$. Indeed, one main thrust of the old articles is that there is a large probability in, say, an even-even nucleus that the protons and neutrons do not couple to zero. Indeed, it has been shown in Ref. [5] that a much better overlap with the realistic interaction can be obtained with a quadrapole-quadrapole interaction (QQ) than with a $J = 0$ pairing interaction. We should also mention here the work on GT by Lawson [6], who invoked a K selection rule to explain why GT matrix elements decrease with neutron excess.

II. WAVE FUNCTIONS AND QUANTUM NUMBERS FOR A $J = 0, T = 1$ PAIRING INTERACTION OF A QQ INTERACTION

In this section we present energy levels and wave functions of ^{43}Sc and ^{44}Ti that have a $J = 0$, $T = 1$ pairing interaction of Flowers and Edmonds [7,8] and a QQ interaction. The wave functions are presented as column vectors of probability amplitudes. To identify the higher isospin states we subtracted 3 MeV from all $T = 0$, two-body matrix elements for the pairing interaction. Doing so does not affect the wave functions of the nondegenerate states, but it does remove degeneracies of states with different isospins. For scandium isotopes we use a star (*) to indicate states with $T = 3/2$. For ^{44}Ti we use a star for $T = 1$ and two stars (**) for $T = 2$. In Tables I and II we present, for selected angular momenta, the energy levels and wave functions of ^{43}Sc and ^{44}Ti with the $J = 0$ $T = 1$ pairing interaction; Tables III and IV for the QQ interaction.

TABLE I. Energies (MeV) and wave functions of ^{43}Sc with a $J = 0$, $T = 1$ pairing interaction.

		$I = 5/2$		
J_p	J_n	1.125	1.125	5.625*
3.5	2.0	0.4210	−0.4600	0.7817
3.5	4.0	0.4695	0.8479	0.2462
3.5	6.0	0.7761	−0.2633	−0.5730

			$I = 7/2$		
J_p	J_n	0.000	1.125	1.125	4.875*
3.5	0.0	0.8660	0.000	0.000	0.500
3.5	2.0	0.2152	−0.8924	−0.1358	0.3727
3.5	4.0	0.2887	0.1565	0.8014	0.500
3.5	6.0	0.3469	0.4232	−0.5826	0.6009

		$I = 9/2$		
J_p	J_n	1.125	1.125	5.625*
3.5	2.0	−0.1015	0.9416	−0.3212
3.5	4.0	0.4930	0.3280	0.08058
3.5	6.0	0.8641	−0.0766	−0.4975

TABLE II. Energies (MeV) and wave functions of ^{44}Ti with a $J = 0$, $T = 1$ pairing interaction.

$I = 0$

J_p	J_n	0.000	0.750**	2.25	2.25
0.0	0.0	0.8660	−0.5000	0.000	0.000
2.0	2.0	0.2152	0.3737	0.8863	0.1712
4.0	4.0	0.2887	0.5000	−0.1244	−0.8070
6.0	6.0	0.3469	0.6009	−0.4461	0.5652

$I = 1$

J_p	J_n	1.500*	2.250*	2.250*
2.0	2.0	0.1992	0.9258	0.3212
4.0	4.0	0.4879	−0.3780	0.7868
6.0	6.0	0.8498	0.0000	−0.5270

$I = 2$

J_p	J_n	1.000	1.250	1.750	2.250	2.250	2.250	2.250	2.250	2.250
0.0	2.0	0.6455	0.7071	−0.2887	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.0	0.0	0.6455	−0.7071	−0.2887	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.0	2.0	−0.1205	0.0000	−0.2694	0.6032	0.3665	−0.0549	−0.0618	−0.3799	0.5134
2.0	4.0	0.1730	0.0000	0.3869	−0.1407	−0.4053	−0.0033	0.2281	−0.7442	0.1746
4.0	2.0	0.1730	0.0000	0.3869	0.6458	0.1122	0.1532	0.1480	−0.0348	−0.5867
4.0	4.0	−0.0193	0.0000	−0.0431	0.0193	0.0946	−0.5433	0.8105	0.1821	0.0569
4.0	6.0	0.1403	0.0000	0.3138	0.3245	−0.4415	−0.5108	−0.3715	0.3276	0.2746
6.0	4.0	0.1403	0.0000	0.3138	0.0626	−0.0068	0.5991	0.2948	0.3981	0.5230
6.0	6.0	0.2292	0.0000	0.5125	−0.2997	0.6964	−0.2418	−0.2013	−0.0407	0.0973

$I = 2$ (with shift in energy to remove degeneracies)

J_p	J_n	1.000	2.250	2.250	2.250	4.250*	5.250*	5.250*	10.750**	11.250**
0.0	2.0	0.6455	0.0000	0.0000	0.0000	0.7071	0.0000	0.0000	−0.2887	0.0000
2.0	0.0	0.6455	0.0000	0.0000	0.0000	−0.7071	0.0000	0.0000	−0.2887	0.0000
2.0	2.0	−0.1205	0.1561	0.6065	0.6391	0.0000	0.0000	0.0000	−0.2694	−0.3350
2.0	4.0	0.1730	−0.3895	−0.1445	0.3056	0.0000	−0.6977	0.1151	0.3869	−0.2333
4.0	2.0	0.1730	−0.3895	−0.1445	0.3056	0.0000	0.6977	−0.1151	0.0869	−0.2333
4.0	4.0	−0.0193	0.1797	−0.3647	0.6726	0.0000	0.0000	0.0000	−0.0431	0.6623
4.0	6.0	0.1403	−0.0861	0.4752	−0.0728	0.0000	0.1151	0.6977	0.3138	0.3785
6.0	4.0	0.1403	−0.0861	0.4752	−0.0728	0.0000	0.1151	0.6977	0.3138	0.3785
6.0	6.0	0.2292	0.7906	−0.0757	0.0165	0.0000	0.0000	0.0000	0.5125	−0.2318

TABLE III. Energies (MeV) and wave functions of ^{43}Sc with a QQ interaction.

$I = 5/2$

J_p	J_n	2.8243	3.0148	5.1306*
3.5	2.0	0.5053	0.7817	−0.3655
3.5	4.0	0.2885	0.2462	0.9253
3.5	6.0	0.8133	−0.5730	−0.1011

$I = 7/2$

J_p	J_n	0.000	3.3016*	3.7874	5.4618
3.5	0.0	0.7069	−0.5000	0.4402	0.2376
3.5	2.0	0.6864	0.3727	−0.4393	−0.4439
3.5	4.0	0.1694	0.5000	−0.1549	0.8350
3.5	6.0	0.0216	0.6009	0.7676	−0.2218

$I = 9/2$

J_p	J_n	1.4765	4.1843	5.6367*
3.5	2.0	0.9032	−0.2847	−0.3212
3.5	4.0	0.4186	0.4188	0.8058
3.5	6.0	0.0949	0.8623	−0.4975

TABLE IV. Energies (MeV) and wave functions of ^{44}Ti with a QQ interaction.

		$I = 0$			
J_p	J_n	0.000	6.6031**	7.5748	10.9236
0.0	0.0	0.7069	−0.5000	0.4402	0.2376
2.0	2.0	0.6864	0.3727	−0.4393	−0.4439
4.0	4.0	0.1694	0.5000	−0.1549	0.8350
6.0	6.0	0.0216	0.6009	0.7676	−0.2218

		$I = 1$	
J_p	J_n	4.3648 *	7.3405 *	10.5620*
2.0	2.0	0.9109	−0.2082	−0.3563
4.0	4.0	0.3967	0.2040	0.8950
6.0	6.0	0.1137	0.9566	−0.2684

					$I = 2$					
J_p	J_n	0.9665	4.6015*	6.4691	7.7501	7.7502**	8.5695*	10.4893	10.6179**	10.7351*
0.0	2.0	0.5807	−0.5255	0.2263	0.08223	−0.2887	−0.4146	0.1466	0.0000	0.2280
2.0	0.0	0.5807	0.5255	0.2263	0.08223	−0.2887	0.4146	0.1466	0.0000	−0.2280
2.0	2.0	−0.4331	0.0000	0.7001	−0.2689	−0.2694	0.0000	0.2554	−0.3350	0.0000
2.0	4.0	0.2513	−0.4562	0.1629	−0.35010	0.3869	0.3535	−0.2881	−0.2333	0.4085
4.0	2.0	0.2513	0.4562	0.1629	−0.35010	0.3869	−0.3535	−0.2880	−0.2333	0.4805
4.0	4.0	−0.0916	0.0000	0.4892	0.1211	−0.0431	0.0000	−0.5451	0.6623	0.0000
4.0	6.0	0.0403	−0.1255	0.0802	−0.2115	0.3138	0.4507	0.4533	0.3785	0.5302
6.0	4.0	0.0403	0.1255	0.0802	−0.2115	0.3138	−0.4507	0.4533	0.3785	−0.5302
6.0	6.0	−0.0099	0.0000	0.3198	0.75087	0.5125	0.0000	0.1334	−0.2318	0.0000

III. ASSIGNING QUANTUM NUMBERS FOR $J = 0, T = 1$ PAIRING

Despite the fact that we have the energies and wave functions of the $J = 0^+$ and $J = 1^+$ states from an explicit matrix diagonalization. It is convenient to add a constant so that the states that are not collective are at zero energy. When this is done, the energies of the $J = 0^+$ ^{44}Ti states are

$$-2.25, -1.5, 0, \quad \text{and} \quad 0\ \text{MeV}$$

and the energies of the 1^+ states are

$$-0.75, 0, \quad \text{and} \quad 0\ \text{MeV}.$$

We then fit these with the formula of Flowers and Edmonds [7,8], as given in Talmi's book [9],

$$E = C\left\{\left(\frac{n-v}{4}\right)(4j+8-n-v) - T(T+1) + t(t+1)\right\}. \tag{1}$$

TABLE V. Quantum numbers for ^{43}Sc with a pairing interaction.

Energy	J	T	t	v
0	5/2	1/2	1/2	3
0	5/2	1/2	1/2	3
0	5/2	3/2	3/2	3
–1.125	7/2	1/2	1/2	1
–0.75	7/2	3/2	1/2	1
0	7/2	1/2	1/2	3
0	7/2	1/2	1/2	3
0	9/2	1/2	1/2	3
0	9/2	1/2	1/2	3
0	9/2	3/2	3/2	3

C is most easily determined by the isospin splitting of the $T = 2$ state at –1.5 MeV relative to the –2.25 ground state (in the shifted energies). We set $-0.75 = 2 \cdot 3C$, so that $C = -0.125$ (in general, $C = \frac{-1}{2j+1}$). The quantum numbers are shown in Tables V and VI. Previously, Neergaard [10] used this method to obtain quantum numbers in his study of $N = Z$ nuclei.

TABLE VI. Quantum numbers for ^{44}Ti with a pairing interaction.

	$J = 0$		
Energy	T	t	v
–2.25	0	0	0
–1.5	2	0	0
0	0	0	4
0	0	0	4

	$J = 1$		
Energy	T	t	v
0	1	0	2
0	1	1	4
0	1	1	4

	$J = 2$		
Energy	T	t	v
–1.25	0	1	2
0	0	0	4
0	0	0	4
0	0	0	4
–1.0	1	1	2
0	1	1	4
0	1	1	4
–0.5	2	1	2
0	2	2	4

TABLE VII. Gamow-Teller matrix elements for selected scandium isotopes.

7/2-7/2	$E(0)$	MBZE	$E(7)$	QQ
^{43}Sc	0.3849	−0.2088	−0.101 60	0.1207
^{45}Sc	0.2666	0.0927	−0.0027	0.0255
7/2-5/2(43)	0	0.2020	0.2902	0.2763
7/2-5/2(45)	0	0.0459	−0.0022	0.000 792
7/2-9/2(43)	0	−0.0818	0.0168	0.008 380
7/2-9/2(45)	0	0.0008	−0.0028	−0.023 99

TABLE VIII. $B(M1)$ values in ^{44}Ti for $I = 1$ to $I = 0$ pairing interaction.

State (v,T,t)	000	400	400	020
210	2.699 63	8.0995	1.929 94	0.898 554
411	0	7.6793	0.111 74	0
411	0	1.918 66	2.892 21	0

IV. RESULTS

The GT operator is $C\sigma t_+$. The wave functions for the scandium isotopes are of the form

$$\sum D(J_n v)[j_p, J_n]^I, \tag{2}$$

with j_p, the angular momentum of the single proton, equal to 7/2. Here $D(J_n v)$ is the probability amplitude that the neutrons couple to J_n. The matrix element from McCullen *et al.* [5] is

$$M_{ij} = \sum D^i(j, J_n) D^f(j, J_n) U(1 j J_f J_n; j J_i). \tag{3}$$

We put the results of the calculated matrix elements in Table VII.

The results for the $7/2^+$ to $7/2^-$ transitions are shown in the first two rows above. We see that a $J = 0, T = 1$ pairing gives the largest matrix element, MBZE is in the middle, and $E(J_{\max})$ is the smallest. Thus, we have the systematic that deviations for a $J = 0, T = 1$ pairing lead to reduced Gamow-Teller matrix elements. It is not surprising that the realistic case, MBZE, is in the middle because the two-body interaction used in that calculation has both a low-lying $J = 0$ part and a low-lying $J = 7$ part. Of perhaps greatest interest is the fact that the matrix elements of GT for the $E(0)$ interaction vanish when J_f is different than J_i. We have here considered the cases $J_i = (7/2)_1$ and $J_f = 5/2$ or 9/2, both for ^{43}Sc and ^{45}Sc. It was noted in Ref. [4] that the matrix elements from experiment for the 7/2 to 7/2 decay and the 7/2 to 5/2 decay are almost the same, in agreement with MBZE here and in disagreement with $E(0)$.

There is considerable discussion of the pairing interaction in the 1993 book by Talmi [9]. He has a discussion of odd tensor operators in space and spin but not isospin. It is there shown that these operators conserve seniority. In this work on GT we have a product of an odd tensor operator in spin and an odd tensor operator in isospin. The general selection rules for overall isospin are that T_f can be equal to T_i, $T_i + 1$, or $T_i - 1$. We will soon see that in general the GT operator does not conserve seniority. For the $J = 0$, $T = 1$ pairing interaction the lowest state in ^{43}Sc with $J_i = j = 7/2$ has seniority $v = 1$. All other states for this and all other angular momenta have $v = 3$ except the $T = 3/2$, $J = j$ state which also has $v = 1$. In the $f_{7/2}$ shell the latter state is unique. We see from Table VII that if our initial state is a $v = 1$ state with $J = j$ (7/2 in this case) and isospin $T = 1/2$ there is a nonvanishing matrix element to a $v = 1$, $T = 3/2$ state and $J_f = j$. However, with a $J = 0, T = 1$ pairing interaction the matrix element from the $v = 1$ state to the $v = 3$ states with $J = j + 1$ or $J = j - 1$ vanishes. It should be noted that, although one constructs a $J = j$, $v = 1$ state in say ^{43}Sc by first adding two neutrons coupled to $J_n = 0$ to the single proton, that is not the end of the story. One must introduce isospin wave functions and antisymmetrize. The values of $D(J_n)$ for the $v = 1$, $J = j$, $T = 1/2$ state for $J_n = 0, 2, 4,$ and 6 are, respectively, 0.8660, 0.2152, 2887, and 0.3469. Consider the matrix element

$$M' = N\Big(\psi^{J_f} \sum \sigma t_+ (1 - P_{12} - P_{13})\{j(1)[j(2)j(3)]^0\}^j \times p(1)n(2)n(3)\Big), \tag{4}$$

where $t_z = -1/2$ for a proton and $+1/2$ for a neutron. We can replace $\sum \sigma t_+$ by $3\sigma(1)t_+(1)$. Because $t_+ n = 0$ we see that the $(-P_{12} - P_{13})$ terms will not contribute. We are left with $3N(j[\sigma j]^j)(\psi^{J_f}\{j(1)[j(2)j(3)]^0\}^j)$ and we can write $\psi^{J_f} = \sum D^{J_f}(J_n v)[j_p, J_n]^{J_f}$. Hence the last factor is simply $D^{J_f}(0)$. However, for a seniority $v = 3$ final state, $D^{J_f}(0)$ is equal to zero. As mentioned before the only $T = 3/2$ state with seniority $v = 1$ is the one with $J_f = j$. The $J = 5/2$ and 9/2 states all have $v = 3$ and hence the matrix element M' vanishes for those cases, but there is a problem. The state on the right is a mixture of $J = 7/2$, $v = 1$, $T = 1/2$ and $J = 7/2$, $v = 1$, $T = 3/2$. We next show that the $T = 3/2$ part also vanishes and this implies that the $T = 1/2$ part will also vanish. Consider a transition from $J = 7/2^-$, $v = 1$, $T = 3/2$ in ^{43}Sc to $J = 5/2^-$ or $9/2^-$ with $v = 3$ in ^{43}Ca. There is a close relation between GT transitions and isovector magnetic dipole ($M1$) transitions. If one removes the orbital part of the $M1$, keeping only the spin, there is an isospin relation between the two transitions. We can transform the GT problem to one of $M1$ transitions in ^{43}Ca. This is shown in the Appendix. We note some recent interest in the $J = 0$ pairing interaction by

TABLE IX. $B(M1)$ values in ^{44}Ti for $I = 1$ to $I = 2$ pairing interaction.

State (v,T,t)	201	400	400	400	211	411	411	221	422
210	1.0286	17.5613	0.0476	2.296 34	0	0	0	5.1433	0
411	0.1819	1.4508	0.0331	1.8904	0	0	0	0.9091	8.2364
411	0.5256	1.4562	2.0713	3.325 67	0	0	0	2.6275	0.4653

TABLE X. $B(M1)$ values in ^{44}Ti for $I=1$ to $I=0$ QQ interaction.

I	0_1	0_2	0_3	0_4
1_1	1.3174	1.8021	0.1833	0.0414
1_2	0.0015	6.1454	9.0414	0.0577
1_3	0.0007	0.1535	0.9530	0.2052

Neergard [10] who studied the symplectic group structure of various $N=Z$ nuclei.

V. RESULTS IN ^{44}Ti

We show calculated values of $B(M1)$ for ^{44}Ti for the pairing interaction in Tables VIII and IX; for QQ in Tables X and XI.

We see that with the $J=0$ pairing interaction there is a nonzero transition from a $J=0$, $v=0$ state to a $J=1$, $v=2$ state; i.e., the $M1$ (or GT) operator does not conserve seniority. We can, in analogy with what we did for scandium, form a ^{44}Ti state $[[jj]^0[jj]^0]^0$ and antisymmetrize. This will be an admixture of $J=0$, $v=0$, $T=0$ and $J=0$, $v=0$, $T=2$. We now have to show that the $T=2$ part vanishes when we overlap with a $J=1$, $v=4$, $T=1$ state and this will lead to the desired result that the $T=0$ part vanishes. It is easier to use an isospin transformation and consider the transition between a unique $J=0$, $v=0$, $T=2$ state in ^{44}Ca to a $v=4$, $T=1$ state in ^{44}Sc. The $T=2$ state can be obtained by forming the four neutron state $[[jj]^0[jj]^0]^0$ and antisymmetrizing. However, as shown before, we do not have to antisymmetrize in the matrix element. Clearly, the $v=4$, $T=1$, $J=1^+$ state will, even after antisymmetrization, not have any $[[jj]^1[jj]^0]^1$ component. Thus, the $T=2$ part vanishes and so will the $T=0$ part. We discuss the selection rules more systematically in the next section.

VI. MORE RESULTS: A SYSTEMATIC LOOK AT $B(M1)$ SELECTION RULES FOR ^{44}Ti and ^{46}Ti

We gather the vanishing $B(M1)$ values in ^{44}Ti from Tables VIII and IX. The selection rules for $I=1$ to 0 are presented in Table XII and for 1 to 2 in Table XIII. The initial and final quantum numbers (v,T,t) are shown for these vanishings.

We next consider ^{46}Ti. In Table XIV we consider 1 to 0 transitions and the selection rules are discussed in the Table XV. In Table XVI we show the $B(M1)$'s from 1 to 2 in ^{46}Ti and show the seletion rules in Table XVII. Note that for ^{46}Ti each column correponds to a different $I=1$ state.

TABLE XII. Selection rules for vanishing $B(M1)$'s: ^{44}Ti $I=1$ to $I=0$.

Selection rule	$I=1$	$I=0$
$\Delta v=4$	411	000
$\Delta v=4$	411	020

VII. DISCUSSION OF THE TABLES

We observe that $B(M1)$'s vanish in the following cases:

(a) in ^{44}Ti and only for $N=Z$ nuclei, from $T=1$ to $T=1$. For all nuclei here considered ^{43}Sc, ^{45}Sc, ^{44}Ti, ^{46}Ti
(b) $\Delta T=2$ or more,
(c) $\Delta v=4$ or 6,
(d) $\Delta v=2$ and $\Delta t\neq 0$.

The selection rule for case (a) is well known. It is discussed in several places including the book by Talmi [9]. It can be explained by the vanishing of the Clebsh-Gordan coefficient $(1,1,0,0|1,0)$.

Case (b), where the change of isospin is 2 or more units, is also easy to explain. These $B(M1)$'s obviously are zero because the $M1$ operator is of rank 1 in isospin. Some examples are $(411)^2\to(030)$ and $(611)^2\to(030)$.

In case (c) the change of seniority is more that 2 units, i.e., 4 or 6. The $B(M1)$'s for these cases are also obviously zero because the one-body $M1$ operator can only uncouple one $J=0$ pair. Some examples are $(411)^2\to(010)$, $(421)^2\to(010)$, $(611)^2\to(010)$, $(220)^2\to(611)$, and $(421)^2\to(030)$.

In case (d) we get vanishing $B(M1)$'s when seniority and the reduced isospin simultaneously change. The $M1$ operator can attack a $J=0$ pair and increase the isospin but that will not affect the particles not coupled to zero whose isospin is indeed the reduced isospin. Examples of this are in the Sc isotopes where $J=7/2$, $T=1/2$ transitions to $J=5/2$ or $9/2$ states with $T=3/2$ are forbidden with a pairing interaction. Also in the Ti isotopes $(611)^2\to(412),(410)^3,(422)$, and $(220)\to(412),(411)^2,(422),(421)^2$.

There is one ambiguity—the case $I=1^+$ to 2^+; there are two (421) states. One has a nonzero $B(M1)$ to (211) and the other does not. However, when there is a twofold degeneracy one can take arbitrary linear combinations of the two states and so get neither of the two $B(M1)$'s to be zero. We note that there are some very large $B(M1)$'s, e.g., 12.13 for the transition (410) to (220), $I=1$ to 2 in ^{46}Ti.

We mention briefly that Zamick [11] had previously considered $M1$ transitions from $J=0^+$ ground states in Ti isotopes to $J=1^+$ excited states in the context of scissor modes. These transitions are sometimes called spin scissors

TABLE XI. $B(M1)$ values in ^{44}Ti for $I=1$ to $I=2$ QQ interaction.

I	2_1	2_2	2_3	2_4	2_5	2_6	2_7	2_8	2_9
1_1	0.885 292	0	5.0018	0.030 127 3	0.053 353 9	0	0.078 158 1	3.36014	0
1_2	0.012 688 2	0	3.301 66	18.1444	8.086 02	0	0.033 980 1	0.347936	0
1_3	0.000 092 473 5	0	0.180 103	0.276 92	0.534 714	0	5.13135	8.26883	0

TABLE XIII. Selection rules for vanishing $B(M1)$'s: ^{44}Ti $I=1$ to $I=2$.

Selection rule	$I=1$	$I=0$
$T=1 \Rightarrow T=1$	210	211
$T=1 \Rightarrow T=1$	210	411
$\Delta v=2, \Delta t \neq 0$	210	422
$T=1 \Rightarrow T=1$	411	211
$T=1 \Rightarrow T=1$	411	211

excitations and have a fair amount of orbital content—not just spin. They bear some analogy to the scissors modes in deformed nuclei such as ^{156}Gd [12].

In the Appendix we give detailed expressions for $B(M1)$'s and B(GT). It should be noted that such a relation between them has been previously discussed by Zamick and Zheng [13], but not in such a complete way.

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APPENDIX

1. Formulas for B(GT)

The following formulas are

$$X_1 = \sum_{J_p J_n} D^f(J_p J_n) D^i(J_p J_n) U(1 J_p I_f J_n; J_p I_i) \times \sqrt{J_p(J_p+1)}, \tag{A1}$$

$$X_2 = \sum_{J_p J_n} D^f(J_p J_n) D^i(J_p J_n) U(1 J_n I_f J_p; J_n I) \times \sqrt{J_n(J_n+1)}, \tag{A2}$$

$$B(\mathrm{GT}) = 0.5 \frac{2I_f+1}{2I_i+1} f(j)^2 \left[\frac{\langle 1 T_i 1 M_{T_i} | T_f M_{T_f} \rangle}{\langle 1 T_i 0 M_{T_i} | T_f M_{T_i} \rangle} \right]^2 \times [X_1 - (-1)^{I_f - I_i} X_2]^2, \tag{A3}$$

TABLE XIV. $B(M1)$'s in ^{46}Ti: $I=1$ to $I=0$ pairing interaction.

State (v,T,t)	411	411	611	611	220	421	421
010	0	0	0	0	1.0799	0	0
410	2.8794	0.0491	0	0	2.4344	0.5611	0.4150
410	0.7573	5.7648	0	0	0.3947	0.1157	2.0588
611	1.0423	0.0987	2.3989	0.6317	0	3.1539	0.2640
611	0.0049	0.1721	0.0001	1.7267	0	0.0858	0.4450
030	0	0	0	0	9.7201	0	0

TABLE XV. Selection rules for vanishing $B(M1)$'s: ^{46}Ti $I=1$ to $I=0$.

Selection rule	$I=1$	$I=0$
$\Delta T=2,\ \ \Delta v=4$	411	030
$\Delta T=2,\ \ \Delta v=4$	611	030
$\Delta v=4$	411	010
$\Delta v=4$	421	010
$\Delta v=6$	611	010
$\Delta v=4$	220	611
$\Delta v=4$	421	030?
$\Delta v=2, \Delta t \neq 0$	611	410

TABLE XVI. $B(M1)$'s in ^{46}Ti: $I=1$ to $I=2$ pairing interaction.

State (v,T,t)	411	411	611	611	220	421	421
211	0.9874	0.3326	0	0	1.3712	0.0272	0.0019
211	0.4367	0.1472	0	0	0.1715	0	0.3238
412	0.0916	1.5360	0	0	0	0.0607	0.4819
411	0.0847	0.0914	0.4365	0.0065	0	0.0374	0.0261
411	0.0041	0.0186	1.5191	0.0152	0	0.0846	0.0668
410	0.0646	1.6850	0	0	12.1303	0.0832	0.5004
410	3.5617	0.1189	0	0	2.9785	0.6431	0.5838
410	0.4668	2.4445	0	0	5.3986	0.0273	0.9432
611	2.1377	0.2523	2.3618	0.0555	0	2.9801	0.4370
611	0.2654	0.0135	0.1597	0.8390	0	0.0329	0.2333
611	0.0616	0.1344	7.1099	1.4178	0	1.4482	0.5082
611	0.0375	0.0024	0.0873	0.0461	0	0.0123	0.0127
611	0.1215	1.3291	0.0001	5.7321	0	0.0315	4.0036
221	2.2323	0.7524	0	0	2.5716	0.0398	0.0883
422	0.2746	4.6069	0	0	0	0.1821	1.4454
421	0.1804	0.0338	0.6123	0.0630	0	0.3563	0.0188
421	0.0862	0.2962	5.2534	0.0019	0	1.2615	0.2597
231	0	0	0	0	2.0572	0.5125	4.5230

TABLE XVII. Selection rules for vanishing $B(M1)$'s: ^{46}Ti $I=1$ to $I=2$.

Selection rule	$I=1$	$I=2$
$\Delta T=2$	411	231
$\Delta T=2,\ \ \Delta v=4$	611	231
$\Delta v=4$	611	211
$\Delta v=4$	220	611
$\Delta v=2, \Delta t \neq 0$	611	412
$\Delta v=2, \Delta t \neq 0$	611	410
$\Delta v=2, \Delta t \neq 0$	611	422
$\Delta v=2, \Delta t \neq 0$	220	412
$\Delta v=2, \Delta t \neq 0$	220	411
$\Delta v=2, \Delta t \neq 0$	220	422
$\Delta v=2, \Delta t \neq 0$	220	421

where

$$f(j) = \begin{cases} \frac{1}{j} & \text{if } j = l + 1/2, \text{ e.g., } f_{7/2}, \\ \frac{-1}{j+1} & \text{if } j = l - 1/2, \text{ e.g., } f_{5/2}, \end{cases} \tag{A4}$$

$$ft = \frac{6177}{B(F) + 1.583B(\mathrm{GT})}. \tag{A5}$$

2. Formulas for $B(M1)$

The following is a formula for $B(M1)$:

$$B(M1) = \frac{3}{4\pi}\frac{2I_f + 1}{2I_i + 1}\left[g_{j_p}X_1 + (-1)^{I_f - I_i} g_{j_n} X_2\right]^2. \tag{A6}$$

Here

$$g_j = g_l \pm \left\{\frac{g_s - g_l}{2l + 1}\right\}, \tag{A7}$$

$$g_{s_p} = 5.586, \quad g_{l_p} = 1, \tag{A8}$$

$$g_{s_n} = -3.826, \quad g_{sl_n} = 0. \tag{A9}$$

For the case where T_f is not equal to T_i we find the following:

$$X_1 = (-1)^{I_f - I_i + 1} X_2, \tag{A10}$$

$$B(M1) = \frac{3}{4\pi}\frac{2I_f + 1}{2I_i + 1}(g_{jp} - g_{jn})^2 X_1^2, \tag{A11}$$

$$B(\mathrm{GT}) = 2\frac{2I_f + 1}{2I_i + 1} f(j)^2 \left[\frac{\langle 1T_i 1M_{T_i} | T_f M_{T_f}\rangle}{\langle 1T_i 0M_{T_i} | T_f M_{T_i}\rangle}\right]^2 (X_1)^2. \tag{A12}$$

With this simplification we see that $B(\mathrm{GT})$ is proportional to $B(M1)$. Using bare values we find $B(\mathrm{GT})/B(M1) = 0.1411$ for $j = 7/2$ in ^{44}Ti. The magnetic moment is

$$\frac{\mu}{I} = \frac{g_{j_p} + g_{j_n}}{2} + \frac{g_{j_p} - g_{j_n}}{2(I+1)} \times \left[\sum_{J_p J_n} |D(J_p J_n)|^2 [J_p(J_p + 1) - J_n(J_n + 1)]\right]. \tag{A13}$$

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